SOME WEAK TYPE ESTIMATES FOR MAXIMAL SINGULAR INTEGRALS

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Abstract. We consider some maximal singular integral operators having variable kernels on $\mathbb{R}^n$ with doubling measures and prove L^p and weak type estimates for them under certain conditions. Also, certain weighted weak type estimates are shown for maximal singular integrals with A_1 weights of Muckenhoupt for the Lebesgue measure.

1. Introduction

Let $T : L^2(\mathbb{R}^n, d\mu) \to L^2(\mathbb{R}^n, d\mu)$ be a linear operator, where μ is a regular Borel measure on $\mathbb{R}^n$ (see [10, p. 205]) such that there exists a positive constant C satisfying

$$\mu(B(x, r)) \le C\mu(B(x, r/2)) \quad \text{for all } x \in \mathbb{R}^n \text{ and } r > 0$$

(the doubling condition) and such that $\mu(\mathbb{R}^n) = \infty$ and $\mu(E) < \infty$ when E is a compact set, where $B(x, r)$ denotes a ball with radius r centered at x:

$$B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}.$$

It is known that $C_0^\infty(\mathbb{R}^n)$ (the set of infinitely differentiable functions on $\mathbb{R}^n$ with compact support) is dense in $L^p(\mathbb{R}^n, d\mu)$ for $1 \le p < \infty$ (see Section 5.5). Let $L_0^\infty(\mathbb{R}^n)$ be the set of bounded measurable functions f on $\mathbb{R}^n$ for which there exists a compact set E such that $f(x) = 0$ for a.e. $x \in \mathbb{R}^n \setminus E$ with respect to μ (μ-a.e. x); the smallest such compact set is defined to be $\mathrm{supp}(f)$. If $f \in L_0^\infty(\mathbb{R}^n)$, then $T(f) \in L^2(\mathbb{R}^n, d\mu)$ and we have values $T(f)(x)$ meaningful for μ-a.e. x. We assume that there exists a kernel $K(x, y)$ which is locally integrable in $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ with respect to the product measure $d\mu \otimes d\mu$, where $\Delta = \{(x, x) : x \in \mathbb{R}^n\}$, such that if $f \in L_0^\infty(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)\, d\mu(y) \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n \setminus \mathrm{supp}(f).$$

For the kernel K we assume that the limit

$$\lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} K(x, y)f(y)\, d\mu(y)$$

exists and equals $Tf(x)$ for μ-a.e. x when $f \in C_0^\infty(\mathbb{R}^n)$. Also, we consider the following conditions.

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(K.1)
$$\int_{\alpha<|x-y|<2\alpha} |K(x,y)|\, d\mu(x) \le C_1$$
for all $y \in \mathbb{R}^n$ and $\alpha > 0$.

(K.2)
$$\int_{\alpha<|x-y|<2\alpha} |K(x,y)|\, d\mu(y) \le C_2$$
for all $x \in \mathbb{R}^n$ and $\alpha > 0$.

(K.3)
$$\int_{|x-y_0|\ge 2|y-y_0|} |K(x,y) - K(x,y_0)|\, d\mu(x) \le C_3$$
for all y_0, $y \in \mathbb{R}^n$.

(K.4)
$$\int_{|y-x_0|\ge 2|x-x_0|} |K(x,y) - K(x_0,y)|\, d\mu(y) \le C_4$$
for all x_0, $x \in \mathbb{R}^n$.

The following result is known.

Theorem A. *Suppose that the kernel K satisfies the condition $(K.3)$. Then the operator T extends to a bounded linear operator from $L^p(\mathbb{R}^n, d\mu)$ to $L^p(\mathbb{R}^n, d\mu)$ for every $p \in (1,2]$ and from $L^1(\mathbb{R}^n, d\mu)$ to $L^{1,\infty}(\mathbb{R}^n, d\mu)$ (T is of weak type $(1,1)$), which means that*
$$\mu\left(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}\right) \le C\lambda^{-1}\|f\|_1 = C\lambda^{-1}\int_{\mathbb{R}^n} |f(x)|\, d\mu(x), \quad \forall\lambda > 0.$$

For Theorem A see Theorem (2.4) on Coifman-Weiss [6, pp. 74–75] and also Theorem 1.2 on [18, p. 30]. In Theorem (2.4) of [6], the kernel K of T is assumed to be in $L^2(\mathbb{R}^n \times \mathbb{R}^n, d\mu \otimes d\mu)$, but the proof given there can be applied to prove Theorem A. When μ is the Lebesgue measure, we can find in [5, Chap. IV] results related to operators T with standard kernels. See also [15, Chap. 4].

Let $T_*f(x) = \sup_{\alpha>0} |T_\alpha f(x)|$, where
$$T_\alpha f(x) = \int_{|x-y|>\alpha} K(x,y) f(y)\, d\mu(y).$$
Then in this note we shall prove the following theorem.

Theorem 1.1. *Suppose that the kernel K satisfies the conditions $(K.2)$, $(K.3)$ and $(K.4)$. Then T_* extends to a bounded operator on $L^p(\mathbb{R}^n, d\mu)$ for every $p \in (1,2)$ and extends to an operator of weak type $(1,1)$.*

Let
$$T^{(\beta)} f(x) = \text{p.v.} \int_{\mathbb{R}^n} K_\beta(x,y) f(y)\, d\mu_\beta(y),$$
where

(1.1) $$K_\beta(x,y) = k(x-y)(|x|^\beta - |y|^\beta), \quad k(x) = |x|^{-n}\Omega(x'), \quad x' = x/|x|,$$

and $d\mu_\beta(y) = |y|^{-\beta}\, dy$ with $0 \le \beta < n$ (dy denotes the Lebesgue measure). We assume that Ω is continuous on S^{n-1} and $\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0$, where $d\sigma$ denotes the Lebesgue surface measure on S^{n-1}. We further assume that Ω satisfies the Dini condition:

$$\int_0^1 \omega(t)\,\frac{dt}{t} < \infty,$$

where

$$\omega(t) = \sup\left\{|\Omega(\theta) - \Omega(\zeta)| : |\theta - \zeta| < t, \quad \theta, \zeta \in S^{n-1}\right\}.$$

We note that ω is non-decreasing and $\omega(t) \le 2\|\Omega\|_\infty$ for $t > 0$. As an application of Theorem 1.1, we can show the following.

Theorem 1.2. *Let $n \ge 2$. We consider the maximal operator $T_*^{(\beta)} f$. Then $T_*^{(\beta)}$ is bounded on $L^p(\mathbb{R}^n, d\mu_\beta)$ for $p \in (1, 2]$ and of weak type $(1, 1)$.*

The $L^2(\mathbb{R}^n, d\mu_\beta)$ boundedness of $T_*^{(\beta)}$ in Theorem 1.2 follows from Theorem 4.1 below in Section 4. When Ω satisfies a Lipschitz condition on S^{n-1}, see [6, p. 76] about a result for $T^{(\beta)}$ analogous to Theorem 1.2.

Also, we consider weighted weak type estimates for the maximal singular integrals. From now on, through this section, we assume that the measure $d\mu$ is the Lebesgue measure dx. Let $K(x, y)$ be locally integrable in $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$. When $0 \le \alpha < \beta \le \infty$, let

$$A(x; \alpha, \beta) = \{z : \alpha < |x - z| < \beta\};$$
$$\Delta(\alpha, \beta) = \{(y, z) \in \mathbb{R}^n \times \mathbb{R}^n : \alpha \le |y - z| \le \beta\}.$$

Let $1 \le r < \infty$, $0 < t \le 1$ and $R > 0$. We define

$$\omega_{r,R}(t) = \sup_{(y,z)\in\Delta(Rt/4, Rt/2)} \left(R^{-n} \int\limits_{A(z;R,2R)} |R^n (K(x, y) - K(x, z))|^r \, dx \right)^{1/r}.$$

We say that the kernel K satisfies the D_r-condition if

$$B_r = \sum_{k=0}^{\infty} \omega_r(2^{-k}) < \infty, \qquad \text{where} \quad \omega_r(t) = \sup_{R>0} \omega_{r,R}(t) = \sup_{R>0} \omega_{r,t^{-1}R}(t).$$

By the usual modifications we can also define the D_∞-condition. The D_r condition is equivalent to the (D_r) condition defined in [18] (see Section 5.1 below). We see that the D_s condition follows from the D_r condition if $s < r$. It is easily shown that the D_1 condition implies $(K.3)$.

In [16] weighted weak type estimates were proved for certain singular integrals T under Dini conditions. At present, for certain singular integrals, weighted weak type $(1, 1)$ estimates can be shown without Dini conditions (see [9]), while if we focus our attention on maximal singular integrals T_*, we see that even at present stage of research certain Dini conditions are still needed to prove weighted weak type $(1, 1)$ estimates analogous for T. We shall prove the following results on weighted weak type estimates for T_*.

Theorem 1.3. *Let $1 < r < \infty$. Suppose that T is bounded on L^p for some $p \in [r, \infty)$. Suppose that the kernel K of T satisfies $(K.2)$, $(K.4)$ and the D_r condition and that a weight w satisfies $w^{r'} \in A_1$, where $1/r + 1/r' = 1$. Then T_* is bounded from L^1_w to $L^{1,\infty}_w$, which means that there exists a constant $C > 0$ such that*

$$\sup_{\lambda>0} \lambda w\left(\{x \in \mathbb{R}^n : T_* f(x) > \lambda\}\right) \le C\|f\|_{L^1_w},$$

where $w(E) = \int_E w(x)\,dx$ *and* $\|f\|_{L^1_w} = \int_{\mathbb{R}^n} |f(x)|w(x)\,dx$.

Proposition 1.4. *Let* $w \in A_1$. *Suppose that* T *is bounded from* L^1_w *to* $L^{1,\infty}_w$ *and that the kernel* K *of* T *satisfies* $(K.2)$ *and* $(K.4)$ *conditions. Then,* T_* *uniquely extends to a positive sublinear operator on* $L^1_w \cap L^\infty$ *such that*

$$\sup_{\lambda > C_1\|f\|_\infty} \lambda w\left(\{x \in \mathbb{R}^n : T_*f(x) > \lambda\}\right) \le C_2\|f\|_{L^1_w}$$

for some constants $C_1, C_2 > 0$.

See [11] for the weight class A_1 of Muckenhoupt. As an application of Proposition 1.4 and a result of [9], we have the following result for maximal singular integrals with homogeneous convolution kernels.

