LITTLEWOOD-PALEY FUNCTIONS UNDER SHARP KERNEL CONDITIONS

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ABSTRACT. We prove L^p-estimates for Littlewood-Paley functions under sharp kernel conditions without assuming compactness of support by applying extrapolation arguments.

1. INTRODUCTION

Let

$$g_\psi(f)(x) = \left(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t}\right)^{1/2}$$

be the Littlewood-Paley function on $\mathbb{R}^n$, where $\psi \in L^1(\mathbb{R}^n)$ and $\psi_t(x) = t^{-n}\psi(t^{-1}x)$. We assume that

$$\int_{\mathbb{R}^n} \psi(x)\, dx = 0. \tag{1.1}$$

Let

$$\psi(x) = |x|^{-n+1}\Omega(x')\chi_{(0,1]}(|x|), \quad x' = x/|x|, \text{ for } x \in \mathbb{R}^n \setminus \{0\},$$

where $\Omega \in L^1(S^{n-1})$, $\int_{S^{n-1}} \Omega\, d\sigma = 0$ with $d\sigma$ denoting the Lebesgue surface measure on the unit sphere S^{n-1}, and χ_E denotes the characteristic function of a set E. Then, $g_\psi(f)$ is the Marcinkiewicz function $\mu_\Omega(f)$ in Stein [18] (see also Hörmander [10, pp. 135–137]). The Marcinkiewicz function was introduced by [11] in the one dimensional case. For recent results on applications of the square functions of Marcinkiewicz type to characterization of the Sobolev spaces we refer to [16] and [17].

When considering g_ψ, we always assume (1.1). A well-known theorem for L^p boundedness of g_ψ is the following result.

Theorem A (Benedek, Calderón and Panzone [3])**.** *If there exists $\epsilon > 0$ such that*

$$|\psi(x)| \le C(1+|x|)^{-n-\epsilon}, \tag{1.2}$$

$$\int_{\mathbb{R}^n} |\psi(x-y) - \psi(x)|\, dx \le C|y|^\epsilon, \tag{1.3}$$

then g_ψ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$.

We also recall the following result of [9].

Theorem B. *We assume that the function ψ satisfies the following conditions*:

(1) $\psi\chi_\epsilon \in L^1(\mathbb{R}^n)$ *for some* $\epsilon > 0$, *where* $\chi_\epsilon(x) = (1+|x|)^\epsilon$;
(2) $\psi L_0 \in L^u(\mathbb{R}^n)$ *for some* $u > 1$, *where* $L_0(x) = \chi_{(0,1]}(|x|)$;

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(3) *there exist non-negative functions* h *on* $(0,\infty)$ *and* Ω *on* S^{n-1} *such that*
  (a) $|\psi(x)| \le h(|x|)\Omega(x')$, $\forall x \in \mathbb{R}^n \setminus \{0\}$,
  (b) $h(r)$ *is non-increasing on* $(0,\infty)$ *and* $h(|x|) \in L^1(\mathbb{R}^n)$,
  (c) $\Omega \in L^q(S^{n-1})$ *for some* $q \in (1,\infty)$.

Then g_ψ *is bounded on* $L^p(\mathbb{R}^n)$ *for all* $p \in (1,\infty)$.

Theorem B implies, in particular, that in Theorem A the condition (1.3) is not needed for the L^p boundedness of g_ψ; the condition (1.2) only is sufficient (see [5, p. 148] for the L^2 case).

Furthermore, we recall results of [13] (Theorem C and Theorem E). We say that a function Ω on S^{n-1} belongs to the class $L(\log L)^\alpha(S^{n-1})$, $\alpha > 0$, if

$$\int_{S^{n-1}} |\Omega(\theta)| \left(\log(2 + |\Omega(\theta)|)\right)^\alpha \, d\sigma(\theta) < \infty.$$

The class $L(\log L)^\alpha(\mathbb{R}^n)$ of functions on $\mathbb{R}^n$ is defined similarly.

Theorem C. *Let* $\Omega \in L(\log L)^{1/2}(S^{n-1})$ *and* $\Omega \ge 0$. *Suppose that* $|\psi(x)| \le h(|x|)\Omega(x')$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$, *where* h *is a non-negative, non-increasing function on* $(0,\infty)$ *supported in* $(0,1]$. *We further assume that* $h(|x|) \in L^q(\mathbb{R}^n)$ *for some* $q > 1$. *Then* g_ψ *is bounded on* $L^p(\mathbb{R}^n)$ *for all* $1 < p < \infty$.

As an application of Theorem C we have the following result of Al-Salman, Al-Qassem, Cheng and Pan [1].

Theorem D. *If* $\Omega \in L(\log L)^{1/2}(S^{n-1})$, *then* μ_Ω *is bounded on* $L^p(\mathbb{R}^n)$ *for all* $p \in (1,\infty)$.

The case $p = 2$ of Theorem D is due to Walsh [20]. See [6] for a generalization to homogeneous groups including the Heisenberg groups.

Theorem E. *Suppose that* ψ *is compactly supported and that* ψ *is in* $L(\log L)^{1/2}(\mathbb{R}^n)$. *Then* g_ψ *is bounded on* $L^p(\mathbb{R}^n)$ *for all* p *in the range* $2 \le p < \infty$.

Let ψ be compactly supported. Then it is known that if $\psi \in L^2(\mathbb{R}^n)$, g_ψ is bounded on $L^p(\mathbb{R}^n)$ for every $p \in (1,\infty)$ and that if $\psi \in L^q(\mathbb{R}^n)$ for some $q \in (1,2]$, then g_ψ is bounded on $L^p(\mathbb{R}^n)$ provided that $0 < 1/p < 1/2 + 1/q'$ (see [7] and also [4]), where q' denotes the conjugate exponent to q. The optimality of the result is also shown in [7].

