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# Unified Theory Based on Parameter Scaling for Derivation of Nonlinear Wave Equations in Bubbly Liquids*

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**Abstract**

We propose a systematic derivation method of the Korteweg–de Vries–Burgers (KdVB) equation and nonlinear Schrödinger (NLS) equation for nonlinear waves in bubbly liquids on the basis of appropriate choices of scaling relations of physical parameters. The basic equations are composed of a set of conservation equations for mass and momentum and the equation of bubble dynamics in a two-fluid model. The scaling of parameters is related to the wavelength, frequency, propagation speed, and amplitude of waves concerned. With the help of the method of multiple scales, appropriate choices of the parameter scaling allow us to derive various nonlinear wave equations systematically from a set of basic equations. The result shows that the one-dimensional nonlinear propagation of a long wave with a low frequency is described by the KdVB equation, and that of an envelope of a carrier wave with a high frequency by the NLS equation. Thus, we establish a unified theory of derivation of nonlinear wave equations in bubbly liquids.

*Key words* : Weakly Nonlinear Waves, Dispersive Waves, Bubbly Liquids, Bubble Dynamics, KdV–Burgers Equation, Nonlinear Schrödinger Equation

## 1. Introduction

Pressure wave propagation in bubbly liquids has long been one of the most fundamental topics in the field of multiphase flows. Its characteristics are considerably different from those in single phase fluids. For instance, the dispersion in the sense that waves of different wavelengths propagate with different phase velocities is usually caused by bubble oscillations(1). The decrease in sound speed of a long wave is also well known(2).

For more than 40 years, a number of theoretical papers on weakly nonlinear waves in bubbly liquids have been published intensively(3)−(14). Especially, a pioneering work by van Wijngaarden is well known, which derived the Korteweg–de Vries (KdV) equation(3) and the KdV–Burgers (KdVB) equation(4) from a set of basic equations for bubbly flows, on the basis of a perturbation method, with conditions of long wavelength, low frequency, weak dispersion, incompressible liquid, and so on. Subsequently, the KdV, KdVB, and nonlinear Schrödinger (NLS) equations have also been rederived or derived for the waves in bubbly liquids on the basis of various mathematical techniques, assumptions, and basic equations (5), (6), (8), (9), (11), (12), (14). However, some of these studies are closely connected and others are not. Individual derivation procedures rely on specific methodologies or brilliant mathematical

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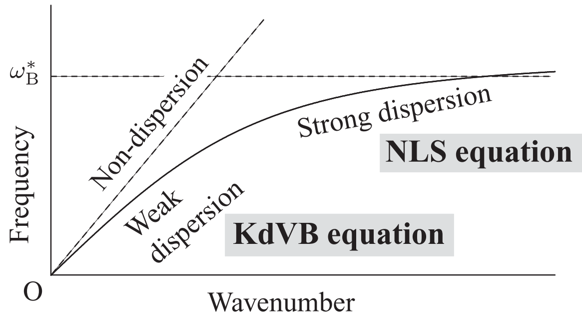


Fig. 1 The dispersion relation in bubbly liquids[(3),(4),(8),(9)]. Weakly nonlinear propagation of pressure waves in two frequency bands, i.e., the low frequency band and high frequency band are described by the KdVB and NLS equations, respectively. Here, $\omega_B^*$ denotes the natural frequency of a single bubble.

techniques, and hence the explicit relations among them have remained vague. This situation has prevented us from developing theoretical study on the nonlinear waves in bubbly liquids and enhancement of their applications.

The aim of this paper is to propose a unified theory for the derivation of nonlinear wave equations in bubbly liquids. By the unified theory, we can derive different types of nonlinear wave equations from a set of generic equations of bubbly flows. The essence of this theory lies in the fact that there exists a scaling of some physical parameters appropriate to a specific wave phenomenon. The parameter scaling means measurement of the nondimensional magnitudes of some physical quantities in terms of a typical nondimensional amplitude of the wave concerned. To be precise, the nondimensional magnitudes of physical quantities are the ratio of a typical propagation speed of the wave to the sound speed in the liquid, that of a typical bubble radius to a typical wavelength, and that of a typical frequency of the wave to an natural frequency of the bubble. Since these ratios have physically clear implications, the unified theory can offer a perspective for understanding various complex nonlinear wave phenomena in bubbly liquids.

Now, let us show the parameter scaling considered in this paper. The scaling relations appropriate to the low frequency and long wavelength band (weak dispersion band) and the high frequency and short wavelength band (strong dispersion band) shown in Fig. 1[(3),(4),(8),(9)], are, respectively, defined by

$$\left(\frac{U^*}{c_{L0}^*}, \frac{R_0^*}{L^*}, \frac{\omega^*}{\omega_B^*}\right) \equiv \begin{cases} \left(O(\sqrt{\epsilon}), O(\sqrt{\epsilon}), O(\sqrt{\epsilon})\right), & \text{(for KdVB)}, \\ \left(O(\epsilon^2), O(1), O(1)\right), & \text{(for NLS)}, \end{cases} \tag{1}$$

where $\epsilon\,(\ll 1)$ is a nondimensional amplitude of the waves, $U^*$ and $L^*$ are a typical propagation speed of the waves and a typical wavelength, respectively, $R_0^*$ is a bubble radius in an initially unperturbed state, $c_{L0}^*$ is a sound speed in the unperturbed liquid, $\omega^*$ is an angular frequency of the waves, and $\omega_B^*$ is a natural angular frequency of the bubble. The superscript $^*$ denotes a dimensional quantity throughout this paper. More detailed explanations of parameter scaling (1) are presented in §3 and §4.

By the use of parameter scaling (1) and method of multiple scales[(15),(16)], we derive the KdVB and NLS equations from a set of two-fluid averaged equations for bubbly flows[(17),(18)], in which we take account of liquid compressibility responsible for wave attenuation due to acoustic radiation from oscillating bubbles[(19)]. These two nonlinear wave equations have been intensively examined in the field of the nonlinear wave theory[(16),(20)], apart from the theory of multiphase flows. The KdVB equation describes the nonlinear propagation of a long wave with a low frequency and the NLS equation does that of an envelope of a carrier wave with a high frequency. This can readily be recognized from Fig. 1 and parameter scaling (1). It should be emphasized that, in the past, the derivations of KdV, KdVB, and NLS equations for waves in bubbly liquids were limited to those from basic equations of a bubble–liquid mixture model, and the present result is the first demonstration of derivation from a two-fluid model. The coefficients in the KdVB and NLS equations derived here reflect the effect of extension of basic equations from the mixture model to the two-fluid model, and this effect can easily be examined since the unified theory presented here is applicable to the both models. Incidentally, the liquid compressibility is not essential for the wave motions characterized by the dispersion relation shown in Fig. 1 (see Egashira *et al.*[17]), except for the effect of wave attenuation.

The contents of this paper are as follows: in §2, we introduce the basic equations for bubbly flows and the method of multiple scales. The main parts of this paper are §3 and §4; the derivations of the KdVB and NLS equations based on the parameter scaling are systematically demonstrated there. Section 5 is devoted to conclusions.

## 2. Formulation of the problem

We shall examine one-dimensional nonlinear dispersive waves in a mixture of a compressible liquid and a number of small spherical gas bubbles. At an initial state, the mixture is assumed to be uniform and at rest. A pressure wave is generated from a sound source placed in the bubbly liquid. The amplitude of pressure wave is sufficiently small compared with the pressure in the ambient bubbly liquid. The present analysis aims to derive the nonlinear wave equations governing the asymptotic behaviors of wave motions with respect to the finite but small amplitude (weakly nonlinear problem).

Let us here summarize the main assumptions: (i) the bubbles are spherically symmetric. (ii) The bubbles do not coalesce, break up, extinct, and appear. (iii) The effect of bubble–bubble interaction is ignored. (iv) The volume fraction of gas phase (void fraction) in the bubbly liquid is uniform at the initial state. (v) The compressibility of the liquid is taken into account. (vi) The viscosity of the liquid is considered at surface of bubbles, although that of the gas is omitted. (vii) The bulk viscosities of the gas and liquid are neglected. (viii) The gas inside bubbles is composed of only a non-condensable gas, and hence the phase change across the bubble–liquid interface does not occur. (ix) The thermal conductivities of the gas and liquid, Reynolds stress, and gravitation, are dismissed.

The wave attenuation is caused by the three effects[4], i.e., the liquid viscosity, liquid compressibility, and thermal conductivity. It is well known[8],[10] that the thermal process inside bubbles with the heat exchange at the bubble–liquid interface induces a significant attenuation of waves in bubbly liquids. In this paper, however, we neglect the thermal effect for simplicity, and we take into account the wave attenuation due to only the liquid viscosity and liquid compressibility.

### 2.1. Basic equations for bubbly flows

We shall use basic equations for bubbly flows recently proposed by our group[17],[18], which are composed of conservation equations of mass and momentum for the gas and liquid phases, the equation of motion for the bubble wall, the equations of state for the gas and liquid phases, the mass conservation equation inside the bubble, and the balance of normal stresses at the bubble–liquid interface. For one-dimensional flows, firstly, the conservation equations of mass and momentum based on a two-fluid model are given by

$$\frac{\partial}{\partial t^*}(\alpha\rho_{\mathrm{G}}^*) + \frac{\partial}{\partial x^*}(\alpha\rho_{\mathrm{G}}^* u_{\mathrm{G}}^*) = 0, \tag{2}$$

$$\frac{\partial}{\partial t^*}\left[(1-\alpha)\rho_{\mathrm{L}}^*\right] + \frac{\partial}{\partial x^*}\left[(1-\alpha)\rho_{\mathrm{L}}^* u_{\mathrm{L}}^*\right] = 0, \tag{3}$$

$$\frac{\partial}{\partial t^*}(\alpha\rho_{\mathrm{G}}^* u_{\mathrm{G}}^*) + \frac{\partial}{\partial x^*}\left(\alpha\rho_{\mathrm{G}}^* u_{\mathrm{G}}^{*2}\right) + \alpha\frac{\partial p_{\mathrm{G}}^*}{\partial x^*} = F^*, \tag{4}$$

$$\frac{\partial}{\partial t^*}\left[(1-\alpha)\rho_{\mathrm{L}}^* u_{\mathrm{L}}^*\right] + \frac{\partial}{\partial x^*}\left[(1-\alpha)\rho_{\mathrm{L}}^* u_{\mathrm{L}}^{*2}\right] + (1-\alpha)\frac{\partial p_{\mathrm{L}}^*}{\partial x^*} + P^*\frac{\partial \alpha}{\partial x^*} = -F^*, \tag{5}$$

where $t^*$ is the time, $x^*$ is the space coordinate normal to the wave front, $\alpha$ is the void fraction ($0 < \alpha < 1$), $\rho^*$ is the density, $u^*$ is the fluid velocity, $p^*$ is the pressure, and the subscripts G and L denote volume-averaged variables in the gas and liquid phases, respectively. In addition to the volume-averaged pressures $p_G^*$ and $p_L^*$, the liquid pressure averaged on the bubble–liquid interface[21], $P^*$, is introduced.