Corollary 1.5. *Let* $n \ge 2$ *and define*

$$Tf(x) = \text{p.v.} \int f(x-y)\frac{\Omega(y')}{|y|^n}\,dy,$$

where Ω *is homogeneous of degree* 0 *and* $\Omega \in L^r(S^{n-1})$ *for some* $r > 1$ *and* $\int_{S^{n-1}} \Omega(\theta)\,d\sigma(\theta) = 0$. *Suppose that* Ω *satisfies the* L^1*-Dini condition on* S^{n-1} *and suppose that* $w^{r'} \in A_1$. *Then, there exist positive constants* C_1 *and* C_2 *such that for* $f \in L^1_w \cap L^\infty$ *we have*

$$\sup_{\lambda > C_1\|f\|_\infty} \lambda w\left(\{x \in \mathbb{R}^n : T_*f(x) > \lambda\}\right) \le C_2\|f\|_{L^1_w}.$$

For the L^r-Dini condition for Ω, see [16]. In [16] the L^1_w-$L^{1,\infty}_w$ boundedness of T is shown under the assumptions that $\Omega \in L^r$ and that Ω satisfies the L^r-Dini condition, when $w^{r'} \in A_1$. In [9], the same boundedness is proved under the condition that $\Omega \in L^r$ without the L^r-Dini condition (see [8, p. 267] for the case when $\Omega \in L^\infty$); the proof given in [9] is based on results in [23] and [26]. An analogous result is expected for T_*. We note that in Corollary 1.5 the L^r-Dini condition is relaxed to the L^1-Dini condition in comparison with the result of [16] for T but the range of λ for which the supremum is taken in the conclusion of the corollary is restricted to $\lambda > C_1\|f\|_\infty$. See [4, 13, 20, 21, 22, 25] for singular integrals with rough kernels; in [21, 22, 25] results on homogeneous groups can be found.

We see an application of Theorem 1.3 to singular integrals with convolution kernels. Let

$$(1.2)\quad Tf(x) = \text{p.v.} \int f(x-y)K(y)\,dy, \quad T_*f(x) = \sup_{\epsilon>0}\left|\int_{|y|>\epsilon} f(x-y)K(y)\ dy\right|,$$

for $f \in C_0^\infty(\mathbb{R}^n)$, where K satisfies the following.

$$(1.3)\qquad \sup_{t>0}\int_{A(0;t,2t)} |K(x)|\,dx < \infty;$$

$$(1.4)\qquad \sup_{y\in\mathbb{R}^n}\int_{A(0;2|y|,\infty)} |K(x-y) - K(x)|\,dx < \infty;$$

$$(1.5)\qquad \sup_{0<s<t<\infty}\left|\int_{A(0;s,t)} K(x)\,dx\right| < \infty;$$

$$(1.6)\qquad \text{the limit } \lim_{\epsilon\to 0}\int_{|y|>\epsilon} f(x-y)K(y)\,dy \text{ exists for a.e. } x \text{ when } f \in C_0^\infty(\mathbb{R}^n).$$

It is known that T and T_* extend to bounded operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and to operators of weak type $(1,1)$ on $\mathbb{R}^n$ (see [1] and [17] for T; for T_* see [17] and also [18, pp. 25–26], [2, p. 72]). We note that the D_r condition, which is stated above for variable kernels, can be formulated in the case of convolution kernels as follows.

$$B_r = \sum_{k=0}^{\infty} \omega_r(2^{-k}) < \infty, \qquad \text{where} \quad \omega_r(t) = \sup_{R>0} \omega_{r,R}(t)$$

and

$$\omega_{r,R}(t) = \sup_{w \in A(0;Rt/4,Rt/2)} \left(R^{-n} \int\limits_{A(0;R,2R)} \left| R^n \left(K(x-w) - K(x)\right)\right|^r \, dx \right)^{1/r}.$$

Theorem 1.3 immediately implies the following weighted weak type estimates for the maximal singular integrals T_*.

Corollary 1.6. *Let $r > 1$. Let w be a weight such that $w^{r'} \in A_1$. Suppose that the kernel K satisfies* (1.3), (1.5), (1.6) *and the D_r condition. Then we have*

$$\sup_{\lambda>0} \lambda w\left(\{x \in \mathbb{R}^n : T_* f(x) > \lambda\}\right) \le C\|f\|_{L^1_w},$$

where T_* is as in (1.2).

We note that the D_r condition in Corollary 1.6 implies (1.4). When K is a homogeneous kernel of the form $K = |x|^{-n}\Omega(x')$, see [3] for a relation between (1.4) (the Hörmander condition [14]) and the L^1 Dini condition for Ω.

We shall prove Theorem 1.1 in Section 2. The proofs of Theorem 1.3, Proposition 1.4 and Corollary 1.5 will be given in Section 3. In proving Theorems 1.1, 1.3 and Proposition 1.4, we shall apply methods of Rivière [17] and also methods of [5, Chap. IV] for standard kernels. In proving Corollary 1.5, we shall also use a result of [9]. To prove Theorem 1.3 we shall apply the D_r condition to estimate $T_*(b)$ along with Hölder's inequality, where b is the bad part arising from the Calderón-Zygmund decomposition $f = g + b$.

The proof of Theorem 1.2 will be provided in Section 4. To establish the theorem we need to prove the condition (K.3), which is in Lemma 4.2. We shall state the proof of the lemma in detail. Finally, in Section 5, we shall give proofs for some results which have been stated without proofs before.

2. Proof of Theorem 1.1

We need the following lemmas (Lemmas 2.1, 2.3 and 2.4).

Lemma 2.1. *Let $f \in L_0^\infty(\mathbb{R}^n)$ and $0 < \delta < 1$. Then*

$$|T_\alpha f(x)| \le N^{(1)}_{\mu,\alpha}(f)(x) + N^{(2)}_{\mu,\alpha}(f)(x) + C_\delta M_{\mu,\delta}(Tf)(x) + C_\delta M_\mu f(x) \quad \mu\text{-a.e.},$$

where $M_\mu f$ denotes the Hardy-Littlewood maximal function with respect to the measure μ:

$$M_\mu f(x) = \sup_{x\in B} \mu(B)^{-1} \int_B |f(y)|\, d\mu(y),$$

with the supremum being taken over all balls B containing x, and $M_{\mu,\delta}(f) = (M_u(|f|^\delta)^{1/\delta}$; also

$$N^{(1)}_{\mu,\alpha}(f)(x) = \sup_{z\in B(x,\alpha/3)} \left| \int_{|x-y|>\alpha} (K(x,y) - K(z,y))f(y)\, d\mu(y) \right|,$$

$$N^{(2)}_{\mu,\alpha}(f)(x) = \sup_{z\in B(x,\alpha/3)} \int_{2\alpha/3<|z-y|<2\alpha} |K(z,y)||f(y)|\, d\mu(y).$$

Proof. First we assume that $f \in C_0^\infty(\mathbb{R}^n)$. Let $\bar{B}(x,r) = \{y \in \mathbb{R}^n : |y - x| \le r\}$ be the closure of a ball $B(x,r)$. Let $\varphi_{x,\alpha} \in C_0^\infty(\mathbb{R}^n)$, $0 \le \varphi_{x,\alpha} \le 1$, $\varphi_{x,\alpha} = 1$ on $B(x,\alpha)$ and $\mathrm{supp}(\varphi_{x,\alpha}) \subset B(x, 3\alpha/2)$. For $z \in \mathbb{R}^n$, we have

$$\begin{aligned} |T_\alpha f(x)| \le \Bigg| &\int_{|x-y|>\alpha} K(x,y)f(y)\, d\mu(y) \\ &- \int_{|z-y|>\epsilon} K(z,y)(f(y) - f(y)\varphi_{x,\alpha}(y))\, d\mu(y) \Bigg| \\ &+ |T_\epsilon f(z)| + |T_\epsilon(f\varphi_{x,\alpha})(z)|. \end{aligned} \tag{2.1}$$

We note that

$$\begin{aligned} &\int_{|z-y|>\epsilon} K(z,y)(f(y) - f(y)\varphi_{x,\alpha}(y))\, d\mu(y) \\ &= \int_{\substack{|z-y|>\epsilon \\ |x-y|>\alpha}} K(z,y)f(y)\, d\mu(y) - \int_{|z-y|>\epsilon} K(z,y)f(y)(\varphi_{x,\alpha}(y) - \chi_{B(x,\alpha)}(y))\, d\mu(y). \end{aligned} \tag{2.2}$$

If $|z - x| < \alpha/3$ and $|x - y| > \alpha$, then $|z - y| > 2\alpha/3$. So, if $|z - x| < \alpha/3$ and $\epsilon < 2\alpha/3$, we have

$$\int_{\substack{|z-y|>\epsilon \\ |x-y|>\alpha}} K(z,y)f(y)\, d\mu(y) = \int_{|x-y|>\alpha} K(z,y)f(y)\, d\mu(y). \tag{2.3}$$

Also, we observe that if $|z - x| < \alpha/3$,

$$\begin{aligned} &\left| \int_{|z-y|>\epsilon} K(z,y)(\varphi_{x,\alpha}(y) - \chi_{B(x,\alpha)}(y))f(y)\, d\mu(y) \right| \\ &\le \int_{\substack{|z-y|>\epsilon \\ \alpha<|x-y|<3\alpha/2}} |K(z,y)||f(y)|\, d\mu(y) \le \int_{2\alpha/3<|z-y|<2\alpha} |K(z,y)||f(y)|\, d\mu(y). \end{aligned} \tag{2.4}$$

Combining (2.1), (2.2), (2.3) and (2.4) and letting $\epsilon \to 0$, we have, if $|z - x| < \alpha/3$,

$$\begin{aligned} |T_\alpha f(x)| \le &\left| \int_{|x-y|>\alpha} (K(x,y) - K(z,y))f(y)\, d\mu(y) \right| \\ &+ \int_{2\alpha/3<|z-y|<2\alpha} |K(z,y)||f(y)|\, d\mu(y) + |Tf(z)| + |T(f\varphi_{x,\alpha})(z)|. \end{aligned} \tag{2.5}$$

To prove (2.5) for $f \in L_0^\infty(\mathbb{R}^n)$, we take a sequence $\{f_k\}_{k=1}^\infty$ in $C_0^\infty(\mathbb{R}^n)$ such that $f_k \to f$ in $L^2(d\mu)$ and μ-a.e. and such that $\{f_k\}$ is uniformly bounded: $|f_k| \le M$ and $\mathrm{supp}(f_k) \subset E$ for a compact set E independent of k (for a sequence which satisfies the $L^2(d\mu)$ convergence, see Section 5.5 and then it is easily seen that we can choose $\{f_k\}$ which also complies with the other requirements). Next, we apply the inequality (2.5) to each f_k. Then by a limiting arguments in letting $k \to \infty$, we get (2.5) for f.

Therefore, for $f \in L_0^\infty(\mathbb{R}^n)$ we see that

$$(2.6)\quad |T_\alpha f(x)| \le N^{(1)}_{\mu,\alpha}(f)(x) + N^{(2)}_{\mu,\alpha}(f)(x) + \inf_{z\in B(x,\alpha/3)} \left(|Tf(z)| + |T(f\varphi_{x,\alpha})(z)|\right).$$

We estimate the last term as follows. Let $0 < \delta < 1$. Then

$$(2.7)\quad \begin{aligned} &\inf_{z\in B(x,\alpha/3)} \left(|Tf(z)| + |T(f\varphi_{x,\alpha})(z)|\right) \\ &\le \left(\inf_{z\in B(x,\alpha/3)} \left(|Tf(z)|^\delta + |T(f\varphi_{x,\alpha})(z)|^\delta\right)\right)^{1/\delta} \\ &\le \left(⨍_{B(x,\alpha/3)} |Tf(z)|^\delta \, d\mu(z) + ⨍_{B(x,\alpha/3)} |T(f\varphi_{x,\alpha})(z)|^\delta \, d\mu(z)\right)^{1/\delta} \\ &\le C_\delta \left(⨍_{B(x,\alpha/3)} |Tf(z)|^\delta \, d\mu(z)\right)^{1/\delta} \\ &\qquad + C_\delta \left(⨍_{B(x,\alpha/3)} |T(f\varphi_{x,\alpha})(z)|^\delta \, d\mu(z)\right)^{1/\delta}, \end{aligned}$$

where $⨍_E g\, d\mu = \mu(E)^{-1} \int_E g\, d\mu$. To estimate the last integral, we apply the following well-known result (see Section 5.3 for the proof).

