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ABSTRACT

Theoretical investigation of the effects of a translation of bubbles and a drag force acting on bubbles on the wave propagation in bubbly flows has long been lacking. In this study, we theoretically and numerically investigate the weakly nonlinear (i.e., finite but small amplitude) propagation of plane progressive pressure waves in compressible water flows that contain uniformly distributed spherical gas bubbles with translation and drag forces. First, we assume that the gas and liquid phases flow at independent velocities. Then, the drag force and virtual mass force are introduced in an interfacial transport across the bubble–liquid interface in the momentum conservation equations. Furthermore, we consider the translation and spherically symmetric oscillations as bubble dynamics and deploy a two-fluid model to introduce the translation and drag forces. Bubbles do not coalesce, break up, extinct, or appear. For simplicity, the gas viscosity, thermal conductivities of the gas and liquid, and phase change and mass transport across the bubble–liquid interface are ignored. The following results are then obtained: (i) Using the method of multiple scales, two types of Korteweg–de Vries–Burgers equations with a correction term due to the drag force are derived. (ii) The translation of bubbles enhances the nonlinear effect of waves, and the drag force acting on bubbles contributes the nonlinear and dissipation effects of waves. (iii) The results of long-period numerical analysis verify that the temporal evolution of the wave (not flow) dissipation due to the drag force differs from that caused by the acoustic radiation.

I. INTRODUCTION

Many forces act on gas bubbles, such as drag, lift, gravity, and virtual mass force. The drag force acting on translational bubbles, in particular, is one of the most important factors in terms of the dynamics of single- or multi-bubble flows. Many theoretical studies and subsequent numerical and experimental works on the drag force and bubble dynamics have been carried out.[1–6] Furthermore, the translation and drag forces are quite important factors when assuming a high-velocity water flow accompanied by cavitation in hydraulic machinery.

In cavitating water flows, pressure variation is always generated by bubble oscillations and evolves into pressure waves (non-void waves[7]). As in nonlinear acoustics or nonlinear wave theory for pure (non-bubbly) water flows, the pressure wave evolves into a shock wave owing to the competition between the nonlinear effect and dissipation effect of the waves. Contrarily, for bubbly flows, the volumetric oscillations of bubbles induce a dispersion effect of waves. Owing to the competition between the nonlinear and dispersion effects, the pressure wave evolves into a stable wave, the so-called (acoustic) soliton. The shock wave and the soliton are exact solutions of the Burgers and Korteweg–de Vries (KdV) equations,[8] respectively. Hence, it is important to explore the relative ratios of the nonlinear, dissipation, and dispersion effects, because the pressure wave in bubbly flows may evolve into a shock wave or a soliton,[9] which have quite different properties. In the framework of weakly nonlinear (i.e., finite but small amplitude[10]) waves in bubbly liquids, the Korteweg–de Vries–Burgers (KdVB) equation[11] is one of the most famous nonlinear wave equations. As the KdVB equation comprises a linear combination of the nonlinear, dissipation, and dispersion terms [see (55)], the relative magnitude of these effects determines whether the pressure waves evolve into the shock wave or the soliton. Although estimating the magnitude of these three effects is essential to predict the evolution of the pressure wave, it cannot be obtained directly from experiments or numerical analysis of the governing equations. Therefore, the theoretical derivation of weakly nonlinear wave equations, such as the KdVB equation, is an effective method of estimating the relative strength of the above-mentioned effects.

Since the pioneering work of van Wijngaarden,[12] many studies have derived the KdVB equation from the gas–liquid mixture model[13] to describe pressure waves in bubbly liquids. Furthermore, Kuznetsov *et al.* showed that a waveform obtained by the KdVB equation agrees with the experimentally observed propagation of weakly nonlinear pressure waves in a bubbly flow.[14] However, all existing theoretical studies have ignored the drag force acting on the bubbles. Furthermore, only some studies[15,16] incorporated the translation of bubbles and the initial velocity of bubbly liquids. This may stem from the preconception that the effect of the non-oscillating components (i.e., translation and drag forces) on the oscillating components (i.e., bubble oscillations and pressure waves) is negligible. Because momentum transport across the bubble–liquid interface should be formulated to incorporate the drag force, such complex basic equations (e.g., two-fluid model equations) are required to resolve the weakly nonlinear (or linear) wave problem. Recently, based on the two-fluid model equations,[17] our group theoretically investigated the linear[17,18] and weakly nonlinear[19,20] waves and derived the KdVB equation[19] under the assumption that the translation and drag forces are negligible. Later, the initial nonuniform flow velocities were also incorporated.[21] On the other hand, our group numerically solved the original KdVB equation[19] and studied the temporal evolution of waveforms taking into account the nonlinear, dissipation, and dispersion effect; however, the translation and drag forces were not considered.[22]

In this study, we employ the two-fluid model[17] that accounts for the interfacial momentum transport and introduce the translation and drag forces. Unlike our previous work in which the initial flow was assumed to be at rest,[19] herein, at the initial state, the gas and liquid phases have velocities in this study as our preceding study.[21] The remainder of this paper is organized as follows: In Sec. II, we introduce the basic equations of the two-fluid model and bubble dynamics, including the drag force and translation, respectively. In Sec. III, we derive two types of KdVB equations and point out that the translation of bubbles increases the nonlinearity, while the drag force affects both the nonlinearity and dissipation. In Sec. IV, the evolution of waveforms is calculated using numerical analysis, and the effects of the translation and drag force are discussed quantitatively. Then, the causes of dissipation are decomposed and compared using long-period calculation. Finally, Sec. V concludes this paper.

II. FORMULATION OF THE PROBLEM

A. Problem statement

This paper theoretically investigates the weakly nonlinear (i.e., finite but small amplitude) propagation of one-dimensional (plane) pressure progressive waves in a flowing compressible water. The water containing distributed small spherical gas bubbles is shown in Fig. 1. Initially, the gas and liquid phases flow with a constant velocity. We newly introduce drag force and translation to the bubble dynamics. The bubbles do not coalesce, break up, extinct, or appear. For simplicity, the gas viscosity, Reynolds stress, thermal conduction[23,24] of the gas and liquid, and phase change and mass transport across the bubble–liquid interface are ignored. We assume a laminar flow and do not use turbulent models.

