# Holographic Teleportation in Quantum Critical Spin Systems 

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#### Abstract

According to the AdS/CFT correspondence, certain quantum many-body systems in $d$-dimensions are equivalent to gravitational theories in $(d+1)$-dimensional asymptotically AdS spacetimes. When a massless particle is sent from the AdS boundary to the bulk curved spacetime, it reaches another point of the boundary after a time lag. In the dual quantum system, it should appear as if quasiparticles have teleported between two separated points. We theoretically demonstrate that this phenomenon, which we call "holographic teleportation," is actually observed in the dynamics of the one-dimensional transverse-field Ising model near the quantum critical point. This result suggests that the experimental probing of the emergent extra-dimension is possible by applying a designed stimulus to a quantum many-body system, which is holographically equivalent to sending a massless particle through the higher-dimensional curved bulk geometry. We also discuss possible experimental realizations using Rydberg atoms in an optical tweezers array.


Introduction.- The AdS/CFT correspondence [1-3] is a holographic duality between the gravitational theory in the $(d+1)$-dimensional anti-de Sitter spacetime (AdS) and the conformal field theory (CFT) living in the $d$ dimensional boundary of the AdS. Although the correspondence is originally proposed for a supersymmetric large- $N$ Yang-Mills theory, its idea has been also applied to more realistic systems such as condensed matter systems [4-8]. Those studies indicate that there may be "materials" having their gravitational duals in our world. Experimenting with such materials, if realized, opens a new path for tabletop experiments of quantum gravity.

As a tool of probing the dual spacetime, some of the authors of this Letter and others have proposed a way to create a null geodesic in the asymptotically AdS spacetime by the manipulation of the source in the quantum field theory (QFT) [9-16]. Once a null geodesic is created in AdS, it bounces repeatedly at the AdS boundary $[9,16]$. In the viewpoint of the dual QFT, while the energy flux locally propagates following the conservation law, the operator expectation value coupled to the source behaves as if the signal teleports: We refer to this phenomenon as "holographic teleportation." The holographic teleportation is naturally understood with the knowledge of the dual spacetime as schematically drawn in Fig.1, but is highly non-trivial in terms of the QFT.

In this Letter, we investigate the occurrence of the holographic teleportation behavior in the transverse-field Ising model on a lattice ring, aiming at its experimental realization. This model is known to be described by a CFT at the critical point. While it may not have a gravitational dual due to its small central charge, we demonstrate that the Jordan-Wigner fermions, generated by a specific form of local perturbation, do exhibit the teleportation behavior in the dynamics. This should be attributed to the fact that the retarded propagators in CFTs are determined only from conformal dimensions


FIG. 1. Schematic picture of the holographic teleportation in a spin system on a lattice ring. The response to a local perturbation suddenly appears at a spatially separated point after a certain time lag.
of operators [14-17]. Furthermore, we observe that the teleportation behavior persists, even in small lattice systems, well outside the QFT limit, and in cases where the parameters deviate away from the critical point. These observations suggest the universality of the holographic teleportation behavior for general quantum many-body models that possess an (approximate) CFT description, regardless of the existence of dual gravitational theories.

We also propose experimental realizations using a wellcontrollable quantum system of Rydberg atoms in an optical tweezers array [18-22] in the case of the transversefield Ising model. The observation of this phenomenon, inspired by the AdS/CFT correspondence, in a laboratory setting can serve as a foundation for future experiments with holographic materials. This study not only predicts the nontrivial phenomenon of holographic teleportation, which is interesting in its own right, but also offers an experimental method to see the motion of particles in the dual higher-dimensional spacetime. This represents a crucial first step in exploring semi-classical gravitational duals for condensed matter systems in tabletop experiments.


FIG. 2. (a) A typical orbit of the null geodesic in $\mathrm{AdS}_{3}$. It bounces repeatedly at the AdS boundary depicted as the yellow cylinder. The radial distance $\rho \equiv r / \sqrt{1+r^{2}}$ and the azimuthal angle $\phi$ correspond to space coordinates, while the axial coordinate corresponds to the time variable $t$. The AdS boundary is located at $\rho=1(r \rightarrow \infty)$. (b) Points at which the null geodesic collides with the AdS boundary on the $(t, \phi)$ plane.

Null geodesics in $A d S_{3}$ - We consider the null geodesic in the global $\mathrm{AdS}_{3}$ spacetime:

$$
\begin{equation*}
d s^{2}=-\left(1+r^{2}\right) d t^{2}+\frac{d r^{2}}{1+r^{2}}+r^{2} d \phi^{2} \tag{1}
\end{equation*}
$$

where we take the unit of $\operatorname{AdS}$ radius $=1$. We have the exact solution of the null geodesic equation in the $\mathrm{AdS}_{3}$ as

$$
\begin{equation*}
t=\frac{\pi}{2}+\tan ^{-1} \lambda, r^{2}=\frac{m^{2}+\lambda^{2}}{1-m^{2}}, \phi=\frac{\pi}{2}+\tan ^{-1} \frac{\lambda}{m} \tag{2}
\end{equation*}
$$

where $\lambda$ is the affine parameter and $m$ is the angular momentum per unit energy. A typical orbit of the null geodesic is shown in Fig. 2(a). Suppose that a null geodesic is injected into the $\mathrm{AdS}_{3}$ from the AdS boundary at $(t, \phi)=(0,0)$. Such a particle arrives at the antipodal point $(t, \phi)=(\pi, \pi)$, bounces back there, and returns to the original position $(t, \phi)=(2 \pi, 0)$. Figure 2(b) shows the points at which the particle reaches the AdS boundary on $(t, \phi)$-plane. These points are independent of the angular momentum $M$ of the null geodesic, although the trajectory does depend on it.