Lemma 2.2. *Let (E,ν) be a measure space with $\nu(E) < \infty$. Let $0 < \delta < 1$. For a non-negative measurable function F on E, suppose that*

$$\nu\{x \in E : F(x) > \lambda\} \le \frac{1}{\lambda} A \quad \text{for all } \lambda > 0.$$

Then

$$\int_E F(x)^\delta \, d\nu(x) \le \frac{1}{1-\delta} A^\delta \nu(E)^{1-\delta}.$$

Since T is of weak type $(1,1)$ by Theorem A, using Lemma 2.2 we see that

$$(2.8)\quad \begin{aligned} &⨍_{B(x,\alpha/3)} |T(f\varphi_{x,\alpha})(z)|^\delta \, d\mu(z) \\ &\qquad \le C_\delta \left(⨍_{B(x,3\alpha/2)} |f(z)| \, d\mu(z)\right)^\delta \le C_\delta (M_\mu f(x))^\delta. \end{aligned}$$

By (2.6), (2.7) and (2.8), we have the conclusion of Lemma 2.1. $\square$

Lemma 2.3. *Let $\{Q_m\}_{m=1}^\infty$ be a family of non-overlapping dyadic cubes. Let B_m be the smallest ball such that $Q_m \subset \bar{B}_m$. Let $\{h_m\}$ be a sequence of functions in $L_0^\infty(\mathbb{R}^n)$ such that*

(1) $\mathrm{supp}(h_m) \subset Q_m$;
(2) $\int h_m(x)\, d\mu(x) = 0$;
(3) $\|h_m\|_1 \le C\mu(B_m)$.

Let $B_m^* = B(x_m, 8r_m)$, *where* $B_m = B(x_m, r_m)$. *Let* $E = \cup B_m^*$. *Suppose that the kernel* K *of* T *satisfies the* $(K.2)$ *and* $(K.3)$ *conditions. Let* $h = \sum h_m$. *Then there exists a constant* $C_0 > 0$ *such that*

$$\mu\left(\{x \in E^c : T_*(h)(x) > C_0\}\right) \le C \sum_{m=1}^{\infty} \mu(B_m),$$

where $E^c = \mathbb{R}^n \setminus E$.

Proof. We consider the integral

$$T_\alpha(h_m)(x) = \int_{|x-y|>\alpha} K(x,y)h_m(y)\,d\mu(y) \quad \text{for } x \notin E.$$

Fix $x \in E^c$ and $\alpha > 0$. We divide the set $\mathbb{N}_0(x,\alpha)$ of positive integers m for which $T_\alpha(h_m)(x) \neq 0$ into three pieces $\mathbb{N}_1(x,\alpha)$, $\mathbb{N}_2(x,\alpha)$, $\mathbb{N}_3(x,\alpha)$ as follows.

$$\mathbb{N}_1(x,\alpha) = \{m \in \mathbb{N}_0(x,\alpha) : \alpha \le r_m\},$$
$$\mathbb{N}_2(x,\alpha) = \{m \in \mathbb{N}_0(x,\alpha) : r_m < \alpha, x \notin B(x_m, 2\alpha)\},$$
$$\mathbb{N}_3(x,\alpha) = \{m \in \mathbb{N}_0(x,\alpha) : r_m \le \alpha/4, x \in B(x_m, 2\alpha)\}.$$

We observe that the case $\alpha/4 < r_m < \alpha$ and $x \in B(x_m, 2\alpha)$ is excluded, since if $\alpha/4 < r_m < \alpha$, then $B(x_m, 2\alpha) \subset B(x_m, 8r_m)$, and so $x \notin B(x_m, 2\alpha)$.

Let $m \in \mathbb{N}_1(x,\alpha)$. If $y \in \bar{B}(x_m, r_m)$, we have $|x-y| > \alpha$, since

$$|x-y| \ge |x-x_m| - |x_m - y| \ge 8r_m - r_m = 7r_m > \alpha.$$

Therefore

$$\int_{|x-y|>\alpha} K(x,y)h_m(y)\,d\mu(y) = \int K(x,y)h_m(y)\,d\mu(y) \tag{2.9}$$
$$= \int (K(x,y) - K(x,x_m))h_m(y)\,d\mu(y).$$

Let $m \in \mathbb{N}_2(x,\alpha)$. Then we have $|x-y| > \alpha$ for $y \in \bar{B}(x_m, r_m)$, since

$$|x-y| \ge |x-x_m| - |x_m-y| \ge 2\alpha - r_m > 2\alpha - \alpha = \alpha.$$

Thus we also have (2.9) in this case.

Let $m \in \mathbb{N}_3(x,\alpha)$. Then for $y \in \bar{B}(x_m, r_m)$ we have $|x-y| < (4/9)\alpha$, since

$$|x-y| \le |x-x_m| - |x_m-y| < 2\alpha + r_m \le 2\alpha + \alpha/4 = 9\alpha/4.$$

Therefore

$$\int_{|x-y|>\alpha} K(x,y)h_m(y)\,d\mu(y) = \int_{\alpha<|x-y|<9\alpha/4} K(x,y)h_m(y)\,d\mu(y). \tag{2.10}$$

For $x \in E^c$ and $\alpha > 0$, we decompose

$$T_\alpha(h)(x) = \sum_{m\in\mathbb{N}_1(x,\alpha)\cup\mathbb{N}_2(x,\alpha)} T_\alpha(h_m)(x) + \sum_{m\in\mathbb{N}_3(x,\alpha)} T_\alpha(h_m)(x). \tag{2.11}$$

We first estimate $\sum_{m\in\mathbb{N}_3(x,\alpha)} T_\alpha(h_m)(x)$. We observe that

$$Q_m \subset A(x;\alpha/2, 9\alpha/4) \quad \text{for } m \in \mathbb{N}_3(x,\alpha). \tag{2.12}$$

We have already seen that $Q_m \subset B(x, 9\alpha/4)$. Since $T_\alpha(h_m)(x) \neq 0$, there is $y_0 \in Q_m$ such that $|x-y_0| > \alpha$. Therefore, if $y \in Q_m$, then

$$|x-y| \ge |x-y_0| - |y_0-y| > \alpha - 2r_m \ge \alpha - \alpha/2 = \alpha/2.$$

This completes the proof of (2.12).

Let

$$m_{x,\alpha}(h_m) = \mu(Q_m)^{-1} \int_{A(x;\alpha,9\alpha/4)} h_m(y)\, d\mu(y).$$

Then

$$|m_{x,\alpha}(h_m)| \le \mu(Q_m)^{-1}\|h_m\|_1 \le C\mu(Q_m)^{-1}\mu(B_m) \le C.$$

We write

$$\begin{aligned}
&\int_{\alpha<|x-y|<9\alpha/4} K(x,y)h_m(y)\, d\mu(y) \\
&= \int\limits_{Q_m} K(x,y)\left(\chi_{A(x;\alpha,9\alpha/4)}(y)h_m(y) - m_{x,\alpha}(h_m)\right) d\mu(y) + m_{x,\alpha}(h_m)\int\limits_{Q_m} K(x,y)\, d\mu(y) \\
&= \int\limits_{Q_m} (K(x,y) - K(x,x_m))\left(\chi_{A(x;\alpha,9\alpha/4)}(y)h_m(y) - m_{x,\alpha}(h_m)\right) d\mu(y) \\
&\quad + m_{x,\alpha}(h_m)\int\limits_{Q_m} K(x,y)\, d\mu(y)
\end{aligned}$$

Then we see that

$$\begin{aligned}
&\left|\int_{\alpha<|x-y|<9\alpha/4} K(x,y)h_m(y)\, d\mu(y)\right| \\
&\le \int\limits_{Q_m} |K(x,y) - K(x,x_m)|\,(|h_m(y)| + C)\, d\mu(y) + C\int\limits_{Q_m} |K(x,y)|\, d\mu(y).
\end{aligned}$$

Applying (2.12), we see that

$$\begin{aligned}
(2.13)\qquad &\sum_{m\in\mathbb{N}_3(x,\alpha)}\left|\int_{\alpha<|x-y|<9\alpha/4} K(x,y)h_m(y)\, d\mu(y)\right| \\
&\le \sum_{m\in\mathbb{N}_3(x,\alpha)}\int\limits_{Q_m} |K(x,y) - K(x,x_m)|\,(|h_m(y)| + C)\, d\mu(y) \\
&\qquad + C\int\limits_{A(x;\alpha/2,9\alpha/4)} |K(x,y)|\, d\mu(y) \\
&\le \sum_{m\in\mathbb{N}_3(x,\alpha)}\int\limits_{Q_m} |K(x,y) - K(x,x_m)|\,(|h_m(y)| + C)\, d\mu(y) + B,
\end{aligned}$$

where the last inequality follows from (K.2).

Let $x \in E^c$. Then, using (2.10) and (2.13), we have

$$\begin{aligned}
(2.14)\qquad &\sup_{\alpha>0}\sum_{m\in\mathbb{N}_3(x,\alpha)} |T_\alpha(h_m)(x)| \\
&\le \sum_{m=1}^{\infty}\int_{Q_m} |K(x,y) - K(x,x_m)|\,(|h_m(y)| + C)\, d\mu(y) + B.
\end{aligned}$$

By (2.14) we see that

(2.15)
$$
\begin{aligned}
&\mu\left(\left\{x \in E^c : \sup_{\alpha>0} \sum_{m\in\mathbb{N}_3(x,\alpha)} |T_\alpha(h_m)(x)| > 1 + B\right\}\right)\\
&\le \mu\left(\left\{x \in E^c : \sum_{m=1}^{\infty} \int_{Q_m} |K(x,y) - K(x,x_m)| \left(|h_m(y)| + C\right) d\mu(y) > 1\right\}\right)\\
&\le \sum_{m=1}^{\infty} \int_{Q_m} \int_{B(x_m,8r_m)^c} |K(x,y) - K(x,x_m)|\, d\mu(x) \left(|h_m(y)| + C\right) d\mu(y)\\
&\le C \sum_{m=1}^{\infty} \int_{Q_m} \left(|h_m(y)| + C\right) d\mu(y)\\
&\le C \sum_{m=1}^{\infty} \left(\|h_m\|_1 + \mu(Q_m)\right) \le C \sum_{m=1}^{\infty} \mu(B_m),
\end{aligned}
$$

where the third inequality follows from (K.3).

