WEIGHTED INEQUALITIES FOR FOURIER MULTIPLIER OPERATORS OF BOCHNER-RIESZ TYPE ON $\mathbb{R}^2$

SHUICHI SATO

ABSTRACT. We consider Fourier multipliers in $\mathbb{R}^2$ with singularities on certain curves, which are closely related to the Bochner-Riesz Fourier multipliers. We prove weighted inequalities and vector valued inequalities for the Fourier multiplier operators, which generalize some known results.

1. INTRODUCTION

Let

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^2} f(x)e^{-2\pi i\langle x,\xi\rangle}\, dx$$

be the Fourier transform on $\mathbb{R}^2$, where $\langle x,\xi\rangle = x_1\xi_1 + x_2\xi_2$, $x = (x_1,x_2)$, $\xi = (\xi_1,\xi_2)$, denotes the inner product, and let

$$S_R^\lambda f(x) = \int_{|\xi|<R} \hat{f}(\xi)(1 - |R^{-1}\xi|^2)_+^\lambda e^{2\pi i\langle x,\xi\rangle}\, d\xi$$

be the Bochner-Riesz operator of order λ on $\mathbb{R}^2$, where $g_+(\xi) = g(\xi)$ if $g(\xi) > 0$ and $g_+(\xi) = 0$ otherwise, for a real valued function g.

The following is known ([4]).

Theorem A. *If $\lambda > 0$, S_1^λ is bounded on $L^4(\mathbb{R}^2)$:*

$$\|S_1^\lambda f\|_4 \leq C_\lambda \|f\|_4$$

for $f \in \mathcal{S}(\mathbb{R}^2)$, where $\mathcal{S}(\mathbb{R}^2)$ denotes the Schwartz class of infinitely differentiable, rapidly decreasing functions on $\mathbb{R}^2$.

By duality and interpolation Theorem A implies the L^p boundedness of S_1^λ for $4/3 \leq p \leq 4$. The L^4 boundedness for the maximal function $S_*^\lambda(f) = \sup_{R>0} |S_R^\lambda(f)|$ is proved in [1]. In [7] the following weighted inequality for S_1^λ is shown.

Theorem B. *There exists a bounded operator U_λ on $L^2(\mathbb{R}^2)$ such that*

$$\left| \int_{\mathbb{R}^2} (S_1^\lambda f(x))^2 w(x)\, dx \right| \leq \int_{\mathbb{R}^2} |f(x)|^2 U_\lambda(w)(x)\, dx \quad \text{for } f \in \mathcal{S}(\mathbb{R}^2).$$

The operator U_λ is defined constructively by using the Kakeya maximal functions. Theorem A follows from Theorem B. We refer to [2] and [12] for related results. In [2], the following result is shown.

Theorem C. *Let $q \in [2, \infty)$. Then, there exists a bounded operator $P_{q,\lambda}$ on $L^q(\mathbb{R}^2)$ such that*

$$\int_{\mathbb{R}^2} (S_*^\lambda f(x))^2 w(x) \, dx \leq \int_{\mathbb{R}^2} |f(x)|^2 P_{q,\lambda}(w)(x) \, dx, \quad f \in \mathcal{S}(\mathbb{R}^2),$$

for non-negative functions w in $L^q(\mathbb{R}^2)$.

See also [3], [5] and [13] for related results.

For $b \in L^\infty(\mathbb{R}^2)$, define a Fourier multiplier operator T_b by

$$T_b f(x) = \int_{\mathbb{R}^2} b(\xi) \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} \, d\xi,$$

where $f \in \mathcal{S}(\mathbb{R}^2)$. Let I be a compact interval in $\mathbb{R}$. Let $a \in C_0^\infty(\mathbb{R}^2)$ be supported in $I^\circ \times \mathbb{R}$, where I° denotes the interior of I and $C_0^\infty(\mathbb{R}^2)$ the set of infinitely differentiable functions on $\mathbb{R}^2$ with compact support. Let $\sigma_\lambda(\xi) = a(\xi)(\xi_2 - \psi(\xi_1))_+^\lambda$, where ψ is in $C^\infty(I)$ and real valued. We need to introduce an admissible class of curves ψ, which will be used to construct Fourier multipliers in this note.

Definition 1.1. *Let $\psi \in C^\infty(I)$ be real valued. We say $\psi \in \mathcal{C}(I)$ if ψ satisfies that*
  (1) $\psi'' \neq 0$ on I
      or that
  (2) *if $\psi''(t_0) = 0$ for some $t_0 \in I$, then t_0 is a finite order zero of ψ''.*

Then by [11], [14] the following result is known.

Theorem D. *Suppose that $\psi \in \mathcal{C}(I)$. Then, for $\lambda > 0$, T_{σ_λ} is bounded on $L^4(\mathbb{R}^2)$:*

$$\|T_{\sigma_\lambda} f\|_4 \leq C_\lambda \|f\|_4.$$

This can be considered as a generalization of Theorem A.

In this note we shall prove a weighted inequality for T_{σ_λ} which can be considered as a generalization of Theorem B and from which Theorem D will be derived. It is stated as follows.

Theorem 1.2. *Suppose that $\psi \in \mathcal{C}(I)$. Then there exists a bounded operator U on $L^2(\mathbb{R}^2)$ such that*

$$\int_{\mathbb{R}^2} |T_{\sigma_\lambda}(f)(x)|^2 w(x) \, dx \leq \int_{\mathbb{R}^2} |f(x)|^2 U(w)(x) \, dx$$

for $w \in L^2(\mathbb{R}^2)$ with $w \geq 0$.

We shall give a constructive proof for the existence of $U(w)$.

Remark 1.3. Let $q \in [2, \infty)$ and $U_q(w) = U(w^{q/2})^{2/q}$, where U is as in Theorem 1.2 and $w \geq 0$. Then $\|U_q(w)\|_q \leq C\|w\|_q$ and

$$\int_{\mathbb{R}^2} |T_{\sigma_\lambda}(f)(x)|^2 w(x) \, dx \leq \int_{\mathbb{R}^2} |f(x)|^2 U_q(w)(x) \, dx$$

for non-negative functions w in $L^q(\mathbb{R}^2)$, where σ_λ is as in Theorem 1.2. As in [2], this can be shown by applying interpolation with change of measures between the estimates

$$\int_{\mathbb{R}^2} |T_{\sigma_\lambda}(f)(x)|^2 w(x)^j \, dx \leq C_j \int_{\mathbb{R}^2} |f(x)|^2 U(w)(x)^j \, dx \quad \text{for } j = 0, 1.$$

This remark is also the case for Theorem 1.9.

Also, we shall prove vector valued inequalities.

Theorem 1.4. *Let $\{R_\ell\}_{\ell=1}^{\infty}$ be a sequence of positive numbers. Let σ_λ, $\lambda > 0$, be as in Theorem 1.2. Let $\sigma_\lambda^{(R)}(\xi) = \sigma_\lambda(R^{-1}\xi)$, $R > 0$. Then for $p \in [4/3, 4]$ we have*

$$\left\| \left(\sum_{\ell=1}^{\infty} \left| T_{\sigma_\lambda^{(R_\ell)}}(f_\ell) \right|^2 \right)^{1/2} \right\|_p \leq C \left\| \left(\sum_{\ell=1}^{\infty} |f_\ell|^2 \right)^{1/2} \right\|_p.$$

Corollary 1.5. *Let $\sigma_\lambda^{(R)}$ be as in Theorem 1.4. Then for any non-negative $w \in L^2(\mathbb{R}^2)$ there exists a non-negative $W \in L^2(\mathbb{R}^2)$ such that $\|W\|_2 \leq C\|w\|_2$ and*

$$\sup_{R>0} \int_{\mathbb{R}^2} \left| T_{\sigma_\lambda^{(R)}}(f)(x) \right|^2 w(x)\, dx \leq \int_{\mathbb{R}^2} |f(x)|^2 W(x)\, dx.$$

This follows from Theorem 1.4 and a result of [12]. In the case of the Bochner-Riesz operator, analogues of Theorem 1.4 and Corollary 1.5 are shown in [8] and [12], respectively. Here we mention that in this note we do not have an analogue of Theorem C for the functions σ_λ as above. To prove a result analogous to Theorem C for those σ_λ in detail is yet to be done.

Theorems 1.2 and 1.4 for the case $\psi'' \neq 0$ on I will be derived by applications of more general results. Let

$$\Gamma_\psi = \{(\xi_1, \psi(\xi_1)) \in \mathbb{R}^2 : \xi_1 \in I\}.$$

Let $\sigma : \mathbb{R}^2 \to \mathbb{R}$ be such that

    (1) $\sigma \in C^\infty(\mathbb{R}^2 \setminus \Gamma_\psi)$;
    (2) σ is compactly supported in $\{(\xi_1, \xi_2) : \xi_1 \in I^\circ \text{ and } \psi(\xi_1) \leq \xi_2\}$.

Let $\xi \in \Gamma_\psi$. Let $t(\xi) = (t_1(\xi), t_2(\xi))$ be the unit vector such that $t(\xi) = (1, \psi'(\xi_1))/(1 + \psi'(\xi_1)^2)^{1/2}$ and let $n(\xi) = (-t_2(\xi), t_1(\xi))$. Define the differential operators $\partial t(\xi)$ and $\partial n(\xi)$ by

$$(\partial t(\xi)g)(\eta) = t_1(\xi)\frac{\partial}{\partial \eta_1}g(\eta) + t_2(\xi)\frac{\partial}{\partial \eta_2}g(\eta),$$

$$(\partial n(\xi)g)(\eta) = n_1(\xi)\frac{\partial}{\partial \eta_1}g(\eta) + n_2(\xi)\frac{\partial}{\partial \eta_2}g(\eta),$$

where $\eta = (\eta_1, \eta_2)$. We also write $t(\xi_1)$, $n(\xi_1)$, $\partial t(\xi_1)$ and $\partial n(\xi_1)$ for $t(\xi)$, $n(\xi)$, $\partial t(\xi)$ and $\partial n(\xi)$ with $\xi = (\xi_1, \psi(\xi_1))$, respectively.