If a source is applied appropriately in the CFT living on the AdS boundary, a null geodesic can be produced in the bulk as a localized configuration of the probe field [9]. The null geodesic with energy $\Omega$ and angular momentum $M$ is created by the source,

$$
\begin{equation*}
\mathcal{J}(t, \phi)=A \exp \left[-\frac{t^{2}}{2 \sigma_{t}^{2}}-\frac{\phi^{2}}{2 \sigma_{\phi}^{2}}-i \Omega t+i M \phi\right] \tag{3}
\end{equation*}
$$

where $A, \sigma_{t}$, and $\sigma_{\phi}$ are the amplitude and the widths in $t$ and $\phi$, respectively, of the Gaussian part. This source modifies the action of the CFT as $S \rightarrow S+\int d t d \phi \mathcal{J O}$ where $\mathcal{O}$ is an operator in the CFT. Since the subleading term of the asymptotic expansion of the bulk probe field corresponds to the response to $\mathcal{J}$ in the CFT, it is zero while the null geodesic is inside the bulk. However, it
suddenly stands up just at the time the geodesic reaches the boundary. Thus, under the source (3), we can expect sharp signals of the response to be observed at discrete points, as shown in Fig. 2(b).

Transverse-field Ising model- We consider the Ising model in transverse magnetic field on the $L$-site ring:

$$
\begin{equation*}
H=-J \sum_{i=1}^{L} \sigma_{i}^{z} \sigma_{i+1}^{z}-h \sum_{i=1}^{L} \sigma_{i}^{x} \tag{4}
\end{equation*}
$$

where $\sigma_{i}^{a}(a=x, y, z)$ is the Pauli matrix which acts on the $i$-th $\operatorname{spin}\left(\sigma_{L+1}=\sigma_{1}\right)$. This Hamiltonian is explicitly diagonalizable. (See Refs. [23, 24] for nice reviews.) The one-dimensional spin- $1 / 2$ chain can be mapped onto a fermionic system by the Jordan-Wigner transformation

$$
\begin{equation*}
c_{i}=\frac{1}{2} \prod_{j=1}^{i-1} \sigma_{j}^{z}\left(-\sigma_{i}^{z}+i \sigma_{i}^{y}\right) \tag{5}
\end{equation*}
$$

Under the transformation Eq. (5), the Hamiltonian (4) is rewritten as

$$
\begin{align*}
H=-J & \sum_{i=1}^{L}\left(c_{i}^{\dagger} c_{i+1}+c_{i+1}^{\dagger} c_{i}+c_{i}^{\dagger} c_{i+1}^{\dagger}+c_{i+1} c_{i}\right) \\
& \left.-h \sum_{i=1}^{L}\left(1-2 c_{i}^{\dagger} c_{i}\right) \quad \text { (with } c_{L+1}=-c_{1}\right) \tag{6}
\end{align*}
$$

where we assumed that the total number of fermions is even $N=\sum_{j=1}^{L} c_{j}^{\dagger} c_{j} \in 2 \boldsymbol{Z}$ since the ground state is always in the even- $N$ sector [23].

Before showing the behavior of holographic teleportation, let us see the QFT description of the lattice model (6) in the continuum limit. To this end, we introduce the fermion field $\Psi\left(x_{j}\right)=c_{j} / \sqrt{a}$ where $a$ is the lattice spacing. In the continuum limit $a \rightarrow 0$ while keeping the total length of the ring, $\ell=L a$, finite, the Hamiltonian becomes

$$
\begin{equation*}
H=-\int_{0}^{\ell} d x\left[\frac{v}{2}\left(\Psi^{\dagger} \frac{d}{d x} \Psi^{\dagger}-\Psi \frac{d}{d x} \Psi\right)+\delta \Psi^{\dagger} \Psi\right] \tag{7}
\end{equation*}
$$

with $v=2 J a$ and $\delta=2(J-h)$. This is just a field theory for the free Majorana fermion with the mass $\delta / v^{2}$. In the critical case $J=h$, the above Hamiltonian describes the CFT with central charge $c=1 / 2$. Therefore, one can anticipate the occurrence of holographic teleportation in the spin model (4) based on the AdS/CFT correspondence, particularly for a sufficiently large size $L$ and when $J \approx h$, with the caveat that the CFT is not strongly coupled (as will be mentioned later).

Linear response theory- Let us consider the linear response of the transverse-field Ising model with a finite $L$, whose Hamiltonian is given by Eq. (4) or (6). The perturbation of the Hamiltonian is

$$
\begin{equation*}
\delta H(t)=-\sum_{l=1}^{L} \mathcal{J}_{l}(t) n_{l} \tag{8}
\end{equation*}
$$



FIG. 3. Response for parameters $\delta=0, \sigma_{t}=0.4, \sigma_{\phi}=0.4, \Omega=5$ and $M=0$. The number of sites is varied as $L=8,16,32,64$. The color bar corresponds to $\left|\delta\left\langle n_{j}\right\rangle\right|$. The rightmost panel shows time-dependence of the response for fixed $\phi$-slices at $\phi=0$ $(j=0)$ and $\phi=\pi(j=L / 2)$. The number of sites is fixed as $L=32$.
where $n_{j}=c_{j}^{\dagger} c_{j}$ is the number operator of the fermion at the $j$-th site and $\mathcal{J}_{l}(t)=\mathcal{J}\left(t, \phi_{l}\right)$ is the source function of the form Eq. (3). Here, we introduced the spacial coordinate of the $l$-th site on the ring as $\phi_{l}=(2 \pi / L)(j-$ $L / 2$ ). From Eq. (5), the number operator $n_{j}$ is written as $n_{j}=\left(1-\sigma_{j}^{x}\right) / 2$. Thus, $\mathcal{J}_{l}(t)$ is regarded as the transverse magnetic field which depends on the time and space.