Next we estimate $\sum_{m\in\mathbb{N}_1(x,\alpha)\cup\mathbb{N}_2(x,\alpha)} T_\alpha(h_m)(x)$. Let $x \in E^c$. If $m \in \mathbb{N}_1(x,\alpha) \cup \mathbb{N}_2(x,\alpha)$, we have (2.9). It follows that

$$
\sup_{\alpha>0} \sum_{m\in\mathbb{N}_1(x,\alpha)\cup\mathbb{N}_2(x,\alpha)} |T_\alpha(h_m)(x)| \le \sum_{m=1}^{\infty} \int_{Q_m} |K(x,y) - K(x,x_m)||h_m(y)|\, d\mu(y).
$$

Therefore, arguing as in the proof of (2.15), we have

$$
\mu\left(\left\{x \in E^c : \sup_{\alpha>0} \sum_{m\in\mathbb{N}_3(x,\alpha)} |T_\alpha(h_m)(x)| > 1\right\}\right) \le C \sum_{m=1}^{\infty} \mu(B_m). \tag{2.16}
$$

Combining (2.15) and (2.16) and recalling (2.11), we arrive at the estimate

$$
\mu\left(\{x \in E^c : T_*(h)(x) > 2 + B\}\right) \le C \sum_{m=1}^{\infty} \mu(B_m).
$$

This completes the proof of Lemma 2.3. □

Lemma 2.4. *Let $1 \le p < \infty$, $f \in L^\infty_0(\mathbb{R}^n)$. Let v be a weight function. Then $f = g + b$, where g and b have the following properties.*

1. $|g(x)| \le 1 \quad \mu$*-a.e.*;
2. $\|g\|_{L^p(v\, d\mu)} \le C\|f\|_{L^p(M(v)\, d\mu)}$;
3. $b = \sum_{m=1}^{\infty} b_m$;
4. there exists a family $\{Q_m\}_{m=1}^{\infty}$ of non-overlapping dyadic cubes such that
$$\mathrm{supp}(b_m) \subset Q_m;$$
5. $\int b_m(x)\, d\mu(x) = 0$;
6. $\|b_m\|_1 \le C\mu(Q_m)$;
7. $\sum_{m=1}^{\infty} \int_{Q_m} v(x)\, d\mu(x) \le C\|f\|^p_{L^p(M_\mu(v)\, d\mu)}$.

This lemma is stated in a more general form as weighted estimates than needed in the proof of Theorem 1.1; the weighted version will be applied in proving Theorem 1.3.

Proof of Lemma 2.4. Decompose $f = f_1 + f_2$, where

$$f_1(x) = \begin{cases} f(x) & \text{if } |f(x)| > 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

We apply the ordinary Calderón-Zygmund decomposition at height $1/2$ with measure μ to f_1 to get the following.

(i) $f_1 = k + b$;
(ii) $|k(x)| \le 1/2 \quad \mu$-a.e.;
(iii) $\|k\|_{L^1(v\,d\mu)} \le C\|f_1\|_{L^1(M_\mu(v)\,d\mu)}$;
(iv) $b = \sum_{m=1}^\infty$, where b_m satisfies the properties (4), (5), (6) of Lemma 2.4 with a family $\{Q_m\}_{m=1}^\infty$ of non-overlapping dyadic cubes;
(v) $\sum_{m=1}^\infty \int_{Q_m} v(x)\,d\mu(x) \le C\|f_1\|_{L^1(M_\mu(v)\,d\mu)}$.

Proof is similar to the case where μ is the Lebesgue measure (see [11, pp. 141–144] and [6, Chap. III, §2]; see also Section 5.6).

Let $f = g + b$, where $g = k + f_2$ and b, k are as above. Then by (ii) we have $|g| \le |k| + |f_2| \le 1$, which is part (1). Also by (ii) and (iii) we see that

$$\begin{aligned} \|k\|^p_{L^p(v\,d\mu)} &\le (1/2)^{p-1}\|k\|_{L^1(v\,d\mu)} \\ &\le C\|f_1\|_{L^1(M_\mu(v\,d\mu))} \le C\|f_1\|^p_{L^p(M_\mu(v)\,d\mu)} \le C\|f\|^p_{L^p(M_\mu(v)\,d\mu)}. \end{aligned}$$

Since, clearly, $\|f_2\|_{L^p(v\,d\mu)} \le \|f\|_{L^p(v\,d\mu)} \le \|f\|_{L^p(M_\mu(v)\,d\mu)}$, we see that

$$\|g\|_{L^p(v\,d\mu)} \le \|k\|_{L^p(v\,d\mu)} + \|f_2\|_{L^p(v\,d\mu)} \le C\|f\|_{L^p(M_\mu(v)\,d\mu)},$$

which proves part (2). Applying (v), we have

$$\begin{aligned} \sum_{m=1}^\infty \int_{Q_m} v(x)\,d\mu(x) &\le C\|f_1\|_{L^1(M_\mu(v)\,d\mu)} \\ &\le C2^{p-1}\|f_1\|^p_{L^p(M_\mu(v)\,d\mu)} \le C2^{p-1}\|f\|^p_{L^p(M_\mu(v)\,d\mu)}, \end{aligned}$$

which proves part (7). □

Now we can complete the proof of Theorem 1.1. For $f \in L_0^\infty(\mathbb{R}^n)$, by Lemma 2.1 we have

$$|T_*f(x)| \le N_\mu^{(1)}(f)(x) + N_\mu^{(2)}(f)(x) + C_\delta M_{\mu,\delta}(Tf)(x) + C_\delta M_\mu f(x), \tag{2.17}$$

where $N_\mu^{(i)}(f)(x) = \sup_{\alpha>0} N_{\mu,\alpha}^{(i)} f(x)$ for $i = 1, 2$. From (K.4) it follows that

$$\|N_\mu^{(1)}(f)\|_\infty \le C_1\|f\|_\infty. \tag{2.18}$$

Also, (K.2) implies that

$$\|N_\mu^{(2)}(f)\|_\infty \le C_2\|f\|_\infty. \tag{2.19}$$

To estimate $M_{\mu,\delta}(Tf)$ we need the following result (see Section 5.4 for the proof).

Lemma 2.5. *Suppose that a weight w satisfies that*

$$\int_{\{x\in\mathbb{R}^n : M_\mu(f)(x)>\lambda\}} w(x)\,d\mu(x) \le C\lambda^{-1} \int |f(x)|w(x)\,d\mu(x)$$

for all $\lambda > 0$. Then we see that M_μ is of Riesz weak type (see [12, p. 111]):

$$\int_{\{x\in\mathbb{R}^n : M_\mu(f)(x)>\lambda\}} w(x)\, d\mu(x) \le C\lambda^{-1} \int_{\{x\in\mathbb{R}^n : M_\mu(f)(x)>\lambda\}} |f(x)|w(x)\, d\mu(x), \quad \forall\lambda > 0.$$

This lemma is stated more generally with a weight w than needed in the proof of Theorem 1.1; the weighted version will be used in Section 3.

Using Lemma 2.2 with the estimates in Theorem A

$$\mu\left(\{|Tf| > \lambda\}\right) \le C\lambda^{-1}\|f\|_1, \quad \forall\lambda > 0$$

and Lemma 2.5 with $w = 1$, we have

$$\begin{aligned}\mu\left(\{M_{\mu,\delta}(Tf) > \lambda\}\right) &\le C\lambda^{-\delta} \int_{M_{\mu,\delta}(Tf)>\lambda} |Tf|^\delta\, d\mu(x)\\ &\le C_\delta\lambda^{-\delta}\mu\left(\{M_{\mu,\delta}(Tf) > \lambda\}\right)^{1-\delta} \|f\|_1^\delta,\end{aligned}$$

which implies that

$$\mu\left(\{M_{\mu,\delta}(Tf) > \lambda\}\right) \le C\lambda^{-1}\|f\|_1, \quad \forall\lambda > 0. \tag{2.20}$$

We note that Lemma 2.5 can be applied with $w = 1$, since M_μ is of weak type $(1,1)$.

Let $f \in L^\infty_0$ and $f = g + b$, $b = \sum b_m$, and cubes $\{Q_m\}$ be as in Lemma 2.4 with $p = 1$ and $v = 1$. Let B_m be the ball with the same center and diameter as Q_m. Then by (2.17), (2.18), (2.19) and (2.20) we see that

$$\begin{aligned}\mu\left(\{T_*(g) > C_1 + C_2 + 2\}\right) &\le \mu\left(\{C_\delta M_{\mu,\delta}(Tg) > 1\}\right) + \mu\left(\{C_\delta M_\mu(g) > 1\}\right)\\ &\le C\|g\|_1 \le C\|f\|_1.\end{aligned} \tag{2.21}$$

Let $B_m = B(x_m, r_m)$ and $E = \cup_{m=1}^\infty B(x_m, 8r_m)$. Then by applying Lemma 2.3, we have

$$\begin{aligned}\mu\left(\{T_*(b) > C_0\}\right) &\le \mu(E) + \mu\left(\{x \in E^c : T_*(b)(x) > C_0\}\right)\\ &\le C\sum_{m=1}^\infty \mu(B_m) \le C\|f\|_1,\end{aligned} \tag{2.22}$$

where the last inequality follows from part (7) of Lemma 2.4 with $v = 1$. Combining (2.21) and (2.22), we see that

$$\mu\left(\{T_*(f) > C_0 + C_1 + C_2 + 2\}\right) \le C\|f\|_1. \tag{2.23}$$

Next, let us apply Lemma 2.4 with $p = 2$, $v = 1$ and decompose $f = g + b$. Then arguing as in (2.21), by Chebyshev's inequality, the L^2 boundedness of T, the L^p boundedness of M_μ, $1 < p < \infty$, and part (2) of Lemma 2.4 with $v = 1$, we have

$$\begin{aligned}&\mu\left(\{T_*(g) > C_1 + C_2 + 2\}\right)\\ &\quad\le \mu\left(\{C_\delta M_{\mu,\delta}(Tg) > 1\}\right) + \mu\left(\{C_\delta M_\mu(g) > 1\}\right) \le C\|g\|_2^2 \le C\|f\|_2^2.\end{aligned} \tag{2.24}$$

Let E be as in (2.22). Then by Lemma 2.3 and part (7) of Lemma 2.4 with $v = 1$ and $p = 2$, we see that

$$\begin{aligned}\mu\left(\{T_*(b) > C_0\}\right) &\le \mu(E) + \mu\left(\{x \in E^c : T_*(b)(x) > C_0\}\right)\\ &\le C\sum_{m=1}^\infty \mu(B_m) \le C\|f\|_2^2.\end{aligned} \tag{2.25}$$

Using (2.24) and (2.25), we have

$$\mu\left(\{T_*(f) > C_0 + C_1 + C_2 + 2\}\right) \le C\|f\|_2^2. \tag{2.26}$$

From (2.23) and (2.26) we can deduce that T_* extends to a sublinear operator of weak type $(1,1)$ and of weak type $(2,2)$. Interpolating these two estimates, we see that T_* is bounded on L^r, $1 < r < 2$. This completes the proof of Theorem 1.1.

Remark 2.6. Let $2 < s < \infty$. If we further assume in Theorem 1.1 that T is bounded on $L^s(\mathbb{R}^n, d\mu)$, then we can prove the L^r boundedness of T_* for $1 < r < s$, since then we can apply Lemma 2.4 with $p = s$ in the proof given above for Theorem 1.1, where Lemma 2.4 has been applied with $p = 2$, to get the weak type (s,s) boundedness of T_*.

