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ABSTRACT

To clarify the effect of the drag force acting on bubbles and translation of bubbles on pressure waves, the weakly nonlinear (i.e., finite but small-amplitude) propagation of plane pressure waves with a thermal conduction in compressible water flows containing many spherical bubbles is theoretically investigated for moderately high-frequency and short-wavelength case. This work is an extension of our previous report [Yatabe *et al.*, Phys. Fluids, **33**, 033315 (2021)], wherein we elucidated the same for low-frequency and long-wavelength case. Based on our assumptions, the main results of this study are as follows: (i) using the method of multiple scales, the nonlinear Schrödinger type equation was derived; (ii) as in the previous long wave case, the translation of bubbles increased the nonlinear effect of waves, and the drag force acting on the bubbles resulted in the dissipation effect of waves; (iii) the increase in the nonlinear effect of the waves owing to the translation in the present short wavelength case is larger than that in the previous long wavelength case; (iv) the dissipation effect caused by the drag force was smaller than that caused by the liquid viscosity, acoustic radiation (i.e., liquid compressibility), and thermal conduction; (v) we then succeeded the comparison of the four dissipation factors (i.e., liquid viscous damping, thermal conduction, acoustic radiation, and drag force) on pressure waves.

I. INTRODUCTON

The drag force acting on translational bubbles in water flows is one of the most important forces that may alter the dynamics of bubbly flows. Many theoretical, numerical, and experimental reports for drag forces acting on a bubble have been published.[1–7] In particular, in the case of a high-velocity water flow accompanied by cavitation in hydraulic machinery, the physics of translation and drag force are significant. Furthermore, the volumetric oscillation of bubbles induces a pressure variation that evolves into a pressure wave (and not a void wave[8]). As is well known, in the nonlinear acoustics or nonlinear wave theory,[9,10] for a pure (and not bubbly) water flow, the pressure wave evolves into a shock wave owing to the competition between a nonlinear effect and a dissipation effect of the waves. By contrast, for a bubbly flow, oscillations of bubbles induce a dispersion effect of the waves, and the pressure wave then evolves into a stable wave, or the so-called (acoustic) soliton, because of the competition between the nonlinear and dispersion effects. Hence, it is important to understand the relative ratios of the dissipation and dispersion effects to the nonlinear effect, because the pressure wave in bubbly flows may evolve into both the shock wave and the soliton, which have completely different properties.

In the field of weakly nonlinear (i.e., finite but small-amplitude[10]) dispersive waves (not restricted to waves in bubbly flows), the Korteweg–de Vries–Burgers (KdVB) (or KdV) equation for a weakly dispersive case and the nonlinear Schrödinger (NLS) (or Ginzburg–Landau) equation for a strongly dispersive case are popular nonlinear wave equations (or nonlinear evolution equations).[9,10] Because the nonlinear wave equation is composed of a linear combination of the nonlinear, dissipation, and dispersion terms, the relative magnitudes of these effects determine whether a pressure wave will evolve into the shock wave or the soliton. Although estimating the magnitude of these three effects is essential for predicting the evolution of a pressure wave, this magnitude cannot be obtained directly from an experimental observation nor from a direct numerical simulation of the basic equations. Therefore, a theoretical derivation of nonlinear wave equations, such as the KdVB and NLS equations, could be an effective method for estimating the relative strength of these effects. In many previous studies on nonlinear pressure waves in bubbly liquids, the waveform obtained by solving the KdVB (or KdV) equation[11–13] has matched the waveform observed in experiments.[14–16] However, there are no reports of experimental observations of the propagation of a moderately high-frequency (near the eigenfrequency of single bubble oscillations in Fig. 1) short wave that corresponds to the solution of NLS equation, and some NLS equations have been derived that focus on a quite high-frequency (not moderately high-frequency) band that is well above the curve in Fig. 1, induced by considering a liquid compressibility.[17–20] Therefore, theoretical predictions of NLS equations for the moderately high-frequency case have long been strongly desired.