For $\delta \in (0, 1]$ and $\xi_1 \in I$, let

$$E(\psi, \delta, \xi_1) = \{\eta \in \mathbb{R}^2 : |\xi_1 - \eta_1| \leq \delta^{1/2}, \psi(\eta_1) + \delta \leq \eta_2 \leq \psi(\eta_1) + 2\delta, \eta_1 \in I\}.$$

Definition 1.6. Let σ be as above. Let $\{\Theta(2^{-m})\}_{m=1}^{\infty}$ be a sequence of positive numbers. We say that $\sigma \in \mathcal{M}(\psi, \Theta)$ if

$$(1.1) \qquad \sup_{\xi_1 \in I} \sup_{\eta \in E(\psi, \delta, \xi_1)} \left| \left((\partial t(\xi_1))^\alpha (\partial n(\xi_1))^\beta \sigma \right)(\eta) \right| \leq C\Theta(\delta)\delta^{-(1/2)\alpha - \beta}$$

for all $\delta \in \{2^{-m}\}_{m=1}^{\infty}$ and all non-negative integers α, β such that $0 \leq \alpha, \beta \leq 3$ with a positive constant C independent of δ.

Examples 1.7. Let $a(\xi) \in C_0^\infty(\mathbb{R}^2)$ be as in the definition of σ_λ in Theorem 1.2 and $\lambda(\xi) \in C^\infty(\mathbb{R}^2)$ with $\inf_{\xi \in \mathbb{R}^2} \lambda(\xi) \geq \lambda_0$ for some positive constant λ_0. Let $\sigma(\xi) = a(\xi)(\xi_2 - \psi(\xi_1))_+^{\lambda(\xi)}$. Then $\sigma \in \mathcal{M}(\psi, \Theta)$ with $\Theta(\delta) = \delta^{\lambda_0}$ (see (3.3) below in Section 3).

Examples1.8. Let $\sigma(\xi) = a(\xi)(\xi_2 - \psi(\xi_1))_+^{\lambda(\xi)}$ if $\xi_2 - \psi(\xi_1) > 0$, $\xi_1 \in I$, and $\sigma(\xi) = 0$ otherwise, where a is as in Example 1.7 and $\lambda(\xi) = (\log(1/(\xi_2 - \psi(\xi_1))_+))^{-\mu}$ with $0 < \mu < 1$. Then we can see that $\sigma \in \mathcal{M}(\psi, \Theta)$ with $\Theta(\delta) = \delta^{(\log(1/\delta))^{-\mu}}$. Suppose that $\psi'' \neq 0$ on I. Although $\lambda(\xi) \to 0$ as $\xi \to \bar{\xi} \in \Gamma_\psi$ under the condition that $\xi_2 - \psi(\xi_1) > 0$, by Theorem 1.9 below we can see that T_σ is bounded on L^p for $4/3 \le p \le 4$. This will be of interest if we recall the following result: If $\lambda(\xi)$ is identically 0 in the definition of σ above, then $\sigma(\xi) = a(\xi)\chi_E(\xi)$, where $E = \{\xi : \xi_2 > \psi(\xi_1), \xi_1 \in I\}$. Suppose that there is $\xi_0 \in \Gamma_\psi$ such that $a(\xi_0) \neq 0$. Then by the methods of [9] we can see that T_σ is bounded on L^p, $1 \le p < \infty$, only for $p = 2$.

We have the following theorem.

Theorem 1.9. *Let $\psi \in \mathcal{C}(I)$ satisfy $\psi'' \neq 0$ on I. Let $\sigma \in \mathcal{M}(\psi, \Theta)$. Suppose that there exists a sequence $\{a_m\}_{m=1}^\infty$ of positive real numbers such that*

$$\sum_{m=1}^\infty a_m < \infty, \qquad \sum_{m=1}^\infty \Theta(2^{-m})^2 m^{1/2} a_m^{-1} < \infty.$$

Then there exists a bounded operator U on $L^2(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} |T_\sigma(f)(x)|^2 w(x)\, dx \le \int_{\mathbb{R}^2} |f(x)|^2 U(w)(x)\, dx,$$

where $w \in L^2(\mathbb{R}^2)$ and $w \ge 0$.

We also have a vector valued inequality under a stronger condition on σ.

Theorem 1.10. *Let ψ be as in Theorem 1.9 and let $\sigma \in \mathcal{M}(\psi, \Theta)$. We assume that*

$$\sum_{m=1}^\infty \Theta(2^{-m}) m^{3/2} < \infty.$$

Let $\{R_\ell\}_{\ell=1}^\infty$ be a sequence of positive numbers. Let $\sigma^{(R)}(\xi) = \sigma(R^{-1}\xi)$, $R > 0$. Then we have

$$\left\| \left(\sum_{\ell=1}^\infty |T_{\sigma^{(R_\ell)}}(f_\ell)|^2 \right)^{1/2} \right\|_p \le C \left\| \left(\sum_{\ell=1}^\infty |f_\ell|^2 \right)^{1/2} \right\|_p$$

for $p \in [4/3, 4]$.

Corollary 1.11. *Let $T_{\sigma^{(R)}}$ be as in Theorem 1.10. Then for any non-negative $w \in L^2(\mathbb{R}^2)$ we can find a non-negative $W \in L^2(\mathbb{R}^2)$ such that $\|W\|_2 \le C\|w\|_2$ and*

$$\sup_{R>0} \int_{\mathbb{R}^2} |T_{\sigma^{(R)}}(f)(x)|^2\, w(x)\, dx \le \int_{\mathbb{R}^2} |f(x)|^2 W(x)\, dx.$$

This can be shown by Theorem 1.10 in the same way as Corollary 1.5 is proved by Theorem 1.4.

We shall prove Theorem 1.9 in Section 3. To prove Theorem 1.9 we need a certain geometrical result related to decomposition of a neighborhood of Γ_ψ when $\psi'' \neq 0$ on I, which will be given in Section 2 (see Lemma 2.3 below). We refer to [10] for related results in the case of the circle. The proof of Lemma 2.3 is based on methods of [14]. In Section 4 we shall prove Theorem 1.2 by applying Theorem 1.9

and an idea of [11, p. 8] for the resolution of a singularity caused by finite order zeros of ψ''.

Theorem 1.10 will be shown in Section 6 by adapting arguments in [8]. We shall prove Theorem 1.4 also in Section 6 by applying Theorem 1.10 as Theorem 1.2 is shown from Theorem 1.9. To prove Theorem 1.10 we need a geometrical result slightly different from Lemma 2.3, which will be given in Section 5 (see Lemma 5.3).

2. Decomposition of a neighborhood of curve

Let $I = [a, b]$ be a compact interval in $\mathbb{R}$ and $\psi \in C^\infty(I)$. Let $0 < \delta \leq 1$. We are interested in the case where δ is much smaller than $|I| = b - a$. Suppose that $|\psi''| > 0$ on I. We define a partition $\{\omega_1, \omega_2, \ldots, \omega_K\}$ of I consisting of subintervals of I as follows: $\omega_j = [a_{j-1}, a_j]$, $1 \leq j \leq K$, with $a = a_0 < a_1 < \cdots < a_{K-1} < a_K = b$, $|\omega_j| = \delta^{1/2}$, $1 \leq j \leq K - 1$, $|\omega_K| \leq \delta^{1/2}$. Then we have $K \leq |I|\delta^{-1/2} + 1$.

We divide the intervals $\{\omega_j\}$ into 4 families:

$$(2.1) \qquad \begin{aligned} \mathcal{F}_1 &= \{\omega_1, \omega_5, \omega_9, \ldots\}, \quad \mathcal{F}_2 = \{\omega_2, \omega_6, \omega_{10}, \ldots\}, \\ \mathcal{F}_3 &= \{\omega_3, \omega_7, \omega_{11}, \ldots\}, \quad \mathcal{F}_4 = \{\omega_4, \omega_8, \omega_{12}, \ldots\}. \end{aligned}$$

We write $\{I_1, I_2, I_3, \ldots, I_L\} = \{\omega_1, \omega_5, \omega_9, \ldots\}$ and consider this family of intervals.

Let

$$E_{i_k, j_k} = \{(\xi_k - \eta_k, \psi(\xi_k) - \psi(\eta_k)) : \xi_k \in I_{i_k}, \eta_k \in I_{j_k}\},$$

for $i_k, j_k = 1, 2, \ldots, L$, $k = 1, 2$.

Lemma 2.1. *Let $m_0 = \inf_{t \in I} |\psi''(t)|$. Suppose that $m_0 > 0$. There exists a positive constant c_0 such that if $(i_1, j_1) \neq (i_2, j_2)$ and $i_k < j_k$, $k = 1, 2$, then*

$$\mathrm{d}(E_{i_1, j_1}, E_{i_2, j_2}) \geq c_0 \delta,$$

where $\mathrm{d}(E, F) = \inf_{x \in E, y \in F} |x - y|$.

To prove Lemma 2.1, we apply the following.

Lemma 2.2. *Let ψ be as in Lemma 2.1. There exist positive constants c_1, c_2 such that $c_1 < 2$ and if $\xi_k \in I_{i_k}$, $\eta_k \in I_{j_k}$, $k = 1, 2$, $(i_1, j_1) \neq (i_2, j_2)$, $i_k < j_k$, $k = 1, 2$, and $|\xi_1 - \eta_1 - (\xi_2 - \eta_2)| \leq c_1 \delta$, then*

$$J := |\psi(\xi_1) - \psi(\eta_1) - (\psi(\xi_2) - \psi(\eta_2))| \geq c_2 \delta.$$

Lemma 2.1 follows from Lemma 2.2 since Lemma 2.2 implies that

$$\begin{aligned} &|(\xi_1 - \eta_1, \psi(\xi_1) - \psi(\eta_1)) - (\xi_2 - \eta_2, \psi(\xi_2) - \psi(\eta_2))| \\ &\geq \max\left(|\xi_1 - \eta_1 - (\xi_2 - \eta_2)|, |\psi(\xi_1) - \psi(\eta_1) - (\psi(\xi_2) - \psi(\eta_2))|\right) \\ &\geq \min(c_1, c_2)\delta. \end{aligned}$$

Let ψ and c_0 be as in Lemma 2.1 and

$$(2.2) \quad E_{i_k, j_k}^* = \{(\xi_k - \eta_k, \xi_k' - \eta_k') : \psi(\xi_k) \leq \xi_k' \leq \psi(\xi_k) + 10^{-1}c_0\delta,$$
$$\psi(\eta_k) \leq \eta_k' \leq \psi(\eta_k) + 10^{-1}c_0\delta, \quad \xi_k \in I_{i_k}, \eta_k \in I_{j_k}\},$$

for $i_k, j_k = 1, \ldots, L$, $k = 1, 2$. We note that $E_{i_k, j_k}^* = -E_{j_k, i_k}^*$, where $-E = \{-\xi : \xi \in E\}$.