The linear response of the ground state to the dynamic perturbation $\delta H(t)$ is given by

$$
\begin{equation*}
\delta\left\langle n_{j}(t)\right\rangle=-\sum_{l=1}^{L} \int_{-\infty}^{\infty} d t^{\prime} G_{R}\left(t-t^{\prime}, j-l\right) \mathcal{J}_{l}\left(t^{\prime}\right), \tag{9}
\end{equation*}
$$

where $G_{R}\left(t-t^{\prime}, j-l\right)=-i \theta\left(t-t^{\prime}\right)\left\langle\left[n_{j}(t), n_{l}\left(t^{\prime}\right)\right]\right\rangle$ is the retarded propagator with $n_{j}(t)=e^{i H t} n_{j} e^{-i H t}$, and $\langle\cdots\rangle$ represents the expectation value with respect to the ground state. We have an explicit expression for the retarded propagator as

$$
\begin{align*}
G_{R}(t, j)=\frac{2}{L^{2}} \theta(t) & \sum_{k, k^{\prime} \in K} u_{k} v_{k^{\prime}}\left(u_{k} v_{k^{\prime}}+u_{k^{\prime}} v_{k}\right) \\
& \times \sin \left[\left(\epsilon_{k}+\epsilon_{k^{\prime}}\right) t-\left(k-k^{\prime}\right) j\right], \tag{10}
\end{align*}
$$

where $\epsilon_{k}=2 J\left\{(\cos k-h / J)^{2}+\sin ^{2} k\right\}^{1 / 2}$ is the energy of the "single particle state". We have also defined $\left(u_{k}, v_{k}\right)=\mathcal{N}\left(\epsilon_{k}+z_{k}, i y_{k}\right)$ with $z_{k}=2(h-J \cos k)$ and $y_{k}=2 J \sin k$ where $\mathcal{N}$ is the normalization constant to make $\left|u_{k}\right|^{2}+\left|v_{k}\right|^{2}=1$. See supplemental material for the derivation of the above expression. Using Eqs.(3) and (10) in Eq. (9), we can compute the linear response.

Note that, in the linear response theory, we simultaneously describe the formulations for the cosine (real) and sine (imaginary) parts of the source field in terms of the "complex" form of $\mathcal{J}_{l}(t)$. In the experiments, one has to measure the responses $\delta\left\langle n_{j}(t)\right\rangle$ against $\operatorname{Re}\left[\mathcal{J}_{l}\right]$ and $\operatorname{Im}\left[\mathcal{J}_{l}\right]$ separately, and then combined the results into the form of $\left|\delta\left\langle n_{j}(t)\right\rangle\right|$.

Holographic teleportation- Taking units of $v=1$ and
$\ell=2 \pi$, we have

$$
\begin{equation*}
a=\frac{2 \pi}{L}, \quad J=\frac{L}{4 \pi}, \quad h=J-\frac{\delta}{2} \tag{11}
\end{equation*}
$$

The free parameters of the Hamiltonian are now given by $L$ and $\delta$. We take the amplitude of the source as $A=J \sqrt{2 /\left(\sigma_{t} \sigma_{\phi} L\right)}$ so that $\sum_{j} \int d t|\mathcal{J}|^{2} \simeq J^{2}$ is satisfied.

Figure 3 shows the response $\left|\delta\left\langle n_{j}(t)\right\rangle\right|$ for $\delta=0$, $\sigma_{t}=0.4, \sigma_{\phi}=0.4, \Omega=5$ and $M=0$. The number of sites are varied as $L=8,16,32,64$. As shown in the figures, the response suddenly stands up around at the points indicated in Fig. 2(b). This result shows that the holographic teleportation can indeed be observed in the realistic spin model on a finite-size lattice. Note that the behavior of the holographic teleportation is already seen even for a small $L$ (around for $L \sim 16$ ). For $\delta=0$ and $L \rightarrow \infty$, the transverse-field Ising model is described by the free CFT as in Eq. (7). Generally in CFTs, a twopoint function is universally determined by the conformal dimension of the operator regardless of whether the theory is strongly coupled or not. It follows that the linear response is also universally determined only by the conformal dimension of the operator coupling to the source $\mathcal{J}_{l}$. This is the reason why we obtained the consistent result with the classical gravity even though the CFT is not strongly coupled.

We show the time dependence of the response for fixed $\phi=0, \pi$ in the rightmost panel of Fig. 3, in which the holographic teleportation is clearly seen. According to the geodesic motion in the $\mathrm{AdS}_{3}$, the peaks of the response should be at $t=0,2 \pi, 4 \pi, \cdots$ for $\phi=0$ and $t=\pi, 3 \pi, 5 \pi \cdots$ for $\phi=\pi$, but they seem to appear a little later. The shift of the peak positions is caused by the finite- $L$ effect, and it actually gets smaller as $L$ increases.