For the interfacial momentum transport $F^*$, we adopt the following model of virtual mass force[18]

$$F^* = -\beta_1 \alpha \rho_L^* \left( \frac{\mathrm{D_G} u_G^*}{\mathrm{D}t^*} - \frac{\mathrm{D_L} u_L^*}{\mathrm{D}t^*} \right) \\ -\beta_2 \rho_L^* (u_G^* - u_L^*) \frac{\mathrm{D_G} \alpha}{\mathrm{D}t^*} - \beta_3 \alpha (u_G^* - u_L^*) \frac{\mathrm{D_G} \rho_L^*}{\mathrm{D}t^*}, \tag{6}$$

where the values of coefficients $\beta_1$, $\beta_2$, and $\beta_3$ may be set as $1/2$ for the spherical bubble, although we proceed without explicitly showing these values to clarify the contribution of each term in the right-hand side of Eq. (6) to the final result. Equation (6) is suggested by the analysis of virtual mass force in a compressible liquid[22],[23].

The Keller equation[19] for spherical oscillations of a bubble in a compressible liquid is given by

$$\left(1 - \frac{1}{c_{L0}^*} \frac{\mathrm{D_G} R^*}{\mathrm{D}t^*}\right) R^* \frac{\mathrm{D_G^2} R^*}{\mathrm{D}t^{*2}} + \frac{3}{2}\left(1 - \frac{1}{3c_{L0}^*} \frac{\mathrm{D_G} R^*}{\mathrm{D}t^*}\right)\left(\frac{\mathrm{D_G} R^*}{\mathrm{D}t^*}\right)^2 \\ = \left(1 + \frac{1}{c_{L0}^*} \frac{\mathrm{D_G} R^*}{\mathrm{D}t^*}\right) \frac{P^*}{\rho_{L0}^*} + \frac{R^*}{\rho_{L0}^* c_{L0}^*} \frac{\mathrm{D_G}}{\mathrm{D}t^*}(p_L^* + P^*), \tag{7}$$

where $R^*$ is the averaged bubble radius, $\rho_{L0}^*$ is the liquid density in the initial unperturbed state, and the definitions of operators $\mathrm{D_G}/\mathrm{D}t^*$ and $\mathrm{D_L}/\mathrm{D}t^*$ are

$$\frac{\mathrm{D_G}}{\mathrm{D}t^*} \equiv \frac{\partial}{\partial t^*} + u_G^* \frac{\partial}{\partial x^*}, \quad \frac{\mathrm{D_L}}{\mathrm{D}t^*} \equiv \frac{\partial}{\partial t^*} + u_L^* \frac{\partial}{\partial x^*}. \tag{8}$$

The second term in the right-hand side of Eq. (7) embodies a damping effect, which is mainly responsible for the wave attenuation due to the acoustic radiation from oscillating bubbles; the first term in the right-hand side also results in the wave attenuation due to the liquid viscosity $\mu^*$ through Eq. (12) below.

Equations (2)–(7) are closed by the following equations: (i) Tait equation of state for liquid,

$$p_L^* = p_{L0}^* + \frac{\rho_{L0}^* {c_{L0}^*}^2}{n} \left[ \left( \frac{\rho_L^*}{\rho_{L0}^*} \right)^n - 1 \right], \tag{9}$$

where $n$ is the material constant; e.g., $n = 7.15$ for water, (ii) the polytropic equation of state for gas,

$$\frac{p_G^*}{p_{G0}^*} = \left( \frac{\rho_G^*}{\rho_{G0}^*} \right)^\gamma, \tag{10}$$

where $\gamma$ is the polytropic exponent, (iii) the conservation equation of mass inside the bubble,

$$\frac{\rho_G^*}{\rho_{G0}^*} = \left( \frac{R_0^*}{R^*} \right)^3, \tag{11}$$

(iv) the balance of normal stresses across the bubble–liquid interface,

$$p_G^* - (p_L^* + P^*) = \frac{2\sigma^*}{R^*} + \frac{4\mu^*}{R^*} \frac{\mathrm{D_G} R^*}{\mathrm{D}t^*}, \tag{12}$$

where $\sigma^*$ is the surface tension. The effect of liquid viscosity $\mu^*$ is neglected except at the bubble–liquid interface. The physical quantities in the initial unperturbed state are signified by the subscript 0, and they are all constants.

### 2.2. Method of multiple scales

In the case of weakly nonlinear problems, the nonlinear effect manifests itself in a far field, which is defined as a region at a large distance from the sound source compared with a typical wavelength. On the other hand, a near field denotes a region whose distance from the sound source is comparable with a wavelength. Therefore, the problem involves different length scales, and the method of multiple scales[15],[16] is generally effective for such a problem. In the method of multiple scales, various phenomena characterized by different time and length scales can be described by independent variables,

$$t_m = \epsilon^m t, \quad x_m = \epsilon^m x, \quad (m = 0, 1, 2, ...), \tag{13}$$

and

$$t = \frac{t^*}{T^*}, \quad x = \frac{x^*}{L^*}, \tag{14}$$

where $T^*$ and $L^*$ are, respectively, typical periods in time and space of the wave concerned, and $\epsilon$ is a nondimensional wave amplitude which is sufficiently small compared with unity. In Eq. (13), $t_0$ and $x_0$ describe a near field and are sometimes called fast scales, whereas $t_m$ and $x_m$ for $m \geqq 1$ describe far fields and are called slow scales. As a result, dependent variables should now be regarded as functions of these extended independent-variables. Thus, differential operators can be expanded as follows:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon\frac{\partial}{\partial t_1} + \epsilon^2\frac{\partial}{\partial t_2} + \cdots, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial x_0} + \epsilon\frac{\partial}{\partial x_1} + \epsilon^2\frac{\partial}{\partial x_2} + \cdots. \tag{15}$$

Dependent variables are nondimensionalized and expanded in power series of $\epsilon$:

$$\alpha/\alpha_0 = 1 + \epsilon\alpha_1 + \epsilon^2\alpha_2 + \cdots, \tag{16}$$

$$u_G^*/U^* = \epsilon u_{G1} + \epsilon^2 u_{G2} + \cdots, \tag{17}$$

$$u_L^*/U^* = \epsilon u_{L1} + \epsilon^2 u_{L2} + \cdots, \tag{18}$$

$$R^*/R_0^* = 1 + \epsilon R_1 + \epsilon^2 R_2 + \cdots, \tag{19}$$

where $\alpha_0$ is the initial constant void fraction and $U^*$ is a typical propagation speed of the wave. The propagation speed of the wave $U^*$, wavelength $L^*$, and period $T^*$ are related by $L^* \equiv U^*T^*$; they are determined in the following sections. Although the initial void fraction $\alpha_0$ in Eq. (16) should be small compared with unity by the assumptions listed in the second paragraph in §2, it is treated as a quantity of the order of unity because the asymptotic behavior with respect to the small amplitude $\epsilon$ is considered.

Furthermore, the expansion of the liquid density $\rho_L^*$ in $\epsilon$ is given by

$$\begin{aligned}\rho_L^*/\rho_{L0}^* &= 1 + \epsilon^\kappa\rho_{L1} + \epsilon^{\kappa+1}\rho_{L2} + \cdots \\ &\equiv \begin{cases} 1 + \epsilon^2\rho_{L1} + \epsilon^3\rho_{L2} + \cdots, & \text{(for KdVB)}, \\ 1 + \epsilon^5\rho_{L1} + \epsilon^6\rho_{L2} + \cdots, & \text{(for NLS)}, \end{cases}\end{aligned} \tag{20}$$

where $\kappa$ ($\geqq 2$) is an integer number. It is determined as $\kappa = 2$ for the KdVB equation in §3 and $\kappa = 5$ for the NLS equation in §4. Substitution of Eq. (20) into Eq. (9) provides the expansion of the liquid pressure $p_L^*$,

$$p_L \equiv \frac{p_L^*}{\rho_{L0}^* U^{*2}} = \frac{p_{L0}^*}{\rho_{L0}^* U^{*2}} + \epsilon^{\kappa-2\zeta}\frac{\rho_{L1}}{V^2} + \epsilon^{\kappa-2\zeta+1}\frac{\rho_{L2}}{V^2} + \cdots. \tag{21}$$

Here, we have introduced $V\epsilon^\zeta$ as a measure of the ratio of $U^*$ and $c_{L0}^*$, as

$$\frac{U^*}{c_{L0}^*} \equiv O\left(\epsilon^\zeta\right) \equiv V\epsilon^\zeta = \begin{cases} V\epsilon^{1/2}, & \text{(for KdVB)}, \\ V\epsilon^2, & \text{(for NLS)}, \end{cases} \tag{22}$$

where a nondimensional parameter $V$ ($= O(1)$) and a real number $\zeta$ are to be determined. It should be emphasized that $V$ implies the magnitude of liquid compressibility in the sense that $V \to 0$ corresponds to $c_{L0}^* \to \infty$. Now, we impose

$$\kappa - 2\zeta = 1, \tag{23}$$

because the perturbation of the liquid pressure should begin with the term of $O(\epsilon)$ in Eq. (21) for the pressure waves concerned. Equation (21) can then be rewritten as

$$p_{\mathrm{L}} = p_{\mathrm{L}0} + \epsilon p_{\mathrm{L}1} + \epsilon^2 p_{\mathrm{L}2} + \cdots, \tag{24}$$

where the expansion coefficients are defined by

$$p_{\mathrm{L}i} = \frac{\rho_{\mathrm{L}i}}{V^2} \quad (i = 1, 2), \quad p_{\mathrm{L}3} = \frac{\rho_{\mathrm{L}3}}{V^2} + \frac{(n-1)\rho_{\mathrm{L}1}^2}{2V^2}, \quad \text{(for KdVB)}, \tag{25}$$

$$p_{\mathrm{L}i} = \frac{\rho_{\mathrm{L}i}}{V^2} \quad (i = 1, 2, 3, 4, 5), \quad p_{\mathrm{L}6} = \frac{\rho_{\mathrm{L}6}}{V^2} + \frac{(n-1)\rho_{\mathrm{L}1}^2}{2V^2}, \quad \text{(for NLS)}. \tag{26}$$

The remaining variables, $p_{\mathrm{G}}^*$, $\rho_{\mathrm{G}}^*$, and $P^*$, can also be nondimensionalized and expanded in $\epsilon$, and their expansion coefficients can be written in terms of $R_i$ and $p_{\mathrm{L}i}$ $(i = 1, 2, ...)$ from Eqs. (10)–(12). Since the asymptotic expansions (16)–(20) and (24) should be uniformly valid, all expansion coefficients should be determinable as bounded functions of $t_m$ and $x_m$ $(m = 0, 1, 2, ...)$ once initial and boundary conditions are specified. Such a requirement, which leads to the nonlinear wave equation in a far field, is called the non-secular condition.