A critical concern is that both the translation of bubbles and the drag forces acting on bubbles were not considered in all previous studies on weakly nonlinear waves.[11–13,17–34] This may be due to the preconception that the effect of the non-oscillating components (i.e., translation and drag force) on the oscillating components (i.e., bubble oscillation and pressure wave) is negligible. To incorporate a drag force, although a momentum transport across the bubble–liquid interface should be formulated, such complex basic equations (e.g., two-fluid model equations) are not required in the weakly nonlinear (or linear) wave problem. In fact, all previous studies,[11–13,17–20,22,23,28,31] except for Biesheuvel and van Wijngaarden,[21] utilized the gas–liquid mixture model[22,23] as the basic equations. The first study in the literature to have derived nonlinear wave equations based on the two-fluid model equations is our original report,[26] wherein we proposed a unified theoretical framework in which the low-frequency long wave is described by the KdVB equation and the moderately (not quite) high-frequency short wave is by the NLS equation. As shown in the linear dispersion relation in quiescent bubbly liquids (Fig. 1),[11,12,26] our KdVB and NLS equations focused on the low and moderately high frequencies, respectively. Hence, our original paper[26] can be considered to report the only study that has derived the NLS equation by focusing on the moderately high frequency band. Thereafter, we extended our work to diffractive beams with a nonuniform number density,[29]

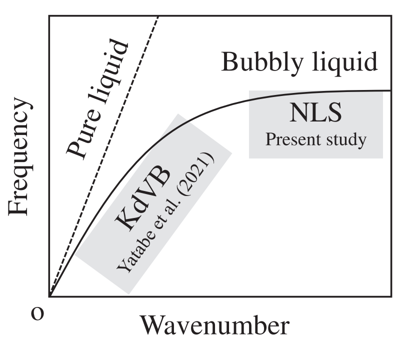

FIG. 1. Conceptual diagram of the linear dispersion relation for pressure waves in bubbly liquids.[11,12] The KdVB equation was derived for low-frequency long waves, and the NLS equation was derived for (moderately) high-frequency short waves.[26,28–30] Our previous paper[36] focused on the KdVB equation.

bubbly flows with nonuniform initial flow velocities,[30] and polydisperse bubbly liquids.[33] However, owing to the aforementioned preconception, our group[24–34] has previously ignored the drag force and translation. Although there have been studies where the linear wave propagation based on a two-fluid model incorporated translation[21] and the numerical analyses incorporated translation and drag force,[35–37] derivation of nonlinear wave equations based on a two-fluid model has not incorporated the drag force and translation. The importance of bubble translation was highlighted by a numerical result that revealed that bubble translation (bubble slip) slightly affected the waveforms.[35]

The study conducted by Yatabe *et al.*,[38] which was our previous work, was the first attempt at a consistent consideration of the translation, drag force, and initial flow velocity; in this work, we derived the KdVB equation for low-frequency waves and successfully indicated that the translation increased the nonlinearity of pressure waves, and that the drag force increased the dissipation of waves. The purpose of this study is to extend the work performed previously to the moderately high frequency case, that is, the derivation of the NLS equation. The remainder of this paper is organized as follows. In Sec. II, the basic equations based on a two-fluid model including the drag force, and bubble dynamics equation including the translation, are introduced. In Sec. III, we derive the NLS equation and indicate that the translation increases the nonlinearity, and the drag force increases the dissipation. We further discuss the four factors of dissipation, that is, the liquid viscosity, acoustic radiation (i.e., liquid compressibility), thermal conduction,[13,39] and drag force.[38] Section IV is devoted to the conclusions of the study.

II. FORMULATION OF THE PROBLEM

A. Problem statement

This study theoretically investigates the weakly nonlinear (i.e., finite but small-amplitude) propagation of one-dimensional (plane) pressure progressive waves in flowing compressible water that uniformly contain many small spherical gas bubbles. Initially, the gas and liquid phases flow with independent constant velocities. We newly introduce a drag force as the force acting on the bubbles and a translation as the bubble dynamics. The bubbles do not coalesce, breakup, disappear, or appear.[24–34,38] For simplicity, the gas viscosity, the Reynolds stress, and the phase change and mass transport across the bubble–liquid interface,[40] are ignored.