Figure 4 shows the response for $L=32, \sigma_{t}=0.4$, $\sigma_{\phi}=0.4, \Omega=5$ and $M=0$. Now the mass of the Majorana fermion $\delta$ is changed as $\delta=-2,0,2$. Although the continuum limit of the system is not a CFT for $\delta \neq 0$, we can still see the behavior of the holographic telepor-


FIG. 4. Response for parameters $L=32, \sigma_{t}=0.4, \sigma_{\phi}=0.4$, $\Omega=5$ and $M=0 . \delta=-2,0,2$.


FIG. 5. Response for parameters $L=32, \delta=0, \sigma_{t}=0.4$, $\sigma_{\phi}=0.4, \Omega=5$ and $m=M / \Omega=0,0.2,0.4,0,6,0.8$.
tation, although it gets blurred as $|\delta|$ increases. We can also observe the shift of peak points of the response for $\delta \neq 0$.

Figure 5 shows the response for $L=32, \delta=0, \sigma_{t}=$ $0.4, \sigma_{\phi}=0.4, \Omega=5$. The angular momentum of the null geodesic is varied as $m=M / \Omega=0,0.2,0.4,0.6,0.8$. The position of the peak points does not depend on $m$ so much. This is consistent with the result of null geodesics in $\mathrm{AdS}_{3}$. In the view of the gravity side, as the value of $|m|$ approaches 1, the null geodesic passes closer to the boundary. (See Fig. 2(a).) Since the null geodesic is realized as a localized configuration of a probe field in $\mathrm{AdS}_{3}$, it actually has a tail. When the particle is close to the AdS boundary, the response has a non-zero value because of the tail. This is the origin of the right moving tail in Fig. 5.

In this Letter, we have taken the unit of $v=\ell /(2 \pi)=$ $\hbar=1$. We can easily restore the dimensions of the quantities as $\left(t, \Omega, \sigma_{t}, \delta\right) \rightarrow\left(t / T, \Omega T, \sigma_{t} / T, T \delta / \hbar\right)$ where $T=\hbar L /(4 \pi J)$. Note that $M$ and $\sigma_{\phi}$ are dimensionless quantities as they are. For example, in Figs. 3, 4 and 5, we set $\Omega=5$. This implies $\Omega=5 \times 4 \pi J /(\hbar L)$.

On experimental realization - The experimental realization and detection of our theoretical findings are feasible using Rydberg atoms trapped in an optical tweezers array [18-22]. The state-of-the-art techniques devel-
oped in recent years have enabled us to simulate a programmable Ising-type quantum spin model with tunable interactions [19], system sizes of up to hundreds [21, 22], and arbitrary lattice geometries [20, 22]. The holographic teleportation can be tested in a ring-shaped lattice [20] of atoms near the quantum critical regime $(J \approx h)$, achieved by global laser light that introduces the coupling between the ground and Rydberg states. The initial null geodesics on the critical vacuum can be created by temporary focused lasers in the shape of Eq. (3), and the response in $\left\langle\sigma_{i}^{x}(t)\right\rangle=1-2\left\langle n_{i}(t)\right\rangle$ of the individual atoms can be monitored at each time slice via the fluorescence imaging after inserting a global $\pi / 2$ pulse.

Conclusion and future directions - In this Letter, we introduced the concept of "holographic teleportation," a phenomenon where a particle appears to "teleport," travelling in the dual higher-dimensional geometry, and showed that this phenomenon actually takes place in the transverse-field Ising model on a lattice ring. This phenomenon is expected to be ubiquitous and working as a probe of the dual higher-dimensional geometry.

Although holographic teleportation can be clearly understood in the gravity side, its physical interpretation in the spin system is not trivial. This would result form the long range correlation of the ground state of the spin system near the critical point. Revealing the physical interpretation of holographic teleportation is an interesting future challenge.

The transverse-field Ising model reduces to the free QFT in the continuum limit as in Eq. (7). There has been attempts for dual descriptions of free quantum theories. One of them is to consider a higher spin gravity equivalent to a coset CFT with $\left(S O(2 N)_{k} \times\right.$ $\left.S O(2 N)_{1}\right) / S O(2 N)_{k+1}[25,26]$. Taking the limit $k \rightarrow \infty$ gives a model with $S O(2 N)_{1}$, which admits a free $N$ Majorana fermion representation. Therefore, at least some higher spin gravity models may be a target for the experimental probe of the dual higher-dimensional spacetime.

Studying finite temperature effects is one of the most important future directions. For a free QFT such as Eq.(7), its finite temperature effects is trivial. On the other hand, for the $S U(N)$ Heisenberg model for example, the continuum limit is a Wess-Zumino-Witten model [27] and can give non-trivial thermal effects. It would allows us to probe quantum black hole spacetimes through tabletop experiments.

Finally, let us suggest possible directions for the application of the holographic teleportation. 1) It may provide a new method in spintronics or magnonics. The teleportation might be used for carrying spin-wave packets from place to place, bypassing undesired operating elements existing on the way. Topological materials whose edge states are gapless CFT would be a suitable test ground for it. 2) It is tempting to suggest a similarity between the holographic teleportation and a time crystal. Figure 3 evidently forms a spacetime crystal-like structure in the
two-dimensional spacetime. Although our teleportation is not a spontaneous breaking of time translation, the crystaline pattern formation would provide some novel holographic understanding of critical materials. 3) Once at a finite temperature a holographic quantum black hole is realized, the quantum matter ring would serve as a quantum "trash can", when this ring is connected to quantum circuits. Black holes are the fastest scramblers [28], and information is effectively lost, which could be efficiently used in quantum information science. These are just a list of interesting suggestions, and we like to explore them in the forthcoming papers.
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# Supplemental material for Holographic Teleportation in Quantum Critical Spin Systems 

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#### Abstract

The transverse-field Ising mode is a solvable spin model, which reduces to a conformal field theory (CFT) at the critical point. In this supplemental material, we give a brief review of the diagonalization of the Hamiltonian of the transverse-field Ising model. Analytical calculations of the two-point function and linear response are also demonstrated. We take the CFT limit of the two point function and find that it coincides with that from the the general argument of the CFT.