The nondimensional pressures for the gas and liquid phases in the unperturbed state $p_{\mathrm{G}0}$ and $p_{\mathrm{L}0}$ are introduced as

$$p_{\mathrm{G}0} \equiv \frac{p_{\mathrm{G}0}^*}{\rho_{\mathrm{L}0}^* U^{*2}} \equiv O(1), \quad p_{\mathrm{L}0} \equiv \frac{p_{\mathrm{L}0}^*}{\rho_{\mathrm{L}0}^* U^{*2}} \equiv O(1), \tag{27}$$

respectively.

The ratio of initial densities of the gas and liquid phases is assumed to be small as

$$\frac{\rho_{\mathrm{G}0}^*}{\rho_{\mathrm{L}0}^*} \equiv O\left(\epsilon^3\right), \tag{28}$$

and hence the density ratio does not affect the final result of the present analysis.

We also define the scaling relation of the liquid viscosity as

$$\frac{\mu^*}{\rho_{\mathrm{L}0}^* U^* L^*} \equiv \begin{cases} O(\epsilon) \equiv \mu\epsilon, & \text{(for KdVB)}, \\ O(\epsilon^2) \equiv \mu\epsilon^2, & \text{(for NLS)}, \end{cases} \tag{29}$$

where $\mu$ is the nondimensional liquid viscosity.

The natural angular frequency of linear spherical symmetric oscillations of a single bubble is also an important parameter, which is given by

$$\omega_{\mathrm{B}}^* \equiv \sqrt{\frac{3\gamma(p_{\mathrm{L}0}^* + 2\sigma^*/R_0^*) - 2\sigma^*/R_0^*}{\rho_{\mathrm{L}0}^* R_0^{*2}}}. \tag{30}$$

Note that the effects of the liquid viscosity and liquid compressibility are not included in Eq. (30).

## 3. Korteweg–de Vries–Burgers equation

We shall derive the KdVB equation for long range propagation of nonlinear waves in the low frequency and long wavelength band shown in Fig. 1. This band can be characterized as weakly dispersive compared with the high frequency and short wavelength band investigated in §4.

Scaling relations appropriate to this case are

$$\frac{U^*}{c_{\mathrm{L}0}^*} \equiv O\left(\sqrt{\epsilon}\right) \equiv V\sqrt{\epsilon}, \tag{31}$$

$$\frac{R_0^*}{L^*} \equiv O\left(\sqrt{\epsilon}\right) \equiv \Delta\sqrt{\epsilon}, \tag{32}$$

$$\frac{\omega^*}{\omega_{\mathrm{B}}^*} \equiv \frac{1}{T^*\omega_{\mathrm{B}}^*} \equiv O\left(\sqrt{\epsilon}\right) \equiv \Omega\sqrt{\epsilon}, \tag{33}$$

where $V$, $\Delta$, and $\Omega$ are constants of $O(1)$, $\omega^* \equiv 1/T^*$ is an angular frequency of the sound source, and $\Omega$ is a normalized angular frequency. The appropriateness of Eqs. (31)–(33) is demonstrated in the following derivation process of the KdVB equation.

Equations (31)–(33) mean that we concentrate on a specific wave motion, where the propagation speed is small compared with the sound speed, the wavelength is large compared with the bubble radius, and the frequency is low compared with the natural frequency of the bubble.

### 3.1. Linear propagation in a near field

Substituting Eqs. (15)–(20), (24), and (31)–(33) into Eqs. (2)–(12), and then equating the coefficients of like powers of $\epsilon$ in the resultant equations, we have the following set of linearized equations as the first-order equations:

(i) mass conservation law in gas phase,

$$\frac{\partial \alpha_1}{\partial t_0} - 3\frac{\partial R_1}{\partial t_0} + \frac{\partial u_{G1}}{\partial x_0} = 0, \tag{34}$$

(ii) mass conservation law in liquid phase,

$$\alpha_0\frac{\partial \alpha_1}{\partial t_0} - (1-\alpha_0)\frac{\partial u_{L1}}{\partial x_0} = 0, \tag{35}$$

(iii) momentum conservation law in gas phase,

$$\beta_1\frac{\partial u_{G1}}{\partial t_0} - \beta_1\frac{\partial u_{L1}}{\partial t_0} - 3\gamma p_{G0}\frac{\partial R_1}{\partial x_0} = 0, \tag{36}$$

(iv) momentum conservation law in liquid phase,

$$(1-\alpha_0+\beta_1\alpha_0)\frac{\partial u_{L1}}{\partial t_0} - \beta_1\alpha_0\frac{\partial u_{G1}}{\partial t_0} + (1-\alpha_0)\frac{\partial p_{L1}}{\partial x_0} = 0, \tag{37}$$

(v) Keller equation,

$$R_1 + \frac{\Omega^2}{\Delta^2}p_{L1} = 0. \tag{38}$$

For use in a later stage, we rewrite Eqs. (34)–(38) into the matrix form:

$$\mathcal{L}_\mathrm{a}\,[R_1 \;\; \alpha_1 \;\; u_{G1} \;\; u_{L1} \;\; p_{L1}]^\mathrm{T} = \mathbf{0}, \qquad \mathcal{L}_\mathrm{a} \equiv \boldsymbol{A}_\mathrm{a}\frac{\partial}{\partial t_0} + \boldsymbol{B}_\mathrm{a}\frac{\partial}{\partial x_0} + \boldsymbol{C}_\mathrm{a},$$

$$\boldsymbol{A}_\mathrm{a} \equiv \begin{pmatrix} -3 & 1 & 0 & 0 & 0 \\ 0 & \alpha_0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & -\beta_1 & 0 \\ 0 & 0 & -\beta_1\alpha_0 & (1-\alpha_0+\beta_1\alpha_0) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{B}_\mathrm{a} \equiv \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -(1-\alpha_0) & 0 \\ -3\gamma p_{G0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-\alpha_0) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{C}_\mathrm{a} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \Omega^2/\Delta^2 \end{pmatrix}, \tag{39}$$

where the superscript T denotes the transpose.

Eliminating $\alpha_1$, $u_{G1}$, $u_{L1}$, and $p_{L1}$ from Eqs. (34)–(38) (or Eq. (39)), we can derive the linear wave equation for the first-order perturbation of the bubble radius, $R_1$,

$$\frac{\partial^2 R_1}{\partial t_0^2} - v_\mathrm{p}^2\frac{\partial^2 R_1}{\partial x_0^2} = 0, \tag{40}$$

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where the phase velocity $v_\mathrm{p}$ is given by

$$v_\mathrm{p} = \sqrt{\frac{3\alpha_0(1-\alpha_0+\beta_1)\gamma p_{\mathrm{G}0} + \beta_1(1-\alpha_0)\Delta^2/\Omega^2}{3\beta_1\alpha_0(1-\alpha_0)}}. \tag{41}$$

That is, the wave motion described in terms of $t_0$ and $x_0$ is linear and non-dispersive, and the propagation speed (phase velocity) $v_\mathrm{p}$ is in proportion to $1/\sqrt{\alpha_0(1-\alpha_0)}$, as is the classical sound speed in bubbly liquids[(3), (4)]. Now, we can choose $U^*$ in Eq. (31) as

$$U^* = \sqrt{\frac{3\alpha_0(1-\alpha_0+\beta_1)\gamma p^*_{\mathrm{G}0}/\rho^*_{\mathrm{L}0} + \beta_1(1-\alpha_0)R_0^{*2}\omega_\mathrm{B}^{*\,2}}{3\beta_1\alpha_0(1-\alpha_0)}}, \tag{42}$$

and then we immediately have $v_\mathrm{p} \equiv 1$, which is a choice making the final result simple.

From now on, we restrict ourselves to the right-running wave in the leading-order of approximation, and a phase function $\varphi_0$ can therefore be introduced as

$$\varphi_0 \equiv x_0 - t_0, \tag{43}$$

where $v_\mathrm{p} = 1$ has been used. Putting $R_1 \equiv f(\varphi_0; t_1, x_1)$, we can reduce Eq. (40) to

$$\frac{\partial f}{\partial t_0} + \frac{\partial f}{\partial x_0} = 0. \tag{44}$$

In the near field characterized by $t_0$ and $x_0$, all the first-order perturbations $\alpha_1$, $u_{\mathrm{G}1}$, $u_{\mathrm{L}1}$, $p_{\mathrm{L}1}$, and $R_1$ are governed by Eq. (44). Rewriting Eqs. (34)–(38) by $\varphi_0$ and integrating them with respect to $\varphi_0$, we can express $\alpha_1$, $u_{\mathrm{G}1}$, $u_{\mathrm{L}1}$, and $p_{\mathrm{L}1}$ in terms of the function $f(\varphi_0)$:

$$\begin{gathered} \alpha_1 = s_1 f, \quad u_{\mathrm{G}1} = s_2 f, \quad u_{\mathrm{L}1} = s_3 f, \quad p_{\mathrm{L}1} = s_4 f, \\ s_4 = -\frac{\Delta^2}{\Omega^2}, \quad s_1 = \frac{(1-\alpha_0)[3\beta_1\alpha_0 - (1-\alpha_0)s_4]}{\alpha_0(1-\alpha_0+\beta_1)}, \quad s_2 = s_1 - 3, \quad s_3 = -\frac{\alpha_0 s_1}{1-\alpha_0}. \end{gathered} \tag{45}$$

Here, constants of integration are dropped because of the boundary conditions at $x_0 \to \infty$, where the bubbly liquid is uniform and at rest.