When $n = 1$, we also recall related results. Let

$$\psi^{(\alpha)}(x) = \alpha |1 - |x||^{\alpha-1} \chi_{(-1,1)}(x) \operatorname{sgn}(x), \quad \alpha \in (0, 1/2).$$

Then $g_{\psi^{(\alpha)}} = \mu_\alpha$ is the generalized Marcinkiewicz function. Let $1 < q < 2$ and $q(1-\alpha) < 1$. Then $\psi^{(\alpha)} \in L^q(\mathbb{R})$. Let $1 < p < 2$. It is known that if $2/(2\alpha+1) > p$, $g_{\psi^{(\alpha)}}$ is not bounded on $L^p(\mathbb{R})$ and that if $2/(2\alpha+1) < p$, $g_{\psi^{(\alpha)}}$ is bounded on $L^p(\mathbb{R})$, and also that if $p = 2/(2\alpha+1)$, $g_{\psi^{(\alpha)}}$ is of weak type (p,p) (see [9, pp. 578–579]). For any $q \in (1,2)$, if $1 > 1/p > 1/2 + 1/q'$, we can find $\alpha \in (0,1/2)$ such that $\alpha > 1 - 1/q$ and $1/p > 1/2 + \alpha$. Then $g_{\psi^{(\alpha)}}$ is not bounded on $L^p(\mathbb{R})$.

In this note we shall generalize Theorems C and E by removing the compactness assumption on the support of the function ψ. Let χ_ϵ, L_0 be as in Theorem B. Our theorems will be stated by using χ_ϵ, L_0.

We shall prove L^p estimates for g_ψ that are useful in extrapolation arguments to obtain a minimum condition on ψ for L^p boundedness of g_ψ.

Theorem 1.1. *Suppose that* $\psi\chi_\epsilon \in L^1(\mathbb{R}^n)$ *for some* $\epsilon > 0$ *and that* $|\psi(x)| \le h(|x|)\Omega(x')$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$*, where* h *is a non-negative, non-increasing function on* $(0, \infty)$ *and* Ω *is a non-negative function on* S^{n-1}*. We assume the following*:

(1) $H \in L^1(\mathbb{R}^n)$, *where* $H(x) = h(|x|)$;
(2) *there exists* $q > 1$ *such that*
  (a) $\Omega \in L^q(S^{n-1})$,
  (b) $\psi L_0 \in L^q(\mathbb{R}^n)$.

Then we have

$$\|g_\psi(f)\|_p \le C_{p,\epsilon}\, (q/(q-1))^{1/2} \left(\|\psi L_0\|_q + \|H\|_1\|\Omega\|_q + \|\psi\chi_\epsilon\|_1\right) \|f\|_p$$

for all $p \in (1, \infty)$*, where the constant* $C_{p,\epsilon}$ *is independent of* q*,* ψ*,* h *and* Ω*.*

By applying Theorem 1.1 and extrapolation, we have the following.

Theorem 1.2. *Let* $\Omega \in L(\log L)^{1/2}(S^{n-1})$ *and* $\Omega \ge 0$*. Suppose that* $|\psi(x)| \le h(|x|)\Omega(x')$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$*, where* h *is as in Theorem* 1.1*. We further assume that* $HL_0 \in L^q(\mathbb{R}^n)$ *for some* $q > 1$ *and that* $H\chi_\epsilon \in L^1(\mathbb{R}^n)$ *for some* $\epsilon > 0$*. Then* g_ψ *is bounded on* $L^p(\mathbb{R}^n)$ *for all* $p \in (1, \infty)$*.*

From the estimates in the proof of Theorem 1.2 with another extrapolation we have the following.

Theorem 1.3. *Suppose that* $\Omega \in L(\log L)^{1/2}(S^{n-1})$*,* $\Omega \ge 0$*, and that* $|\psi(x)| \le h(|x|)\Omega(x')$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$*, where* h *is a non-negative, non-increasing function on* $(0, \infty)$ *such that* $H\chi_\epsilon \in L^1(\mathbb{R}^n)$ *for some* $\epsilon > 0$*. We further assume that* $HL_0 \in L(\log L)^{3/2}(\mathbb{R}^n)$*. Then* g_ψ *is bounded on* $L^p(\mathbb{R}^n)$ *for all* $1 < p < \infty$*.*

Let $\Omega \in L(\log L)^{1/2}(S^{n-1})$ with $\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0$. Set

$$\psi(x) = \frac{\Omega(x')}{|x|^n}(\log(2 + |x|^{-1}))^{-\alpha}(1 + |x|)^{-\beta} b(|x|),$$

where $\alpha > 5/2$, $\beta > 0$ and $b \in L^\infty((0, \infty))$. Then Theorem 1.3 implies that g_ψ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$.

Also, by applying Theorem 1.2 and Theorem E we have the following.

Theorem 1.4. *Suppose that* $\psi L_0 \in L(\log L)^{1/2}(\mathbb{R}^n)$*. Further, we assume that* $|\psi(x)| \le h(|x|)\Omega(x')$ *for all* $|x| \ge 1$*, where* h *is a non-negative, non-increasing function on* $(0, \infty)$ *and* Ω *is a non-negative function on* S^{n-1} *such that* $H \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*,* $H\chi_\epsilon \in L^1(\mathbb{R}^n)$ *for some* $\epsilon > 0$ *and* $\Omega \in L(\log L)^{1/2}(S^{n-1})$*. Then* g_ψ *is bounded on* $L^p(\mathbb{R}^n)$ *for all* $p \ge 2$*.*

It is easily seen that in Theorems 1.1, 1.2, 1.3 and 1.4, we can replace the conditions stated by using L_0 by the conditions with the function $\chi_{(0,a]}(|x|)$ in place of L_0 for any $a > 0$ to have analogous results.