## I. TRANSVERSE-FIELD ISING MODEL

The Hamiltonian of the transverse-field Ising model on the $L$-site ring is

$$
\begin{equation*}
H=-J \sum_{i=1}^{L} \sigma_{i}^{z} \sigma_{i+1}^{z}-h \sum_{i=1}^{L} \sigma_{i}^{x} \tag{1}
\end{equation*}
$$

where $\sigma_{i}^{a}(a=x, y, z)$ is the Pauli matrix which acts on the $i$-th spin and $\sigma_{L+1}=\sigma_{1}$. In this section, we give a brief review of the diagonalization of the transverse-field Ising model. (See also Refs. [1, 2] for nice reviews.) The one-dimensional spin- $1 / 2$ chain can be mapped onto the fermion system by the Jordan-Wigner transformation

$$
\begin{equation*}
\sigma_{i}^{x}=1-2 c_{i}^{\dagger} c_{i}, \quad-\sigma_{i}^{z}+i \sigma_{i}^{y}=2 \prod_{j=1}^{i-1}\left(1-2 c_{j}^{\dagger} c_{j}\right) c_{i} \tag{2}
\end{equation*}
$$

Its inverse transformation is given by

$$
\begin{equation*}
c_{i}=\frac{1}{2} \prod_{j=1}^{i-1} \sigma_{j}^{z}\left(-\sigma_{i}^{z}+i \sigma_{i}^{y}\right) \tag{3}
\end{equation*}
$$

The operator $c_{i}$ satisfies the canonical anti-commutation relation, $\left\{c_{i}, c_{j}^{\dagger}\right\}=\delta_{i j}$. By the Jordan-Wigner transformation, the transverse field Ising model reduces to the system of the fermions as

$$
\begin{align*}
H=-J \sum_{i=1}^{L}\left(c_{i}^{\dagger} c_{i+1}+c_{i+1}^{\dagger} c_{i}+\right. & \left.c_{i}^{\dagger} c_{i+1}^{\dagger}+c_{i+1} c_{i}\right) \\
& -h \sum_{i=1}^{L}\left(1-2 c_{i}^{\dagger} c_{i}\right) \tag{4}
\end{align*}
$$

where $c_{i}^{\dagger}$ and $c_{i}$ are the creation and annihilation operators of fermions at $i$-th site. We assume that the total number of fermions is even: $N=\sum_{j=1}^{L} c_{j}^{\dagger} c_{j} \in 2 \boldsymbol{Z}$. Then, $c_{i}$ satisfies the anti-periodic boundary condition $c_{L+1}=-c_{1}$.

We apply the Fourier transformation of the operator $c_{j}$ as

$$
\begin{equation*}
c_{j}=\frac{1}{\sqrt{L}} \sum_{k \in K} e^{i k j} c_{k} \tag{5}
\end{equation*}
$$

From the anti-periodic boundary condition in Eq. (4), the domain of the wave number $k$ is given by

$$
\begin{equation*}
K=\left\{\left.\frac{2 \pi}{L}\left(n-\frac{1}{2}\right) \right\rvert\, n=-\frac{L}{2}+1, \cdots, \frac{L}{2}\right\} \tag{6}
\end{equation*}
$$

In the momentum space, the Hamiltonian becomes

$$
\begin{array}{r}
H=-J \sum_{k \in K}\left(2 \cos k c_{k}^{\dagger} c_{k}+e^{i k} c_{k}^{\dagger} c_{-k}^{\dagger}+e^{-i k} c_{-k} c_{k}\right) \\
+h \sum_{k \in K}\left(2 c_{k}^{\dagger} c_{k}-1\right) \tag{7}
\end{array}
$$

Rewriting the Hamiltonian (4) in terms of $c_{k}$, we find the coupling between modes with $k$ and $-k$. However, after the Bogoliubov transformation

$$
\binom{c_{k}}{c_{-k}^{\dagger}}=\left(\begin{array}{cc}
u_{k} & -v_{k}^{*}  \tag{8}\\
v_{k} & u_{k}^{*}
\end{array}\right)\binom{\gamma_{k}}{\gamma_{-k}^{\dagger}}
$$

we obtain the diagonalized Hamiltonian as

$$
\begin{equation*}
H=E_{0}+\sum_{k \in K} \epsilon_{k} \gamma_{k}^{\dagger} \gamma_{k} \tag{9}
\end{equation*}
$$

where

$$
\begin{align*}
& \epsilon_{k}=2 J \sqrt{\left(\cos k-\frac{h}{J}\right)^{2}+\sin ^{2} k} \\
& \binom{u_{k}}{v_{k}}=\frac{1}{\sqrt{2 \epsilon_{k}\left(\epsilon_{k}+z_{k}\right)}}\binom{\epsilon_{k}+z_{k}}{i y_{k}} \tag{10}
\end{align*}
$$

with $z_{k}=2(h-J \cos k)$ and $y_{k}=2 J \sin k$. The constant term in Eq. (9) is the energy of the ground state given by $E_{0}=-\sum_{k \in K} \epsilon_{k} / 2$.