### 3.2. Nonlinear propagation in a far field

The system of the second-order equations is given as

$$\mathcal{L}_\mathrm{a}\,[R_2\ \ \alpha_2\ \ u_{\mathrm{G}2}\ \ u_{\mathrm{L}2}\ \ p_{\mathrm{L}2}]^\mathrm{T} = (K_1\ \ K_2\ \ K_3\ \ K_4\ \ K_5)^\mathrm{T}, \tag{46}$$

where the inhomogeneous terms $K_i\,(1 \leqq i \leqq 5)$ are composed of the partial derivatives of the first-order perturbations with respect to $\varphi_0$, $x_1$, and $t_1$; they are explicitly presented in Appendix 1. By the use of the same procedure as that used in the derivation of Eq. (40) from Eq. (39), we have the following inhomogeneous equation for $R_2$,

$$\begin{aligned} \frac{\partial^2 R_2}{\partial t_0^2} - \frac{\partial^2 R_2}{\partial x_0^2} = K(f;\varphi_0,t_1,x_1) &= \frac{1}{3}\frac{\partial K_1}{\partial \varphi_0} - \frac{1}{3\alpha_0}\frac{\partial K_2}{\partial \varphi_0} \\ &+ \frac{1-\alpha_0+\beta_1}{3\beta_1(1-\alpha_0)}\frac{\partial K_3}{\partial \varphi_0} + \frac{1}{3\alpha_0(1-\alpha_0)}\frac{\partial K_4}{\partial \varphi_0} - \frac{\Delta^2}{3\alpha_0\Omega^2}\frac{\partial^2 K_5}{\partial \varphi_0^2}. \end{aligned} \tag{47}$$

From the solvability condition of the inhomogeneous equation (47), which is equivalent to the non-secular condition for expansions (16)–(20) and (24), we have

$$K = 2\frac{\partial}{\partial\varphi_0}\left(\frac{\partial f}{\partial t_1} + \frac{\partial f}{\partial x_1} + \Pi_0\frac{\partial f}{\partial \varphi_0} + \Pi_1 f\frac{\partial f}{\partial \varphi_0} + \Pi_2\frac{\partial^2 f}{\partial \varphi_0^2} + \Pi_3\frac{\partial^3 f}{\partial \varphi_0^3}\right) = 0. \tag{48}$$

With the use of Eqs. (15) and (44), the independent variables $t_m$ and $x_m$ $(m = 0, 1)$ in Eq. (48) can be restored into $t$ and $x$, and this yields

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} + \epsilon\left(\Pi_0\frac{\partial f}{\partial x} + \Pi_1 f\frac{\partial f}{\partial x} + \Pi_2\frac{\partial^2 f}{\partial x^2} + \Pi_3\frac{\partial^3 f}{\partial x^3}\right) = 0. \tag{49}$$

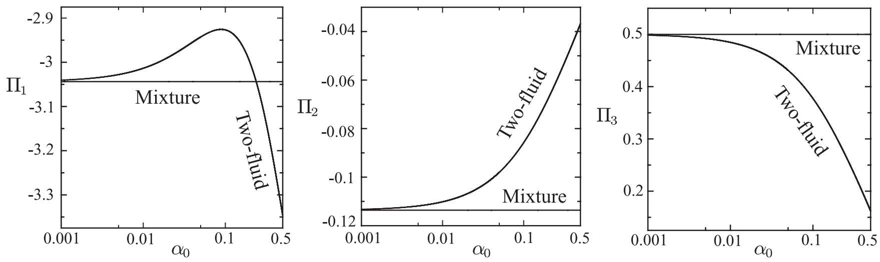


Fig. 2 The nonlinear coefficient $\Pi_1$, dissipation coefficient $\Pi_2$, and dispersion coefficient $\Pi_3$ as functions of the initial void fraction $\alpha_0$ in the case of $\Omega = 1$, $\sqrt{\epsilon} = 0.15$, $R_0^* = 10\,\mu$m, $p_{L0}^* = 101325$ Pa, $\rho_{L0}^* = 10^3$ kg/m$^3$, $\sigma^* = 0.0728$ N/m, $c_{L0}^* = 1.5 \times 10^3$ m/s, $\mu^* = 10^{-3}$ Pa · s, $\gamma = 1$, and $\beta_1 = \beta_2 = 1/2$. The curves represent the coefficients $\Pi_1$, $\Pi_2$, and $\Pi_3$ in the two-fluid model, and the corresponding coefficients in a bubble–liquid mixture model are shown by the straight lines. The initial number density of bubbles varies with the variation of $\alpha_0$ since $R_0^*$ is fixed.

Finally, the KdVB equation is obtained,

$$\frac{\partial f}{\partial \tau} + \Pi_1 f \frac{\partial f}{\partial \xi} + \Pi_2 \frac{\partial^2 f}{\partial \xi^2} + \Pi_3 \frac{\partial^3 f}{\partial \xi^3} = 0, \tag{50}$$

through the variable transformation

$$\tau \equiv \epsilon t, \quad \xi \equiv x - (1 + \epsilon \Pi_0)t, \tag{51}$$

where the coefficients $\Pi_0$, $\Pi_2$, and $\Pi_3$ are, respectively, given by

$$\Pi_0 = -\frac{(1-\alpha_0)\Delta^2 V^2}{6\alpha_0 \Omega^2} \leqq 0, \tag{52}$$

$$\Pi_2 = -\frac{1}{6\alpha_0}\left(4\mu + \frac{\Delta^3 V}{\Omega^2}\right) \leqq 0, \tag{53}$$

$$\Pi_3 = \frac{\Delta^2}{6\alpha_0} \geqq 0. \tag{54}$$

The explicit form of the nonlinear coefficient $\Pi_1$ is shown in Appendix 2. Figure 2 shows $\Pi_1$, $\Pi_2$, and $\Pi_3$ as functions of the initial void fraction $\alpha_0$. In Eqs. (52)–(54) and Fig. 2, $\Pi_0$, $\Pi_1$, and $\Pi_2$, are negative, whereas $\Pi_3$ is positive; these signs affect the waveform of a solution of the KdVB equation as mentioned below. The absolute value of the dissipation coefficient $\Pi_2$ is smaller than that of the nonlinear coefficient $\Pi_1$ and that of the dispersion coefficient $\Pi_3$ in all ranges of $\alpha_0$ in Fig. 2. For comparison, the coefficients deduced by applying the present unified theory to a set of basic equations in a bubble–liquid mixture model are shown by straight lines in Fig. 2. The main equations in the mixture model are the conservation equations[(8),(9)] and Keller's equation. The straight lines are drawn on the same parameters as the curves based on the two-fluid model, as shown in the caption in Fig. 2, although $\beta_i$ ($i = 1, 2, 3$) are not included in the mixture model. Clearly, the coefficients based on the mixture model are independent of $\alpha_0$; note that if the unknown function $f = R_1$ in Eq. (50) is replaced by other unknown functions, the coefficient of the nonlinear term in the resulting KdVB equation is a function of $\alpha_0$. More detailed discussion on the relation between the coefficients and the basic equations will be provided in a forthcoming paper.

Equations (44) and (50) govern the behavior of the first-order perturbation $R_1$ in the temporal and spatial scales of $O(1)$ and $O(1/\epsilon)$, respectively. The KdVB equation (50) describes the wave motion in the far field characterized by $\tau$ and $\xi$, where the weak dissipation and weak dispersion effects caused by the bubble oscillations grow and compete with the weak nonlinear effect. The second, third, and fourth terms in Eq. (50) represent the nonlinear, dissipation, and dispersion effects, respectively. In Eq. (51), $\epsilon\Pi_0$ in $\xi$ (i.e., the advection term in Eq. (49)) is a small correction to the propagation speed in the far field by the weak liquid

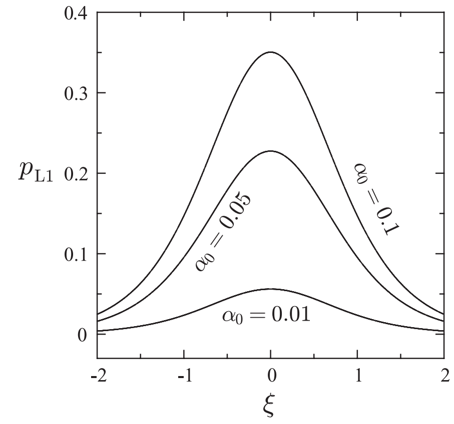


Fig. 3 Solitary wave profiles of the liquid pressure $p_{L1}$ for $|\xi| \leqq 2$ given by Eq. (56). The initial void fraction $\alpha_0$ is chosen as 0.1, 0.05, and 0.01, and the values of other physical quantities are the same as those used in Fig. 2. The height of soliton decreases with the decrease in $\alpha_0$.

compressibility. This correction and the wave attenuation due to acoustic radiation vanish in the limit of incompressible liquid $V \to 0$.

As a simple example, we consider the nondissipative case $\Pi_2 = 0$ (i.e., $\mu = V = 0$). Then, Eq. (50) reduces to the KdV equation. As a particular solution of the KdV equation, a steady traveling wave solution, soliton, is well known[(16), (20)]

$$p_{L1}(\tau, \xi) = -\frac{12\Pi_3}{\Pi_1\Omega^2/\Delta^2} C^2 \mathrm{sech}^2 \left[ C\left(\xi - 4C^2\Pi_3\tau - \xi_0\right)\right], \tag{55}$$

where $C$ is an arbitrary positive constant, $\xi_0$ is an arbitrary constant, and Eq. (45) is used for the transformation from $f$ to $p_{L1}$. It should be noted that Eq. (55) is valid for satisfying both $\Pi_1 \leqq 0$ and $\Pi_3 \geqq 0$. Figure 3 presents typical wave profiles of the liquid pressure $p_{L1}$ at $\tau = 0$, for $|\xi| \leqq 2$, $\xi_0 = 0$, and $C = 1$, i.e.,

$$p_{L1} = -\frac{12\Pi_3}{\Pi_1\Omega^2/\Delta^2} \mathrm{sech}^2 \xi. \tag{56}$$

As being clear from Fig. 3, the height of soliton decreases with the decrease in $\alpha_0$. The variation of $\alpha_0$ corresponds to that of the initial number density of bubbles since $R_0^*$ is fixed.