We shall prove Theorem 1.1 in Section 2. In the proof we apply a Littlewood-Paley decomposition adapted to a suitable lacunary sequence with the Hadamard gap depending on q in (2) of Theorem 1.1.

The method of appropriately choosing a lacunary sequence in defining a Littlewood-Paley decomposition was used in [2] in studying singular integrals. Also, using method inspired by [2], [14] proved some estimates which are useful in studying singular integrals by applying extrapolation method and which were foreseen in [2] (see [2, p. 156]). The method has been extended to the case of square functions in [13], so that we can prove Theorem C and Theorem E in [13]. It is not known whether the method of [1] also can prove these theorems. The argument of [13] is adapted in this note for the case when the function ψ is not assumed to have compact support.

We require some vector valued inequalities, which are based on Lemmas 2.1 and 2.2 below. The proof of Lemma 2.1 is the same as that of [13, Lemma 1], since [13, Lemma 1] holds true without compactness assumption for the support of ψ and the proof in [13] is also available under the conditions of Theorem 1.1.

To prove Lemma 2.2, we apply estimates in the proof of [15, Lemma 3.4]. See also the proof of [12, Lemma 3] for relevant estimates.

Theorems 1.2 and 1.3 will be shown in Section 3 by applying extrapolation methods (see, e.g., Zygmund [21, Chap. XII, pp. 119–120]) and using Theorem 1.1.

Theorem 1.4 will be shown in Section 4 by applying Theorem 1.2 and Theorem E. In this note the letter C will be used to stand for non-negative constants, which may vary in different places.

2. Proof of Theorem 1.1

We denote by $\mathbb{Z}$ the set of integers and by $\mathcal{H}$ the Hilbert space $L^2((0,\infty), dt/t)$. Let $k \in \mathbb{Z}$ and $\rho \geq 2$. We consider operators mapping functions on $\mathbb{R}^n$ to $\mathcal{H}$-valued functions on $\mathbb{R}^n$; define T_k by

$$(T_k(f)(x))\,(t) = T_k(f)(x,t) = (\psi_t * f)(x)\chi_{[1,\rho)}(\rho^{-k}t).$$

Then we have

$$|T_k(f)(x)|_{\mathcal{H}} = \left(\int_{\rho^k}^{\rho^{k+1}} |\psi_t * f(x)|^2 \,\frac{dt}{t}\right)^{1/2}.$$

Lemma 2.1. *If ψ is as in Theorem 1.1 with (1.1) and $m_\psi(x) = h(|x|)\Omega(x')$, then we have*

$$\left\| \left(\sum_{k=-\infty}^{\infty} |T_k(f_k)|_{\mathcal{H}}^2\right)^{1/2} \right\|_s \leq C_s \|\psi\|_1^{1/2} \|m_\psi\|_1^{1/2} (\log \rho)^{1/2} \left\| \left(\sum_{k=-\infty}^{\infty} |f_k|^2\right)^{1/2} \right\|_s$$

for all $s \in (1,\infty)$ with a positive constant C_s independent of ρ, ψ, h and Ω.

Proof. The proof is the same as that for Lemma 1 of [13], since the compactness of the support of ψ is not used there. Let $M_\psi(f)$ be a maximal function defined by

$$M_\psi(f)(x) = \sup_{t>0} \left||\psi|_t * f(x)\right|.$$

Then the method of rotations implies $\|M_\psi(f)\|_r \leq C_r \|m_\psi\|_1 \|f\|_r$ for all $r > 1$. Thus, arguing as in the proofs of Lemmas 1 and 2 of [9] and applying the maximal inequality for M_ψ and checking the constants, we can get Lemma 2.1. For the sake of completeness we give the proof more specifically in what follows.

First, let $2 \leq s < \infty$, $r = (s/2)' = s/(s-2)$. Choose a non-negative $g \in L^r$ satisfying $\|g\|_r \leq 1$ and

$$I := \left\| \left(\sum_k |T_k(f_k)|_{\mathcal{H}}^2\right)^{1/2} \right\|_s^2 = \int \left(\sum_k |T_k(f_k)|_{\mathcal{H}}^2\right) g\,dx.$$

We note that

$$|T_k(f_k)(x)|_{\mathcal{H}}^2 \le \|\psi\|_1 \int_{\mathbb{R}^n} |p_{\rho^k}(y)| f_k(x-y)|^2\, dy,$$

where $p_{\rho^k}(y) = \int_1^\rho |\psi_{t\rho^k}(y)|\, dt/t$. Thus we have

$$\begin{aligned} I &\le \|\psi\|_1 \sum_k \int |f_k(x)|^2 \left(\int_{\mathbb{R}^n} p_{\rho^k}(y) g(x+y))\, dy \right) dx \\ &\le (\log \rho)\|\psi\|_1 \sum_k \int |f_k(x)|^2 M_{\tilde{\psi}}(g)(x)\, dx. \end{aligned}$$

where $\tilde{\psi}(x) = \psi(-x)$. Applying Hölder's inequality, we see that

$$\begin{aligned} \sum_k \int |f_k|^2 M_{\tilde{\psi}}(g)\, dx &\le \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_s^2 \|M_{\tilde{\psi}}(g)\|_r \\ &\le C_r \|m_\psi\|_1 \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_s^2, \end{aligned}$$

where we have used the L^r-boundedness of $M_{\tilde{\psi}}$. Collecting results, we reach the conclusion.