Lemma 2.3. *Suppose that ψ and c_0 are as in Lemma 2.1. If $(i_1, j_1) \neq (i_2, j_2)$ and $i_k < j_k$, $k = 1, 2$, then we have*

$$\mathrm{d}\left(E^*_{i_1, j_1}, E^*_{i_2, j_2}\right) \geq (3/5)c_0\delta.$$

Proof. Let $\psi(\xi_k) \leq \xi'_k \leq \psi(\xi_k) + 10^{-1}c_0\delta$, $\psi(\eta_k) \leq \eta'_k \leq \psi(\eta_k) + 10^{-1}c_0\delta$, $\xi_k \in I_{i_k}, \eta_k \in I_{j_k}$, $k = 1, 2$. Then by Lemma 2.1 we see that

$$\begin{aligned}
&|(\xi_1 - \eta_1, \xi'_1 - \eta'_1) - (\xi_2 - \eta_2, \xi'_2 - \eta'_2)| \\
&\geq |(\xi_1 - \eta_1, \psi(\xi_1) - \psi(\eta_1)) - (\xi_2 - \eta_2, \psi(\xi_2) - \psi(\eta_2))| \\
&\quad - |(\xi_1 - \eta_1, \xi'_1 - \eta'_1) - (\xi_1 - \eta_1, \psi(\xi_1) - \psi(\eta_1))| \\
&\quad - |(\xi_2 - \eta_2, \xi'_2 - \eta'_2) - (\xi_2 - \eta_2, \psi(\xi_2) - \psi(\eta_2))| \\
&\geq c_0\delta - (2/10)c_0\delta - (2/10)c_0\delta = (3/5)c_0\delta.
\end{aligned}$$

This implies the conclusion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In the proof above we have used Lemma 2.1, which follows from Lemma 2.2. Thus to complete the proof of Lemma 2.3 it remains to show Lemma 2.2.

Proof of Lemma 2.2. If $c_1 < 2$ and $|\xi_1 - \eta_1 - (\xi_2 - \eta_2)| \leq c_1\delta$, then we have $i_1 \neq i_2$. To see this, suppose that $i_1 = i_2$. Then $j_1 \neq j_2$, since $(i_1, j_1) \neq (i_2, j_2)$. So we have

$$|\xi_1 - \eta_1 - (\xi_2 - \eta_2)| \geq |\eta_1 - \eta_2| - |\xi_1 - \xi_2| \geq 3\delta^{1/2} - \delta^{1/2} \geq 2\delta.$$

This contradicts the assumption that $c_1 < 2$. Thus we have $i_1 \neq i_2$.

We may assume that $i_1 < i_2$ without loss of generality. Then $j_1 < j_2$. This can be shown as follows by arguing as above. Suppose that $j_2 \leq j_1$. Then

$$|\xi_1 - \eta_1 - (\xi_2 - \eta_2)| = \xi_2 - \xi_1 + (\eta_1 - \eta_2) \geq 3\delta^{1/2} - \delta^{1/2} \geq 2\delta,$$

which leads to a contradiction.

Therefore, to prove the lemma it suffices to estimate J under the condition that $(i_1, j_1) \neq (i_2, j_2)$, $i_1 < j_1$, $i_2 < j_2$, $i_1 < i_2$, $j_1 < j_2$. We see that

$$J = \left|\int_{\xi_1}^{\xi_2} \psi'(t)\,dt - \int_{\eta_1}^{\eta_2} \psi'(t)\,dt\right|.$$

Put $\tau = \min(\xi_2 - \xi_1, \eta_2 - \eta_1)$. Then

$$J \geq \left|\int_{\xi_1}^{\xi_1 + \tau} \psi'(t)\,dt - \int_{\eta_1}^{\eta_1 + \tau} \psi'(t)\,dt\right| - \left|\int_{\xi_1 + \tau}^{\xi_2} \psi'(t)\,dt\right| - \left|\int_{\eta_1 + \tau}^{\eta_2} \psi'(t)\,dt\right| =: J^*.$$

Let $D_0 = \sup_{t \in I} |\psi'(t)|$. Then

$$\left|\int_{\xi_1 + \tau}^{\xi_2} \psi'(t)\,dt\right| \leq D_0(\xi_2 - \xi_1 - \tau), \quad \left|\int_{\eta_1 + \tau}^{\eta_2} \psi'(t)\,dt\right| \leq D_0(\eta_2 - \eta_1 - \tau).$$

We note that

$$\xi_2 - \xi_1 + \eta_2 - \eta_1 - 2\tau = |\xi_2 - \xi_1 - (\eta_2 - \eta_1)| \leq c_1\delta.$$

Therefore

$$
\begin{aligned}
J^* &\geq \left| \int_{\xi_1}^{\xi_1+\tau} \psi'(t)\,dt - \int_{\eta_1}^{\eta_1+\tau} \psi'(t)\,dt \right| - D_0|\xi_2 - \xi_1 - (\eta_2 - \eta_1)| \\
&\geq \left| \int_{\xi_1}^{\xi_1+\tau} \psi'(t)\,dt - \int_{\eta_1}^{\eta_1+\tau} \psi'(t)\,dt \right| - c_1 D_0 \delta \\
&= \left| \int_{\eta_1}^{\eta_1+\tau} (\psi'(t + \xi_1 - \eta_1) - \psi'(t))\,dt \right| - c_1 D_0 \delta \\
&= \int_{\eta_1}^{\eta_1+\tau} \left(\int_{t+\xi_1-\eta_1}^{t} |\psi''(s)|\,ds \right) dt - c_1 D_0 \delta =: J^{**}.
\end{aligned}
$$

We note that $\xi_1 - \eta_1 \leq -3\delta^{1/2}$ and $\tau \geq 3\delta^{1/2}$. Thus $J^{**} \geq 9m_0\delta - c_1 D_0 \delta$. So, if $c_1 D_0 < 9m_0$, $0 < c_2 \leq 9m_0 - c_1 D_0$ and $0 < c_1 < 2$, the constants c_1, c_2 satisfy what is needed. This completes the proof of Lemma 2.2. $\square$

3. Proof of Theorem 1.9

Let $\phi \in C_0^\infty(\mathbb{R})$ be supported in $[1/2, 2]$ and $\sum_{n=-\infty}^{\infty} \phi(2^n t) = 1$ for $t > 0$. Decompose σ as $\sigma = \sigma_0 + \sigma_1$ with

$$
\sigma_0(\xi) = \sum_{n=1}^{\infty} \sigma(\xi)\phi(2^n \kappa^{-1}(\xi_2 - \psi(\xi_1))),
$$

where $\kappa = c_0/30$ and c_0 is as in Lemma 2.3. Let $n \geq 1$. Let $\{\omega_1, \ldots, \omega_K\}$ be a partition of I as in Section 2 with $\delta = 2^{-n}$. We decompose

$$
\sigma(\xi)\phi(2^n \kappa^{-1}(\xi_2 - \psi(\xi_1))) = \sum_{j=1}^{K} \sigma(\xi)\phi(2^n \kappa^{-1}(\xi_2 - \psi(\xi_1)))\chi_{\omega_j \times \mathbb{R}}(\xi).
$$

Let $\mathcal{F}_1 = \{I_1, I_2, \ldots, I_L\}$ be the subcollection of $\{\omega_1, \ldots, \omega_K\}$ as in Lemma 2.3. We consider

$$
\sum_{j=1}^{L} \sigma(\xi)\phi(2^n \kappa^{-1}(\xi_2 - \psi(\xi_1)))\chi_{I_j \times \mathbb{R}}(\xi)
$$

and write

$$
\sum_{j=1}^{L} \sigma(\xi)\phi(2^n \kappa^{-1}(\xi_2 - \psi(\xi_1)))\chi_{I_j \times \mathbb{R}}(\xi) = \sum_{j=1}^{L} s_j^{(n)}(\xi) = \sum_{j=1}^{L} \sigma_j^{(n)}(\xi)\chi_{I_j \times \mathbb{R}}(\xi),
$$

with

$$
(3.1) \qquad s_j^{(n)}(\xi) = \sigma(\xi)\phi(2^n \kappa^{-1}(\xi_2 - \psi(\xi_1)))\chi_{I_j \times \mathbb{R}}(\xi),
$$

and

$$
(3.2) \qquad \sigma_j^{(n)}(\xi) = \sigma(\xi)\phi(2^n \kappa^{-1}(\xi_2 - \psi(\xi_1)))\Phi(2^{n/2}(\xi_1 - c_j)),
$$

where c_j is the center of I_j and Φ is in $C_0^\infty(\mathbb{R})$ such that $\Phi(2^{n/2}(\xi_1 - c_j)) = 1$ for $\xi_1 \in I_j$ for every j.