B. Governing equations

To introduce the drag force in the interfacial momentum transport, we first utilize the conservation laws of mass and momentum for the gas and liquid phases based on a two-fluid model,[17]

$$\frac{\partial}{\partial t^*}(\alpha\rho_{\mathrm{G}}^*) + \frac{\partial}{\partial x^*}(\alpha\rho_{\mathrm{G}}^* u_{\mathrm{G}}^*) = 0, \tag{1}$$

$$\frac{\partial}{\partial t^*}\left[(1-\alpha)\rho_{\mathrm{L}}^*\right] + \frac{\partial}{\partial x^*}\left[(1-\alpha)\rho_{\mathrm{L}}^* u_{\mathrm{L}}^*\right] = 0, \tag{2}$$

$$\frac{\partial}{\partial t^*}(\alpha\rho_{\mathrm{G}}^* u_{\mathrm{G}}^*) + \frac{\partial}{\partial x^*}(\alpha\rho_{\mathrm{G}}^* u_{\mathrm{G}}^{*2}) + \alpha\frac{\partial p_{\mathrm{G}}^*}{\partial x^*} = F^* + D^*, \tag{3}$$

$$\begin{aligned}\frac{\partial}{\partial t^*}\left[(1-\alpha)\rho_{\mathrm{L}}^* u_{\mathrm{L}}^*\right] &+ \frac{\partial}{\partial x^*}\left[(1-\alpha)\rho_{\mathrm{L}}^* u_{\mathrm{L}}^{*2}\right] \\ &+ (1-\alpha)\frac{\partial p_{\mathrm{L}}^*}{\partial x^*} + P^*\frac{\partial \alpha}{\partial x^*} = -F^* - D^*,\end{aligned} \tag{4}$$

where t^* is the time, x^* is the space coordinate, α is the void fraction ($0 < \alpha < 1$), ρ^* is the density, u^* is the velocity, p^* is the pressure, and P^* is the averaged liquid pressure on the bubble–liquid interface. The subscripts G and L denote the volume averaged variables in the gas and liquid phases, respectively, and the superscript $*$ denotes a dimensional quantity. The following model for the virtual mass force[18] is introduced as the interfacial momentum transport F^*:

$$\begin{aligned}F^* = &-\beta_1\alpha\rho_{\mathrm{L}}^*\left(\frac{\mathrm{D_G}u_{\mathrm{G}}^*}{\mathrm{D}t^*} - \frac{\mathrm{D_L}u_{\mathrm{L}}^*}{\mathrm{D}t^*}\right) \\ &- \beta_2\rho_{\mathrm{L}}^*(u_{\mathrm{G}}^* - u_{\mathrm{L}}^*)\frac{\mathrm{D_G}\alpha}{\mathrm{D}t^*} - \beta_3\alpha(u_{\mathrm{G}}^* - u_{\mathrm{L}}^*)\frac{\mathrm{D_G}\rho_{\mathrm{L}}^*}{\mathrm{D}t^*},\end{aligned} \tag{5}$$

where β_1, β_2, and β_3 are constants and may be set as 1/2 for the spherical bubble. The Lagrange derivatives $\mathrm{D_G}/\mathrm{D}t^*$ and $\mathrm{D_L}/\mathrm{D}t^*$ are defined by

$$\frac{\mathrm{D_G}}{\mathrm{D}t^*} \equiv \frac{\partial}{\partial t^*} + u_{\mathrm{G}}^*\frac{\partial}{\partial x^*}, \quad \frac{\mathrm{D_L}}{\mathrm{D}t^*} \equiv \frac{\partial}{\partial t^*} + u_{\mathrm{L}}^*\frac{\partial}{\partial x^*}. \tag{6}$$

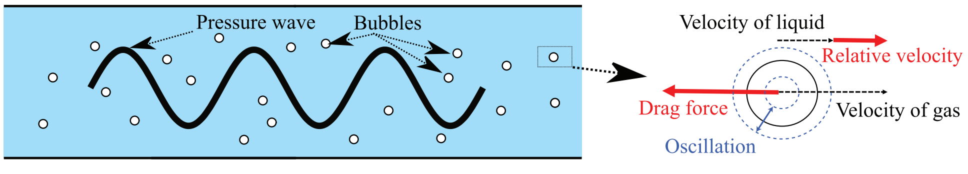

FIG. 1. Pressure wave propagation in bubbly flows.

Furthermore, we introduce a model for the drag force term D^* for spherical bubbles,

$$D^* = -\frac{3}{8R^*}\alpha C_D \rho_L^*(u_G^* - u_L^*)|u_G^* - u_L^*|, \tag{7}$$

where R^* is the radius of a representative bubble and C_D is the drag coefficient for a single spherical bubble (see Sec. II D). The gas viscosity and Reynolds stress induced by volume averaging are ignored. However, for simplicity, we use a Stokes-type drag force. We intend to validate these assumptions in a forthcoming paper.

The bubble dynamics is described as

$$\left(1 - \frac{1}{c_{L0}^*}\frac{\mathrm{D_G}R^*}{\mathrm{D}t^*}\right)R^*\frac{\mathrm{D_G^2}R^*}{\mathrm{D}t^{*2}} + \frac{3}{2}\left(1 - \frac{1}{3c_{L0}^*}\frac{\mathrm{D_G}R^*}{\mathrm{D}t^*}\right)\left(\frac{\mathrm{D_G}R^*}{\mathrm{D}t^*}\right)^2$$
$$= \left(1 + \frac{1}{c_{L0}^*}\frac{\mathrm{D_G}R^*}{\mathrm{D}t^*}\right)\frac{P^*}{\rho_{L0}^*} + \frac{R^*}{\rho_{L0}^* c_{L0}^*}\frac{\mathrm{D_G}}{\mathrm{D}t^*}(p_L^* + P^*)$$
$$+ \frac{(u_G^* - u_L^*)^2}{4}, \tag{8}$$

where c_{L0}^* is the speed of sound in pure water. We introduced the translation of bubbles[15,16] in the third term of the right-hand side of (8) into the original Keller equation.[25] Hence, the translation and volumetric oscillations are described by (8) as a linear combination.