Next, let $1 < s < 2$, $r = (s'/2)' = s/(2-s)$. For a function h on $\mathbb{R}^n \times (0,\infty)$, we define an $\mathcal{H}$-valued function $P_k(h)$ by

$$(P_k(h)(x))(t) = P_k(h)(x,t) = h(x,t)\chi_{[1,\rho)}(\rho^{-k}t).$$

Also, let T_k act on such h by $(T_k(h)(x))(t) = T_k(h)(x,t) = (T_k(h(\cdot,t))(x))(t)$. Then for a sequence $\{h_k(x,t)\}$ we have

$$\left\| \left(\sum_k |T_k(h_k)|_{\mathcal{H}}^2 \right)^{1/2} \right\|_{s'} \le C_s \|\psi\|_1^{1/2} \|m_\psi\|_1^{1/2} \left\| \left(\sum_k |P_k(h_k)|_{\mathcal{H}}^2 \right)^{1/2} \right\|_{s'}. \tag{2.1}$$

To prove this, as in the proof of the lemma for $s \in [2,\infty)$, take a non-negative $g \in L^r$ such that $\|g\|_r \le 1$ and

$$I := \left\| \left(\sum_k |T_k(h_k)|_{\mathcal{H}}^2 \right)^{1/2} \right\|_{s'}^2 = \int \left(\sum_k |T_k(h_k)|_{\mathcal{H}}^2 \right) g\, dx.$$

We have

$$\int |T_k(h_k)|_{\mathcal{H}}^2 g\, dx \le \|\psi\|_1 \int M_{\tilde{\psi}}(g) |P_k(h_k)|_{\mathcal{H}}^2\, dx.$$

Therefore, by the Hölder inequality we see that

$$\begin{aligned} I &\le \|\psi\|_1 \left\| \left(\sum_k |P_k(h_k)|_{\mathcal{H}}^2 \right)^{1/2} \right\|_{s'}^2 \|M_{\tilde{\psi}}(g)\|_r \\ &\le C_r \|\psi\|_1 \|m_\psi\|_1 \left\| \left(\sum_k |P_k(h_k)|_{\mathcal{H}}^2 \right)^{1/2} \right\|_{s'}^2. \end{aligned}$$

This completes the proof of (2.1).

Now we can prove Lemma 2.1 when $1 < s < 2$. If $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ denotes the inner product in $\mathcal{H}$, then

$$\int \langle T_k(f_k)(x,\cdot), h_k(x,\cdot)\rangle_{\mathcal{H}}\, dx = \int \langle P_k(f_k)(x,\cdot), \tilde{T}_k(h_k)(x,\cdot)\rangle_{\mathcal{H}}\, dx,$$

where

$$\tilde{T}_k(h)(x,t) = \chi_{[1,\rho)}(\rho^{-k}t)\int_{\mathbb{R}^n} \bar{\psi}_t(y) h(x+y,t)\, dy,$$

and $P_k(f_k)(x,t) = f_k(x)\chi_{[1,\rho)}(\rho^{-k}t)$. We note that $|P_k(f_k)|_{\mathcal{H}} = (\log\rho)^{1//2}|f_k|$. Thus by Hölder's inequality and (2.1) we have

$$\begin{aligned}&\left|\int \sum_k \langle T_k(f_k)(x,\cdot), h_k(x,\cdot)\rangle_{\mathcal{H}}\, dx\right| \\ &\quad \le C_s(\log\rho)^{1/2}\|\psi\|_1^{1/2}\|m_\psi\|_1^{1/2}\left\|\left(\sum_k |f_k|^2\right)^{1/2}\right\|_s \left\|\left(\sum_k |P_k(h_k)|_{\mathcal{H}}^2\right)^{1/2}\right\|_{s'}.\end{aligned}$$

Taking the supremum over $\{h_k(x,t)\}$ such that $\left\|\left(\sum_k |P_k(h_k)|_{\mathcal{H}}^2\right)^{1/2}\right\|_{s'} \le 1$, we have the conclusion of Lemma 2.1 for $s \in (1,2)$. □

Let

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle x,\xi\rangle}\, dx, \quad \langle x,\xi\rangle = \sum_{j=1}^n x_j\xi_j,$$

be the Fourier transform of f.

Lemma 2.2. *Let ψ, H, Ω, q, ϵ be as in Theorem 1.1. Let χ_ϵ, L_0 be as in Theorem B. Set $G(\psi, H, \Omega, q, \epsilon) = \|\psi L_0\|_q^2 + \|\psi\|_1\|H\|_1\|\Omega\|_q + \|\psi\chi_\epsilon\|_1^2$. Then*

$$\int_{\rho^k}^{\rho^{k+1}} \left|\hat{\psi}(t\xi)\right|^2 \frac{dt}{t} \le C(\log\rho)G(\psi,H,\Omega,q,\epsilon)\min(1, |\rho^{k+1}\xi|, |\rho^k\xi|^{-1})^{\min(2\epsilon, 1/(3q'))},$$

where the constant C is independent of $k \in \mathbb{Z}$, $\rho \ge 2$, $q > 1$, $\epsilon > 0$, H, Ω and ψ.

To prove Lemma 2.2 we need the following.

Lemma 2.3. *Let ψ, H, Ω, q be as in Lemma 2.2. Then*

$$\int_1^2 |\hat{\psi}(t\xi)|^2 \frac{dt}{t} \le C\left(\|\psi L_0\|_q^2 + \|\psi\|_1\|H\|_1\|\Omega\|_q\right)|\xi|^{-1/(3q')}.$$

Proof. When $n \ge 2$, by [15, Lemma 3.3] and the estimates of the proof of [15, Lemma 3.4] we have

$$\int_1^2 |\hat{\psi}(t\xi)|^2 \frac{dt}{t} \le C\left(\|\psi L_0\|_q^2 + \|\psi\|_1\|HL_1\|_1\|\Omega\|_q\right)|\xi|^{-1/(3q')},$$

where $L_1 = 1 - L_0$, which implies the conclusion of the lemma.