We see that

$$
|(\partial \xi)^\gamma \sigma_j^{(n)}(\xi)| \leq C\Theta(2^{-n})2^{n|\gamma|},
$$

where $\gamma = (\gamma_1, \gamma_2)$, $\gamma_1, \gamma_2 \in \mathbb{Z} \cap [0, 3]$, $|\gamma| = \gamma_1 + \gamma_2$ and $(\partial \xi)^\gamma = (\partial/\partial \xi_1)^{\gamma_1}(\partial/\partial \xi_2)^{\gamma_2}$. Here $\mathbb{Z}$ denotes the set of integers. We observe that

$$\partial t(c_j)(\xi_2 - \psi(\xi_1)) = t_1(c_j)(-\psi'(\xi_1)) + t_2(c_j)$$
$$= -(1 + \psi'(c_j)^2)^{-1/2}(\psi'(\xi_1) - \psi'(c_j))$$

and hence taking the support of $\sigma_j^{(n)}$ into account we have

$$\left| (\partial t(c_j))^\alpha \sigma_j^{(n)}(\xi) \right| \leq C\Theta(2^{-n}) 2^{\alpha n/2}.$$

Similarly, by direct computation, we also see that

$$(3.3) \qquad \left| (\partial t(c_j))^\alpha (\partial n(c_j))^\beta \sigma_j^{(n)}(\xi) \right| \leq C\Theta(2^{-n}) 2^{\alpha n/2} 2^{\beta n} \quad \text{for } \alpha, \beta \in \mathbb{Z} \cap [0, 3].$$

Define the Fourier multiplier operators $S_j^{(n)}$ and $T_j^{(n)}$ by

$$S_j^{(n)}(f) = T_{s_j^{(n)}}(f), \quad T_j^{(n)}(f) = T_{\sigma_j^{(n)}}(f), \quad f \in \mathcal{S}(\mathbb{R}^2),$$

where $s_j^{(n)}$ and $\sigma_j^{(n)}$ are as in (3.1) and (3.2), respectively. We have the following result.

Proposition 3.1. *There exists a sequence of L^2 bounded operators $\{A_n\}_{n=1}^\infty$ such that for $w \geq 0$*

$$\int_{\mathbb{R}^2} \left| \sum_{j=1}^L S_j^{(n)}(f)(x) \right|^2 w(x)\, dx \leq \int_{\mathbb{R}^2} |f(x)|^2 A_n(w)(x)\, dx$$

and $\|A_n(w)\|_2 \leq C\Theta(2^{-n})^2 n^{1/2} \|w\|_2$.

Proof. We see that

$$\int_{\mathbb{R}^2} \left| \sum_{j=1}^L S_j^{(n)}(f)(x) \right|^2 w(x)\, dx = \sum_{j=1}^L \sum_{k=1}^L \int_{\mathbb{R}^2} S_j^{(n)}(f)(x)\overline{S_k^{(n)}(f)(x)} w(x)\, dx$$

$$= \sum_{j=1}^L \int_{\mathbb{R}^2} |S_j^{(n)}(f)(x)|^2 w(x)\, dx + \sum_{j \neq k} \int_{\mathbb{R}^2} S_j^{(n)}(f)(x)\overline{S_k^{(n)}(f)(x)} w(x)\, dx$$

$$= \sum_{j=1}^L \int_{\mathbb{R}^2} |S_j^{(n)}(f)(x)|^2 w(x)\, dx + \sum_{j \neq k} \int_{\mathbb{R}^2} \mathcal{F}(S_j^{(n)}(f)) * \mathcal{F}\left(\overline{S_k^{(n)}(f)}\right)(\xi)\hat{w}(-\xi)\, d\xi.$$

Let $E_j^{(n)} = \operatorname{supp}\left(s_j^{(n)}\right)$ and $\widetilde{s_k^{(n)}}(\xi) = s_k^{(n)}(-\xi)$. Then, $\operatorname{supp}\left(\widetilde{s_j^{(n)}}\right) = -E_j^{(n)}$. We note that

$$\operatorname{supp}\left(\mathcal{F}(S_j^{(n)}(f)) * \mathcal{F}\left(\overline{S_k^{(n)}(f)}\right)\right) \subset E_j^{(n)} + (-E_k^{(n)}) =: F_{j,k}.$$

Thus, setting $V_{j,k} = T_{\chi_{-F_{j,k}}}$, we see that

$$\int_{\mathbb{R}^2} \left| \sum_{j=1}^{L} S_j^{(n)}(f)(x) \right|^2 w(x)\, dx$$

$$= \sum_{j=1}^{L} \int_{\mathbb{R}^2} |S_j^{(n)}(f)(x)|^2 w(x)\, dx$$

$$+ \sum_{j \neq k} \int_{\mathbb{R}^2} \mathcal{F}(S_j^{(n)}(f)) * \mathcal{F}\left(\overline{S_k^{(n)}(f)}\right)(\xi) \mathcal{F}(V_{j,k}(w))(-\xi)\, d\xi$$

$$= \sum_{j=1}^{L} \int_{\mathbb{R}^2} |S_j^{(n)}(f)(x)|^2 w(x)\, dx + \sum_{j \neq k} \int_{\mathbb{R}^2} S_j^{(n)}(f)(x)\overline{S_k^{(n)}(f)(x)} V_{j,k}(w)(x)\, dx$$

$$\leq \int_{\mathbb{R}^2} \left(\sum_{j=1}^{L} |S_j^{(n)}(f)(x)|^2 \right) \left(w(x) + \left(\sum_{j \neq k} |V_{j,k}(w)(x)|^2 \right)^{1/2} \right) dx =: J,$$

where the last inequality follows by the Schwarz inequality.

We write $T_j^{(n)}(f) = f * K^{(n,j)}$, where $K^{(n,j)} = \mathcal{F}^{-1}(\sigma_j^{(n)})$. Let O_j be a rotation such that $O_j e_1 = t(c_j)$, $O_j e_2 = n(c_j)$, where $e_1 = (1,0)$, $e_2 = (0,1)$. By (3.3) and the support condition of $\sigma_j^{(n)}$ we have

$$(3.4) \qquad |K^{(n,j)}(O_j x)| \leq C\Theta(2^{-n}) 2^{-n/2} 2^{-n} |2^{-n/2} x_1|^{-\alpha} |2^{-n} x_2|^{-\beta}$$

for all $\alpha, \beta \in \mathbb{Z} \cap [0,3]$. Let

$$R_m^{(n)} = \{ x \in \mathbb{R}^2 : |x_1| \leq 2^m 2^{n/2}, |x_2| \leq 2^m 2^n \}$$

for $m \in \mathbb{Z} \cap [0, \infty)$ and put $R_m^{(n,j)} = O_j R_m^{(n)}$. Then by (3.4) we have

$$(3.5) \qquad |K^{(n,j)}(x)| \leq C\Theta(2^{-n}) \sum_{m=0}^{\infty} 2^{-m} |R_m^{(n,j)}|^{-1} \chi_{R_m^{(n,j)}}(x).$$

Let $P_j = T_{\chi_{I_j \times \mathbb{R}}}$. Then $S_j^{(n)}(f) = T_j^{(n)}(P_j f)$. Thus, putting

$$V_n(w)(x) = w(x) + \left(\sum_{1 \leq j,k \leq L, j \neq k} |V_{j,k}(w)(x)|^2 \right)^{1/2},$$

by (3.5) we have

$$(3.6) \quad J = \int_{\mathbb{R}^2} \left(\sum_{j=1}^{L} |T_j^{(n)}(P_j f)(x)|^2 \right) V_n(w)(x)\, dx$$

$$\leq C\Theta(2^{-n})^2 \sum_{j=1}^{L} \int_{\mathbb{R}^2} |P_j(f)(x)|^2 \sum_{m=0}^{\infty} 2^{-m} |R_m^{(n,j)}|^{-1} \chi_{R_m^{(n,j)}} * V_n(w)(x)\, dx.$$

Define the Kakeya maximal function

$$M_{n,m}(f)(x) = \sup_{1 \leq j \leq L} |R_m^{(n,j)}|^{-1} \left| \chi_{R_m^{(n,j)}} * f(x) \right|.$$

Then by (3.6) we have

$$(3.7) \qquad J \leq C\Theta(2^{-n})^2 \int_{\mathbb{R}^2} \sum_{j=1}^{L} |P_j f(x)|^2 \sum_{m=0}^{\infty} 2^{-m} M_{n,m}(V_n(w))(x)\,dx.$$

Let $W_n = \sum_{m=0}^{\infty} 2^{-m} M_{n,m}(V_n(w))$. Then (3.7) implies that

$$(3.8) \qquad J \leq C\Theta(2^{-n})^2 \int_{\mathbb{R}^2} |f(x)|^2 M_{\vartheta}^{(1)}(W_n)(x)\,dx,$$

where $M_{\vartheta}^{(1)}(g)(x) = M^{(1)}(|g|^{\vartheta})(x)^{1/\vartheta}$, with

$$M^{(1)}(g)(x) = \sup_{h>0} (2h)^{-1} \int_{-h}^{h} |g(x_1 - t, x_2)|\,dt$$

for $\vartheta \in (1,2)$. We note that $W_n \leq M_{\vartheta}^{(1)}(W_n)$ almost everywhere and $M_{\vartheta}^{(1)}(W_n)$ belongs to the weight class A_1 of Muckenhoupt in the x_1 variable uniformly in x_2. It is well-known that the inequality (3.8) follows from (3.7).

We have

$$\|M_{n,m}(f)\|_2 \leq C n^{1/2} \|f\|_2,$$

where C is a constant independent of n and m (see [6, Theorem 2]). We note that $F_{j,k} \subset E_{j,k}^*$, $j, k = 1, \ldots, L$, where $E_{j,k}^*$ is as in (2.2) with $\delta = 2^{-n}$. By applying Lemma 2.3, we have

$$\sum_{j \neq k} \chi_{E_{j,k}^*}(\xi) = \sum_{j<k} (\chi_{E_{j,k}^*}(\xi) + \chi_{E_{j,k}^*}(-\xi)) \leq C.$$

Therefore, we see that

$$\|M_{\vartheta}^{(1)}(W_n)\|_2 \leq C\|W_n\|_2 \leq \sum_{m=0}^{\infty} 2^{-m} \|M_{n,m}(V_n(w))\|_2 \leq C n^{1/2} \|V_n(w)\|_2$$

$$\leq C n^{1/2} \left(\|w\|_2 + \left(\sum_{j \neq k} \|V_{j,k}(w)\|_2^2 \right)^{1/2} \right)$$

$$\leq C n^{1/2} \|w\|_2.$$

Thus we can take $A_n(w) = C\Theta(2^{-n})^2 M_{\vartheta}^{(1)}(W_n)$. This completes the proof of Proposition 3.1. $\qquad \square$

Proof of Theorem 1.9. Recall that $\sigma = \sigma_0 + \sigma_1$. In proving Theorem 1.9, it suffices to show a weighted inequality analogous to that claimed in Theorem 1.9 with T_{σ_0} in place of T_σ, since T_{σ_1} can be handled by the weighted inequality for the Hardy-Littlewood maximal functions. To deal with T_{σ_0} we decompose $T_{\sigma_0} = \sum_{k=1}^{4} S_{\sigma,k}$, where $S_{\sigma,1} = \sum_{n=1}^{\infty} \sum_{j=1}^{L} S_j^{(n)}$ and each $S_{\sigma,k}$, $2 \leq k \leq 4$, is defined similarly to $S_{\sigma,1}$ by using $\mathcal{F}_k$ in (2.1) in place of $\mathcal{F}_1$. Each $S_{\sigma,k}$, $1 \leq k \leq 4$, is estimated similarly. We only give an estimate for $S_{\sigma,1}$, which combined with similar estimates for $S_{\sigma,k}$, $2 \leq k \leq 4$, will complete the estimate needed for T_{σ_0}.