## 4. Nonlinear Schrödinger equation

We next focus on the high frequency and short wavelength band in Fig. 1, which is a strongly dispersive band compared with the weakly dispersive band analyzed in §3. We shall derive the NLS equation for the nonlinear modulation of a quasi-monochromatic wave train in the long range propagation by the nonlinear and strong dispersion effects. Scaling relations in this case are given by

$$\frac{U^*}{c_{L0}^*} \equiv O\left(\epsilon^2\right) \equiv V\epsilon^2, \tag{57}$$

$$\frac{R_0^*}{L^*} \equiv O(1) \equiv \Delta, \tag{58}$$

$$\frac{\omega^*}{\omega_B^*} \equiv T^*\omega^* \equiv O(1) \equiv \Omega, \tag{59}$$

where $T^* \equiv 1/\omega_B^*$ and $L^* \equiv U^*T^*$. Equations (57)–(59) show that a typical propagation speed of the wave is considerably small compared with the sound speed, a typical wavelength is comparable with the bubble radius, and a typical frequency of the wave is comparable with the natural frequency of the bubble (see Fig. 1). While the method of averaged equations is usually prohibited to be applied to such short waves, the plane wave problem may be excluded from the restriction because the average volume can be sufficiently large along the plane parallel to the wave front. Nevertheless, the assumption of spherical symmetry of bubble oscillations should be validated. We will address this problem in a future work.

### 4.1. Quasi-monochromatic wave train

In the same way as §3, the substitution of Eqs. (15)–(20), (24), and (57)–(59) into Eqs. (2)–(12) leads to the linear equations as the first-order equations,

$$\frac{\partial \alpha_1}{\partial t_0} - 3\frac{\partial R_1}{\partial t_0} + \frac{\partial u_{G1}}{\partial x_0} = 0, \tag{60}$$

$$\alpha_0 \frac{\partial \alpha_1}{\partial t_0} - (1-\alpha_0)\frac{\partial u_{L1}}{\partial x_0} = 0, \tag{61}$$

$$\beta_1 \frac{\partial u_{G1}}{\partial t_0} - \beta_1 \frac{\partial u_{L1}}{\partial t_0} - 3\gamma p_{G0}\frac{\partial R_1}{\partial x_0} = 0, \tag{62}$$

$$(1-\alpha_0+\beta_1\alpha_0)\frac{\partial u_{L1}}{\partial t_0} - \beta_1\alpha_0\frac{\partial u_{G1}}{\partial t_0} + (1-\alpha_0)\frac{\partial p_{L1}}{\partial x_0} = 0, \tag{63}$$

$$\frac{\partial^2 R_1}{\partial t_0^2} + R_1 + \frac{p_{L1}}{\Delta^2} = 0. \tag{64}$$

For a later use, we rewrite Eqs. (60)–(64) into the matrix form,

$$\mathcal{L}_{\mathrm{b}}\,[R_1 \;\; \alpha_1 \;\; u_{G1} \;\; u_{L1} \;\; p_{L1}]^{\mathrm{T}} = \mathbf{0}, \qquad \mathcal{L}_{\mathrm{b}} \equiv \boldsymbol{A}_{\mathrm{a}}\frac{\partial}{\partial t_0} + \boldsymbol{B}_{\mathrm{a}}\frac{\partial}{\partial x_0} + \boldsymbol{C}_{\mathrm{b}} + \boldsymbol{D}_{\mathrm{b}}\frac{\partial^2}{\partial t_0^2},$$

$$\boldsymbol{C}_{\mathrm{b}} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1/\Delta^2 \end{pmatrix}, \quad \boldsymbol{D}_{\mathrm{b}} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{65}$$

The second derivative with coefficient $\boldsymbol{D}_{\mathrm{b}}$ in the differential operator $\mathcal{L}_{\mathrm{b}}$ is the essential difference from $\mathcal{L}_{\mathrm{a}}$ in Eq. (39) in §3.1. This comes from the inertia term in the linearized Keller equation (64), and results in the dispersion effect in the first-order equation, which is the consequence from the fact that the wave motion concerned is in a strongly dispersive case as shown in Fig. 1.

The reduction of Eq. (65) to a single linear equation can also be carried out in the same way as in §3.1. The resulting equation is a linear wave equation with a dispersion term,

$$\mathcal{L}_1[R_1] = 0, \;\; \mathcal{L}_1 \equiv \frac{\partial^2}{\partial t_0^2} - \left[\frac{\Delta^2}{3\alpha_0} + \frac{(1-\alpha_0+\beta_1)\gamma p_{G0}}{\beta_1(1-\alpha_0)}\right]\frac{\partial^2}{\partial x_0^2} - \frac{\Delta^2}{3\alpha_0}\frac{\partial^4}{\partial x_0^2 \partial t_0^2}. \tag{66}$$

Owing to the dispersion effect, the wave profile is broken down into each component with its own propagation speed, if an initial wave is a superposition of different harmonic components. We therefore consider a solution of Eq. (66) in the form of a quasi-monochromatic wave train that evolves into a slowly modulated wave packet[(16),(20)]:

$$R_1 = A(t_1, t_2, x_1, x_2)\mathrm{e}^{\mathrm{i}\theta} + \mathrm{c.c.}, \tag{67}$$

with

$$\theta = kx_0 - \Omega(k)t_0, \tag{68}$$

where $A$ is the slowly varying complex amplitude depending on only slow scales and is obviously a constant in the near field characterized by $t_0$ and $x_0$, $k \equiv k^*L^* = k^*U^*/\omega_{\mathrm{B}}^*$ is the normalized wavenumber ($k^*$ is the wavenumber), i is the imaginary unit ($\mathrm{i} \equiv \sqrt{-1}$), and c.c. represents the complex conjugate. Here, $\mathrm{e}^{\mathrm{i}\theta}$ corresponds to the high frequency carrier wave and $A$ to the envelope[(16),(20)]. Equation (68) implies that we focus on only the right-running carrier wave.

Substituting Eq. (67) into Eqs. (60)–(64) and integrating them with respect to $t_0$ and $x_0$ under the boundary condition at $x_0 \to \infty$, where the bubbly liquid is uniform and at rest, we have

$$\alpha_1 = b_1 R_1, \quad u_{\mathrm{G}1} = b_2 R_1, \quad u_{\mathrm{L}1} = b_3 R_1, \quad p_{\mathrm{L}1} = b_4 R_1,$$
$$b_4 = \Delta^2\left(\Omega^2 - 1\right), \quad b_1 = \frac{(1-\alpha_0)\left[3\beta_1\alpha_0 - (1-\alpha_0) b_4 k^2/\Omega^2\right]}{\alpha_0(1-\alpha_0+\beta_1)},$$
$$b_2 = (b_1 - 3)\frac{\Omega}{k}, \quad b_3 = -\frac{\alpha_0 b_1 \Omega}{(1-\alpha_0)k}, \tag{69}$$

where $\Omega$ depends on $k$ through a linear dispersion relation

$$D(k,\Omega) \equiv \frac{\Delta^2 k^2 (1-\Omega^2)}{3\alpha_0} + \frac{(1-\alpha_0+\beta_1)\gamma p_{\mathrm{G}0}}{\beta_1(1-\alpha_0)} k^2 - \Omega^2 = 0, \tag{70}$$

or

$$\Omega(k) = \pm k \sqrt{\frac{\Delta^2}{3\alpha_0 + \Delta^2 k^2} + \frac{3\alpha_0(1-\alpha_0+\beta_1)\gamma p_{\mathrm{G}0}}{\beta_1(1-\alpha_0)(3\alpha_0+\Delta^2 k^2)}}. \tag{71}$$

The right-running carrier wave corresponds to the positive $\Omega$ in Eq. (71). The nondimensional phase velocity $v_{\mathrm{p}}$, group velocity $v_{\mathrm{g}}$, and derivative of the group velocity with respect to the wavenumber, $\mathrm{d}v_{\mathrm{g}}/\mathrm{d}k$, are readily calculated as

$$v_{\mathrm{p}} = \frac{\Omega}{k} \geqq 0, \tag{72}$$
$$v_{\mathrm{g}} = \frac{\mathrm{d}\Omega}{\mathrm{d}k} = \frac{3\alpha_0\Omega}{k(3\alpha_0+\Delta^2k^2)} \geqq 0, \tag{73}$$
$$q \equiv \frac{\mathrm{d}v_{\mathrm{g}}}{\mathrm{d}k} = \frac{\mathrm{d}^2\Omega}{\mathrm{d}k^2} = -\frac{9\alpha_0\Delta^2\Omega}{(3\alpha_0+\Delta^2k^2)^2} \leqq 0. \tag{74}$$

Thus, we can determine the typical propagation speed $U^*$ in Eq. (57). In this case, we choose $U^*$ so that $v_{\mathrm{p}}$ may be equal to unity when $\Omega = 1$. This is satisfied by the choice as

$$U^* \equiv \sqrt{\frac{(1-\alpha_0+\beta_1)\gamma p^*_{\mathrm{G}0}}{\beta_1(1-\alpha_0)\rho^*_{\mathrm{L}0}}}, \tag{75}$$

and then $L^* \equiv U^* T^*$ is simultaneously determined.

### 4.2. Slow variation of wave train

The system of second-order equations is given as

$$\mathcal{L}_{\mathrm{b}}\left[R_2 \;\; \alpha_2 \;\; u_{\mathrm{G}2} \;\; u_{\mathrm{L}2} \;\; p_{\mathrm{L}2}\right]^{\mathrm{T}} = (M_1 \;\; M_2 \;\; M_3 \;\; M_4 \;\; M_5)^{\mathrm{T}}, \tag{76}$$

where the explicit forms of $M_i$ $(1 \leqq i \leqq 5)$ are shown in Appendix 1. A slightly lengthy calculation leads to the following equation for $R_2$,

$$\mathcal{L}_1[R_2] = M(R_1; t_0, t_1, x_0, x_1) = -\frac{1}{3}\frac{\partial M_1}{\partial t_0} + \frac{1}{3\alpha_0}\frac{\partial M_2}{\partial t_0}$$
$$+\frac{1-\alpha_0+\beta_1}{3\beta_1(1-\alpha_0)}\frac{\partial M_3}{\partial x_0} + \frac{1}{3\alpha_0(1-\alpha_0)}\frac{\partial M_4}{\partial x_0} - \frac{\Delta^2}{3\alpha_0}\frac{\partial^2 M_5}{\partial x_0^2}. \tag{77}$$

Only the coefficient of the fifth term in the right-hand side of Eq. (77) differs from the counterpart of Eq. (47). This comes from the difference in the choices $T^* \equiv 1/\omega^*$ in §3 and $T^* \equiv 1/\omega^*_{\mathrm{B}}$ in §4. Substituting Eqs. (67) and (69) into Eq. (77), we have

$$M = \Gamma A^2 \mathrm{e}^{2\mathrm{i}\theta} + \mathrm{i}\left(-\frac{\partial D}{\partial \Omega}\right)\left(\frac{\partial A}{\partial t_1} + v_{\mathrm{g}}\frac{\partial A}{\partial x_1}\right)\mathrm{e}^{\mathrm{i}\theta} + \mathrm{c.c.}, \tag{78}$$

where the nonlinear coefficient $\Gamma$ is a real constant whose explicit form is shown in Appendix 2.