Although these assumptions are the same as those in our previous work,[38] in this study, we focus on the short wave in a moderately high-frequency band (Fig. 1). Furthermore, the thermal conduction at bubble–liquid interface is considered[31] based on Prosperetti's model,[13] along with the temperature gradient model, where the temperature of the liquid phase is assumed to be constant.

B. Governing equations

As in our previous work,[38] to introduce the drag force in the interfacial momentum transport, we first utilize the conservation laws of mass and momentum for the gas and liquid phases based on a two-fluid model,[24,34]

$$\frac{\partial}{\partial t^*}(\alpha \rho_G^*) + \frac{\partial}{\partial x^*}(\alpha \rho_G^* u_G^*) = 0, \tag{1}$$

$$\frac{\partial}{\partial t^*}\left[(1-\alpha)\rho_{\rm L}^*\right]+\frac{\partial}{\partial x^*}\left[(1-\alpha)\rho_{\rm L}^* u_{\rm L}^*\right]=0, \tag{2}$$

$$\frac{\partial}{\partial t^*}(\alpha\rho_{\rm G}^* u_{\rm G}^*)+\frac{\partial}{\partial x^*}(\alpha\rho_{\rm G}^* u_{\rm G}^{*2})+\alpha\frac{\partial p_{\rm G}^*}{\partial x^*}=F^*+D^*, \tag{3}$$

$$\frac{\partial}{\partial t^*}\left[(1-\alpha)\rho_{\rm L}^* u_{\rm L}^*\right]+\frac{\partial}{\partial x^*}\left[(1-\alpha)\rho_{\rm L}^* u_{\rm L}^{*2}\right] + (1-\alpha)\frac{\partial p_{\rm L}^*}{\partial x^*}+P^*\frac{\partial \alpha}{\partial x^*}=-F^*-D^*, \tag{4}$$

where t^* is the time, x^* is the space coordinate, α is the void fraction ($0<\alpha<1$), ρ^* is the density, u^* is the velocity, p^* is the pressure, and P^* is the liquid pressure averaged on the bubble–liquid interface;[24] the subscripts G and L denote volume-averaged variables in the gas and liquid phases, respectively; the superscript $*$ denotes a dimensional quantity. The following model of virtual mass force[25,34] is introduced as the interfacial momentum transport F^*:

$$F^*=-\beta_1\alpha\rho_{\rm L}^*\left(\frac{{\rm D_G}u_{\rm G}^*}{{\rm D}t^*}-\frac{{\rm D_L}u_{\rm L}^*}{{\rm D}t^*}\right) -\beta_2\rho_{\rm L}^*(u_{\rm G}^*-u_{\rm L}^*)\frac{{\rm D_G}\alpha}{{\rm D}t^*}-\beta_3\alpha(u_{\rm G}^*-u_{\rm L}^*)\frac{{\rm D_G}\rho_{\rm L}^*}{{\rm D}t^*}, \tag{5}$$

where β_1, β_2, and β_3 are constants, and may be set as 1/2 for the spherical bubble. Lagrange derivatives ${\rm D_G}/{\rm D}t^*$ and ${\rm D_L}/{\rm D}t^*$ are defined as

$$\frac{\rm D_G}{{\rm D}t^*}\equiv\frac{\partial}{\partial t^*}+u_{\rm G}^*\frac{\partial}{\partial x^*},\qquad \frac{\rm D_L}{{\rm D}t^*}\equiv\frac{\partial}{\partial t^*}+u_{\rm L}^*\frac{\partial}{\partial x^*}. \tag{6}$$

Furthermore, we introduce a model that incorporates the drag force term for spherical bubbles, D^*,[38]

$$D^*=-\frac{3}{8R^*}\alpha C_{\rm D}\rho_{\rm L}^*(u_{\rm G}^*-u_{\rm L}^*)|u_{\rm G}^*-u_{\rm L}^*|, \tag{7}$$

where R^* is the radius of a representative bubble and $C_{\rm D}$ is the drag coefficient for a single spherical bubble.