Let $S_\sigma = S_{\sigma,1}$. Setting $\|(a_n)\|_1 = \sum_{n=1}^{\infty} a_n$, by the Schwarz inequality we have

$$|S_\sigma(f)|^2 \leq \|(a_n)\|_1 \sum_{n=1}^{\infty} a_n^{-1} \left| \sum_{j=1}^{L} S_j^{(n)}(f) \right|^2.$$

Thus, applying Proposition 3.1, we see that

$$\int_{\mathbb{R}^2} |S_\sigma(f)(x)|^2 w(x)\,dx \leq \|(a_n)\|_1 \sum_{n=1}^{\infty} a_n^{-1} \int_{\mathbb{R}^2} \left| \sum_{j=1}^{L} S_j^{(n)}(f)(x) \right|^2 w(x)\,dx$$

$$\leq \|(a_n)\|_1 \sum_{n=1}^{\infty} a_n^{-1} \int_{\mathbb{R}^2} |f(x)|^2 A_n(w)(x)\,dx$$

$$= \int_{\mathbb{R}^2} |f(x)|^2 U(w)(x)\,dx,$$

where $U(w) = \|(a_n)\|_1 \sum_{n=1}^{\infty} a_n^{-1} A_n(w)$. We observe that

$$\|U(w)\|_2 \leq \|(a_n)\|_1 \sum_{n=1}^{\infty} a_n^{-1}\|A_n(w)\|_2$$

$$\leq C\|(a_n)\|_1 \sum_{n=1}^{\infty} a_n^{-1}\Theta(2^{-n})^2 n^{1/2}\|w\|_2 \leq C_1\|w\|_2.$$

This completes the proof of Theorem 1.9.

4. Proof of Theorem 1.2

If $\psi'' \neq 0$ on I, we can apply Theorem 1.9 directly to prove Theorem 1.2, since if $\Theta(\delta) = \delta^\lambda$, then $\sigma_\lambda \in \mathcal{M}(\psi, \Theta)$ and we can see that $\Theta(\delta) = \delta^\lambda$ satisfies the requirement in Theorem 1.9.

To prove Theorem 1.2 in the case when ψ'' has zeros of finite order by applying Theorem 1.9 we need the following result, which can be shown by a straight forward computation.

Lemma 4.1. *Let A be a non-singular linear transformation on $\mathbb{R}^2$. Put $m_A(\xi) = m(A\xi)$ for a bounded function m. Let $B = A^t$ (the transpose of A). Then, for $f \in \mathcal{S}(\mathbb{R}^2)$, we have $T_{m_A}(f)(x) = T_m(f_B)(B^{-1}x)$.*

Proof of Theorem 1.2 *when ψ'' has zeros of finite order.* Applying a suitable partition of unity and change of variables, we may assume that $0 \in I^\circ$ and ψ'' vanishes in I only at 0 and that ψ is of the form

$$(4.1) \qquad \psi(t) = \alpha_0 t^d + O(t^{d+1}), \quad d \geq 3, \quad \alpha_0 \neq 0.$$

Then to prove the theorem, it is sufficient to handle the case when the function $a(\xi)$ in the definition of σ_λ satisfies that $a(\xi) = 1$ near the origin, from which we can deduce the result for the general case.

Define a linear transformation $L_\epsilon(\xi) = (\epsilon^{-1}\xi_1, \epsilon^{-d}\xi_2)$ for $\epsilon > 0$. Let $b(\xi) = a(\xi) - a(L_{2^{-1}}\xi)$. Then $a(\xi) = \sum_{k=0}^{\infty} b(L_{2^{-k}}\xi)$ for $\xi \neq 0$. Using this, we decompose

$$(4.2) \qquad \sigma_\lambda(\xi) = a(\xi)(\xi_2 - \psi(\xi_1))_+^\lambda = \sum_{k=0}^{\infty} b(L_{2^{-k}}\xi)(\xi_2 - \psi(\xi_1))_+^\lambda.$$

Let $\psi^{(\epsilon)}(t) = \epsilon^{-d}\psi(\epsilon t)$. Then $\psi^{(\epsilon)}(t) \to \alpha_0 t^d$ in $C^\infty(I)$ as $\epsilon \to 0$. Thus, since b vanishes near 0, if $\sigma_{\lambda,\epsilon}(\xi) = b(\xi)(\xi_2 - \psi^{(\epsilon)}(\xi_1))_+^\lambda$, by the case when ψ'' has no zeros, we have

$$(4.3) \qquad \int_{\mathbb{R}^2} |T_{\sigma_{\lambda,\epsilon}}(f)(x)|^2 w(x)\,dx \leq \int_{\mathbb{R}^2} |f(x)|^2 U_\epsilon(w)(x)\,dx$$

with a bounded operator U_ϵ on L^2; $\|U_\epsilon(w)\|_2 \le C\|w\|_2$ with the constant C independent of $\epsilon \in (0,1)$. Note that

$$(\sigma_{\lambda,\epsilon})_{L_\epsilon}(\xi) = \epsilon^{-d\lambda} b(L_\epsilon \xi)(\xi_2 - \psi(\xi_1))_+^\lambda.$$

Therefore, if $m_{\lambda,\epsilon}(\xi) = b(L_\epsilon \xi)(\xi_2 - \psi(\xi_1))_+^\lambda$, by Lemma 4.1 and (4.3) we see that

$$(4.4) \qquad \int_{\mathbb{R}^2} |T_{m_{\lambda,\epsilon}}(f)(x)|^2 w(x)\,dx = \epsilon^{2d\lambda} \int_{\mathbb{R}^2} |T_{(\sigma_{\lambda,\epsilon})_{L_\epsilon}}(f)(x)|^2 w(x)\,dx$$

$$= \epsilon^{2d\lambda} \int_{\mathbb{R}^2} |T_{\sigma_{\lambda,\epsilon}}(f_{L_\epsilon})(L_\epsilon^{-1}x)|^2 w(x)\,dx$$

$$= \epsilon^{2d\lambda}\epsilon^{-d-1} \int_{\mathbb{R}^2} |T_{\sigma_{\lambda,\epsilon}}(f_{L_\epsilon})(x)|^2 w(L_\epsilon x)\,dx$$

$$\le \epsilon^{2d\lambda}\epsilon^{-d-1} \int_{\mathbb{R}^2} |f_{L_\epsilon}(x)|^2 U_\epsilon(w_{L_\epsilon})(x)\,dx$$

$$= \epsilon^{2d\lambda} \int_{\mathbb{R}^2} |f(x)|^2 (U_\epsilon(w_{L_\epsilon}))_{L_\epsilon^{-1}}(x)\,dx.$$

By (4.2) and (4.4) we have

$$\|T_{\sigma_\lambda}(f)\|_{L^2(w)} \le \sum_{k=0}^\infty \|T_{m_{\lambda,2^{-k}}}(f)\|_{L^2(w)}$$

$$\le \sum_{k=0}^\infty 2^{-d\lambda k}\|f\|_{L^2\left((U_{2^{-k}}(w_{L_{2^{-k}}}))_{L_{2^{-k}}^{-1}}\right)}$$

$$\le \|f\|_{L^2(\widetilde{U}(w))},$$

where

$$\widetilde{U}(w) = \left(\sum_{k=0}^\infty 2^{-d\lambda k}\right) \sum_{k=0}^\infty 2^{-d\lambda k}(U_{2^{-k}}(w_{L_{2^{-k}}}))_{L_{2^{-k}}^{-1}}.$$

The L^2 boundedness of $\widetilde{U}$ follows from that of $U_{2^{-k}}$. This completes the proof of Theorem 1.2. $\hfill\square$

5. Another result on decomposition of a neighborhood of curve

Suppose that $m_0 = \inf_{t \in I} |\psi''(t)| > 0$. Let $\mathcal{F}_1 = \{I_1, I_2, I_3, \dots, I_L\}$ be as in Section 2. Let $1/2 \le R \le 2$. Define

$$F_{i_k,j_k} = \{(\xi_k + R\eta_k, \psi(\xi_k) + R\psi(\eta_k)) : \xi_k \in I_{i_k}, \eta_k \in I_{j_k}\},$$

for $i_k, j_k = 1, 2, \dots, L$, $k = 1, 2$.

Lemma 5.1. *There exists a positive constant c_0 such that if $(i_1, j_1) \ne (i_2, j_2)$, $i_k \le j_k$, $k = 1, 2$, then*

$$\mathrm{d}(F_{i_1,j_1}, F_{i_2,j_2}) \ge c_0\delta.$$

Lemma 5.2. *There exist positive constants c_1, c_2 such that $c_1 < 1/2$ and if $\xi_k \in I_{i_k}$, $\eta_k \in I_{j_k}$, $i_k \le j_k$, $k = 1, 2$, $(i_1, j_1) \ne (i_2, j_2)$ and $|\xi_1 + R\eta_1 - (\xi_2 + R\eta_2)| \le c_1\delta$, then*

$$N := |\psi(\xi_1) + R\psi(\eta_1) - (\psi(\xi_2) + R\psi(\eta_2))| \ge c_2\delta.$$

This implies Lemma 5.1 as Lemma 2.2 implies Lemma 2.1. We note that in Lemma 5.2 the condition that $i_k < j_k$ is not set, which is assumed in Lemma 2.2.