To complete the set of (1)–(4) and (8), the polytropic equation of state for gas, the Tait equation of state for liquid, the mass conservation law of gas inside the bubbles, and the balance of normal stresses across the bubble–liquid interface are introduced,

$$\frac{p_G^*}{p_{G0}^*} = \left(\frac{\rho_G^*}{\rho_{G0}^*}\right)^\gamma, \tag{9}$$

$$p_L^* = p_{L0}^* + \frac{\rho_{L0}^* c_{L0}^{*2}}{n}\left[\left(\frac{\rho_L^*}{\rho_{L0}^*}\right)^n - 1\right], \tag{10}$$

$$\frac{\rho_G^*}{\rho_{G0}^*} = \left(\frac{R_0^*}{R^*}\right)^3, \tag{11}$$

$$p_G^* - (p_L^* + P^*) = \frac{2\sigma^*}{R^*} + \frac{4\mu^*}{R^*}\frac{\mathrm{D_G}R^*}{\mathrm{D}t^*}, \tag{12}$$

where γ is the polytropic exponent, n is a material constant (e.g., $n = 7.15$ for water), σ^* is the surface tension, and μ^* is the liquid viscosity (see Sec. II D). It should be noted that the effect of liquid viscosity is considered only at the bubble–liquid interface. The physical quantities at the initial state are signified by the subscript 0, and they are constants.

C. Multiple scales analysis

Based on the method of multiple scales,[10] four multiple scales[10] are introduced as extended independent variables by assuming the finite but small nondimensional wave amplitude ε ($\ll 1$),

$$t_0 = t, \quad t_1 = \varepsilon t; \quad x_0 = x, \quad x_1 = \varepsilon x, \tag{13}$$

where the nondimensional independent variables are defined by $t = t^*/T^*$ and $x = x^*/L^*$; T^* is the typical period and L^* is the typical wavelength. Here, the subscripts 0 and 1 correspond to the near and far fields,[10] respectively.

The dependent variables are nondimensionalized and are expanded in power series of ε,

$$R^*/R_0^* = 1 + \varepsilon R_1 + \varepsilon^2 R_2 + O(\varepsilon^3), \tag{14}$$

$$u_G^*/U^* = u_{G0} + \varepsilon u_{G1} + \varepsilon^2 u_{G2} + O(\varepsilon^3), \tag{15}$$

$$u_L^*/U^* = u_{L0} + \varepsilon u_{L1} + \varepsilon^2 u_{L2} + O(\varepsilon^3), \tag{16}$$

$$\alpha/\alpha_0 = 1 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + O(\varepsilon^3), \tag{17}$$

$$\rho_L^*/\rho_{L0}^* = 1 + \varepsilon^2\rho_{L2} + \varepsilon^3\rho_{L3} + O(\varepsilon^4), \tag{18}$$

$$p_L^*/(\rho_{L0}^* U^{*2}) = p_{L0} + \varepsilon p_{L1} + \varepsilon^2 p_{L2} + O(\varepsilon^3), \tag{19}$$

where U^* ($= L^*/T^*$) is the typical propagation speed and the initial nondimensional pressures of $O(1)$ are defined by $p_{G0} = p_{G0}^*/(\rho_{L0}^* U^{*2})$ and $p_{L0} = p_{L0}^*/(\rho_{L0}^* U^{*2})$. Furthermore, the ratio of initial densities of gas and liquid phases is negligibly small.[19]

Similar to our previous study,[19] the set of sizes of the three nondimensional ratios appropriate to the low-frequency long wave is also determined using ε,

$$\frac{U^*}{c_{L0}^*} \equiv O(\sqrt{\varepsilon}) \equiv V\sqrt{\varepsilon}, \tag{20}$$

$$\frac{R_0^*}{L^*} \equiv O(\sqrt{\varepsilon}) \equiv \Delta\sqrt{\varepsilon}, \tag{21}$$

$$\frac{\omega^*}{\omega_B^*} \equiv \frac{1}{T^*\omega_B^*} \equiv O(\sqrt{\varepsilon}) \equiv \Omega\sqrt{\varepsilon}, \tag{22}$$

where V, Δ, and Ω are constants of $O(1)$, and the eigenfrequency of a single bubble ω_B^* is given by

$$\omega_B^* = \sqrt{\frac{3\gamma(p_{L0}^* + 2\sigma^*/R_0^*) - 2\sigma^*/R_0^*}{\rho_{L0}^* R_0^{*2}}}. \tag{23}$$

Equations (20)–(22) represent that the speed of sound in bubbly flows is much smaller than that in pure water, the initial bubble radius is much shorter than the typical wavelength, and the incident frequency of waves is much lower than the eigenfrequency of a single bubble, respectively.

D. Drag force

The nondimensional liquid viscosity μ is defined by

$$\frac{\mu^*}{\rho_{L0}^* U^* L^*} \equiv \begin{cases} O(\varepsilon^2) \equiv \mu\varepsilon^2 & \text{(case A)} \\ O(\varepsilon) \equiv \mu\varepsilon & \text{(case B)}, \end{cases} \tag{24}$$

where cases A and B correspond to low liquid viscosity (e.g., $R_0^* \gtrsim 500\,\mu$m) and high liquid viscosity (e.g., $R_0^* \lesssim 10\,\mu$m), respectively.