The spherically symmetric oscillations of bubbles in compressible water can be expressed as

$$\left(1-\frac{1}{c_{\rm L0}^*}\frac{{\rm D_G}R^*}{{\rm D}t^*}\right)R^*\frac{{\rm D_G^2}R^*}{{\rm D}t^{*2}}+\frac{3}{2}\left(1-\frac{1}{3c_{\rm L0}^*}\frac{{\rm D_G}R^*}{{\rm D}t^*}\right)\left(\frac{{\rm D_G}R^*}{{\rm D}t^*}\right)^2 =\left(1+\frac{1}{c_{\rm L0}^*}\frac{{\rm D_G}R^*}{{\rm D}t^*}\right)\frac{P^*}{\rho_{\rm L0}^*}+\frac{R^*}{\rho_{\rm L0}^*c_{\rm L0}^*}\frac{\rm D_G}{{\rm D}t^*}(p_{\rm L}^*+P^*)+\frac{(u_{\rm G}^*-u_{\rm L}^*)^2}{4}. \tag{8}$$

We introduced the translation of bubbles[21,36,37] in the third term on the right-hand side of (8) into the Keller equation.[41] Hence, the translation and volumetric oscillations can be described using (8) as a linear combination. Equation (8) allows us to treat the translation and spherically symmetric oscillations as the bubble dynamics.

In this study, Prosperetti's equation[13] for thermal conduction at the bubble–liquid interface is also introduced to express the thermal effect inside bubbles,[31]

$$\frac{{\rm D}p_{\rm G}^*}{{\rm D}t^*}=\frac{3}{R^*}\left[(\kappa-1)\lambda_{\rm G}^*\left.\frac{\partial T_{\rm G}^*}{\partial r^*}\right|_{r^*=R^*}-\kappa p_{\rm G}^*\frac{{\rm D}R^*}{{\rm D}t^*}\right], \tag{9}$$

where $T_{\rm G}^*$ is the temperature of the gas phase, κ is the ratio of specific heats, r^* is the radial distance from the center of the bubble, and $\lambda_{\rm G}^*$ is the thermal conductivity of the gas inside the bubble. Some models[31,42–45] have proposed the use of the temperature-gradient as the first term on the right-hand side of (9); herein, we utilize the following model by Sugiyama *et al.*:[45]

$$\left.\frac{\partial T_{\rm G}^*}{\partial r^*}\right|_{r^*=R^*}=\frac{{\rm Re}(\widetilde{L_{\rm P}^*})(T_0^*-T_{\rm G}^*)}{|\widetilde{L_{\rm P}^*}|^2}+\frac{{\rm Im}(\widetilde{L_{\rm P}^*})}{\omega_{\rm B}^*|\widetilde{L_{\rm P}^*}|^2}\frac{{\rm D_G}T_{\rm G}^*}{{\rm D}t^*}, \tag{10}$$

where T_0^* is the initial temperature; Re and Im denote the real and imaginary parts, respectively; noting that the physical quantities in the initial state are denoted by the subscript 0, and these are constants. Furthermore, some symbols are defined as follows:[31]

$$\omega_{\rm B}^*=\sqrt{\frac{3\gamma_{\rm e}(p_{\rm L0}^*+2\sigma^*/R_0^*)-2\sigma^*/R_0^*}{\rho_{\rm L0}^*R_0^{*2}}-\left(\frac{2\mu_{\rm e0}^*}{\rho_{\rm L0}^*R_0^{*2}}\right)^2}, \tag{11}$$

$$\gamma_{\rm e}={\rm Re}\left(\frac{\Gamma_{\rm N}}{3}\right), \tag{12}$$

$$\mu_{\rm e0}^*=\mu_{\rm L}^*+{\rm Im}\left(\frac{p_{\rm G0}^*\Gamma_{\rm N}}{4\omega_{\rm B}^*}\right), \tag{13}$$