Define

$$(5.1) \quad F^*_{i_k,j_k} = F^*_{i_k,j_k}(R) = \{(\xi_k + R\eta_k, \xi'_k + R\eta'_k) : \psi(\xi_k) \le \xi'_k \le \psi(\xi_k) + 10^{-1}c_0\delta,$$

$$\psi(\eta_k) \le \eta'_k \le \psi(\eta_k) + 10^{-1}c_0\delta, \quad \xi_k \in I_{i_k}, \eta_k \in I_{j_k}\},$$

for $i_k, j_k = 1, \dots, L, \ k = 1, 2$, where ψ, c_0 are as in Lemma 5.1. We note that $F^*_{j_k,i_k}(R) = RF^*_{i_k,j_k}(R^{-1})$.

Lemma 5.3. *If* $(i_1,j_1) \ne (i_2,j_2)$, $i_k \le j_k$, $k = 1, 2$, *then we have*

$$\mathrm{d}\left(F^*_{i_1,j_1}, F^*_{i_2,j_2}\right) \ge (2/5)c_0\delta.$$

Proof. Let $\psi(\xi_k) \le \xi'_k \le \psi(\xi_k) + 10^{-1}c_0\delta$, $\psi(\eta_k) \le \eta'_k \le \psi(\eta_k) + 10^{-1}c_0\delta$, $\xi_k \in I_{i_k}, \eta_k \in I_{j_k}$, $k = 1, 2$. Then, using Lemma 5.1, we have

$$|(\xi_1 + R\eta_1, \xi'_1 + R\eta'_1) - (\xi_2 + R\eta_2, \xi'_2 + R\eta'_2)|$$
$$\ge |(\xi_1 + R\eta_1, \psi(\xi_1) + R\psi(\eta_1)) - (\xi_2 + R\eta_2, \psi(\xi_2) + R\psi(\eta_2))|$$
$$- |(\xi_1 + R\eta_1, \xi'_1 + R\eta'_1) - (\xi_1 + R\eta_1, \psi(\xi_1) + R\psi(\eta_1))|$$
$$- |(\xi_2 + R\eta_2, \xi'_2 + R\eta'_2) - (\xi_2 + R\eta_2, \psi(\xi_2) + R\psi(\eta_2))|$$
$$\ge c_0\delta - (3/10)c_0\delta - (3/10)c_0\delta = (2/5)c_0\delta.$$

This will imply the result claimed. $\qquad\square$

Collecting results above, we see that to prove Lemma 5.3 it remains only to show Lemma 5.2.

Proof of Lemma 5.2. Under the assumptions of the lemma, we have $i_1 \ne i_2$. To see this, suppose that $i_1 = i_2$. Then $j_1 \ne j_2$, since $(i_1,j_1) \ne (i_2,j_2)$. Therefore

$$|\xi_1 + R\eta_1 - (\xi_2 + R\eta_2)| \ge R|\eta_1 - \eta_2| - |\xi_1 - \xi_2| \ge 3R\delta^{1/2} - \delta^{1/2} \ge (1/2)\delta^{1/2} \ge (1/2)\delta.$$

This contradicts the assumption that $c_1 < 1/2$. Consequently, $i_1 \ne i_2$.

We may assume that $i_1 < i_2$ without loss of generality. Then we have $j_2 < j_1$. This can be shown as follows by arguing as above. Suppose that $j_1 \le j_2$, then

$$|\xi_1 + R\eta_1 - (\xi_2 + R\eta_2)| = \xi_2 - \xi_1 + R(\eta_2 - \eta_1) \ge 3\delta^{1/2} - 2\delta^{1/2} = \delta^{1/2},$$

from which a contradiction will follow.

Consequently, to estimate N we may assume that $(i_1,j_1) \ne (i_2,j_2)$, $i_1 \le j_1$, $i_2 \le j_2$, $i_1 < i_2$ and $j_2 < j_1$. We have

$$N = \left|\int_{\xi_1}^{\xi_2} \psi'(t)\, dt - R\int_{\eta_2}^{\eta_1} \psi'(t)\, dt\right| = \left|\int_{\xi_1}^{\xi_2} \psi'(t)\, dt - \int_{R\eta_2}^{R\eta_1} \psi'(t/R)\, dt\right|.$$

Put $\tau = \min(\xi_2 - \xi_1, R(\eta_1 - \eta_2))$. Then

$$N \ge \left|\int_{\xi_1}^{\xi_1+\tau} \psi'(t)\, dt - \int_{R\eta_2}^{R\eta_2+\tau} \psi'(t/R)\, dt\right| - \left|\int_{\xi_1+\tau}^{\xi_2} \psi'(t)\, dt\right| - \left|\int_{R\eta_2+\tau}^{R\eta_1} \psi'(t/R)\, dt\right|$$
$$=: N^*.$$

If $D_0 = \sup_{t \in I} |\psi'(t)|$, then

$$\left|\int_{\xi_1+\tau}^{\xi_2} \psi'(t)\, dt\right| \le D_0(\xi_2 - \xi_1 - \tau), \quad \left|\int_{R\eta_2+\tau}^{R\eta_1} \psi'(t)\, dt\right| \le D_0(R(\eta_1 - \eta_2) - \tau).$$

We note that

$$\xi_2 - \xi_1 + R(\eta_1 - \eta_2) - 2\tau = |\xi_2 - \xi_1 - R(\eta_1 - \eta_2)| \le c_1 \delta.$$

Thus

$$N^* \ge \left| \int_{\xi_1}^{\xi_1+\tau} \psi'(t)\,dt - \int_{R\eta_2}^{R\eta_2+\tau} \psi'(t/R)\,dt \right| - D_0 |\xi_2 - \xi_1 - R(\eta_1 - \eta_2)|$$

$$\ge \left| \int_{\xi_1}^{\xi_1+\tau} \psi'(t)\,dt - \int_{R\eta_2}^{R\eta_2+\tau} \psi'(t/R)\,dt \right| - c_1 D_0 \delta$$

$$= \left| R \int_{\eta_2}^{\eta_2+\tau/R} (\psi'(Rt + \xi_1 - R\eta_2) - \psi'(t))\,dt \right| - c_1 D_0 \delta$$

$$= R \int_{\eta_2}^{\eta_2+\tau/R} \left(\int_{Rt+\xi_1-R\eta_2}^{t} |\psi''(s)|\,ds \right) dt - c_1 D_0 \delta =: N^{**}.$$

Let $\eta_2 \le t \le \eta_2 + \tau/R$. If $1/2 \le R < 1$, then

$$t(1-R) - \xi_1 + R\eta_2 \ge \eta_2(1-R) - \xi_1 + R\eta_2 = \eta_2 - \xi_1 \ge 3\delta^{1/2}.$$

On the other hand, if $1 \le R \le 2$, then

$$t(1-R) - \xi_1 + R\eta_2 \ge (\eta_2 + \tau/R)(1-R) - \xi_1 + R\eta_2$$
$$= \eta_2 - \xi_1 + \tau(1/R - 1)$$
$$\ge \eta_2 - \xi_1 + (\eta_1 - \eta_2)(1-R)$$
$$= \eta_1 - \xi_1 - R(\eta_1 - \eta_2)$$
$$= \eta_1 - \xi_2 + (\xi_2 - \xi_1 - R(\eta_1 - \eta_2))$$
$$\ge \eta_1 - \xi_2 - c_1\delta \ge c\delta^{1/2}$$

for some $c > 0$. Also, $\tau \ge c\delta^{1/2}$ with $c > 0$. Therefore $N^{**} \ge c^2 m_0 \delta - c_1 D_0 \delta$ for some $c > 0$. Thus if c_1 is small enough, we have the desired estimate. This completes the proof of Lemma 5.2. $\qquad\square$

6. Proofs of Theorems 1.10 and 1.4

Let $s_j^{(\ell,n)}(\xi) = s_j^{(n)}(R_\ell^{-1}\xi)$, where $s_j^{(n)}$ is as in (3.1). Let $S_j^{(\ell,n)} = T_{s_j^{(\ell,n)}}$. Then to prove Theorem 1.10 we need the following.

Proposition 6.1. *We have*

$$\int_{\mathbb{R}^2} \left(\sum_{\ell=1}^{\infty} \left| \sum_{j=1}^{L} S_j^{(\ell,n)}(f_\ell)(x) \right|^2 \right)^2 dx \le C\Theta(2^{-n})^4 n^6 \int_{\mathbb{R}^2} \left(\sum_{\ell=1}^{\infty} |f_\ell(x)|^2 \right)^2 dx,$$

where C is independent of n.