3. Proofs of Theorem 1.3, Proposition 1.4 and Corollary 1.5

In this section we assume that the measure μ is the Lebesgue measure. For $t > 0$, let $M_t(f) = (M(|f|^t))^{1/t}$, where M denotes the Hardy-Littlewood maximal operator with respect to the Lebesgue measure.

Lemma 3.1. *Let $1 < r < \infty$. Suppose that K satisfies the D_r condition. Let $u > 0$. Then*

$$\sup_{y\in B(y_0,u)} \int_{|x-y_0|>2u} |K(x,y) - K(x,y_0)||g(x)|\,dx \le C \inf_{z\in B(y_0,u)} M_{r'}(g)(z).$$

Proof. Let $y \in B(y_0,u)$, $y \ne y_0$. Then, using Hölder's inequality, we have

$$\begin{aligned}
&\int_{|x-y_0|>2u} |K(x,y) - K(x,y_0)||g(x)|\,dx \\
&= \sum_{k=1}^{\infty} \int_{A(y_0;2^k u,2^{k+1}u)} |K(x,y) - K(x,y_0)||g(x)|\,dx \\
&\le \sum_{k=1}^{\infty} \left(\int\limits_{A(y_0;2^k u,2^{k+1}u)} |K(x,y) - K(x,y_0)|^r\,dx \right)^{1/r} \left(\int\limits_{A(y_0;2^k u,2^{k+1}u)} |g(x)|^{r'}\,dx \right)^{1/r'} \\
&\le C \sum_{k=1}^{\infty} \omega_r\left(22^{-k}u^{-1}|y-y_0|\right) \left((2^k u)^{-n} \int_{B(y_0,2^{k+1}u)} |g(x)|^{r'}\,dx \right)^{1/r'}.
\end{aligned}$$

To estimate $\omega_r\left(22^{-k}u^{-1}|y-y_0|\right)$, we apply the following result (see Section 5.2 for the proof).

Lemma 3.2. *Let $0 < t \le s \le 2t \le 1$. Then*

$$\omega_r(s) \le C(\omega_r(t) + \omega_r(2t)).$$

Let $2^{-m-1} \le u^{-1}|y-y_0| < 2^{-m}$, $m \ge 0$, $m \in \mathbb{Z}$ (the set of integers). Then by Lemma 3.2,

$$\omega_r\left(22^{-k}u^{-1}|y-y_0|\right) \le C(\omega_r(2^{-k-m}) + \omega_r(2^{-k-m+1})),$$

which implies

$$\sum_{k=1}^{\infty} \omega_r\left(22^{-k}u^{-1}|y-y_0|\right) \le C \sum_{k=0}^{\infty} \omega_r(2^{-k}).$$

Thus we have

$$\sum_{k=1}^{\infty} \omega_r \left(2^{-k+1}u^{-1}|y-y_0|\right) \left((2^k u)^{-n} \int_{B(y_0,2^{k+1}u)} |g(x)|^{r'}\,dx\right)^{1/r'}$$
$$\le C \inf_{z\in B(y_0,u)} M_{r'}g(z) \left(\sum_{k=1}^{\infty} \omega_r \left(2^{-k+1}u^{-1}|y-y_0|\right)\right) \le C \inf_{z\in B(y_0,u)} M_{r'}g(z).$$

This completes the proof of Lemma 3.1. □

Lemma 3.3. *Let families $\{Q_m\}_{m=1}^{\infty}$ of non-overlapping dyadic cubes and $\{B_m\}_{m=1}^{\infty}$ of balls be as in Lemma 2.3. Let $\{h_m\}$ be a sequence of functions in $L_0^{\infty}(\mathbb{R}^n)$ related to Q_m and B_m as in Lemma 2.3 with the Lebesgue measure in place of the measure μ; so h_m satisfies that*

(1) $\operatorname{supp}(h_m) \subset Q_m$;
(2) $\int h_m(x)\,dx = 0$;
(3) $\|h_m\|_1 \le C|B_m|$.

Also, let $B_m^* = B(x_m, 8r_m)$, $B_m = B(x_m, r_m)$ and $E = \cup B_m^*$, as in Lemma 2.3. Suppose that the kernel K of T satisfies the (K.2) and the D_r condition for some $r > 1$. Let v be a weight function and $h = \sum h_m$. Then there exists a constant $C_0 > 0$ such that

$$v\left(\{x \in E^c : T_*(h)(x) > C_0\}\right) \le C \sum_{m=1}^{\infty} \inf_{z\in B_m} M_{r'}(v)(z)|B_m|.$$

Proof. We take care of the integral

$$T_\alpha(h_m)(x) = \int_{|x-y|>\alpha} K(x,y)h_m(y)\,dy \quad \text{for } x \notin E.$$

Fixing $x \in E^c$ and $\alpha > 0$, we consider sets of positive integers $\mathbb{N}_1(x,\alpha)$, $\mathbb{N}_2(x,\alpha)$ and $\mathbb{N}_3(x,\alpha)$ as in the proof of Lemma 2.3 with the Lebesgue measure in place of μ. Then we have

$$T_\alpha(h)(x) = \sum_{m\in\mathbb{N}_1(x,\alpha)\cup\mathbb{N}_2(x,\alpha)} T_\alpha(h_m)(x) + \sum_{m\in\mathbb{N}_3(x,\alpha)} T_\alpha(h_m)(x). \tag{3.1}$$

As in the proof of Lemma 2.3, using (K.2), we have

$$\sup_{\alpha>0} \sum_{m\in\mathbb{N}_3(x,\alpha)} |T_\alpha(h_m)(x)| \le \sum_{m=1}^{\infty} \int_{Q_m} |K(x,y)-K(x,x_m)|\left(|h_m(y)|+C\right)\,dy + B. \tag{3.2}$$

By (3.2) we see that

$$
\begin{aligned}
(3.3)\quad & v\left(\left\{x\in E^c : \sup_{\alpha>0}\sum_{m\in\mathbb{N}_3(x,\alpha)}|T_\alpha(h_m)(x)|>1+B\right\}\right)\\
&\le v\left(\left\{x\in E^c : \sum_{m=1}^{\infty}\int_{Q_m}|K(x,y)-K(x,x_m)|\left(|h_m(y)|+C\right)\,dy>1\right\}\right)\\
&\le \sum_{m=1}^{\infty}\int_{Q_m}\int_{B(x_m,8r_m)^c}|K(x,y)-K(x,x_m)|\,v(x)\,dx\left(|h_m(y)|+C\right)\,dy\\
&\le C\sum_{m=1}^{\infty}\inf_{z\in B(x_m,r_m)}M_{r'}(v)(z)\int_{Q_m}\left(|h_m(y)|+C\right)\,dy\\
&\le C\sum_{m=1}^{\infty}\inf_{z\in B(x_m,r_m)}M_{r'}(v)(z)|B_m|,
\end{aligned}
$$

where the third inequality follows from Lemma 3.1.

We now consider $\sum_{m\in\mathbb{N}_1(x,\alpha)\cup\mathbb{N}_2(x,\alpha)}T_\alpha(h_m)(x)$, $x\in E^c$. If $m\in\mathbb{N}_1(x,\alpha)\cup\mathbb{N}_2(x,\alpha)$, as in the proof of Lemma 2.3, we have

$$\sup_{\alpha>0}\sum_{m\in\mathbb{N}_1(x,\alpha)\cup\mathbb{N}_2(x,\alpha)}|T_\alpha(h_m)(x)|\le\sum_{m=1}^{\infty}\int_{Q_m}|K(x,y)-K(x,x_m)||h_m(y)|\,dy.$$

Thus, arguing as in the proof of (3.3), we see that

$$
\begin{aligned}
(3.4)\quad & v\left(\left\{x\in E^c : \sup_{\alpha>0}\sum_{m\in\mathbb{N}_1(x,\alpha)\cup\mathbb{N}_2(x,\alpha)}|T_\alpha(h_m)(x)|>1\right\}\right)\\
&\qquad\le C\sum_{m=1}^{\infty}\inf_{z\in B(x_m,r_m)}M_{r'}(v)(z)|B_m|.
\end{aligned}
$$

By (3.1), (3.3) and (3.4), we have

$$v\left(\{x\in E^c : T_*(h)(x)>2+B\}\right)\le C\sum_{m=1}^{\infty}\inf_{z\in B(x_m,r_m)}M_{r'}(v)(z)|B_m|.$$

This proves Lemma 3.3. □

The proof of Theorem 1.3 is as follows. Let $f\in L_0^\infty$. By Lemma 2.1 we have

$$(3.5)\qquad |T_*f(x)|\le N^{(1)}(f)(x)+N^{(2)}(f)(x)+C_\delta M_\delta(Tf)(x)+C_\delta Mf(x),$$

where $N^{(i)}f$ is $N_\mu^{(i)}(f)$ with the measure μ replaced by the Lebesgue measure, $i=1,2$. From (K.4) it follows that

$$(3.6)\qquad \|N^{(1)}(f)\|_\infty\le C_1\|f\|_\infty.$$

Also, (K.2) implies that

$$(3.7)\qquad \|N^{(2)}(f)\|_\infty\le C_2\|f\|_\infty.$$

Let $w\in A_1$ be as in Theorem 1.3. Using Lemma 2.5 and Lemma 2.2 with the estimates

$$w\left(\{|Tf|>\lambda\}\right)\le C\lambda^{-1}\|f\|_{1,w},\quad \forall\lambda>0,$$

which can be found in [18, III, Theorem 1.2], where $\|f\|_{1,w} = \|f\|_{L^1_w}$, we have

$$w\left(\{M_\delta(Tf) > \lambda\}\right) \le C\lambda^{-\delta} \int_{M_\delta(Tf)>\lambda} |Tf(x)|^\delta w(x)\,dx$$
$$\le C_\delta \lambda^{-\delta} \|f\|_{1,w}^\delta w\left(\{M_\delta(Tf) > \lambda\}\right)^{1-\delta},$$

which implies that

$$w\left(\{M_\delta(Tf) > \lambda\}\right) \le C\lambda^{-1}\|f\|_{1,w}, \quad \forall \lambda > 0. \tag{3.8}$$

Let $f \in L_0^\infty$ and $f = g + b$, $b = \sum b_m$, and $\{Q_m\}$ be as in Lemma 2.4 with μ replaced by the Lebesgue measure and with $p = 1$ and $v = w$. Then by (3.5), (3.6), (3.7) and (3.8) we see that

$$\begin{aligned} w\left(\{T_*(g) > C_1 + C_2 + 2\}\right) &\le w\left(\{C_\delta M_\delta(Tg) > 1\}\right) + w\left(\{C_\delta M(g) > 1\}\right) \\ &\le C\|g\|_{1,w} \le C\|f\|_{1,w}. \end{aligned} \tag{3.9}$$

Let B_m be a ball with the same center and diameter as Q_m. Let $B_m^* = B(x_m, r_m)$ and $E = \cup_{m=1}^\infty B(x_m, 8r_m)$. Then by applying Lemma 3.3 with b in place of h, we have

$$\begin{aligned} w\left(\{T_*(b) > C_0\}\right) &\le w(E) + w\left(\{x \in E^c : T_*(b)(x) > C_0\}\right) \\ &\le C\sum_{m=1}^\infty \inf_{z\in B_m} M_{r'}(w)(z)|B_m| \le C\|f\|_{1,w}, \end{aligned} \tag{3.10}$$

where the last inequality follows from part (7) of Lemma 2.4 with $p = 1$ and the fact that $M_{r'}(w) \le Cw$ a.e. Combining (3.9) and (3.10), we see that

$$w\left(\{T_*(f) > C_0 + C_1 + C_2 + 2\}\right) \le C\|f\|_{1,w}. \tag{3.11}$$

From (3.11) and the sublinearity of T_*, we can deduce that T_* extends to a sublinear operator of weak type $(1,1)$ with respect to weight w. This completes the proof of Theorem 1.3.