In this paper, the drag coefficient is assumed to be $O(\sqrt{\varepsilon})$ and is defined as

$$C_D \equiv O(\sqrt{\varepsilon}) \equiv \begin{cases} 8\mu^*/(|u_G^* - u_L^*|\rho_L^* R^*) & \text{(case A)} \\ \Lambda\sqrt{\varepsilon} & \text{(case B)}, \end{cases} \tag{25}$$

where Λ is the constant of $O(1)$. In case A, C_D depends on the Reynolds number Re ($C_D = 16/\mathrm{Re}$).[1]

III. THEORETICAL RESULT

This section focuses on the derivation of the two types of KdVB equations and theoretically discusses their physical meaning.

A. Linear propagation

Substituting (13)–(25) into (1)–(4) and (8), the set of linear equations can be derived from the leading-order of approximation with the aid of (9)–(12),

$$\frac{\mathrm{D_G}\alpha_1}{\mathrm{D}t_0} - 3\frac{\mathrm{D_G}R_1}{\mathrm{D}t_0} + \frac{\partial u_{\mathrm{G1}}}{\partial x_0} = 0, \tag{26}$$

$$\alpha_0\frac{\mathrm{D_L}\alpha_1}{\mathrm{D}t_0} - (1-\alpha_0)\frac{\partial u_{\mathrm{L1}}}{\partial x_0} = 0, \tag{27}$$

$$\beta_1\left(\frac{\mathrm{D_G}u_{\mathrm{G1}}}{\mathrm{D}t_0} - \frac{\mathrm{D_L}u_{\mathrm{L1}}}{\mathrm{D}t_0}\right) + \beta_2(u_{\mathrm{G0}} - u_{\mathrm{L0}})\frac{\mathrm{D_G}\alpha_1}{\mathrm{D}t_0} - 3\gamma p_{\mathrm{G0}}\frac{\partial R_1}{\partial x_0} = 0, \tag{28}$$

$$(1-\alpha_0)\frac{\mathrm{D_L}u_{\mathrm{L1}}}{\mathrm{D}t_0} - \alpha_0\beta_1\left(\frac{\mathrm{D_G}u_{\mathrm{G1}}}{\mathrm{D}t_0} - \frac{\mathrm{D_L}u_{\mathrm{L1}}}{\mathrm{D}t_0}\right) - \alpha_0\beta_2(u_{\mathrm{G0}} - u_{\mathrm{L0}})\frac{\mathrm{D_G}\alpha_1}{\mathrm{D}t_0} - \alpha_0 u_{\mathrm{L0}}\frac{\mathrm{D_L}\alpha_1}{\mathrm{D}t_0} + u_{\mathrm{L0}}(1-\alpha_0)\frac{\partial u_{\mathrm{L1}}}{\partial x_0} + (1-\alpha_0)\frac{\partial p_{\mathrm{L1}}}{\partial x_0} = 0, \tag{29}$$

$$R_1 + \frac{\Omega^2}{\Delta^2}p_{\mathrm{L1}} = 0. \tag{30}$$

Combining (26)–(30) results in the linear wave equation for the first-order variation of the bubble radius, R_1,

$$\frac{\mathrm{D}^2 R_1}{\mathrm{D}t_0^2} - v_{\mathrm{p}}^2\frac{\partial^2 R_1}{\partial x_0^2} = 0, \tag{31}$$

where v_{p} is the phase velocity given by

$$v_{\mathrm{p}} = \sqrt{\frac{3\alpha_0(1-\alpha_0+\beta_1)\gamma p_{\mathrm{G0}} + \beta_1(1-\alpha_0)\Delta^2/\Omega^2}{3\beta_1\alpha_0(1-\alpha_0)}}. \tag{32}$$

The definition of the linear Lagrange derivative $\mathrm{D}/\mathrm{D}t_0$ is

$$\frac{\mathrm{D}}{\mathrm{D}t_0} = \frac{\partial}{\partial t_0} + v_{\mathrm{p}}\frac{\partial}{\partial x_0}. \tag{33}$$

It should be noted that the initial velocities of both phases are assumed to be the same ($u_{\mathrm{G0}} = u_{\mathrm{L0}} \equiv u_0$) for simplicity; however, the perturbations of each velocity are not the same ($u_{\mathrm{G1}} \neq u_{\mathrm{L1}}$). Setting $v_{\mathrm{p}} = 1$ gives the explicit form of U^* as

$$U^* = \sqrt{\frac{3\alpha_0(1-\alpha_0+\beta_1)\gamma p_{\mathrm{G0}}^*/\rho_{\mathrm{L0}}^* + \beta_1(1-\alpha_0)R_0^{*2}\omega_{\mathrm{B}}^{*2}}{3\beta_1\alpha_0(1-\alpha_0)}}. \tag{34}$$

By focusing on the right-running wave (i.e., introducing a moving coordinate $x_0 - v_{\mathrm{p}}t_0$), α_1, u_{G1}, u_{L1}, and p_{L1} can be expressed in terms of R_1,

$$\alpha_1 = s_1 R_1, \tag{35}$$

$$u_{\mathrm{G1}} = s_2 R_1, \tag{36}$$

$$u_{\mathrm{L1}} = s_3 R_1, \tag{37}$$

$$p_{\mathrm{L1}} = s_4 R_1 \tag{38}$$

with

$$s_1 = \frac{(1-\alpha_0)}{\alpha_0(1-\alpha_0+\beta_1)}\left[3\alpha_0\beta_1 - \frac{(1-\alpha_0)s_4}{v_{\mathrm{p}}^2}\right], \tag{39}$$

$$s_2 = s_1 - 3, \tag{40}$$

$$s_3 = -\frac{\alpha_0 s_1}{1-\alpha_0}, \tag{41}$$

$$s_4 = -\frac{\Delta^2}{\Omega^2}. \tag{42}$$

The effects of the translation and drag force are not evident in the near field. Therefore, in the absence of an initial flow velocity (i.e., $u_{\mathrm{G0}} = u_{\mathrm{L0}} = u_0 = 0$), $\mathrm{D}/\mathrm{D}t_0$ becomes $\partial/\partial t_0$, and the present result coincides with the results of our previous work.[19]