From the solvability condition of the inhomogeneous equation (77), the coefficient of $e^{i\theta}$ in the right-hand side of Eq. (78) should vanish(16), and we have

$$\frac{\partial A}{\partial t_1} + v_g \frac{\partial A}{\partial x_1} = 0. \tag{79}$$

The complex conjugate of Eq. (79) also holds. As mentioned in §4.1, the complex amplitude $A$ is only constant throughout the near field. In the far field characterized by $t_1$ and $x_1$, however, it is a constant along the characteristic curve $dx_1/dt_1 = v_g$. This implies a slow variation of the wave train. The nonlinear and dissipation effects appear in the next-order analysis.

Applying Eq. (79) to Eqs. (77) and (78) yields

$$\mathcal{L}_1[R_2] = \Gamma A^2 e^{2i\theta} + \text{c.c.}, \tag{80}$$

and a solution of Eq. (80) uniformly valid up to the far field concerned is given by

$$R_2 = c_0 A^2 e^{2i\theta} + \text{c.c.}, \quad c_0 \equiv \frac{\Gamma}{D_{22}}, \quad D_{22} \equiv D(2k, 2\Omega) = -\frac{4\Delta^2\Omega^2 k^2}{\alpha_0}. \tag{81}$$

Some comments should be made on the second-order solution (81): the complex component of complementary function (a solution of homogeneous equation) is dropped because it can be included in $A$(24), (25). Furthermore, the real component of complementary function is eliminated by the boundary conditions at $x_1 \to \infty$; if it could not be eliminated, however, it should be determined by higher-order solvability conditions(16), (26), (27).

Substituting Eq. (81) into Eq. (76) and taking account of the boundary condition at infinity, we obtain

$$\begin{pmatrix} \alpha_2 \\ u_{G2} \\ u_{L2} \\ p_{L2} \end{pmatrix} = \begin{pmatrix} c_1 & d_1 & 0 \\ c_2 & d_2 & 0 \\ c_3 & d_3 & 0 \\ c_4 & d_4 & c_s \end{pmatrix} \begin{pmatrix} A^2 e^{2i\theta} + \text{c.c.} \\ i\,\partial A/\partial t_1\, e^{i\theta} + \text{c.c.} \\ |A|^2 \end{pmatrix}, \tag{82}$$

where $c_i$, $d_i$ $(1 \leqq i \leqq 4)$, and $c_s$ are real constants given by

$$\begin{aligned}
c_4 &= \Delta^2 \left[ c_0 \left( 4\Omega^2 - 1 \right) + m_5 \right], \quad c_3 = \left( c_4 - \frac{3\gamma p_{G0}\alpha_0 c_0}{1-\alpha_0} \right) \frac{k}{\Omega} - \frac{\alpha_0 m_3 + m_4}{2(1-\alpha_0)\Omega}, \\
c_1 &= -\frac{(1-\alpha_0)c_3 k}{\alpha_0 \Omega} - \frac{m_2}{2\alpha_0\Omega}, \quad c_2 = (c_1 - 3c_0)\frac{\Omega}{k} + \frac{m_1}{2k}, \\
d_4 &= 2\Delta^2\Omega, \quad d_1 = \frac{d_4}{3\alpha_0 v_p^2}[b_1 - 3(1-\alpha_0)], \quad d_2 = \frac{d_4}{v_p}\left(1 + \frac{b_2}{6\alpha_0 v_p}\right), \\
d_3 &= \frac{d_4}{v_p}\left(1 + \frac{b_3}{6\alpha_0 v_p}\right), \quad c_s = \Delta^2\left(2 - \Omega^2 - 2b_2\Omega k\right) + 3\gamma(3\gamma - 1)p_{G0}.
\end{aligned} \tag{83}$$

Here, the expressions of $m_i$ $(1 \leqq i \leqq 5)$ are presented in Eq. (A.4) in Appendix 2.

### 4.3. NLS equation and nonlinear propagation of envelope wave

Let us proceed to the next-order calculation in order to determine the behavior of the slowly modulated wave packet as a result of long range propagation with the weak nonlinear, weak dissipation, and strong dispersion effects.

In the third-order, we have

$$\mathcal{L}_b \left[ R_3 \;\; \alpha_3 \;\; u_{G3} \;\; u_{L3} \;\; p_{L3} \right]^T = (N_1 \;\; N_2 \;\; N_3 \;\; N_4 \;\; N_5)^T, \tag{84}$$

where $N_i$ $(1 \leqq i \leqq 5)$ are also explicitly presented in Appendix 1. Equation (84) can be reduced to the equation for $R_3$,

$$\mathcal{L}_1[R_3] = N(t_0, t_1, t_2, x_0, x_1, x_2) = \Lambda_1 e^{3i\theta} + \Lambda_2 e^{2i\theta} + \Lambda_3 e^{i\theta} + \text{c.c.}, \tag{85}$$

where $\Lambda_i$ $(1 \leqq i \leqq 3)$ are the complex variables consisting of $A$. Here, $\Lambda_1$ and $\Lambda_2$ are

$$\Lambda_1 = \lambda_1 A^3, \quad \Lambda_2 = i\lambda_2 A \frac{\partial A}{\partial x_1}, \tag{86}$$

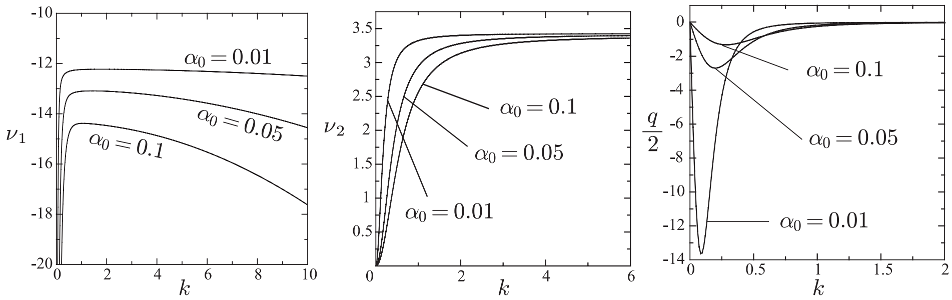


Fig. 4 The nonlinear coefficient $\nu_1$, dissipation coefficient $\nu_2$, and dispersion coefficient $q/2$ versus the wavenumber $k$ in the case that the physical quantities except $\epsilon$ and $\Omega$ are the same as those used in Fig. 2, where $\epsilon = 0.07$ and $\Omega$ varies with the variation of $k$ as shown in Eq. (71). Long waves around $k = 0$ are outside the applicability of the present analysis based on parameter scaling (57)–(59).

where the explicit forms of real constants $\lambda_1$ and $\lambda_2$ are not shown since they are not essential for the following discussion.

Imposing the non-secular condition to Eq. (85), we have

$$\Lambda_3 = \left(-\frac{\partial D}{\partial \Omega}\right)\left[\mathrm{i}\left(\frac{\partial A}{\partial t_2} + v_{\mathrm{g}}\frac{\partial A}{\partial x_2}\right) + \frac{q}{2}\frac{\partial^2 A}{\partial x_1^2} + \nu_1|A|^2 A + \mathrm{i}\nu_2 A\right] = 0, \tag{87}$$

where the nonlinear coefficient $\nu_1$ is a real constant and its explicit form is presented in Appendix 2, $q/2$ denotes the dispersion coefficient, and $\nu_2$ is given as

$$\nu_2 = \frac{\left(4\mu + \Delta^3 V\right)k^2}{2\left(3\alpha_0 + \Delta^2 k^2\right)} \geqq 0. \tag{88}$$

Combining Eqs. (79) and (87) with the help of Eq. (15), we obtain

$$\mathrm{i}\left(\frac{\partial A}{\partial t} + v_{\mathrm{g}}\frac{\partial A}{\partial x}\right) + \frac{q}{2}\frac{\partial^2 A}{\partial x^2} + \epsilon^2(\nu_1|A|^2 A + \mathrm{i}\nu_2 A) = 0, \tag{89}$$

in the same manner as in the derivation of Eq. (49). Since $\nu_2$ is positive, $\mathrm{i}\nu_2 A$ in Eq. (89) acts as a dissipation term, which is resulted from the liquid viscosity $\mu$ and liquid compressibility $V$ as in the case of $\Pi_2$ in Eq. (53). Thus, Eq. (89) may be called the NLS equation with a dissipation term. Finally, the NLS equation can be rewritten as

$$\mathrm{i}\frac{\partial A}{\partial \tau} + \frac{q}{2}\frac{\partial^2 A}{\partial \xi^2} + \nu_1|A|^2 A + \mathrm{i}\nu_2 A = 0, \tag{90}$$

through the variable transformation

$$\tau \equiv \epsilon^2 t, \quad \xi \equiv \epsilon(x - v_{\mathrm{g}} t). \tag{91}$$

Figure 4 shows the nonlinear, dissipation, and dispersion coefficients, $\nu_1$, $\nu_2$, and $q/2$, as functions of the wavenumber $k$. We can see that $\nu_1$ is negative for the range shown in Fig. 4.

As a simple explanation of the solution of the NLS equation, we shall consider the nondissipative case $\nu_2 = 0$ (i.e., $\mu = V = 0$) in Eq. (90). An exact solution of Eq. (90) with $\nu_2 = 0$, the envelope soliton solution, is well known(16),(20)

$$A(\tau, \xi) = A_0\sqrt{\frac{q(k)}{\nu_1(k)}}\,\mathrm{sech}\left[A_0(\xi - \xi_0)\right]\exp\left[\frac{\mathrm{i}A_0^2 q(k)\tau}{2}\right], \tag{92}$$

where $A_0$ and $\xi_0$ are real arbitrary constants. We should emphasize that Eq. (92) is valid for $\nu_1 \leqq 0$ and $q \leqq 0$. Then, the real amplitude $|A|$ is given by

$$|A| = \sqrt{\frac{q(k)}{\nu_1(k)}}\,\mathrm{sech}\,\xi, \tag{93}$$

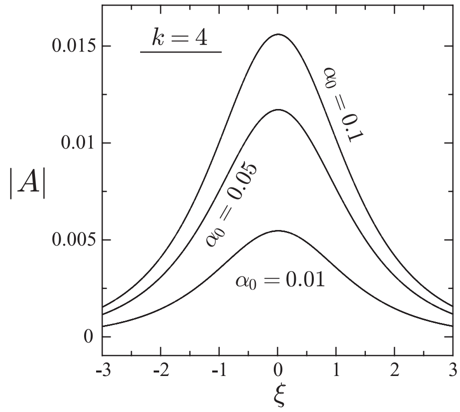


Fig. 5 The real amplitude $|A|$ for $|\xi| \leqq 3$ expressed by Eq. (93). The initial void fraction is chosen as $\alpha_0 = 0.1$, 0.05, and 0.01, the wavenumber is fixed as $k = 4$, and other quantities are the same as those used in Fig. 4. The height of the envelope soliton decreases with the decrease in $\alpha_0$.

where the arbitrary constants are chosen as $A_0 = 1$ and $\xi_0 = 0$. The typical profiles of the real amplitude $|A|$ for three cases of the initial void fraction $\alpha_0$ are illustrated in Fig. 5. The height of the envelope soliton decreases with the decrease in $\alpha_0$, as in the case of the soliton solution of the KdV equation shown in Fig. 3.