Proofs of Proposition 1.4 and Corollary 1.5 will be given in what follows. Let T and w be as in Proposition 1.4. Let $f \in L_0^\infty(\mathbb{R}^n)$. We recall (3.5). Since the L^1_w-$L^{1,\infty}_w$ boundedness of T is assumed, we have (3.8). By the conditions $(K.4)$ and $(K.2)$ we have (3.6) and (3.7), respectively. Therefore we see that for $\lambda > 0$

$$\begin{aligned} &w\left(\{T_*(f) > (C_1 + C_2)\|f\|_\infty + 2\lambda\}\right) \\ &\quad\le w\left(\{C_\delta M_\delta(Tf) > \lambda\}\right) + w\left(\{C_\delta M(f) > \lambda\}\right) \\ &\quad\le C\lambda^{-1}\|f\|_{1,w}. \end{aligned}$$

This implies that if $\lambda > 2(C_1 + C_2)\|f\|_\infty$,

$$w\left(\{T_*(f) > \lambda\}\right) \le 4C\lambda^{-1}\|f\|_{1,w}.$$

This completes the proof of Proposition 1.4 for $f \in L_0^\infty(\mathbb{R}^n)$. The sublinear operator T_* can be uniquely extended to $L^1_w \cap L^\infty$. The proof is by standard methods. We omit the details.

Let Ω and w be as in Corollary 1.5. Let $K(x,y) = |x-y|^{-n}\Omega((x-y)')$. Then the condition $(K.2)$ obviously holds and the L^1 Dini condition of Ω implies $(K.4)$. Further, the L^1_w-$L^{1,\infty}_w$ boundedness follows from [9, Corollary 1]. Thus we can apply Proposition 1.4 to get the conclusion of Corollary 1.5.

4. Proof of Theorem 1.2

Let $w \in A_2$ (see [11] for the Muckenhoupt weight class A_2). Put

$$K(x,y) = k(x-y)(w(x) - w(y)), \qquad k(x) = h(|x|)\frac{\Omega(x')}{|x|^n},$$

where h is a bounded function on $[0,\infty)$ and Ω is a bounded function on S^{n-1} such that $\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0$. Define $d\mu(y) = w(y)^{-1}\, dy$ and

$$Tf(x) = \lim_{\epsilon\to 0} T_\epsilon f(x) = \lim_{\epsilon\to 0} \int_{|x-y|>\epsilon} K(x,y) f(y)\, d\mu(y), \qquad T_* f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|.$$

We have the following result.

Theorem 4.1. *Let $n \ge 2$. Then the maximal operator T_* is bounded on $L^2(\mathbb{R}^n, d\mu)$.*

Proof. Let

$$S_\epsilon f(x) = \int_{|x-y|>\epsilon} k(x-y) f(y)\, dy.$$

Then we see that

$$T_\epsilon f(x) = -S_\epsilon f(x) + w(x) S_\epsilon(w^{-1} f)(x).$$

Let $S_* f(x) = \sup_{\epsilon>0} |S_\epsilon f(x)|$. Then it is known that S_* is bounded on $L^2(\mathbb{R}^n, v\, dx)$ for $v \in A_2$ (see [7, Corollary 4.2]). Thus, since $w, w^{-1} \in A_2$, we have

$$\begin{aligned}\int |T_* f(x)|^2\, d\mu &\le 2\int |S_* f(x)|^2 w(x)^{-1}\, dx + 2\int |S_*(w^{-1}f)|^2 w(x)\, dx \\ &\le C\int |f(x)|^2 w(x)^{-1}\, dx = C\int |f(x)|^2\, d\mu(x).\end{aligned}$$

This completes the proof. □

It is known that $|x|^{-\beta} \in A_1$ for $0 \le \beta < n$ and $A_1 \subset A_2$.

Lemma 4.2. *Let $K_\beta(x,y)$ be as in* (1.1). *Then K_β satisfies the condition* $(K.3)$ *with the measure $d\mu_\beta$:*

$$\int_{|x-y_0|\ge 2|y-y_0|} |K_\beta(x,y) - K_\beta(x,y_0)|\, d\mu_\beta(x) \le C$$

for all y_0, $y \in \mathbb{R}^n$.

When Ω satisfies a Lipschitz condition, a result similar to this is stated on [6, p. 76] without a proof.

Proof of Lemma 4.2. We first observe that

$$\begin{aligned}(4.1)\qquad &\int_{|x-y_0|\ge 2|y-y_0|} |K_\beta(x,y) - K_\beta(x,y_0)|\, d\mu_\beta(x) \\ &\le \int_{|x-y_0|\ge 2|y-y_0|} |k(x-y) - k(x-y_0)|\, dx \\ &\quad + \int_{|x-y_0|\ge 2|y-y_0|} \left| k(x-y)|y|^\beta - k(x-y_0)|y_0|^\beta \right| |x|^{-\beta}\, dx.\end{aligned}$$

The first integral on the right hand side is estimated as in [24, Chap. II, §4]. The second integral on the right hand side is majorized by the sum of the following two integrals.

$$I = \int_{|x-y_0|\ge 2|y-y_0|} |k(x-y)-k(x-y_0)|\,|y|^\beta |x|^{-\beta}\,dx,$$
$$J = \int_{|x-y_0|\ge 2|y-y_0|} |k(x-y_0)|\,\big||y_0|^\beta - |y|^\beta\big|\,|x|^{-\beta}\,dx.$$

We estimate I and J separately. To estimate I, we note that

$$\begin{aligned}(4.2)\quad &|k(z-u)-k(z)| \\ &\le C\omega(c|u|/|z|)|z|^{-n} + C\|\Omega\|_\infty(|u|/|z|)|z|^{-n} \le C\widetilde{\omega}(c|u|/|z|)|z|^{-n}\end{aligned}$$

if $|z| \ge 2|u|$, where $\widetilde{\omega}(t) = \omega(t)+t$ (see [24, Chap. II, §4]). We split the region of integration in I into three parts and decompose I into three pieces accordingly: $I = I_1 + I_2 + I_3$, where

$$I_1 = \int_{\substack{|x-y_0|\ge 2|y-y_0|\\ |x|>2|y_0|}} |k(x-y)-k(x-y_0)|\,|y|^\beta|x|^{-\beta}\,dx,$$
$$I_2 = \int_{\substack{|x-y_0|\ge 2|y-y_0|\\ |x|<|y_0|/2}} |k(x-y)-k(x-y_0)|\,|y|^\beta|x|^{-\beta}\,dx,$$
$$I_3 = \int_{\substack{|x-y_0|\ge 2|y-y_0|\\ |y_0|/2\le|x|\le 2|y_0|}} |k(x-y)-k(x-y_0)|\,|y|^\beta|x|^{-\beta}\,dx.$$

Let $|y_0| > |y|/2$. Then by (4.2) we have

$$\begin{aligned} I_1 &\le C|y|^\beta \int_{|x|>2|y_0|} \widetilde{\omega}(c|y-y_0|/|x-y_0|)|x-y_0|^{-n}|x|^{-\beta}\,dx \\ &\le C|y|^\beta \int_{|x|>2|y_0|} \widetilde{\omega}(2c|y-y_0|/|x|)|x|^{-n-\beta}\,dx \\ &\le C|y|^\beta|y_0|^{-\beta} \int_{|x|>2|y_0|} \widetilde{\omega}(2c|y-y_0|/|x|)|x|^{-n}\,dx \\ &= C|y|^\beta|y_0|^{-\beta} \int_{|x|>c|y_0|/|y-y_0|} \widetilde{\omega}(1/|x|)|x|^{-n}\,dx \\ &= C|y|^\beta|y_0|^{-\beta} \int_0^{c|y-y_0|/|y_0|} \widetilde{\omega}(t)\,dt/t \le C\int_0^c \widetilde{\omega}(t)\,dt/t \end{aligned}$$

for some constants $c, C > 0$. Similarly, I_2 is estimated as follows.

$$\begin{aligned} I_2 &\le C|y|^\beta \int_{|x|<|y_0|/2} \widetilde{\omega}(c|y-y_0|/|x-y_0|)|x-y_0|^{-n}|x|^{-\beta}\,dx \\ &\le C|y|^\beta|y_0|^{-n} \int_{|x|<|y_0|/2} |x|^{-\beta}\,dx \\ &\le C|y|^\beta|y_0|^{-n}|y_0|^{n-\beta} \le C. \end{aligned}$$

Also, we see that

$$\begin{aligned}
I_3 &\le C|y|^\beta \int_{\substack{|x-y_0|\ge 2|y-y_0| \\ |y_0|/2\le |x|\le 2|y_0|}} \widetilde{\omega}(c|y-y_0|/|x-y_0|)|x-y_0|^{-n}|x|^{-\beta}\,dx \\
&\le C|y|^\beta |y_0|^{-\beta} \int_{|z|\ge 2|y-y_0|} \widetilde{\omega}(c|y-y_0|/|z|)|z|^{-n}\,dx \\
&\le C|y|^\beta |y_0|^{-\beta} \int_0^c \widetilde{\omega}(t)\,dt/t \le C.
\end{aligned}$$

Next, we assume that $|y_0| \le |y|/2$ and estimate I_j, $1 \le j \le 3$. To estimate I_1 we note that if $|x-y_0| > 2|y-y_0|$ and $|y_0| \le |y|/2$, then $|x| > |y|/2$. Using this and (4.2), we have

$$\begin{aligned}
I_1 &\le C|y|^\beta \int_{\substack{|x-y_0|\ge 2|y-y_0| \\ |x|>2|y_0|}} \widetilde{\omega}(c|y-y_0|/|x-y_0|)|x-y_0|^{-n}|x|^{-\beta}\,dx \\
&\le C|y|^\beta \int_{|x|\ge |y|/2} \widetilde{\omega}(c|y|/|x|)|x|^{-n-\beta}\,dx \\
&\le C \int_{|x|\ge |y|/2} \widetilde{\omega}(c|y|/|x|)|x|^{-n}\,dx \\
&= C \int_0^c \widetilde{\omega}(t)\,dt/t \le C.
\end{aligned}$$

As above, if $|x-y_0| \ge 2|y-y_0|$ and $|y_0| \le |y|/2$, then $|x| \ge |y|/2$. On the other hand, in the region of integration of I_2 we have $|x| < |y_0|/2$. Thus $|x| < |y|/4$, which is incompatible with $|x| \ge |y|/2$. So, the region of integration of I_2 is empty and we discard I_2.