B. Nonlinear propagation for case A: Large bubble

As in the case of $O(\varepsilon)$, the following set of inhomogeneous equations of $O(\varepsilon^2)$ is derived:

$$\frac{\mathrm{D_G}\alpha_2}{\mathrm{D}t_0} - 3\frac{\mathrm{D_G}R_2}{\mathrm{D}t_0} + \frac{\partial u_{\mathrm{G2}}}{\partial x_0} = K_1, \tag{43}$$

$$\alpha_0\frac{\mathrm{D_L}\alpha_2}{\mathrm{D}t_0} - (1-\alpha_0)\frac{\partial u_{\mathrm{L2}}}{\partial x_0} = K_2, \tag{44}$$

$$\beta_1\left(\frac{\mathrm{D_G}u_{\mathrm{G2}}}{\mathrm{D}t_0} - \frac{\mathrm{D_L}u_{\mathrm{L2}}}{\mathrm{D}t_0}\right) + \beta_2(u_{\mathrm{G0}} - u_{\mathrm{L0}})\frac{\mathrm{D_G}\alpha_2}{\mathrm{D}t_0} - 3\gamma p_{\mathrm{G0}}\frac{\partial R_2}{\partial x_0} = K_{3\mathrm{A}}, \tag{45}$$

$$(1-\alpha_0)\frac{\mathrm{D_L}u_{\mathrm{L2}}}{\mathrm{D}t_0} - \alpha_0\beta_1\left(\frac{\mathrm{D_G}u_{\mathrm{G2}}}{\mathrm{D}t_0} - \frac{\mathrm{D_L}u_{\mathrm{L2}}}{\mathrm{D}t_0}\right) - \alpha_0\beta_2(u_{\mathrm{G0}} - u_{\mathrm{L0}})\frac{\mathrm{D_G}\alpha_2}{\mathrm{D}t_0} - \alpha_0 u_{\mathrm{L0}}\frac{\mathrm{D_L}\alpha_2}{\mathrm{D}t_0} + u_{\mathrm{L0}}(1-\alpha_0)\frac{\partial p_{\mathrm{L2}}}{\partial x_0} + (1-\alpha_0)\frac{\partial u_{\mathrm{L2}}}{\partial x_0} = K_{4\mathrm{A}}, \tag{46}$$

$$R_2 + \frac{\Omega^2}{\Delta^2}p_{\mathrm{L2}} = K_{5\mathrm{A}}. \tag{47}$$

The inhomogeneous terms $K_{3\mathrm{A}}$, $K_{4\mathrm{A}}$, and $K_{5\mathrm{A}}$ are defined as

$$K_{3\mathrm{A}} = K_3 - \frac{3\mu}{\Delta^2}(u_{\mathrm{G1}} - u_{\mathrm{L1}}), \tag{48}$$

$$K_{4\mathrm{A}} = K_4 + \frac{3\mu}{\Delta^2}\alpha_0(u_{\mathrm{G1}} - u_{\mathrm{L1}}), \tag{49}$$

$$K_{5\mathrm{A}} = K_5' + \frac{\Omega^2(u_{\mathrm{G}1} - u_{\mathrm{L}1})^2}{4\Delta^2}, \tag{50}$$

$$K_5' = K_5 + \frac{4\mu\Omega^2}{\Delta^2}\frac{\mathrm{D_G} R_1}{\mathrm{D}t_0}, \tag{51}$$

where K_i ($i = 1, 2, 3, 4, 5$) are the original inhomogeneous terms (see Appendix 1 in Ref. 19) and K_5' is the inhomogeneous term that ignores the effect of liquid viscosity from K_5.

Then, (43)–(47) are combined into a single inhomogeneous equation

$$\frac{\mathrm{D}^2 R_2}{\mathrm{D}t_0^2} - v_\mathrm{p}^2 \frac{\partial^2 R_2}{\partial x_0^2} = K(R_1; x_1, t_1, x_0 - v_\mathrm{p} t_0), \tag{52}$$

with

$$K = -\frac{1}{3}\frac{\mathrm{D}K_1}{\mathrm{D}t_0} + \frac{1}{3\alpha_0}\frac{\mathrm{D}K_2}{\mathrm{D}t_0} + \frac{u_0}{3\alpha_0(1-\alpha_0)}\frac{\partial K_2}{\partial x_0} + \frac{1-\alpha_0+\beta_1}{3(1-\alpha_0)\beta_1}\frac{\partial K_{3\mathrm{A}}}{\partial x_0} + \frac{1}{3\alpha_0(1-\alpha_0)}\frac{\partial K_{4\mathrm{A}}}{\partial x_0} - \frac{\Delta^2}{3\alpha_0\Omega^2}\frac{\partial^2 K_{5\mathrm{A}}}{\partial x_0^2}. \tag{53}$$

From the solvability condition for (51), $K = 0$ is required.[19] From (13), the original independent variables x and t are restored,

$$\frac{\partial R_1}{\partial t} + (v_\mathrm{p} + u_0)\frac{\partial R_1}{\partial x} + \varepsilon\left(\Pi_0 \frac{\partial R_1}{\partial x} + \Pi_{1\mathrm{A}} R_1 \frac{\partial R_1}{\partial x} + \Pi_{2\mathrm{A}}\frac{\partial^2 R_1}{\partial x^2} + \Pi_3 \frac{\partial^3 R_1}{\partial x^3} + \Pi_{4\mathrm{A}} R_1\right) = 0. \tag{54}$$

Finally, we obtain the KdVB-A equation

$$\frac{\partial R_1}{\partial \tau} + \Pi_{1\mathrm{A}} R_1 \frac{\partial R_1}{\partial \xi} + \Pi_{2\mathrm{A}}\frac{\partial^2 R_1}{\partial \xi^2} + \Pi_3 \frac{\partial^3 R_1}{\partial \xi^3} + \Pi_{4\mathrm{A}} R_1 = 0, \tag{55}$$

through the variable transform

$$\tau \equiv \varepsilon t, \quad \xi \equiv x - (v_\mathrm{p} + u_0 + \varepsilon \Pi_0)t, \tag{56}$$