The stability of a uniform wave train solution of the NLS equation in the nondissipative case has been investigated[(11),(20)]: a uniform wave train solution is unstable for $q\nu_1 > 0$ and is stable for $q\nu_1 < 0$. As we can see from Eq. (74) and Fig. 4, the uniform wave train solution is unstable since both $q$ and $\nu_1$ are negative. This implies that disturbances on the uniform solution grow and the whole wave profile develops into the soliton solution (92).

## 5. Conclusions

We have presented the unified theory for derivation of nonlinear wave equations in bubbly liquids, the main ingredients of which are the scaling relations of a set of $U^*/c^*_{L0}$, $R^*_0/L^*$, and $\omega^*/\omega^*_B$ with respect to the nondimensional wave amplitude $\epsilon$ and the method of multiple scales. The applicability of the theory has been demonstrated in the derivation of the KdVB equation for a long wave with a low frequency case and the NLS equation for an envelope of a carrier wave with a high frequency case. Since the physical meaning of the parameter scaling is clear, the theory can make a definite contribution in not only theoretical but also experimental studies for the waves in bubbly liquids.

A wide applicability and high expandability are distinguishing features of the present theory. In fact, as mentioned in §1, the theory is applicable to the system of basic equations of mixture model as well as that of two-fluid model. Furthermore, it is clearly possible to apply the theory to a more general set of basic equations including the thermal effects and various forces exerted on the bubble (e.g., the drag, lift, and so on). The extensions of the theory to a wave phenomenon including a non-uniformity of initial void fraction and to a two-dimensional wave motion are underway.

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## Appendix 1. Inhomogeneous terms

The inhomogeneous terms $K_i$ $(1 \leqq i \leqq 5)$ in Eq. (46) are given by

$$K_1 = -\frac{\partial u_{G1}}{\partial x_1} + \frac{\partial}{\partial t_1}(3R_1 - \alpha_1) + 3\frac{\partial R_1(\alpha_1 - 2R_1)}{\partial t_0} + \frac{\partial}{\partial x_0}[u_{G1}(3R_1 - \alpha_1)],$$

$$K_2 = (1-\alpha_0)\frac{\partial u_{L1}}{\partial x_1} - \alpha_0\frac{\partial \alpha_1}{\partial t_1} - \alpha_0\frac{\partial \alpha_1 u_{L1}}{\partial x_0} + (1-\alpha_0)\frac{\partial \rho_{L1}}{\partial t_0},$$

$$K_3 = 3\gamma p_{G0}\frac{\partial R_1}{\partial x_1} - \beta_1\frac{\partial}{\partial t_1}(u_{G1} - u_{L1}) - \beta_1\left(u_{G1}\frac{\partial u_{G1}}{\partial x_0} - u_{L1}\frac{\partial u_{L1}}{\partial x_0}\right)$$

$$-\beta_1\alpha_1\frac{\partial}{\partial t_0}(u_{G1} - u_{L1}) - \beta_2(u_{G1} - u_{L1})\frac{\partial \alpha_1}{\partial t_0} + 3\gamma p_{G0}\left[\alpha_1\frac{\partial R_1}{\partial x_0} - (3\gamma+1)R_1\frac{\partial R_1}{\partial x_0}\right],$$

$$K_4 = -(1-\alpha_0)\left(\frac{\partial p_{L1}}{\partial x_1} + \frac{\partial u_{L1}}{\partial t_1}\right) + \beta_1\alpha_0\frac{\partial}{\partial t_1}(u_{G1} - u_{L1}) + \alpha_0\frac{\partial \alpha_1 u_{L1}}{\partial t_0}$$

$$+\beta_1\alpha_0\left(u_{G1}\frac{\partial u_{G1}}{\partial x_0} - u_{L1}\frac{\partial u_{L1}}{\partial x_0}\right) + \beta_1\alpha_0\alpha_1\frac{\partial}{\partial t_0}(u_{G1} - u_{L1}) + \beta_2\alpha_0(u_{G1} - u_{L1})\frac{\partial \alpha_1}{\partial t_0}$$

$$+\alpha_0\alpha_1\frac{\partial p_{L1}}{\partial x_0} - (1-\alpha_0)\frac{\partial u_{L1}^2}{\partial x_0} + \alpha_0\left[p_{L1} + \left(\frac{\omega_B^* R_0^*}{U^*}\right)^2 R_1\right]\frac{\partial \alpha_1}{\partial x_0},$$

$$K_5 = \left[1 + \frac{3\gamma(3\gamma-1)\Omega^2 p_{G0}}{2\Delta^2}\right]R_1^2 - \Omega^2\frac{\partial^2 R_1}{\partial t_0^2} - \left(\frac{4\Omega^2\mu}{\Delta^2} + \Delta V\right)\frac{\partial R_1}{\partial t_0}.$$

The explicit representations of $M_i$ $(1 \leqq i \leqq 5)$ in Eq. (76) are

$$M_1 = K_1, \quad M_2 = K_2 - (1-\alpha_0)\frac{\partial \rho_{L1}}{\partial t_0}, \quad M_3 = K_3, \quad M_4 = K_4,$$

$$M_5 = -2\frac{\partial^2 R_1}{\partial t_0 \partial t_1} - R_1\frac{\partial^2 R_1}{\partial t_0^2} - 2u_{G1}\frac{\partial^2 R_1}{\partial t_0 \partial x_0} - \frac{\partial u_{G1}}{\partial t_0}\frac{\partial R_1}{\partial x_0} - \frac{3}{2}\left(\frac{\partial R_1}{\partial t_0}\right)^2$$

$$+\left[1 + \frac{3\gamma(3\gamma-1)p_{G0}}{2\Delta^2}\right]R_1^2.$$

The explicit representations of $N_i$ $(1 \leqq i \leqq 5)$ in Eq. (84) are

$$N_1 = -\frac{\partial u_{G1}}{\partial x_2} + \frac{\partial}{\partial t_2}(3R_1 - \alpha_1) + \frac{\partial}{\partial t_1}[3R_1(\alpha_1 - 2R_1) + 3R_2 - \alpha_2]$$

$$+\frac{\partial}{\partial x_1}[u_{G1}(3R_1 - \alpha_1) - u_{G2}] + \frac{\partial}{\partial x_0}[3(u_{G2}R_1 + u_{G1}R_2) - (\alpha_1 u_{G2} + \alpha_2 u_{G1})]$$

$$+\frac{\partial}{\partial t_0}\left[3(\alpha_1 R_2 + \alpha_2 R_1) - 3R_1R_2 - 6\alpha_1 R_1^2 + 10R_1^3\right] + 3\frac{\partial u_{G1}R_1(\alpha_1 - 2R_1)}{\partial x_0},$$

$$N_2 = (1-\alpha_0)\left(\frac{\partial u_{L1}}{\partial x_2} + \frac{\partial u_{L2}}{\partial x_1}\right) - \alpha_0\left(\frac{\partial \alpha_1}{\partial t_2} + \frac{\partial \alpha_2}{\partial t_1}\right) - \alpha_0\frac{\partial \alpha_1 u_{L1}}{\partial x_1} - \alpha_0\frac{\partial}{\partial x_0}(\alpha_2 u_{L1} + \alpha_1 u_{L2}),$$

$$N_3 = 3\gamma p_{G0}\frac{\partial R_1}{\partial x_2} - \beta_1\frac{\partial}{\partial t_2}(u_{G1} - u_{L1}) - \beta_1\left(u_{G1}\frac{\partial u_{G1}}{\partial x_1} - u_{L1}\frac{\partial u_{L1}}{\partial x_1}\right) + 3\gamma p_{G0}\frac{\partial R_2}{\partial x_1}$$

$$-\beta_1\alpha_1\frac{\partial}{\partial t_1}(u_{G1} - u_{L1}) - \beta_2(u_{G1} - u_{L1})\frac{\partial \alpha_1}{\partial t_1} - \beta_1\frac{\partial}{\partial t_1}(u_{G2} - u_{L2})$$

$$+3\gamma p_{G0}\left[\alpha_1\frac{\partial R_1}{\partial x_1} - (3\gamma+1)R_1\frac{\partial R_1}{\partial x_1}\right] - \beta_2(u_{G1} - u_{L1})\left(u_{G1}\frac{\partial \alpha_1}{\partial x_0} + \frac{\partial \alpha_2}{\partial t_0}\right)$$

$$-\beta_1\left[\alpha_1\frac{\partial}{\partial t_0}(u_{G2} - u_{L2}) + \alpha_2\frac{\partial}{\partial t_0}(u_{G1} - u_{L1})\right] - \beta_2(u_{G2} - u_{L2})\frac{\partial \alpha_1}{\partial t_0}$$

$$-\beta_1\left[\frac{\partial}{\partial x_0}(u_{G1}u_{G2} - u_{L1}u_{L2}) + \alpha_1\left(u_{G1}\frac{\partial u_{G1}}{\partial x_0} - u_{L1}\frac{\partial u_{L1}}{\partial x_0}\right)\right]$$

$$+3\gamma p_{G0}\left[\alpha_1\frac{\partial R_2}{\partial x_0} + \alpha_2\frac{\partial R_1}{\partial x_0} - (3\gamma+1)\left(\frac{\partial R_1 R_2}{\partial x_0} + \alpha_1 R_1\frac{\partial R_1}{\partial x_0} - \frac{3\gamma+2}{6}\frac{\partial R_1^3}{\partial x_0}\right)\right],$$