Finally we estimate I_3. We note that if $|x-y_0| \ge 2|y-y_0|$, $|y_0| \le |y|/2$ and $|x| \le 2|y_0|$, then $|y| < |x-y_0| \le 3|y|/2$. Therefore, we see that

$$\begin{aligned}
I_3 &\le C|y|^\beta \int_{\substack{|x-y_0|\ge 2|y-y_0| \\ |y_0|/2\le |x|\le 2|y_0|}} \widetilde{\omega}(c|y-y_0|/|x-y_0|)|x-y_0|^{-n}|x|^{-\beta}\,dx \\
&\le C|y|^\beta |y_0|^{-\beta} |y|^{-n} \int_{|y_0|/2\le |x|\le 2|y_0|} dx \\
&= C|y|^\beta |y_0|^{-\beta} |y|^{-n} |y_0|^n = C|y|^{\beta-n}|y_0|^{n-\beta} \le C.
\end{aligned}$$

This completes the estimates for I.

We now estimate J. As in the case of I, we decompose J analogously: $J = J_1 + J_2 + J_3$, where

$$\begin{aligned}
J_1 &= \int_{\substack{|x-y_0|\ge 2|y-y_0| \\ |x|>2|y_0|}} |k(x-y_0)|\,\big||y_0|^\beta - |y|^\beta\big|\,|x|^{-\beta}\,dx, \\
J_2 &= \int_{\substack{|x-y_0|\ge 2|y-y_0| \\ |x|<|y_0|/2}} |k(x-y_0)|\,\big||y_0|^\beta - |y|^\beta\big|\,|x|^{-\beta}\,dx, \\
J_3 &= \int_{\substack{|x-y_0|\ge 2|y-y_0| \\ |y_0|/2\le |x|\le 2|y_0|}} |k(x-y_0)|\,\big||y_0|^\beta - |y|^\beta\big|\,|x|^{-\beta}\,dx.
\end{aligned}$$

We first assume that $|y_0| > |y|/2$. Then

$$\begin{aligned}J_1 &\le C\left||y_0|^\beta - |y|^\beta\right| \int_{|x|>2|y_0|} |x-y_0|^{-n}|x|^{-\beta}\,dx \\ &\le C\left||y_0|^\beta - |y|^\beta\right| \int_{|x|>2|y_0|} |x|^{-n-\beta}\,dx \\ &\le C(|y_0|^\beta + |y|^\beta)|y_0|^{-\beta} \le C.\end{aligned}$$

J_2 is estimated as follows.

$$\begin{aligned}J_2 &\le C\left||y_0|^\beta - |y|^\beta\right| \int_{|x|<|y_0|/2} |x-y_0|^{-n}|x|^{-\beta}\,dx \\ &\le C\left||y_0|^\beta - |y|^\beta\right| |y_0|^{-n} \int_{|x|<|y_0|/2} |x|^{-\beta}\,dx \\ &\le C(|y_0|^\beta + |y|^\beta)|y_0|^{-n}|y_0|^{n-\beta} \le C.\end{aligned}$$

To estimate J_3, first we assume that $|y_0| > 2|y|$. Then

$$\begin{aligned}J_3 &\le C\left||y_0|^\beta - |y|^\beta\right| \int_{\substack{|x-y_0|\ge 2|y-y_0| \\ |y_0|/2\le|x|\le 2|y_0|}} |x-y_0|^{-n}|x|^{-\beta}\,dx \\ &\le C\left||y_0|^\beta - |y|^\beta\right| |y_0|^{-\beta} \int_{|y_0|<|x-y_0|\le 3|y_0|} |x-y_0|^{-n}\,dx \le C.\end{aligned}$$

Next, assume that $|y|/2 < |y_0| \le 2|y|$. Then, using the inequality

$$\left||y_0|^\beta - |y|^\beta\right| \le C|y_0 - y||y_0|^{\beta-1},$$

which follows by the mean value theorem, we see that

$$\begin{aligned}J_3 &\le C|y_0-y||y_0|^{\beta-1} \int_{\substack{|x-y_0|\ge 2|y-y_0| \\ |y_0|/2\le|x|\le 2|y_0|}} |x-y_0|^{-n}|x|^{-\beta}\,dx \\ &\le C|y_0|^{\beta-1}|y_0|^{-\beta} \int_{|x-y_0|\le 3|y_0|} |x-y_0|^{-n+1}\,dx \\ &\le C|y_0|^{\beta-1}|y_0|^{-\beta}|y_0| \le C.\end{aligned}$$

Next, assuming $|y_0| \le |y|/2$, J_1, J_2 and J_3 are estimated as follows.

$$\begin{aligned}J_1 &\le C\left||y_0|^\beta - |y|^\beta\right| \int_{\substack{|x-y_0|\ge 2|y-y_0| \\ |x|>2|y_0|}} |x-y_0|^{-n}|x|^{-\beta}\,dx \\ &\le C\left||y_0|^\beta - |y|^\beta\right| \int_{|x-y_0|\ge|y|} |x-y_0|^{-n-\beta}\,dx \\ &\le C(|y_0|^\beta + |y|^\beta)|y|^{-\beta} \le C.\end{aligned}$$

As in the case of I_2, the region of integration of J_2 is empty. So, J_2 is excluded. Finally, we estimate J_3 as follows.

$$\begin{aligned}J_3 &\le C\left||y_0|^\beta - |y|^\beta\right| \int_{\substack{|x-y_0|\ge 2|y-y_0| \\ |y_0|/2\le|x|\le 2|y_0|}} |x-y_0|^{-n}|x|^{-\beta}\,dx \\ &\le C\left||y_0|^\beta - |y|^\beta\right| |y-y_0|^{-n}|y_0|^{-\beta} \int_{|y_0|/2\le|x|\le 2|y_0|} dx \\ &\le C|y|^\beta|y|^{-n}|y_0|^{-\beta}|y_0|^n = C|y|^{\beta-n}|y_0|^{n-\beta} \le C.\end{aligned}$$

This completes the estimates for J.

Combining the estimates for I and J, we have desired bounds for the second integral in (4.1), which finishes the proof of Lemma 4.2. □

Lemma 4.3. *Let K_β be as in Lemma* 4.2*. Then K_β satisfies the condition* $(K.2)$ *with the measure $d\mu_\beta$.*

Proof. Let

$$I_\alpha = \int_{\alpha<|x-y|<2\alpha} \left|k(x-y)(|x|^\beta - |y|^\beta)\right| |y|^{-\beta}\, dy$$
$$= \int_{\alpha<|y|<2\alpha} \left|k(y)(|x|^\beta - |x-y|^\beta)\right| |x-y|^{-\beta}\, dy.$$

We write $I_\alpha = I_{\alpha,1} + I_{\alpha,2} + I_{\alpha,3}$, where

$$I_{\alpha,1} = \int_{\substack{\alpha<|y|<2\alpha\\ |y|<|x|/2}} \left|k(y)(|x|^\beta - |x-y|^\beta)\right| |x-y|^{-\beta}\, dy,$$
$$I_{\alpha,2} = \int_{\substack{\alpha<|y|<2\alpha\\ |y|>2|x|}} \left|k(y)(|x|^\beta - |x-y|^\beta)\right| |x-y|^{-\beta}\, dy,$$
$$I_{\alpha,3} = \int_{\substack{\alpha<|y|<2\alpha\\ |x|/2\le|y|\le2|x|}} \left|k(y)(|x|^\beta - |x-y|^\beta)\right| |x-y|^{-\beta}\, dy.$$

By straightforward computations, we have

$$I_{\alpha,1} \le C\int_{\alpha<|y|<2\alpha} |y|^{-n}\, dy \le C;$$

$$\begin{aligned} I_{\alpha,2} &\le \int_{\substack{\alpha<|y|<2\alpha\\ |y|>2|x|}} |y|^{-n}(|x|^\beta|y|^{-\beta}+1)\, dy \\ &\le C\int_{\alpha<|y|<2\alpha} |y|^{-n}\, dy + \int_{|y|>2|x|} |x|^\beta|y|^{-n-\beta}\, dy \le C; \end{aligned}$$

$$\begin{aligned} I_{\alpha,3} &\le C\int_{\substack{\alpha<|y|<2\alpha\\ |x|/2\le|y|\le2|x|}} |y|^{-n}(|x|^\beta|x-y|^{-\beta}+1)\, dy \\ &\le C\int_{\alpha<|y|<2\alpha} |y|^{-n}\, dy + C|x|^{\beta-n}\int_{|x-y|\le3|x|} |x-y|^{-\beta}\, dy \\ &\le C + C|x|^{\beta-n}|x|^{n-\beta} \le C. \end{aligned}$$

This completes the proof. □

Lemma 4.4. *Let $K_\beta(x,y)$ be as in Lemma* 4.2*. Then K_β satisfies the condition* $(K.4)$ *with the measure $d\mu_\beta$.*

Proof. Let $L(x,y) = K_\beta(y,x)$. Then K_β satisfies the $(K.4)$ condition if and only if L satisfies the $(K.3)$ condition. We note that

$$L(x,y) = \widetilde{k}(x-y)(|x|^\beta - |y|^\beta),$$

where $\widetilde{k}(x) = -k(-x)$. Since $\widetilde{k}$ has properties similar to those of k which are required in Lemma 4.2, L satisfies $(K.3)$ by Lemma 4.2 and hence K_β satisfies $(K.4)$. □

Now we can give the proof of Theorem 1.2. $T^{(\beta)}$ is bounded on $L^2(\mathbb{R}^n, d\mu_\beta)$ by Theorem 4.1. By Lemmas 4.2, 4.3 and 4.4, we see that K_β satisfies $(K.3)$, $(K2)$ and $(K.4)$, respectively. Thus by Theorem 1.1 we have the conclusion of Theorem 1.2 except for the $L^2(\mathbb{R}^n, d\mu_\beta)$ boundedness of $T_*^{(\beta)}$, which is in Theorem 4.1.

5. Appendix

Let $K(x,y)$ be locally integrable in $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ with the Lebesgue measure. In this section we show that the D_r condition for K is equivalent to the (D_r) condition for K on [18, p. 30]. Also, proofs of Lemmas 2.2, 2.5 and 3.2 will be provided. Furthermore, we see two results stated above relative to the regular Borel measure μ as in Section 1; one is approximation by functions of C_0^∞ for functions in $L^p(d\mu)$, $1 \le p < \infty$, and the other is Calderón-Zygmund decomposition for $L^1(d\mu)$.

5.1. **Equivalence of the conditions D_r and (D_r).** For a positive integer k and $1 \le r < \infty$, let

$$c_k = \sup_{y,z\in\mathbb{R}^n} \left(|S_k(y,z)|^{-n} \int_{S_k(y,z)} \left| |S_k(y,z)|^n \left(K(x,y) - K(x,z)\right) \right|^r \, dx \right)^{1/r},$$

where $S_k(y,z) = A(z; 2^k|y-z|, 2^{k+1}|y-z|)$. When $r = \infty$, c_k is defined by usual modifications. We recall that K satisfies the (D_r) condition if

$$\sum_{k=1}^{\infty} c_k < \infty.$$

We also write $\omega_r^*(2^{-k}) = c_k$.