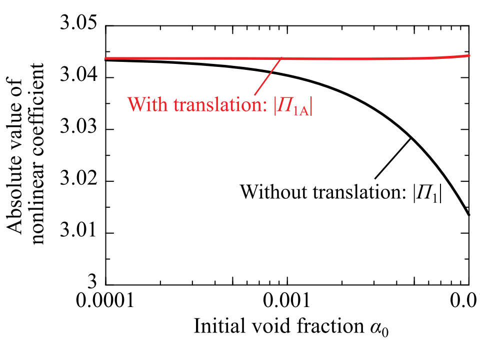

FIG. 2. The absolute value of nonlinear coefficient Π_1 and $\Pi_{1\mathrm{A}}$ vs the initial void fraction α_0 for the case of $\sqrt{\varepsilon} = 0.15$, $R_0^* = 10\ \mu$m, $v_\mathrm{p} = 1$, $\rho_{\mathrm{L}0}^* = 10^3$ kg/m^3, $p_{\mathrm{L}0}^* = 101\,325$ Pa, $\beta_1 = \beta_2 = 1/2$, $c_{\mathrm{L}0}^* = 1500$ m/s, $\gamma = 1$, $\sigma^* = 0.0728$ N/m, $\mu^* = 10^{-3}$ Pa s^{2}s, and $\Omega = 1$. The same conditions except for R_0^* are used in Figs. 3–7.

where Π_0 is the original advection coefficient [see Eq. (52) in Ref. 19] and the other constant coefficients are

$$\Pi_{1\mathrm{A}} = \Pi_1 - \frac{(s_2 - s_3)^2}{12\alpha_0 v_\mathrm{p}} < 0, \tag{57}$$

$$\Pi_{2\mathrm{A}} = -\frac{1}{6\alpha_0}\frac{V\Delta^3}{\Omega^2} < 0, \tag{58}$$

$$\Pi_3 = \frac{\Delta^2}{6\alpha_0} > 0, \tag{59}$$

$$\Pi_{4\mathrm{A}} = \frac{\mu}{2v_\mathrm{p}\beta_1\Delta^2}(s_3 - s_2) > 0. \tag{60}$$

Here, Π_1 is the nonlinear coefficient (see the explicit form in Appendix 1 of Ref. 19) and Π_3 is the dispersion coefficient. These coefficients are the same as the original ones.[19] Furthermore, $\Pi_{1\mathrm{A}}$ is the nonlinear coefficient, $\Pi_{2\mathrm{A}}$ is the dissipation coefficient owing to acoustic radiation, and $\Pi_{4\mathrm{A}}$ is the dissipation coefficient owing to the drag force. For the preceding case without translation,[26] the nonlinear

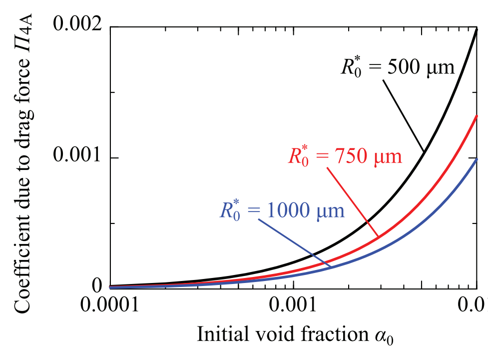

FIG. 3. The coefficient related to the drag force $\Pi_{4\mathrm{A}}$ vs the initial void fraction α_0. The black, red, and blue curves represent $R_0^* = 500\ \mu$m, 750 μm, and 1000 μm, respectively.

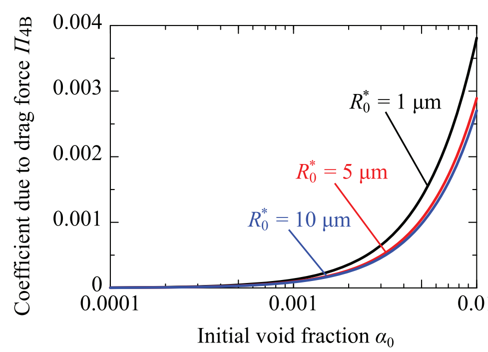

FIG. 4. The coefficient related to the drag force $\Pi_{4\mathrm{B}}$ vs the initial void fraction α_0 for the case of $\Lambda = 1$. The black, red, and blue curves represent $R_0^* = 1\ \mu$m, 5 μm, and 10 μm, respectively.

coefficient Π_{1A} reduces to Π_1, and the other coefficients remain unchanged.

C. Nonlinear propagation for case B: Small bubble

In case B, the following set of inhomogeneous equations of $O(\varepsilon^2)$ can be derived by introducing the new scaling Λ. The mass conservation equations are the same as (43) and (44). However, K_{3A} in (45), K_{4A} in (46), and K_{5A} in (47) become K_{3B}, K_{4B}, and K_{5B}, respectively:

$$K_{3B} = K_3 - \frac{3\Lambda}{8\Delta}(u_{G1} - u_{L1})|u_{G1} - u_{L1}|, \tag{61}$$

$$K_{4B} = K_4 + \frac{3\Lambda}{8\Delta}\alpha_0(u_{G1} - u_{L1})|u_{G1} - u_{L1}|, \tag{62}$$

$$K_{5B} = K_5 + \frac{\Omega^2(u_{G1} - u_{L1})^2}{4\Delta^2}. \tag{63}$$

Finally, we obtain the KdVB-B equation as in the case of KdVB-A equation,

$$\frac{\partial R_1}{\partial \tau} + \Pi_{1A}R_1\frac{\partial R_1}{\partial \xi} + \Pi_2\frac{\partial^2 R_1}{\partial \xi^2} + \Pi_3\frac{\partial^3 R_1}{\partial \xi^3} + \Pi_{4B}|R_1|R_1 = 0. \tag{64}$$