Proof. Let $\delta = 2^{-n}$. Let $\mathbb{Z}_u = \{nm + u : m \in \mathbb{Z}\}$, $u \in [0, n-1] \cap \mathbb{Z}$. Then $\{\mathbb{Z}_0, \mathbb{Z}_1, \ldots, \mathbb{Z}_{n-1}\}$ is a partition of $\mathbb{Z}$. If $m_1, m_2 \in \mathbb{Z}_u$ and $m_1 < m_2$, then $2^{m_1} \le \delta 2^{m_2}$. For $m \in \mathbb{Z}$ let $\mathbb{I}_m = \{\ell : 2^m \le R_\ell < 2^{m+1}\}$. Let $S^{(\ell,n)} = \sum_{j=1}^{L} S_j^{(\ell,n)}$. Fix

$u \in [0, n-1] \cap \mathbb{Z}$ and consider

$$
(6.1) \qquad J_{1,u} := \int_{\mathbb{R}^2} \left(\sum_{\ell \in \mathbb{Z}_u} \sum_{r \in \mathcal{I}_\ell} |S^{(r,n)}(f_r)(x)|^2 \right)^2 dx
$$

$$
= \sum_{\ell \in \mathbb{Z}_u} \sum_{m \in \mathbb{Z}_u} \sum_{r \in \mathcal{I}_\ell} \sum_{s \in \mathcal{I}_m} \int_{\mathbb{R}^2} |S^{(r,n)}(f_r)(x) S^{(s,n)}(f_s)(x)|^2 dx
$$

$$
= \sum_{\ell \in \mathbb{Z}_u} \sum_{r \in \mathcal{I}_\ell} \sum_{s \in \mathcal{I}_\ell} \int_{\mathbb{R}^2} |S^{(r,n)}(f_r)(x) S^{(s,n)}(f_s)(x)|^2 dx
$$

$$
+ \sum_{\ell, m \in \mathbb{Z}_u, \ell \neq m} \sum_{r \in \mathcal{I}_\ell} \sum_{s \in \mathcal{I}_m} \int_{\mathbb{R}^2} |S^{(r,n)}(f_r)(x) S^{(s,n)}(f_s)(x)|^2 dx
$$

$$
= J_2 + J_3, \quad \text{say.}
$$

By the Plancherel theorem, we have

$$
\int_{\mathbb{R}^2} |S^{(r,n)}(f_r)(x) S^{(s,n)}(f_s)(x)|^2 dx = \int_{\mathbb{R}^2} \left| \sum_{j,k=1}^{L} S_j^{(r,n)}(f_r)(x) S_k^{(s,n)}(f_s)(x) \right|^2 dx
$$

$$
= \int_{\mathbb{R}^2} \left| \sum_{j,k=1}^{L} \mathcal{F}(S_j^{(r,n)}(f_r)) * \mathcal{F}(S_k^{(s,n)}(f_s))(\xi) \right|^2 d\xi.
$$

Recall that $E_j^{(n)} = \text{supp}\left(s_j^{(n)} \right)$ and note that $\text{supp}\left(s_j^{(\ell,n)} \right) = R_\ell E_j^{(n)}$. Let $r, s \in \mathcal{I}_\ell$. Then $1/2 \leq R_r^{-1} R_s \leq 2$ and $\mathcal{F}(S_j^{(r,n)}(f_r)) * \mathcal{F}(S_k^{(s,n)}(f_s))$ is supported in

$$
R_r E_j^{(n)} + R_s E_k^{(n)} = R_r \left(E_j^{(n)} + R_r^{-1} R_s E_k^{(n)} \right) \subset R_r F_{j,k}^* \left(R_r^{-1} R_s \right),
$$

where $F_{j,k}^*(R)$ is as in (5.1) with $\delta = 2^{-n}$. Thus applying Lemma 5.3, we see that

$$
\int_{\mathbb{R}^2} |S^{(r,n)}(f_r)(x) S^{(s,n)}(f_s)(x)|^2 dx \leq C \sum_{j,k=1}^{L} \int_{\mathbb{R}^2} \left| \mathcal{F}(S_j^{(r,n)}(f_r)) * \mathcal{F}(S_k^{(s,n)}(f_s))(\xi) \right|^2 d\xi
$$

$$
= C \sum_{j,k=1}^{L} \int_{\mathbb{R}^2} \left| S_j^{(r,n)}(f_r) S_k^{(s,n)}(f_s) \right|^2 dx.
$$

Therefore

$$
(6.2) \qquad J_2 \leq C \sum_{\ell \in \mathbb{Z}_u} \sum_{r \in \mathcal{I}_\ell} \sum_{s \in \mathcal{I}_\ell} \sum_{j,k=1}^{L} \int_{\mathbb{R}^2} \left| S_j^{(r,n)}(f_r) S_k^{(s,n)}(f_s) \right|^2 dx
$$

$$
\leq C \sum_{\ell \in \mathbb{Z}_u} \int_{\mathbb{R}^2} \left(\sum_{r \in \mathcal{I}_\ell} \sum_{j=1}^{L} \left| S_j^{(r,n)}(f_r) \right|^2 \right)^2 dx
$$

$$
\leq C \int_{\mathbb{R}^2} \left(\sum_{\ell \in \mathbb{Z}_u} \sum_{r \in \mathcal{I}_\ell} \sum_{j=1}^{L} \left| S_j^{(r,n)}(f_r) \right|^2 \right)^2 dx =: C J_4.
$$

To estimate J_3, we use that if $r \in \mathcal{I}_\ell$, $s \in \mathcal{I}_m$ and $\ell < m$, then $R_r \leq 2\delta R_s$. Also, we note that $\text{d}(R_s E_k^{(n)}, R_s E_{k'}^{(n)}) \geq \delta^{1/2} R_s$ if $k \neq k'$, and $\text{diam}(\cup_{j=1}^{L} \text{supp}(\mathcal{F}(S_j^{(r,n)}(f_r)))) \leq$

CR_r. Here $\operatorname{diam}(E)$ denotes the diameter of E. Thus

$$
\operatorname{supp}\left(\sum_{j=1}^{L}\mathcal{F}(S_j^{(r,n)}(f_r))*\mathcal{F}(S_k^{(s,n)}(f_s))\right)\bigcap\operatorname{supp}\left(\sum_{j=1}^{L}\mathcal{F}(S_j^{(r,n)}(f_r))*\mathcal{F}(S_{k'}^{(s,n)}(f_s))\right)
$$
$$
=\emptyset,
$$

if $k\neq k'$, when $\delta=2^{-n}$ is small enough. Therefore, if $r\in\mathcal{I}_\ell$, $s\in\mathcal{I}_m$ and $\ell<m$, we have

$$
\int_{\mathbb{R}^2}\left|S^{(r,n)}(f_r)(x)S^{(s,n)}(f_s)(x)\right|^2 dx
$$
$$
\leq C\int_{\mathbb{R}^2}\sum_{k=1}^{L}\left|\sum_{j}^{L}\mathcal{F}(S_j^{(r,n)}(f_r))*\mathcal{F}(S_k^{(s,n)}(f_s))(\xi)\right|^2 d\xi.
$$
$$
=C\int_{\mathbb{R}^2}\left|S^{(r,n)}(f_r)(x)\right|^2\sum_{k=1}^{L}\left|S_k^{(s,n)}(f_s)\right|^2 dx,
$$

where C is independent of n. Thus

$$
(6.3)\qquad J_3\leq C\int_{\mathbb{R}^2}\sum_{\ell\in\mathbb{Z}_u}\sum_{r\in\mathcal{I}_\ell}\left|S^{(r,n)}(f_r)(x)\right|^2\sum_{m\in\mathbb{Z}_u}\sum_{s\in\mathcal{I}_m}\sum_{k=1}^{L}\left|S_k^{(s,n)}(f_s)\right|^2 dx
$$
$$
\leq CJ_{1,u}^{1/2}J_4^{1/2},
$$

where the last inequality follows by the Schwarz inequality. By (6.1), (6.2) and (6.3) we have

$$
J_{1,u}\leq CJ_4+CJ_{1,u}^{1/2}J_4^{1/2},
$$

which implies that

$$
(6.4)\qquad\qquad\qquad J_{1,u}\leq CJ_4.
$$

Let $\sigma_j^{(n)}$ be as in (3.2). Set $\sigma_j^{(\ell,n)}(\xi)=\sigma_j^{(n)}(R_\ell^{-1}\xi)$ and $T_j^{(\ell,n)}=T_{\sigma_j^{(\ell,n)}}$. Let $P_j^{(\ell)}=T_{\chi_{R_\ell I_j\times\mathbb{R}}}$. Then $S_j^{(\ell,n)}(f)=T_j^{(\ell,n)}(P_j^{(\ell)}f)$. We note that $T_{\sigma_j^{(\ell,n)}}*f=K^{(\ell,n,j)}*f$ with $K^{(\ell,n,j)}(x)=R_\ell^2K^{(n,j)}(R_\ell x)$, where we recall $K^{(n,j)}=\mathcal{F}^{-1}(\sigma_j^{(n)})$. Thus by (3.5) we have

$$
(6.5)\qquad|K^{(\ell,n,j)}(x)|\leq C\Theta(2^{-n})\sum_{\mu=0}^{\infty}2^{-\mu}|R_\mu^{(\ell,n,j)}|^{-1}\chi_{R_\mu^{(\ell,n,j)}}(x),
$$

where $R_\mu^{(\ell,n,j)}=R_\ell^{-1}R_\mu^{(n,j)}$.

To estimate J_4, for a non-negative $w\in L^2$ let

$$
J_{r,w}=\int_{\mathbb{R}^2}\left(\sum_{j=1}^{L}\left|S_j^{(r,n)}(f_r)\right|^2\right)w(x)\,dx.
$$

By (6.5) we have

$$(6.6) \quad J_{r,w} \leq C\Theta(2^{-n})^2 \sum_{j=1}^{L} \int_{\mathbb{R}^2} |P_j^{(r)}(f_r)(x)|^2 \sum_{\mu=0}^{\infty} 2^{-\mu} |R_\mu^{(r,n,j)}|^{-1} \chi_{R_\mu^{(r,n,j)}} * w(x) \, dx$$

$$\leq C\Theta(2^{-n})^2 \sum_{j=1}^{L} \int_{\mathbb{R}^2} |P_j^{(r)}(f_r)(x)|^2 M_n^*(w)(x) \, dx,$$

where

$$M_n^*(f)(x) = \sup_{1 \leq j \leq L, t > 0} |t R_\mu^{(n,j)}|^{-1} \left| \chi_{t R_\mu^{(n,j)}} * f(x) \right|.$$

We note that $M_n^*(f)$ is independent of μ. It is known that

$$\|M_n^*(f)\|_2 \leq Cn \|f\|_2,$$

where C is independent of n (see [15]).