We see that $\omega_r^*(2^{-k})$ and $\omega_r(2^{-k})$ are related as follows.

Proposition 5.1. *Let $k \in \mathbb{Z}$. There exists a positive constant c such that*

$$\omega_r^*(2^{-k}) \le c\,\omega_r(2^{-k+1}) \quad \textit{for } k \ge 1; \tag{5.1}$$

$$\omega_r(2^{-k}) \le c(\omega_r^*(2^{-k-1}) + \omega_r^*(2^{-k-2})) \quad \textit{for } k \ge 0. \tag{5.2}$$

Proof. For $y, z \in \mathbb{R}^n$ with $y \ne z$, let $R = 2|y-z|$. Then $(y,z) \in \Delta(R/4, R/2)$ and

$$\begin{aligned}
&|S_k(y,z)|^{-n} \int_{S_k(y,z)} \left| |S_k(y,z)|^n \left(K(x,y) - K(x,z)\right) \right|^r \, dx \\
&= C(2^{k-1}R)^{-n} \int_{A(z;2^{k-1}R, 22^{k-1}R)} \left| (2^{k-1}R)^n \left(K(x,y) - K(x,z)\right) \right|^r \, dx \\
&\le C\omega_r(2^{-k+1}),
\end{aligned}$$

which proves (5.1).

Conversely, let $R > 0$ and $(y,z) \in \Delta(R/4, R/2)$. Then $R/4 \le |y-z| \le R/2$. Thus

$$A(z; 2^k R, 2^{k+1}R) \subset A(z; 2^{k+1}|y-z|, 2^{k+3}|y-z|).$$

Using this, we can easily see that (5.2) holds. □

By Proposition 5.1 we have

$$c\sum_{k=0}^{\infty} \omega_r(2^{-k}) \le \sum_{k=1}^{\infty} \omega_r^*(2^{-k}) \le C\sum_{k=0}^{\infty} \omega_r(2^{-k})$$

for positive constants c, C, which implies the equivalence between the D_r and (D_r) conditions.

5.2. **Proof of Lemma** 3.2. We observe that

$$A(z; s^{-1}R, 2s^{-1}R) \subset \bar{A}(z; (2t)^{-1}R, t^{-1}R) \cup A(z; t^{-1}R, 2t^{-1}R)$$

for $R > 0$ under the assumption of the lemma, where $\bar{A}$ denotes the closure of A in $\mathbb{R}^n$. Using this in the integral of $\omega_{r,s^{-1}R}(s)$ in the definition $\omega_r(s) = \sup_{R>0} \omega_{r,s^{-1}R}(s)$, we get the conclusion of Lemma 3.2.

5.3. **Proof of Lemma** 2.2. We use the formula:

$$\int_E F(x)^\delta \, d\nu(x) = \int_0^\infty \nu\{x \in E : F(x) > \lambda\} \delta\lambda^{\delta-1} \, d\lambda.$$

See Rudin [19, Theorem 8.16 on p. 172]. The proof is straightforward as follows.

$$\begin{aligned}
\int_E F(x)^\delta \, d\nu(x) &\le \int_0^\infty \min(\nu(E), \lambda^{-1}A) \delta\lambda^{\delta-1} \, d\lambda \\
&= \int_0^{A/\nu(E)} \nu(E)\delta\lambda^{\delta-1} \, d\lambda + \int_{A/\nu(E)}^\infty A\delta\lambda^{\delta-2} \, d\lambda \\
&= \nu(E)(A/\nu(E))^\delta + A\frac{\delta}{1-\delta}(A/\nu(E))^{\delta-1} \\
&= \frac{1}{1-\delta} A^\delta \nu(E)^{1-\delta},
\end{aligned}$$

which completes the proof.

5.4. **Proof of Lemma** 2.5. Let $O_\lambda(f) = \{x \in \mathbb{R}^n : M_\mu(f)(x) > \lambda\}$. If $x \in O_\lambda(f)$, there exists a ball B such that $x \in B$ and $⨍_B |f| \, d\mu(y) > \lambda$. Then for $z \in B$, $M_\mu(f)(z) \ge ⨍_B |f| \, d\mu(y) > \lambda$. Therefore, $B \subset O_\lambda(f)$ and hence

$$⨍_B |f| \chi_{O_\lambda(f)} \, d\mu(y) = ⨍_B |f| \, d\mu(y) > \lambda,$$

which implies $x \in O_\lambda(f\chi_{O_\lambda(f)})$. It follows that $O_\lambda(f) \subset O_\lambda(f\chi_{O_\lambda(f)})$. Using this and the assumption for w, we have

$$\begin{aligned}
\int_{\{x\in\mathbb{R}^n : M_\mu(f)(x)>\lambda\}} w(x) \, d\mu(x) &= \int_{O_\lambda(f)} w(x) \, d\mu(x) \le \int_{O_\lambda(f\chi_{O_\lambda(f)})} w(x) \, d\mu(x) \\
&= \int_{\{x\in\mathbb{R}^n : M_\mu(f\chi_{O_\lambda(f)})(x)>\lambda\}} w(x) \, d\mu(x) \\
&\le C\lambda^{-1} \int_{\mathbb{R}^n} |f(x)| \chi_{O_\lambda(f)}(x) w(x) \, d\mu(x) \\
&= C\lambda^{-1} \int_{\{x\in\mathbb{R}^n : M_\mu(f)(x)>\lambda\}} |f(x)| w(x) \, d\mu(x),
\end{aligned}$$

which completes the proof.

5.5. **Approximation by C_0^∞ in $L^p(d\mu)$, $1 \le p < \infty$.** Let μ be the regular Borel measure as in Section 1. Then, the set $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, d\mu)$ for $1 \le p < \infty$. This can be shown as follows. Let $f \in L^p(\mathbb{R}^n, d\mu)$. Given $\epsilon > 0$ we can find a function g which is continuous and compactly supported such that $\|f - g\|_p < \epsilon/2$ (see [10, pp. 210–211]). Let $g^{(\delta)} = g * \phi_\delta$, $0 < \delta < 1$, where $\phi \in C_0^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$ and $\phi_\delta(x) = \delta^{-n}\phi(\delta^{-1}x)$. Then $g^{(\delta)} \in C_0^\infty(\mathbb{R}^n)$ and we easily see that $g^{(\delta)} \to g$ uniformly on $\mathbb{R}^n$ as $\delta \to 0$ and $\operatorname{supp}(g^{(\delta)}) \subset E$ for some compact set E independent of δ. This implies that $\|g - g^{(\delta_0)}\|_p < \epsilon/2$ for some δ_0. Thus

$$\|f - g^{(\delta_0)}\|_p \le \|f - g\|_p + \|g - g^{(\delta_0)}\|_p < \epsilon,$$

which implies what we claimed.

5.6. **Calderón-Zygmund decomposition for $L^1(d\mu)$.** Let $f \in L^1(\mathbb{R}^n, d\mu)$ and $\lambda > 0$, where μ is as in Section 1. As in the case where μ is the Lebesgue measure, using the doubling condition of μ, by the stopping time arguments, we can find a family $\{\widetilde{Q}_m\}_{m=1}^\infty$ of disjoint right open dyadic cubes such that

$$\lambda < \mu(\widetilde{Q}_m)^{-1} \int_{\widetilde{Q}_m} |f(x)|\,d\mu(x) \le C\lambda,$$

where a right open interval has a form $[a_1, b_1) \times \cdots \times [a_n, b_n)$. Let $U = \cup \widetilde{Q}_m$. Then $|f| \le \lambda$ (μ-a.e.) on U^c, which can be shown by applying the weak type $(1,1)$ boundedness of M_μ and the fact that the set of continuous functions with compact support is dense in $L^1(\mathbb{R}^n, d\mu)$ (see [10, pp. 210–211]). Define

$$g(x) = f(x)\chi_{U^c}(x) + \sum_m \left(\mu(\widetilde{Q}_m)^{-1} \int_{\widetilde{Q}_m} |f|\,d\mu\right) \chi_{\widetilde{Q}_m}(x),$$

$$b = \sum_m b_m, \quad b_m(x) = f(x)\chi_{\widetilde{Q}_m}(x) - \left(\mu(\widetilde{Q}_m)^{-1} \int_{\widetilde{Q}_m} |f|\,d\mu\right) \chi_{\widetilde{Q}_m}(x).$$

Let Q_m be the closure of $\widetilde{Q}_m$ in $\mathbb{R}^n$. Then $\operatorname{supp}(b_m) \subset Q_m$, $\int b_m\,d\mu = 0$, $\|b_m\|_1 \le C\lambda\mu(Q_m)$, $f = g + b$ and $\{Q_m\}$ is a family of non-overlapping cubes. Also, for a weight function v, we have

$$\|g\|_{L^1(v\,d\mu)} \le C\|f\|_{L^1(M_\mu(v)\,d\mu)}, \tag{5.3}$$

$$\sum_m \int_{Q_m} v(x)\,d\mu(x) \le C\lambda^{-1}\|f\|_{L^1(M_\mu(v)\,d\mu)}. \tag{5.4}$$

Proof of (5.3). Since $v \le M_\mu(v)$ (μ-a.e.), we have

$$\int |f(x)|\chi_{U^c}(x)v(x)\,d\mu(x) \le \int |f(x)|M_\mu(v)(x)\,d\mu(x).$$

Also, since $\mu(\widetilde{Q}_m)^{-1}\int_{\widetilde{Q}_m} v\,d\mu \le CM_\mu(v)(z)$ for $z \in \widetilde{Q}_m$, which can be shown by the doubling condition for μ, we have

$$\begin{aligned}\sum_m \left(\mu(\widetilde{Q}_m)^{-1}\int_{\widetilde{Q}_m} |f|\,d\mu\right)\int_{\widetilde{Q}_m} v(x)\,d\mu(x) &\le C\sum_m \inf_{z\in\widetilde{Q}_m} M_\mu(v)(z)\int_{\widetilde{Q}_m} |f|\,d\mu \\ &\le C\sum_m \int_{\widetilde{Q}_m} |f|M_\mu(v)\,d\mu \le C\int |f|M_\mu(v)\,d\mu.\end{aligned}$$

Combining results, we get (5.3). □

Proof of (5.4). Since

$$1 < \lambda^{-1}\mu(\widetilde{Q}_m)^{-1}\int_{\widetilde{Q}_m}|f|\,d\mu,$$

using the doubling condition for μ as in the proof of (5.3), we see that

$$\begin{aligned}\sum_m \int_{Q_m} v\,d\mu &\le \sum_m \lambda^{-1}\mu(\widetilde{Q}_m)^{-1}\int_{\widetilde{Q}_m}|f|\,d\mu\int_{Q_m} v\,d\mu \\ &\le \sum_m C\lambda^{-1}\inf_{z\in Q_m} M_\mu(v)(z)\int_{\widetilde{Q}_m}|f|\,d\mu \\ &\le \sum_m C\lambda^{-1}\int_{\widetilde{Q}_m}|f|M_\mu(v)\,d\mu \\ &\le C\lambda^{-1}\|f\|_{L^1(M_\mu(v)\,d\mu)}.\end{aligned}$$

This completes the proof of (5.4). □

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