$$N_4 = -(1-\alpha_0)\left(\frac{\partial p_{L1}}{\partial x_2} + \frac{\partial u_{L1}}{\partial t_2}\right) + \beta_1\alpha_0\frac{\partial}{\partial t_2}(u_{G1} - u_{L1}) + \beta_1\alpha_0\alpha_1\frac{\partial}{\partial t_1}(u_{G1} - u_{L1})$$

$$+\beta_1\alpha_0\left(u_{G1}\frac{\partial u_{G1}}{\partial x_1}-u_{L1}\frac{\partial u_{L1}}{\partial x_1}\right)+\beta_2\alpha_0(u_{G1}-u_{L1})\frac{\partial\alpha_1}{\partial t_1}+\alpha_0\alpha_1\frac{\partial p_{L1}}{\partial x_1}+\alpha_0\frac{\partial\alpha_1 u_{L1}}{\partial t_1}$$
$$-(1-\alpha_0)\left(\frac{\partial u_{L1}^2}{\partial x_1}+\frac{\partial p_{L2}}{\partial x_1}+\frac{\partial u_{L2}}{\partial t_1}\right)+\beta_1\alpha_0\left[\alpha_1\frac{\partial}{\partial t_0}(u_{G2}-u_{L2})+\alpha_2\frac{\partial}{\partial t_0}(u_{G1}-u_{L1})\right]$$
$$+\alpha_0\Delta^2\Omega^2R_1\frac{\partial\alpha_1}{\partial x_1}+\beta_1\alpha_0\left[\frac{\partial}{\partial x_0}(u_{G1}u_{G2}-u_{L1}u_{L2})+\alpha_1\left(u_{G1}\frac{\partial u_{G1}}{\partial x_0}-u_{L1}\frac{\partial u_{L1}}{\partial x_0}\right)\right]$$
$$+\beta_1\alpha_0\frac{\partial}{\partial t_1}(u_{G2}-u_{L2})+\beta_2\alpha_0\left[(u_{G1}-u_{L1})\left(u_{G1}\frac{\partial\alpha_1}{\partial x_0}+\frac{\partial\alpha_2}{\partial t_0}\right)+(u_{G2}-u_{L2})\frac{\partial\alpha_1}{\partial t_0}\right]$$
$$+\alpha_0\frac{\partial\alpha_1 u_{L1}^2}{\partial x_0}+\alpha_0\frac{\partial}{\partial t_0}(\alpha_1u_{L2}+\alpha_2u_{L1})-2(1-\alpha_0)\frac{\partial u_{L1}u_{L2}}{\partial x_0}+\alpha_0\left(\alpha_1\frac{\partial p_{L2}}{\partial x_0}+\alpha_2\frac{\partial p_{L1}}{\partial x_0}\right)$$
$$+\alpha_0\Delta^2\Omega^2R_1\frac{\partial\alpha_2}{\partial x_0}+\alpha_0\left(\Delta^2R_2+p_{L2}\right)\frac{\partial\alpha_1}{\partial x_0}-\alpha_0\left[\Delta^2+3\gamma(3\gamma-1)p_{G0}/2\right]R_1^2\frac{\partial\alpha_1}{\partial x_0},$$
$$N_5=-2\frac{\partial^2R_1}{\partial t_0\partial t_2}-\frac{\partial^2R_1}{\partial t_1^2}-2\frac{\partial^2R_2}{\partial t_0\partial t_1}-2R_1\frac{\partial^2R_1}{\partial t_0\partial t_1}-2u_{G1}\left(\frac{\partial^2R_1}{\partial t_1\partial x_0}+\frac{\partial^2R_1}{\partial t_0\partial x_1}\right)-3\frac{\partial R_1}{\partial t_0}\frac{\partial R_1}{\partial t_1}$$
$$-\frac{\partial u_{G1}}{\partial t_0}\frac{\partial R_1}{\partial x_1}-\frac{\partial u_{G1}}{\partial t_1}\frac{\partial R_1}{\partial x_0}-R_1\frac{\partial^2R_2}{\partial t_0^2}-R_2\frac{\partial^2R_1}{\partial t_0^2}-2u_{G1}\left(R_1\frac{\partial^2R_1}{\partial t_0\partial x_0}+\frac{\partial^2R_2}{\partial t_0\partial x_0}\right)-\frac{\partial u_{G1}}{\partial t_0}\frac{\partial R_2}{\partial x_0}$$
$$-2u_{G2}\frac{\partial^2R_1}{\partial t_0\partial x_0}-u_{G1}^2\frac{\partial^2R_1}{\partial x_0^2}-\frac{\partial R_1}{\partial x_0}\left(u_{G1}\frac{\partial u_{G1}}{\partial x_0}+\frac{\partial u_{G2}}{\partial t_0}+R_1\frac{\partial u_{G1}}{\partial t_0}+3u_{G1}\frac{\partial R_1}{\partial t_0}\right)$$
$$+\left[2+3\gamma(3\gamma-1)p_{G0}/\Delta^2\right]R_1R_2-\left[1+\frac{\gamma(3\gamma-1)(3\gamma+4)p_{G0}}{2\Delta^2}\right]R_1^3-\left(\frac{4\mu}{\Delta^2}+\Delta V\right)\frac{\partial R_1}{\partial t_0}.$$

## Appendix 2. Nonlinear coefficients

We shall present the explicit forms of the nonlinear coefficients, $\Pi_1$, $\Gamma$, and $\nu_1$. Firstly, we show the nonlinear coefficient of the KdVB equation $\Pi_1$ (see Eq. (50) and Fig. 2):

$$\Pi_1=\frac{1}{6}\left[k_1-\frac{k_2}{\alpha_0}+\frac{(1-\alpha_0+\beta_1)k_3}{\beta_1(1-\alpha_0)}+\frac{k_4}{\alpha_0(1-\alpha_0)}-\frac{2\Delta^2k_5}{\alpha_0\Omega^2}\right],\tag{A.1}$$

where

$$k_1=6(2-s_1)+2s_2(3-s_1),\quad k_2=-2\alpha_0s_1s_3,\quad k_5=1+\frac{3\gamma(3\gamma-1)p_{G0}\Omega^2}{2\Delta^2},$$
$$\widehat{k}=(\beta_1+\beta_2)(s_2-s_3)s_1-\beta_1\left(s_2^2-s_3^2\right),\quad k_3=\widehat{k}+3\gamma p_{G0}(s_1-3\gamma-1),$$
$$k_4=-\alpha_0\widehat{k}+\alpha_0s_1s_4-2(1-\alpha_0)s_3^2-2\alpha_0s_1s_3.\tag{A.2}$$

The coefficient $\Gamma$ in Eqs. (78), (80), and (81) is given by

$$\Gamma=-\frac{2}{3}\left[\Omega m_1-\frac{\Omega m_2}{\alpha_0}+\frac{1-\alpha_0+\beta_1}{\beta_1(1-\alpha_0)}km_3+\frac{km_4}{\alpha_0(1-\alpha_0)}-\frac{2\Delta^2k^2m_5}{\alpha_0}\right],\tag{A.3}$$

where

$$m_1=6(2-b_1)\Omega+2b_2(3-b_1)k,\quad m_2=-2\alpha_0b_1b_3k,$$
$$\widehat{m}=(\beta_1+\beta_2)(b_2-b_3)b_1\Omega-\beta_1\left(b_2^2-b_3^2\right)k,\quad m_3=\widehat{m}+3\gamma p_{G0}(b_1-3\gamma-1)k,$$
$$m_4=-\alpha_0\widehat{m}+\alpha_0b_1b_4k-2(1-\alpha_0)b_3^2k-2\alpha_0b_1b_3\Omega+\alpha_0b_1\Delta^2\Omega^2k,$$
$$m_5=1-3b_2\Omega k+\frac{3\gamma(3\gamma-1)p_{G0}}{2\Delta^2}+\frac{5\Omega^2}{2}.\tag{A.4}$$

Finally, the nonlinear coefficient of the NLS equation $\nu_1$ is given as follows (see Eq. (90) and Fig. 4):

$$\nu_1=\frac{1}{3}\frac{1}{\partial D/\partial\Omega}\left[\Omega n_1-\frac{\Omega n_2}{\alpha_0}+\frac{1-\alpha_0+\beta_1}{\beta_1(1-\alpha_0)}kn_3+\frac{kn_4}{\alpha_0(1-\alpha_0)}-\frac{\Delta^2k^2n_5}{\alpha_0}\right],\tag{A.5}$$

where

$$
\begin{aligned}
n_1 &= 3\Omega[c_0(1-b_1) - c_1 + 6b_1 - 10] + k[c_2(3-b_1) + b_2(3c_0 - c_1 + 9b_1 - 18)],\\
n_2 &= -\alpha_0(b_1c_3 + b_3c_1)k,\\
\widehat{n} &= (2\beta_1 - \beta_2)b_1(c_2 - c_3)\Omega - (\beta_1 - 2\beta_2)(b_2 - b_3)c_1\Omega\\
&\quad -kb_1(b_2 - b_3)[\beta_1(b_2 + b_3) + \beta_2 b_2] - \beta_1 k(b_2c_2 - b_3c_3),\\
n_3 &= \widehat{n} + 3\gamma p_{G0}k[2b_1c_0 - c_1 + (3\gamma+1)(1 - b_1 - c_0 + 3\gamma/2)],\\
n_4 &= -\alpha_0\widehat{n} + \Omega n_2/k - 2(1-\alpha_0)b_3c_3k\\
&\quad +\alpha_0 k\left\{b_1c_4 - b_4c_1 + 2\Delta^2\Omega^2c_1 - b_1\left[\Delta^2(c_0+1) - 3b_3^2 + 3\gamma(3\gamma-1)p_{G0}/2\right]\right\},\\
n_5 &= -3 + c_0\left(5\Omega^2 + 2\right) + 2b_2k[b_2k - (1+3c_0)\Omega] + 3\gamma(3\gamma-1)(c_0 - 2 - 3\gamma/2)p_{G0}/\Delta^2.
\end{aligned}
\tag{A.6}
$$

As can be seen, the second coefficient of virtual mass force $\beta_2$ appears in $\widehat{k}$, $\widehat{m}$, and $\widehat{n}$ in the nonlinear coefficients $\Pi_1$, $\Gamma$, and $\nu_1$. The second coefficient $\beta_2$ is included also in the inhomogeneous terms $K_i$, $M_i$, and $N_i$ ($i = 3, 4$) presented in Appendix 1. The third coefficient $\beta_3$ does not appear in the approximation examined in this paper.

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