If $n = 1$, similarly to the arguments above by results of [15] we have

$$\int_1^2 |\hat{\psi}(t\xi)|^2 \frac{dt}{t} \le C\left(\|\psi L_0\|_q^2 + \|\psi\|_1\|\Omega\|_\infty\left(\|HL_1\|_1 + \|HL_1\|_\infty\right)\right)|\xi|^{-1/(3q')},$$

from which the conclusion follows, since $\|\Omega\|_\infty \le C\|\Omega\|_q$ and $\|HL_1\|_\infty \le C\|H\|_1$. □

Proof of Lemma 2.2. Since $\|\hat{\psi}\|_\infty \le \|\psi\|_1$, we see that

$$\int_1^\rho \left|\hat{\psi}(t\rho^k\xi)\right|^2 \, dt/t \le (\log\rho)\|\psi\|_1^2. \tag{2.2}$$

On the other hand, by Lemma 2.3 we see that

$$\begin{aligned} \int_1^\rho \left|\hat{\psi}(t\rho^k\xi)\right|^2 \, dt/t &\le \sum_{0\le m\le(\log\rho)/\log 2} \int_1^2 \left|\hat{\psi}(t2^m\rho^k\xi)\right|^2 \, dt/t \\ &\le \sum_{0\le m\le(\log\rho)/\log 2} C\left(\|\psi L_0\|_q^2 + \|\psi\|_1\|H\|_1\|\Omega\|_q\right)\left|2^m\rho^k\xi\right|^{-1/(3q')} \\ &\le C(\log\rho)\left(\|\psi L_0\|_q^2 + \|\psi\|_1\|H\|_1\|\Omega\|_q\right)\left|\rho^k\xi\right|^{-1/(3q')}. \end{aligned} \tag{2.3}$$

Also, by the proof of Lemma 1 of [12], we have $|\hat{\psi}(\xi)| \le C|\xi|^\epsilon\|\psi\chi_\epsilon\|_1$. Thus

$$\int_1^\rho \left|\hat{\psi}(t\rho^k\xi)\right|^2 \, dt/t \le C\|\psi\chi_\epsilon\|_1^2 \int_1^\rho |t\rho^k\xi|^{2\epsilon}\, dt/t \le C(\log\rho)\|\psi\chi_\epsilon\|_1^2|\rho^{k+1}\xi|^{2\epsilon}. \tag{2.4}$$

Using (2.2), (2.3), (2.4) and the inequality

$$\min(1, |\rho^{k+1}\xi|^{2\epsilon}, |\rho^k\xi|^{-1/(3q')}) \le \min(1, |\rho^{k+1}\xi|, |\rho^k\xi|^{-1})^{\min(2\epsilon, 1/(3q'))},$$

we have the conclusion of the lemma. □

We also need to apply the ordinary Littlewood-Paley theory. Let $\rho \ge 2$. As in [13, p. 433] we take a sequence $\{\Psi_k\}_{-\infty}^\infty$ of non-negative functions in $C^\infty(\mathbb{R})$ such that

$$\operatorname{supp}(\Psi_k) \subset [\rho^{-k-1}, \rho^{-k+1}];$$

$$\sum_{k=-\infty}^\infty \Psi_k(t) = 1, \quad \forall t > 0;$$

$$|(d/dt)^j\Psi_k(t)| \le C_j|t|^{-j}, \quad \forall j \in \mathbb{Z}\cap[1,\infty),$$

with the constants C_j independent of ρ and k.

Lemma 2.4. *Let D_j be the Fourier multiplier operator defined by*

$$\mathcal{F}(D_j(f))(\xi) = \Psi_j(|\xi|)\hat{f}(\xi)$$

for $j \in \mathbb{Z}$. Then we have the Littlewood-Paley inequality:

$$\left\|\left(\sum_{k=-\infty}^\infty |D_k(f)|^2\right)^{1/2}\right\|_p \le C_p\|f\|_p, \quad 1 < p < \infty,$$

where the constant C_p is independent of $\rho \ge 2$.

See also [19], [8, §2], [2, pp. 158–159], [14, pp. 225–226] for the operators D_j.

Decompose

$$(\psi_t * f)(x) = \sum_{j=-\infty}^\infty F_j(x,t),$$

where

$$F_j(x,t) = \sum_{k=-\infty}^\infty D_{j+k}(\psi_t * f)(x)\chi_{[\rho^k,\rho^{k+1})}(t).$$

If

$$g_j(f)(x) = \left(\int_0^\infty |F_j(x,t)|^2 \,\frac{dt}{t}\right)^{1/2} = \left(\sum_{k=-\infty}^{\infty} |T_k(D_{j+k}(f))(x)|_{\mathcal{H}}^2\right)^{1/2},$$

then we have $g_\psi(f)(x) \le \sum_{j=-\infty}^{\infty} g_j(f)(x)$.

For $j \in \mathbb{Z}$, set $E_j = \{\rho^{-1-j} \le |\xi| \le \rho^{1-j}\}$. Then applying the Plancherel theorem, we see that

$$\begin{aligned}\|g_j(f)\|_2^2 &= \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^n} \int_{\rho^k}^{\rho^{k+1}} |D_{j+k}(\psi_t * f)(x)|^2 \,\frac{dt}{t}\,dx \\ &\le \sum_{k=-\infty}^{\infty} C \int_{E_{j+k}} \left(\int_{\rho^k}^{\rho^{k+1}} \left|\hat{\psi}(t\xi)\right|^2 \frac{dt}{t}\right) \left|\hat{f}(\xi)\right|^2 d\xi =: I.\end{aligned}$$