From Eq. (9), we find that the ground state $|0\rangle$ is the state which is annihilated by $\gamma_{k}$ :

$$
\begin{equation*}
\gamma_{k}|0\rangle=0 \quad(\forall k \in K) \tag{11}
\end{equation*}
$$

Excited states are constructed by multiplying the creation operators to the ground state as

$$
\begin{equation*}
|\vec{m}\rangle=\prod_{k \in K}\left(\gamma_{k}^{\dagger}\right)^{m_{k}}|0\rangle \quad\left(m_{k}=0 \text { or } 1\right) \tag{12}
\end{equation*}
$$

where $\vec{m}$ represents the list of $m_{k}$. As assumed in Eq. (4), the total fermion number $\sum_{k \in K} m_{k}$ should be even. Their energy eigenvalues are $E(\vec{m})=E_{0}+\sum_{k \in K} m_{k} \epsilon_{k}$.

## II. RETARDED PROPAGATOR AND LINEAR RESPONSE

We define the two-point function in the transverse-field Ising model as

$$
\begin{equation*}
C_{s}(t)=\left\langle n_{j+s}(t) n_{j}(0)\right\rangle \tag{13}
\end{equation*}
$$

where $\langle\cdots\rangle$ is the expectation value with respect to the ground state. We denoted the number operator of the fermion at the $j$-th site by $n_{j}=c_{j}^{\dagger} c_{j}$, and introduced its Heisenberg picture as $n_{j}(t)=e^{i H t} n_{j} e^{-i H t}$. We can also express $n_{j}$ in terms of $\gamma_{k}$ defined in Eq. (8) as

$$
\begin{align*}
n_{j}=\frac{1}{L} & \sum_{k, k^{\prime} \in K} e^{i\left(k^{\prime}-k\right) j} \\
& \times\left(u_{k}^{*} \gamma_{k}^{\dagger}-v_{k} \gamma_{-k}\right)\left(u_{k^{\prime}} \gamma_{k^{\prime}}-v_{k^{\prime}}^{*} \gamma_{-k^{\prime}}^{\dagger}\right) \tag{14}
\end{align*}
$$

The two point function is rewritten as $C_{s}(t)=$ $\langle 0| n_{j+s} e^{-i\left(H-E_{0}\right) t} n_{j}|0\rangle$. From Eq. (14), we have

$$
\begin{align*}
& n_{j}|0\rangle=-\frac{1}{L} \sum_{k, k^{\prime} \in K} e^{i\left(k^{\prime}-k\right) j} u_{k}^{*} v_{k^{\prime}}^{*}\left|k,-k^{\prime}\right\rangle \\
&+\frac{1}{L} \sum_{k \in K}\left|v_{k}\right|^{2}|0\rangle \tag{15}
\end{align*}
$$

where we defined the two-particle state

$$
\begin{equation*}
\left|k,-k^{\prime}\right\rangle \equiv \gamma_{k}^{\dagger} \gamma_{-k^{\prime}}^{\dagger}|0\rangle \tag{16}
\end{equation*}
$$

This satisfies

$$
\begin{align*}
& \left\langle 0 \mid k,-k^{\prime}\right\rangle=0 \\
& \left\langle p,-p^{\prime} \mid k,-k^{\prime}\right\rangle=\delta_{p k} \delta_{p^{\prime} k^{\prime}}-\delta_{p,-k^{\prime}} \delta_{p^{\prime},-k} \tag{17}
\end{align*}
$$

Since the ground state $|0\rangle$ and the two-particle state $\left|k,-k^{\prime}\right\rangle$ are energy eigenstates, we also have $e^{-i\left(H-E_{0}\right) t}|0\rangle=|0\rangle$ and $e^{-i\left(H-E_{0}\right) t}\left|k,-k^{\prime}\right\rangle=$ $e^{-i\left(\epsilon_{k}+\epsilon_{k^{\prime}}\right) t}\left|k,-k^{\prime}\right\rangle$. From these relations, the two-point
function is computed as

$$
\begin{align*}
& C_{s}(t)=\frac{1}{L^{2}}\left(\sum_{k \in K}\left|v_{k}\right|^{2}\right)^{2} \\
&-\frac{1}{L^{2}} \sum_{k, k^{\prime} \in K} e^{-i\left(\epsilon_{k}+\epsilon_{k^{\prime}}\right) t+i\left(k-k^{\prime}\right) s} \\
& \times u_{k} v_{k^{\prime}}\left(u_{k} v_{k^{\prime}}+u_{k^{\prime}} v_{k}\right) \tag{18}
\end{align*}
$$

The regarded propagator is defined as

$$
\begin{equation*}
G_{R}\left(t-t^{\prime}, j-l\right)=-i \theta\left(t-t^{\prime}\right)\left\langle\left[n_{j}(t), n_{l}\left(t^{\prime}\right)\right]\right\rangle \tag{19}
\end{equation*}
$$

which is computed from the two point function (13) as $G_{R}\left(t-t^{\prime}, j-l\right)=-i \theta\left(t-t^{\prime}\right)\left(C_{j-l}\left(t-t^{\prime}\right)-C_{l-j}\left(t^{\prime}-t\right)\right)$. Thus, we obtain the explicit expression for the regarded propagator as

$$
\begin{align*}
G_{R}\left(t-t^{\prime}, j\right. & -l)=-\frac{i}{L^{2}} \theta\left(t-t^{\prime}\right) \\
& \times \sum_{k, k^{\prime} \in K} u_{k} v_{k^{\prime}}\left(u_{k} v_{k^{\prime}}+u_{k^{\prime}} v_{k}\right) \\
& \times \sum_{r= \pm 1} r e^{i r\left(\epsilon_{k}+\epsilon_{k^{\prime}}\right)\left(t-t^{\prime}\right)-i r\left(k-k^{\prime}\right)(j-l)} \tag{20}
\end{align*}
$$