Here, Π_0, Π_2, and Π_3 are the same as the original coefficients,[19] Π_{1A} is the nonlinear coefficient that is the same as that of the KdVB-A equation, and Π_{4B} is the nonlinear dissipation coefficient owing to the drag force, which is obtained as follows:

$$\Pi_{4B} = \frac{\Lambda}{16\nu_p\beta_1\Delta}(s_3 - s_2)|s_3 - s_2| > 0. \tag{65}$$

D. Comparison of coefficients: Translation to nonlinearity and drag force to dissipation

The second term on the right-hand side of (57) describes the effect of bubble translation, and the translation contributes to the nonlinear effect of waves. Figure 2 shows the dependence of the absolute value of the nonlinear coefficient on the initial void fraction α_0. It can be seen that the incorporation of bubble translation results in an increase in the absolute value of the nonlinear coefficient Π_{1A} and thus the increase of the nonlinear effect.

Then, we focus on Π_{4A} and Π_{4B}, the coefficients related to the drag force. Figures 3 and 4 show the dependence of Π_{4A} and Π_{4B} on the initial void fraction α_0, respectively. As can be seen, both Π_{4A} and Π_{4B} increase with increasing α_0 and with decreasing R_0^*.

E. Discussion

The two types of KdVB equations [(55) and (64)] may be difficult to solve analytically, thus detracting to some extent from the utility of the KdVB formalism. The most important effect is that the drag force and translation affect the dissipation and nonlinearity, respectively, as discussed below.

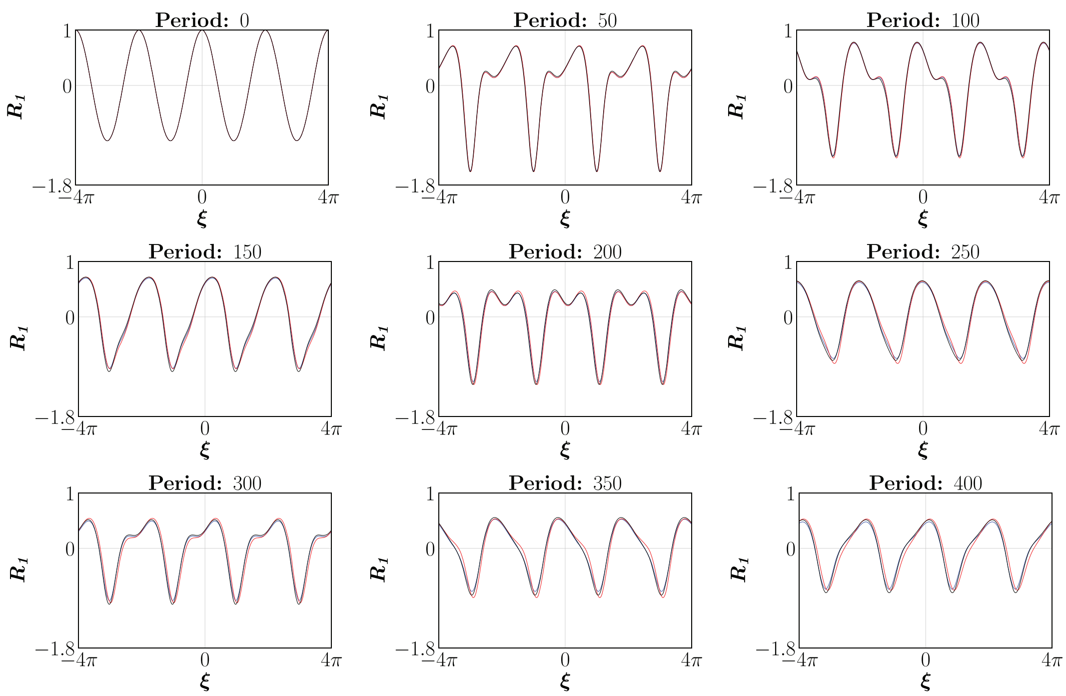

FIG. 5. Temporal evolution of the numerical solutions to the KdVB equation for $\alpha_0 = 0.05$ and $R_0^* = 500\ \mu$m; grid steps $N = 1024$; duration of numerical integration $\Delta\tau = 0.001$; and size of the computational domain $W = 8\pi$. The black, red, and blue curves represent the waveforms for no translation and drag force,[19] with only translation, and with both the translation and drag force (55), respectively.

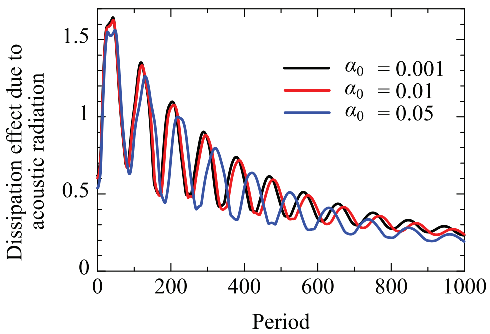

FIG. 6. Dissipation effect due to the acoustic radiation Γ_A vs the period for $R_0^* = 500\ \mu$m.

First, for drag force, by assuming the solution of R_1 to the KdVB equation (55) as a sinusoidal wave, we establish that the drag force (i.e., Π_{4A}) affects the wave dissipation effect. From an intuitive point of view, we also consider the physical reason for the impact of the drag force on dissipation. We consider that the drag force transports the momentum across the bubble–liquid interface against the flow. It is well known that the drag force contributes to the dissipation of the flow, not the pressure wave. In this study, it is newly found that the drag force also contributes to the dissipation of waves from the physico-mathematical point of view. Since this is only a theoretical prediction and is not justified experimentally owing to the complete absence of corresponding experiments and direct numerical simulations, our theory should be verified experimentally.

Second, for translation, the results of a previous numerical study[27] indicated that bubble translation (bubble slip) slightly affects the waveforms. Herein, we demonstrated that bubble translation affects nonlinearity from the physico-mathematical viewpoint.