Using Lemma 2.2, we have

$$\begin{aligned} I \le& \sum_{k=-\infty}^{\infty} C(\log\rho) G(\psi, H, \Omega, q, \epsilon) \\ &\times \int_{E_{j+k}} \min(1, |\rho^{k+1}\xi|, |\rho^k\xi|^{-1})^{\min(2\epsilon, 1/(3q'))} \left|\hat{f}(\xi)\right|^2 d\xi \\ \le& C(\log\rho) G(\psi, H, \Omega, q, \epsilon) \min(1, \rho^{-|j|+2})^{\min(2\epsilon, 1/(3q'))} \sum_{k=-\infty}^{\infty} \int_{E_{j+k}} \left|\hat{f}(\xi)\right|^2 d\xi \\ \le& C(\log\rho) G(\psi, H, \Omega, q, \epsilon) \min(1, \rho^{-|j|+2})^{\min(2\epsilon, 1/(3q'))} \|f\|_2^2,\end{aligned}$$

where the last inequality holds since $\sum_j \chi_{E_j} \le C$ with a constant C independent of ρ. Thus it follows that

$$\|g_j(f)\|_2 \le C(\log\rho)^{1/2} G(\psi, H, \Omega, q, \epsilon)^{1/2} \min(1, \rho^{-|j|+2})^{\min(\epsilon, 1/(6q'))} \|f\|_2. \tag{2.5}$$

By Lemmas 2.1 and 2.4 we have

$$\begin{aligned}\|g_j(f)\|_s &= \left\| \left(\sum_{k=-\infty}^{\infty} |T_k(D_{j+k}(f))|_{\mathcal{H}}^2\right)^{1/2} \right\|_s \\ &\le C\|\psi\|_1^{1/2} \|m_\psi\|_1^{1/2} (\log\rho)^{1/2} \left\| \left(\sum_{k=-\infty}^{\infty} |D_{j+k}(f)|^2\right)^{1/2} \right\|_s \\ &\le C\|\psi\|_1^{1/2} \|m_\psi\|_1^{1/2} (\log\rho)^{1/2} \|f\|_s\end{aligned} \tag{2.6}$$

for $s \in (1,\infty)$. Interpolating between (2.5) and (2.6), and writing $G = G(\psi, H, \Omega, q, \epsilon)$, we get

$$\begin{aligned}&\|g_j(f)\|_p \\ &\le C(\log\rho)^{1/2} \left(G^{1/2} \min(1, \rho^{-|j|+2})^{\min(\epsilon, 1/(6q'))}\right)^{\eta} \left(\|\psi\|_1^{1/2} \|m_\psi\|_1^{1/2}\right)^{1-\eta} \|f\|_p \\ &\le C(\log\rho)^{1/2} G^{\eta/2} \|\psi\|_1^{(1-\eta)/2} \|m_\psi\|_1^{(1-\eta)/2} \min(1, \rho^{-|j|+2})^{\eta \min(\epsilon, 1/(6q'))} \|f\|_p\end{aligned}$$

for some $\eta \in (0, 1]$ depending on p, where $1 < p < \infty$. Thus

$$
\begin{aligned}
\|g_\psi(f)\|_p &\le \sum_{j=-\infty}^{\infty} \|g_j(f)\|_p \\
&\le C \left(\sum_{j=-\infty}^{\infty} \min(1, \rho^{-|j|+2})^{\eta \min(\epsilon, 1/(6q'))} \right) (\log \rho)^{1/2} \\
&\qquad \times G^{\eta/2} \|\psi\|_1^{(1-\eta)/2} \|m_\psi\|_1^{(1-\eta)/2} \|f\|_p \\
&\le C(1 - \rho^{-\eta \min(\epsilon, 1/(6q'))})^{-1} (\log \rho)^{1/2} \\
&\qquad \times G^{\eta/2} \|\psi\|_1^{(1-\eta)/2} \|m_\psi\|_1^{(1-\eta)/2} \|f\|_p \\
&\le C(1 - \rho^{-\eta \min(\epsilon, 1/(6q'))})^{-1} (\log \rho)^{1/2} \\
&\qquad \times (\|\psi L_0\|_q + \|H\|_1 \|\Omega\|_q + \|\psi \chi_\epsilon\|_1) \|f\|_p,
\end{aligned}
\tag{2.7}
$$

where the last inequality follows by

$$G(\psi, H, \Omega, q, \epsilon)^{\eta/2} \|\psi\|_1^{(1-\eta)/2} \|m_\psi\|_1^{(1-\eta)/2} \le C (\|\psi L_0\|_q + \|H\|_1 \|\Omega\|_q + \|\psi \chi_\epsilon\|_1),$$

which can be seen by applying Young's inequality as follows.

$$
\begin{aligned}
&G(\psi, H, \Omega, q, \epsilon)^{\eta/2} \|\psi\|_1^{(1-\eta)/2} \|m_\psi\|_1^{(1-\eta)/2} \\
&\le C \left(\|\psi L_0\|_q^\eta + \|\psi\|_1^\eta + \|H\|_1^\eta \|\Omega\|_q^\eta + \|\psi \chi_\epsilon\|_1^\eta \right) \|H\|_1^{(1-\eta)} \|\Omega\|_1^{(1-\eta)} \\
&\le C (\|\psi L_0\|_q + \|H\|_1 \|\Omega\|_q + \|\psi \chi_\epsilon\|_1).
\end{aligned}
$$

Taking $\rho = 2^{q'}$ in (2.7), we get the conclusion of Theorem 1.1, since

$$
\begin{aligned}
(1 - \rho^{-\eta \min(\epsilon, 1/(6q'))})^{-1} (\log \rho)^{1/2} &= (1 - 2^{-\eta \min(q'\epsilon, 1/6)})^{-1} \left((q \log 2)/(q-1)\right)^{1/2} \\
&\le (1 - 2^{-\eta \min(\epsilon, 1/6)})^{-1} \left((q \log 2)/(q-1)\right)^{1/2}.
\end{aligned}
$$