Under the perturbation of the Hamiltonian,

$$
\begin{equation*}
\delta H=-\sum_{l=1}^{L} \mathcal{J}_{l}(t) n_{l} \tag{21}
\end{equation*}
$$

its linear response is given by

$$
\begin{equation*}
\delta\left\langle n_{j}(t)\right\rangle=-\sum_{l=1}^{L} \int_{-\infty}^{\infty} d t^{\prime} G_{R}\left(t-t^{\prime}, j-l\right) J_{l}\left(t^{\prime}\right) \tag{22}
\end{equation*}
$$

We assume the time dependence of the source as

$$
\begin{equation*}
J_{l}(t)=A_{l} \exp \left[-\frac{t^{2}}{2 \sigma_{t}^{2}}-i \Omega t\right] \tag{23}
\end{equation*}
$$

This is localized in time and oscillates with the frequency $\Omega . A_{l}$ describes the spacial dependence of the source. Then, we can perform the $t^{\prime}$-integration analytically in Eq. (22). Using the error function: $\operatorname{erfc}(z)=$ $2 \pi^{-1 / 2} \int_{z}^{\infty} e^{-t^{2}} d t$, we can write the analytical form of the linear response as

$$
\begin{align*}
& \delta\left\langle n_{j}(t)\right\rangle= \frac{i \sigma_{t}}{L^{2}} \sqrt{\frac{\pi}{2}} \sum_{r= \pm 1, k, k^{\prime}} r u_{k} v_{k^{\prime}}\left(u_{k} v_{k^{\prime}}+u_{k^{\prime}} v_{k}\right) \\
& \times\left(\sum_{l=1}^{L} A_{l} e^{i r\left(k-k^{\prime}\right) l}\right) e^{-i r\left(k-k^{\prime}\right) j} \\
& \times \exp \left[-\frac{\sigma_{t}^{2}}{2}\left\{\Omega+r\left(\epsilon_{k}+\epsilon_{k^{\prime}}\right)\right\}^{2}+i r\left(\epsilon_{k}+\epsilon_{k^{\prime}}\right) t\right] \\
& \times \operatorname{erfc}\left[-\frac{t}{\sqrt{2} \sigma_{t}}-\frac{i \sigma_{t}}{\sqrt{2}}\left\{\Omega+r\left(\epsilon_{k}+\epsilon_{k^{\prime}}\right)\right\}\right] \tag{24}
\end{align*}
$$

For a given space dependence of the source $A_{l}$, we can numerically compute the summation of $k, k^{\prime}, r$ and $l$.

## III. CONFORMAL FIELD THEORY LIMIT OF THE TRANSVERSE-FIELD ISING MODEL

Let us consider the the continuum limit of Eq. (4). We introduce the fermion field $\Psi\left(x_{j}\right)=c_{j} / \sqrt{a}$ where $a$ is the lattice spacing. For $a \rightarrow 0, \Psi(x)$ satisfies $\left\{\Psi(x), \Psi\left(x^{\prime}\right)\right\}=\delta\left(x-x^{\prime}\right)$. Then, the Hamiltonian is written as

$$
\begin{equation*}
H=-\int_{0}^{\ell} d x\left[\frac{v}{2}\left(\Psi^{\dagger} \frac{d}{d x} \Psi^{\dagger}-\Psi \frac{d}{d x} \Psi\right)+\delta \Psi^{\dagger} \Psi\right] \tag{25}
\end{equation*}
$$

where

$$
\begin{equation*}
\ell=L a, \quad v=2 J a=\frac{2 J \ell}{L}, \quad \delta=2(J-h) \tag{26}
\end{equation*}
$$

This describe the theory for the free Majorana fermion with the mass $\delta / v^{2}$. For the critical case $J=h$, above Hamiltonian describes the conformal field theory (CFT) with central charge $c=1 / 2$. Thus, the CFT limit of the transverse-field Ising model is given by

$$
\begin{equation*}
h=J, \quad J \rightarrow \infty, \quad L \rightarrow \infty, \quad a \rightarrow 0 \tag{27}
\end{equation*}
$$

with fixed $v$ and $\ell$ in Eq. (26). In the followings, we take the unit of $v=1$ and $\ell=2 \pi$, i.e, $4 \pi J / L=1$ and $L a=2 \pi$.