Let $W_n^* = M_n^*(w)$. Then as in (3.8) from (6.6) it follows that

$$J_{r,w} \leq C\Theta(2^{-n})^2 \int_{\mathbb{R}^2} |f_r(x)|^2 M_\vartheta^{(1)}(W_n^*)(x) \, dx.$$

Thus

$$\sum_{\ell \in \mathbb{Z}_u} \sum_{r \in \mathcal{I}_\ell} J_{r,w} \leq C\Theta(2^{-n})^2 \int_{\mathbb{R}^2} \sum_{\ell \in \mathbb{Z}_u} \sum_{r \in \mathcal{I}_\ell} |f_r(x)|^2 M_\vartheta^{(1)}(W_n^*)(x) \, dx,$$

and hence by the Schwarz inequality and the estimates $\|M_\vartheta^{(1)}(W_n^*)\|_2 \leq Cn\|w\|_2$ we see that

$$J_4 = \left(\sup_{\|w\|_2 \leq 1} \sum_{\ell \in \mathbb{Z}_u} \sum_{r \in \mathcal{I}_\ell} J_{r,w} \right)^2 \leq C\Theta(2^{-n})^4 n^2 \int_{\mathbb{R}^2} \left(\sum_{\ell \in \mathbb{Z}_u} \sum_{r \in \mathcal{I}_\ell} |f_r(x)|^2 \right)^2 \, dx.$$

Therefore, by (6.4) we have

$$J_{1,u}^{1/4} \leq C\Theta(2^{-n}) n^{1/2} \left\| \left(\sum_{\ell=1}^{\infty} |f_\ell|^2 \right)^{1/2} \right\|_4.$$

Using this and a triangle inequality, we have

$$\left\| \left(\sum_{\ell=1}^{\infty} \left| \sum_{j=1}^{L} S_j^{(\ell,n)}(f_\ell)(x) \right|^2 \right)^{1/2} \right\|_4 \leq \sum_{u=0}^{n-1} J_{1,u}^{1/4} \leq C\Theta(2^{-n}) n^{3/2} \left\| \left(\sum_{\ell=1}^{\infty} |f_\ell|^2 \right)^{1/2} \right\|_4.$$

This completes the proof of Proposition 6.1. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 1.10. As in the proof of Theorem 1.9, to prove Theorem 1.10 it suffices to show the vector valued inequality with $\{S_\sigma^{(\ell)}(f_\ell)\}$ in place of $\{T_{\sigma^{(R_\ell)}}(f_\ell)\}$, where $S_\sigma^{(\ell)} = \sum_{n=1}^{\infty} \sum_{j=1}^{L} S_j^{(\ell,n)}$. Also, we may assume $p = 4$ in proving the vector valued inequality; duality and interpolation will provide the result for the whole range of p, $4/3 \leq p \leq 4$. The proof for $\{S_\sigma^{(\ell)}(f_\ell)\}$ will be accomplished by applying Proposition 6.1 via the triangle inequality.

Proof of Theorem 1.4. If $\psi'' \neq 0$ on I, we can derive Theorem 1.4 directly from Theorem 1.10 as Theorem 1.2 is shown from Theorem 1.9.

When ψ'' has zeros of finite order, we may assume that ψ is as in the proof of Theorem 1.2 with the form (4.1). We argue as in the proof of Theorem 1.2 with notations used there. Recall that $\sigma_{\lambda,\epsilon}(\xi) = b(\xi)(\xi_2 - \psi^{(\epsilon)}(\xi_1))_+^\lambda$, $\psi^{(\epsilon)}(\xi_1) = \epsilon^{-d}\psi(\epsilon\xi_1)$, and let $\sigma_\lambda^{(\epsilon,\ell)}(\xi) = \sigma_{\lambda,\epsilon}(R_\ell^{-1}\xi)$. Then by applying Theorem 1.10 we see that

$$(6.7) \qquad \left\| \left(\sum_{\ell=1}^\infty \left| T_{\sigma_\lambda^{(\epsilon,\ell)}}(f_\ell) \right|^2 \right)^{1/2} \right\|_p \leq C \left\| \left(\sum_{\ell=1}^\infty |f_\ell|^2 \right)^{1/2} \right\|_p$$

for $p \in [4/3, 4]$ with a constant C independent of $\epsilon \in (0,1)$. Recall that $m_{\lambda,\epsilon}(\xi) = b(L_\epsilon\xi)(\xi_2 - \psi(\xi_1))_+^\lambda$ and define $m_{\lambda,\epsilon}^{(\ell)}(\xi) = m_{\lambda,\epsilon}(R_\ell^{-1}\xi)$. Then

$$(6.8) \qquad (\sigma_\lambda^{(\epsilon,\ell)})_{L_\epsilon}(\xi) = \epsilon^{-d\lambda} m_{\lambda,\epsilon}^{(\ell)}(\xi).$$

By Lemma 4.1, (6.7) and (6.8) we see that

$$(6.9) \qquad \left\| \left(\sum_{\ell=1}^\infty \left| T_{m_{\lambda,\epsilon}^{(\ell)}}(f_\ell) \right|^2 \right)^{1/2} \right\|_p = \epsilon^{d\lambda} \left\| \left(\sum_{\ell=1}^\infty \left| T_{(\sigma_\lambda^{(\epsilon,\ell)})_{L_\epsilon}}(f_\ell) \right|^2 \right)^{1/2} \right\|_p$$

$$= \epsilon^{d\lambda}\epsilon^{-(d+1)/p} \left\| \left(\sum_{\ell=1}^\infty \left| T_{\sigma_\lambda^{(\epsilon,\ell)}}((f_\ell)_{L_\epsilon}) \right|^2 \right)^{1/2} \right\|_p$$

$$\leq C\epsilon^{d\lambda}\epsilon^{-(d+1)/p} \left\| \left(\sum_{\ell=1}^\infty |(f_\ell)_{L_\epsilon}|^2 \right)^{1/2} \right\|_p$$

$$= C\epsilon^{d\lambda} \left\| \left(\sum_{\ell=1}^\infty |f_\ell|^2 \right)^{1/2} \right\|_p$$

for $4/3 \leq p \leq 4$. By (4.2) we have

$$\sigma_\lambda^{(R_\ell)}(\xi) = \sum_{k=0}^\infty m_{\lambda,2^{-k}}^{(\ell)}(\xi).$$

Thus by (6.9) for $4/3 \leq p \leq 4$ we see that

$$\left\| \left(\sum_{\ell=1}^\infty \left| T_{\sigma_\lambda^{(R_\ell)}}(f_\ell) \right|^2 \right)^{1/2} \right\|_p \leq \sum_{k=0}^\infty \left\| \left(\sum_{\ell=1}^\infty \left| T_{m_{\lambda,2^{-k}}^{(\ell)}}(f_\ell) \right|^2 \right)^{1/2} \right\|_p$$

$$\leq C \sum_{k=0}^\infty 2^{-kd\lambda} \left\| \left(\sum_{\ell=1}^\infty |f_\ell|^2 \right)^{1/2} \right\|_p$$

$$\leq C \left\| \left(\sum_{\ell=1}^\infty |f_\ell|^2 \right)^{1/2} \right\|_p.$$

This completes the proof of Theorem 1.4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 6.2. Let $\{R_\ell\}_{\ell=-\infty}^\infty$ be a positive lacunary sequence with Hadamard gap $\mathfrak{a} > 1$: $\mathfrak{a} \leq \inf_\ell R_{\ell+1}/R_\ell$. Let $\sigma_\lambda^{(R_\ell)}$ be as in Theorem 1.4. By application of Theorem 1.4 and the Littlewood-Paley theory, we have

$$(6.10) \qquad \left\| \sup_\ell \left| T_{\sigma_\lambda^{(R_\ell)}}(f) \right| \right\|_p \leq C \|f\|_p, \quad 4/3 \leq p \leq 4.$$

Let Γ be a simple, closed C^∞ curve in $\mathbb{R}^2$ with no tangent of infinite order such that the origin is contained in Γ. Then we can consider a summation operator $\widetilde{S}_R^\lambda$ for Fourier integrals analogous to Bochner-Riesz summation operator S_R^λ by replacing the unit circle with Γ (see [14, Theorem 1, Corollary 1]). By applying (6.10) suitably through a partition of unity, we can prove a lacunary convergence of $\widetilde{S}_R^\lambda(f)$ for $f \in L^p(\mathbb{R}^2)$, $4/3 \leq p \leq 4$ (see [8, Theorem 2] for S_R^λ).

References

[1] A. Carbery, The boundedness of the maximal Bochner-Riesz operators on $L^4(\mathbb{R}^2)$, *Duke math. J.*, **50** (1983), 409–416.

[2] A. Carbery, A weighted inequality for the maximal Bochner-Riesz operator on $\mathbb{R}^2$, *Trans. Amer. Math. Soc.*, **287** (1985), 673–680.

[3] A Carbery and A Seeger, Weighted inequalities for Bochner-Riesz means in the plane, *Quart. J. Math.*, **51** (2000), 155–167.

[4] L. Carleson and P. Sjölin, Oscillatory integrals and a multiplier problem for the disc, *Studia Math.*, **44** (1972), 287–299.

[5] L. Cladek, Multiplier Transformations Associated with Convex Domains in $\mathbb{R}^2$, *J Geom Anal*, **26** (2016), 3129–3175.

[6] A. Córdoba, A note on Bochner-Riesz operators, *Duke Math. J.*, **46** (1979), 505–511.

[7] A. Córdoba, An integral inequality for the disc multiplier, *Proc. Amer. Math. Soc.*, **92** (1984), 407–408.

[8] A. Córdoba and B. López-Melero, Spherical summation: a problem of E. M. Stein, *Ann. Inst. Fourier, Grenoble*, **31** (1981), 147–152.

[9] C. Fefferman, The multiplier problem for the ball, *Annals of Math.*, **94** (1971), 330–336.

[10] C. Fefferman, A note on spherical summation multipliers, *Israel J. Math.*, **15** (1973), 44–52.

[11] L. Hörmander, Oscillatory integrals and multipliers on FL^p, *Ark. Mat.*, **11** (1973), 1–11.

[12] J. L. Rubio de Francia, *Weighted norm inequalities and vector valued inequalities*, Lecture notes in Math. Springer-Verlag, Berlin and New York, **908** (1982), 86–101.

[13] A. Seeger and S. Ziesler, Riesz means associated with convex domains in the plane, *Math. Z.*, **236** (2001), 643–676.

[14] P. Sjölin, Fourier multipliers and estimates of the Fourier transform of measures carried by smooth curves in R^2, *Studia Math.*, **51** (1974), 169–182.

[15] Jan-Olov Strömberg, Maximal functions associated to rectangles with uniformly distributed directions, *Annals of Math.*, **107** (1978), 399–402.

Department of Mathematics, Faculty of Education, Kanazawa University, Kanazawa 920-1192, Japan

E-mail address: shuichipm@gmail.com