3. Proofs of Theorems 1.2 and 1.3

We may assume that $1 < q \le 2$ in proving Theorem 1.2. Let $F_k = \{\theta \in S^{n-1} : 2^{k-1} < |\Omega(\theta)| \le 2^k\}$ for $k \in \mathbb{Z} \cap [2, \infty)$ and $F_1 = \{\theta \in S^{n-1} : |\Omega(\theta)| \le 2\}$. Let $\Omega_k(\theta) = \Omega(\theta)\chi_{F_k}(\theta)$ for $k \ge 1$. We define $E_k = \{x \in \mathbb{R}^n \setminus \{0\} : x' \in F_k\}$ for $k = 1, 2, 3, \dots$. Let $B = \{x \in \mathbb{R}^n : |x| \le 1\}$. We decompose ψ as $\psi = \sum_{k=1}^\infty \psi_{(k)}$, where

$$\psi_{(k)} = \psi \chi_{E_k} - |B|^{-1} \left(\int_{E_k} \psi \, dx \right) \chi_B.$$

We can find similar decompositions in [2, pp. 156–157], [14, pp. 229–230], [13, pp. 438–439]. We note that $\int \psi_{(k)} \, dx = 0$ and $|\psi_{(k)}(x)| \le h^*(|x|)\Omega_k^*(x')$ for $x \in \mathbb{R}^n \setminus \{0\}$, where

$$
\begin{aligned}
h^*(|x|) &= (h(|x|) + C\|H\|_1) \chi_{(0,1]}(|x|) + h(|x|)\chi_{(1,\infty)}(|x|), \\
\Omega_k^*(x') &= \Omega_k(x') + \|\Omega_k\|_1.
\end{aligned}
$$

We see that $\|\Omega_k^*\|_r \le C 2^k e_k^{1/r}$ for $1 < r < \infty$, where $e_k = \sigma(F_k)$, for $k \ge 1$, and that $\|H^* L_0\|_q \le C\|H L_0\|_q + C\|H\|_1$, where $H^*(x) = h^*(|x|)$, and that $\|H^* \chi_\epsilon\|_1 \le C\|H \chi_\epsilon\|_1$.

Fix $p \in (1, \infty)$ and f with $\|f\|_p \le 1$. Put $R(\psi) = \|g_\psi(f)\|_p$. Using subadditivity of $R(\psi)$ and applying Theorem 1.1, we have

$$
\begin{aligned}
\|g_\psi(f)\|_p &= R(\psi) \le \sum_{k\ge 1} R(\psi_{(k)}) \\
&\le C(q/(q-1))^{1/2} \sum_{k<1/(q-1)} (\|HL_0\|_q \|\Omega_k^*\|_q + \|H\chi_\epsilon\|_1 \|\Omega_k^*\|_q) \\
&\quad + C(\|HL_0\|_q + \|H\chi_\epsilon\|_1) \sum_{k\ge 1/(q-1)} k^{1/2} \|\Omega_k^*\|_{1+1/k} \\
&\le C(q/(q-1))^{1/2} (\|HL_0\|_q + \|H\chi_\epsilon\|_1) \sum_{k<1/(q-1)} \|\Omega_k^*\|_q \\
&\quad + C(\|HL_0\|_q + \|H\chi_\epsilon\|_1) \sum_{k\ge 1/(q-1)} k^{1/2} 2^k e_k^{k/(k+1)}.
\end{aligned} \tag{3.1}
$$

We note that $\|\Omega_k^*\|_q \le C(1 + \|\Omega\|_1)2^{k/q'}$ and so

$$
\sum_{1\le k\le 1/(q-1)} \|\Omega_k^*\|_q \le C(1 + \|\Omega\|_1)(q-1)^{-1}. \tag{3.2}
$$

Also, applying Young's inequality, we can see that

$$
\begin{aligned}
&\sum_{k\ge 1} k^{1/2} 2^k e_k^{k/(k+1)} \\
&\le 2 \sum_{k\ge 1} (k/(k+1)) \left(k^{(1+1/k)/2} 2^{k(1+1/k)} e_k\right) + 2 \sum_{k\ge 1} 2^{-k-1}/(k+1) \\
&\le C \sum_{k\ge 1} k^{1/2} 2^k e_k + C \\
&\le C \int_{S^{n-1}} |\Omega(\theta)| \left(\log(2 + |\Omega(\theta)|)\right)^{1/2} \, d\sigma(\theta) + C.
\end{aligned} \tag{3.3}
$$

Using (3.2) and (3.3) in (3.1), we have

$$
\begin{aligned}
\|g_\psi(f)\|_p &\le C(\|HL_0\|_q + \|H\chi_\epsilon\|_1) \\
&\quad \times \left((q-1)^{-3/2}(1 + \|\Omega\|_1) + \int_{S^{n-1}} |\Omega(\theta)| \left(\log(2 + |\Omega(\theta)|)\right)^{1/2} \, d\sigma(\theta) \right),
\end{aligned} \tag{3.4}
$$

provided that $\|f\|_p \le 1$, where C is independent of $q \in (1, 2]$. This implies the conclusion of Theorem 1.2.