We consider the CFT limit of the two point function (18). Since the first term of Eq. (18) is a constant, we will omit it in the following expressions. We can decompose the summation of $k$ and $k^{\prime}$ as

$$
\begin{equation*}
C_{s}(t)=-\frac{1}{L^{2}}\left(C_{s}^{u u}(t) C_{-s}^{v v}(t)+C_{s}^{u v}(t) C_{-s}^{u v}(t)\right) \tag{28}
\end{equation*}
$$

where

$$
\begin{align*}
C_{s}^{u u}(t) & \equiv \sum_{k \in K} e^{-i \epsilon_{k} t+i k s} u_{k}^{2} \\
C_{s}^{u v}(t) & \equiv \sum_{k \in K} e^{-i \epsilon_{k} t+i k s} u_{k} v_{k}  \tag{29}\\
C_{s}^{v v}(t) & \equiv \sum_{k \in K} e^{-i \epsilon_{k} t+i k s} v_{k}^{2}
\end{align*}
$$

In the critical case $J=h$, the dispersion relation becomes gapless as

$$
\begin{equation*}
\epsilon_{k}=4 J\left|\sin \frac{k}{2}\right| \tag{30}
\end{equation*}
$$

We also have

$$
\begin{equation*}
u_{k}=\sqrt{\frac{1+\left|\sin \frac{k}{2}\right|}{2}}, \quad v_{k}=\frac{i \cos \frac{k}{2} \operatorname{sgn}(k)}{\sqrt{2\left(1+\left|\sin \frac{k}{2}\right|\right)}} \tag{31}
\end{equation*}
$$

For the regularization, we shift the time coordinate to the complex plane as $t \rightarrow t-i \varepsilon$. For the shift parameter $\varepsilon$, we assume $1 / J \ll \varepsilon$. Although this is
bounded from below, after taking the CFT limit (27), we can eventually take the limit of $\varepsilon \rightarrow+0$. Due to the imaginary part of the time coordinate, only the region of $|k| \lesssim 1 /(J \varepsilon) \ll 1$ contributes in the summation of Eq. (29). In this region, we can write $\epsilon_{k} \simeq 2 J|k|$, $u_{k} \simeq 1 / \sqrt{2}$ and $v_{k} \simeq i \operatorname{sgn}(k) / \sqrt{2}$. Thus, we have

$$
\begin{align*}
& C_{s}^{u u}(t-i \varepsilon) \rightarrow \frac{1}{2} \sum_{k \in K} e^{-2 i J|k|(t-i \varepsilon)+i k s} \\
& =\frac{1}{2} \sum_{n=-\infty}^{\infty} e^{-i|n-1 / 2|(t-i \varepsilon)+i(n-1 / 2) \phi}  \tag{32}\\
& =\frac{1}{4 i}\left(\frac{1}{\sin \frac{(t-i \varepsilon)-\phi}{2}}+\frac{1}{\sin \frac{(t-i \varepsilon)+\phi}{2}}\right)
\end{align*}
$$

where we introduced the coordinate of the $s$-th spin site as $\phi=2 \pi s / L$. Similarly, we also obtain

$$
\begin{align*}
C_{s}^{u v}(t-i \varepsilon) & \rightarrow \frac{1}{4}\left(\frac{1}{\sin \frac{(t-i \varepsilon)-\phi}{2}}-\frac{1}{\sin \frac{(t-i \varepsilon)+\phi}{2}}\right)  \tag{33}\\
C_{s}^{v v}(t-i \varepsilon) & \rightarrow-\frac{1}{4 i}\left(\frac{1}{\sin \frac{(t-i \varepsilon)-\phi}{2}}+\frac{1}{\sin \frac{(t-i \varepsilon)+\phi}{2}}\right)
\end{align*}
$$

Therefore, from Eq. (28), the CFT limit of the two point function becomes

$$
\begin{equation*}
C_{s}(t-i \varepsilon) \rightarrow-\frac{1}{4 L^{2}} \frac{1}{\sin \frac{(t-i \varepsilon)+\phi}{2} \sin \frac{(t-i \varepsilon)-\phi}{2}} \tag{34}
\end{equation*}
$$

Let us consider the two point function from the general argument of CFT. For Euclidean CFT in $\boldsymbol{R}^{2}$, the twopoint function for the operator with conformal weight ( $h, \bar{h}$ ) is given by

$$
\begin{equation*}
\left\langle\mathcal{O}\left(z_{1}, \bar{z}_{1}\right) \mathcal{O}\left(z_{2}, \bar{z}_{2}\right)\right\rangle=\frac{1}{z_{12}^{2 h} \bar{z}_{12}^{2 \bar{h}}}, \tag{35}
\end{equation*}
$$

where $z_{12}=z_{1}-z_{2}$. Here we will consider the spinless field: $h=\bar{h}=\Delta / 2$. We can move to the CFT in the cylinder $\boldsymbol{R} \times S^{1}$ by the conformal transformation, $z=$ $e^{-i w}$. From $\mathcal{O}(z, \bar{z})=(\partial z / \partial w)^{-h}(\partial \bar{z} / \partial \bar{w})^{-\bar{h}} \mathcal{O}(w, \bar{w})$ and Eq. (35), we obtain

$$
\begin{equation*}
\langle\mathcal{O}(\tau, \phi) \mathcal{O}(0)\rangle \propto \frac{1}{\left(\sin \frac{\phi+i \tau}{2} \sin \frac{\phi-i \tau}{2}\right)^{\Delta}} \tag{36}
\end{equation*}
$$

where we write the complex coordinate in $\boldsymbol{R} \times S^{1}$ as $w=\phi+i \tau$. The two-point function in the Lorentzian signature is given by the analytic continuation of the Euclidean time as $\tau=i t+\varepsilon[3,4]$. Thus, we have

$$
\begin{equation*}
\langle\mathcal{O}(t, \phi) \mathcal{O}(0)\rangle \propto \frac{1}{\left(\sin \frac{(t-i \varepsilon)+\phi}{2} \sin \frac{(t-i \varepsilon)-\phi}{2}\right)^{\Delta}} \tag{37}
\end{equation*}
$$

This coincides with Eq. (34) by setting $\Delta=1$.
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