IV. NUMERICAL RESULT

In this section, we discuss the effect of the drag force and bubble translation based on the KdVB-A equation (55).

A. Numerical analysis of waveform

First, (55) is numerically solved via the split-step (spectrum) Fourier–Galerkin method under a periodic boundary condition; the detailed scheme is described in our previous study.[22] Figure 5 shows the temporal evolution of the waveform for the case of $\alpha_0 = 0.05$ and $R_0^* = 500\ \mu$m; the blue, red, and black curves represent the solutions to the present KdVB-A equation (55), the KdVB equation with only translation, and the original KdVB equation,[19] respectively. Note that the liquid viscosity in the original KdVB equation is dropped from the dissipation coefficient for consistency in the comparison.

Consequently, the tendency of the temporal evolution of the waveform remains unchanged qualitatively between the present and previous cases. The amplitude of the present case is, however, slightly smaller than that of the previous case. As the drag force contributes to the wave dissipation, the consideration of the drag force decreases the wave amplitude (i.e., wave attenuation). On the other hand, the nonlinear effect increases owing to bubble translation.

B. Detailed elucidation of effect of drag force and acoustic radiation

This subsection discusses the most important points in this study.

The fifth term on the left-hand side of (55), i.e., $\Pi_{4A}R_1$, as a new term, contributes to wave dissipation.[26] The liquid viscosity does not affect the dissipation coefficient, as shown in (58) for case A, because the nondimensional size of liquid viscosity [see (24)] is changed from our previous study.[19] Hence, Π_{2A} is due to only the acoustic radiation originating from the oscillating bubbles in a compressible liquid. The dissipation effect due to the liquid viscosity then appears as an approximation of $O(\epsilon^3)$, that is, at a region very far from the sound source in this situation.

Therefore, there are two types of dissipation effects, Π_{2A} and Π_{4A}, with different formulas. As it is impossible to know the temporal evolution of the two types of dissipation by only theoretical analysis, we use numerical analysis. Here, we define the magnitude of dissipation as

$$\Gamma_A = \sum_{i=1}^{1024} \frac{8\pi}{1024} \left| \Pi_{2A} \frac{\partial^2 R_1}{\partial \xi^2} \right|_i, \quad \Gamma_D = \sum_{i=1}^{1024} \frac{8\pi}{1024} \left| \Pi_{4A} R_1 \right|_i, \tag{66}$$

where the index i represents the computational mesh of ξ and numerically obtained values of R_1 in Sec. IV A are used. Equation (66) represents the spatial integration of the dissipation terms. Figures 6 and 7 show the temporal evolutions of Γ_A and Γ_D for

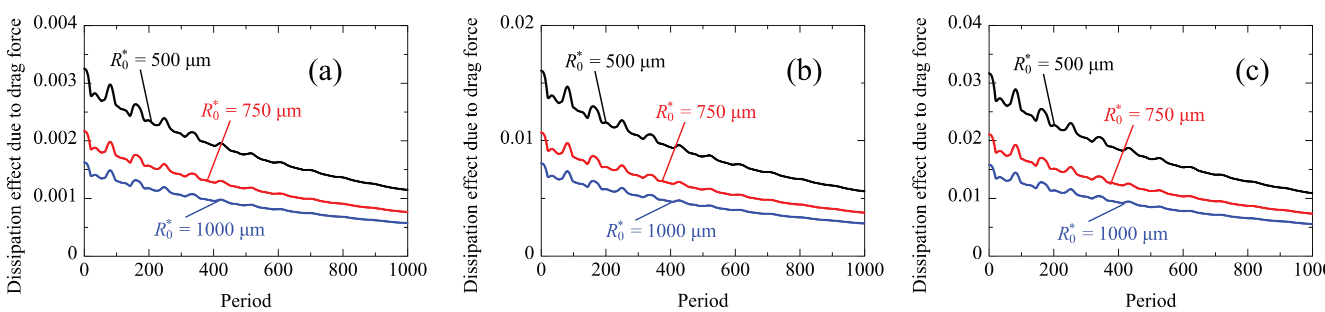

FIG. 7. Dissipation effect due to the drag force Γ_D vs the period: The initial void fraction α_0 is (a) 0.001, (b) 0.005, and (c) 0.01.

various α_0 and R_0^*, respectively. First, while Γ_A decreases with an oscillation (Fig. 6), Γ_D decreases almost monotonically (Fig. 7). Furthermore, Γ_D increases with increasing α_0 and decreasing R_0^*; however, Γ_A is almost independent of R_0^*. Finally, a frequency of Γ_A decreases with increasing α_0.

V. CONCLUSIONS

We have theoretically and numerically examined the weakly nonlinear propagation of pressure waves in bubbly flows, especially focusing on the effects of the translation and drag force acting on the bubbles. The main results are summarized as follows:

(i) From multiple scales analysis, the two types of KdVB equations describing plane progressive pressure waves with low frequency in water flows that contain uniformly distributed translational bubbles were derived. The effect of drag force acting on the bubbles appeared for both the KdVB equation with a correction linear term in (55) and the KdVB equation with a correction nonlinear term in (64).

(ii) The translation of bubbles enhanced the wave nonlinearity and increased the absolute value of the nonlinear coefficient.

(iii) The drag force produced new terms in the KdVB equations and contributed to the dissipation of pressure waves.[26] This term enhanced the dissipation for large bubbles (e.g., $R_0^* \gtrsim 500\,\mu$m) but contributed to both the nonlinearity and dissipation for small bubbles (e.g., $R_0^* \lesssim 10\,\mu$m).

(iv) The temporal evolution of the dissipation effects due to the acoustic radiation and drag force exhibited distinct trends; the acoustic radiation caused a decrease with an oscillation, whereas the drag force resulted in a nearly monotonic decrease.

(v) The dissipation effect due to the drag force strongly depended on both the initial void fraction and the initial bubble radius.

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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