Next, we give the proof of Theorem 1.3. We also fix $p \in (1, \infty)$ and f with $\|f\|_p \le 1$. Let $G_m = \{x \in \mathbb{R}^n : 2^{m-1} < h(|x|) \le 2^m\}$ for $m \in \mathbb{Z} \cap [2, \infty)$, $G_1 = \{x \in \mathbb{R}^n : h(|x|) \le 2\}$. Define

$$
\psi^{(m)} = \psi\chi_{G_m} - |B|^{-1} \left(\int_{G_m} \psi \right) \chi_B.
$$

Then $\psi = \sum_{m=1}^\infty \psi^{(m)}$ and $\int \psi^{(m)} = 0$. Let $h_m(|x|)$ be the least non-increasing radial majorant of $H\chi_{G_m}$. Then

$$
|\psi^{(m)}(x)| \le (h_m(|x|) + C\|H\chi_{G_m}\|_1 L_0(x)) \, (\Omega(x') + \|\Omega\|_1). \tag{3.5}
$$

Let m_0 be a positive integer such that $G_m \cap \{|x| \geq 1\} = \emptyset$ for $m \geq m_0$ and define

$$\Psi = \sum_{m<m_0} \psi^{(m)} = \psi\chi_{\cup_{m<m_0} G_m} - |B|^{-1} \left(\int_{\cup_{m<m_0} G_m} \psi \right) \chi_B.$$

Then $\int \Psi = 0$. Let $H^{(\tau)}$ be the least non-increasing radial majorant of $H(x)\chi_{[\tau,\infty)}(|x|)$, $\tau > 0$. Then

$$|\Psi(x)| \leq \left(H^{(a_0)}(x) + C\|H\|_1 L_0(x) \right) (\Omega(x') + \|\Omega\|_1) \tag{3.6}$$

for some $a_0 > 0$.

We have

$$\|g_\psi(f)\|_p \leq \sum_{m \geq m_0} \|g_{\psi^{(m)}}(f)\|_p + \|g_\Psi(f)\|_p = I + II, \quad \textit{say}. \tag{3.7}$$

By (3.4) for g_Ψ with $q = 2$ and (3.6) we have

$$II \leq C(\|H^{(a_0)}\|_\infty + \|H\chi_\epsilon\|_1)(1 + \|\Omega^*(\log(2+\Omega^*))^{1/2}\|_1), \tag{3.8}$$

where $\Omega^* = \Omega + \|\Omega\|_1$. Let $H_m(x) = h_m(|x|)$. From (3.4) for $g_{\psi^{(m)}}$ and (3.5) it follows that

$$\begin{aligned}
I &\leq C(1 + \|\Omega^*(\log(2+\Omega^*))^{1/2}\|_1) \sum_{m \geq m_0} m^{3/2}(\|H_m\|_{(1+m)/m} + \|H_m\chi_\epsilon\|_1) \\
&\leq C(1 + \|\Omega^*(\log(2+\Omega^*))^{1/2}\|_1) \sum_{m \geq m_0} m^{3/2}\|H_m\|_{(1+m)/m} \\
&\leq C(1 + \|\Omega^*(\log(2+\Omega^*))^{1/2}\|_1) \sum_{m \geq m_0} m^{3/2}2^m |\cup_{\ell \geq m} G_\ell|^{m/(m+1)}.
\end{aligned}$$

Applying Young's inequality, we see that

$$\begin{aligned}
&\sum_{m \geq m_0} m^{3/2}2^m |\cup_{\ell \geq m} G_\ell|^{m/(m+1)} \\
&\leq 2 \sum_{m \geq m_0} m/(m+1) m^{(1+1/m)3/2} 2^{m(1+1/m)} |\cup_{\ell \geq m} G_\ell| + 2 \sum_{m \geq m_0} 2^{-m-1}/(m+1) \\
&\leq C \sum_{m \geq m_0} m^{3/2}2^m |\cup_{\ell \geq m} G_\ell| + 1 \\
&\leq C \sum_{\ell \geq m_0} |G_\ell| \sum_{m=m_0}^{\ell} m^{3/2}2^m + 1 \\
&\leq C \sum_{\ell \geq m_0} |G_\ell| \ell^{3/2}2^\ell + 1 \\
&\leq C\|HL_0(\log(2+HL_0))^{3/2}\|_1 + 1.
\end{aligned}$$

Thus

$$I \leq C(1 + \|\Omega^*(\log(2+\Omega^*))^{1/2}\|_1)(1 + \|HL_0(\log(2+HL_0))^{3/2}\|_1). \tag{3.9}$$

Using (3.8) and (3.9) in (3.7), we have

$$\begin{aligned}
&\|g_\psi(f)\|_p \\
&\leq C(1+\|\Omega^*(\log(2+\Omega^*))^{1/2}\|_1)(1+\|H^{(a_0)}\|_\infty + \|H\chi_\epsilon\|_1 + \|HL_0(\log(2+HL_0))^{3/2}\|_1).
\end{aligned}$$

This completes the proof of Theorem 1.3.

4. Proof of Theorem 1.4

We write $\psi = \Psi + \Phi$, with

$$\Psi = \psi L_0 - |B|^{-1}\left(\int \psi L_0\right) L_0;$$
$$\Phi = |B|^{-1}\left(\int \psi L_0\right) L_0 + \psi L_1,$$

where we recall that $L_1(x) = 1 - L_0(x)$. Then $\int \Psi = 0$, $\Psi \in L(\log L)^{1/2}$ and Ψ is compactly supported. Also, $\int \Phi = 0$ and

$$\begin{aligned}|\Phi(x)| &\le |B|^{-1}\|\psi L_0\|_1 L_0(x) + h(|x|)\Omega(x')L_1(x)\\ &\le (\max(1, \|HL_1\|_\infty)L_0(x) + h(|x|)L_1(x))\left(\Omega(x') + |B|^{-1}\|\psi L_0\|_1\right)\\ &=: h^*(|x|)\Omega^*(x').\end{aligned}$$

We observe that h^* is non-increasing and that $h^*(|x|)\chi_\epsilon(x) \in L^1(\mathbb{R}^n)$, $h^*(|x|)L_0(x) \in L^2(\mathbb{R}^n)$. Also, $\Omega^* \in L(\log L)^{1/2}$ over S^{n-1}.

Furthermore, we recall the subadditivity $g_\psi(f) \le g_\Psi(f) + g_\Phi(f)$. Therefore we can apply Theorem E and Theorem 1.2 to g_Ψ and g_Φ, respectively, to prove Theorem 1.4.

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