where $\omega_{\rm B}^*$ is the eigenfrequency of a single bubble, $\gamma_{\rm e}$ is the effective polytropic exponent, σ^* is the surface tension, $\mu_{\rm e0}^*$ is the initial effective viscosity, and $\mu_{\rm L}^*$ is the liquid viscosity [see also (41) and (42) in Sec. II D]; the explicit form of (11) is different from that used in our previous studies.[26–30,34,38] Complex numbers $\Gamma_{\rm N}$ and $\alpha_{\rm N}$ are as follows:[45]

$$\Gamma_{\rm N}=\frac{3\alpha_{\rm N}^2\kappa}{\alpha_{\rm N}^2+3(\kappa-1)(\alpha_{\rm N}\coth\alpha_{\rm N}-1)}, \tag{14}$$

$$\alpha_{\rm N}=\sqrt{\frac{\kappa\omega_{\rm B}^*p_{\rm G0}^*R_0^{*2}}{2(\kappa-1)T_0^*\lambda_{\rm G}^*}}(1+{\rm i}), \tag{15}$$

where i denotes the imaginary unit. The complex number $\widetilde{L_{\rm P}^*}$ in (10), which represents the dimensions of length, is given by

$$\widetilde{L_{\rm P}^*}=\frac{R_0^*(\alpha_{\rm N}^2-3\alpha_{\rm N}\coth\alpha_{\rm N}+3)}{\alpha_{\rm N}^2(\alpha_{\rm N}\coth\alpha_{\rm N}-1)}. \tag{16}$$

To close the set of (1)–(4), (8), and (9), we introduce the following equations: the equation of state for ideal gas (the polytropic equation of state was used for the previous long wave[38]), the Tait equation of state for liquid, the conservation law of mass inside the bubble, and the balance of normal stresses across the bubble–liquid interface:

$$\frac{p_{\rm G}^*}{p_{\rm G0}^*}=\frac{\rho_{\rm G}^*}{\rho_{\rm G0}^*}\frac{T_{\rm G}^*}{T_0^*}, \tag{17}$$

$$p_{\rm L}^*=p_{\rm L0}^*+\frac{\rho_{\rm L0}^*c_{\rm L0}^{*2}}{n}\left[\left(\frac{\rho_{\rm L}^*}{\rho_{\rm L0}^*}\right)^n-1\right], \tag{18}$$

$$\frac{\rho_{\rm G}^*}{\rho_{\rm G0}^*}=\left(\frac{R_0^*}{R^*}\right)^3, \tag{19}$$

$$p_{\rm G}^*-(p_{\rm L}^*+P^*)=\frac{2\sigma^*}{R^*}+\frac{4\mu_{\rm L}^*}{R^*}\frac{{\rm D_G}R^*}{{\rm D}t^*}, \tag{20}$$

where n is a material constant (e.g., $n=7.15$ for water). It should be noted that the effect of liquid viscosity is considered only at the bubble–liquid interface, as shown in (20).

C. Multiple-scale analysis

First, the independent variables are nondimensionalized as

$$t \equiv \frac{t^*}{T^*}, \quad x \equiv \frac{x^*}{L^*}, \tag{21}$$

where T^* is a typical period of waves and L^* is a typical wavelength. Based on the method of multiple scales,[10] six multiple scales[26,28–30,33,34] as extended independent variables are then introduced using the finite but small nondimensional wave amplitude ε ($\ll 1$),

$$t_0 = t, \quad x_0 = x \quad (\text{near field}), \tag{22}$$

$$t_1 = \varepsilon t, \quad x_1 = \varepsilon x \quad (\text{far field I}), \tag{23}$$

$$t_2 = \varepsilon^2 t, \quad x_2 = \varepsilon^2 x \quad (\text{far field II}). \tag{24}$$

Here, the subscripts 0, 1, and 2 correspond to the near field, far field I, and far field II, respectively.[10] The differential operators are immediately expanded as[10]

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1} + \varepsilon^2\frac{\partial}{\partial t_2}, \tag{25}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_0} + \varepsilon\frac{\partial}{\partial x_1} + \varepsilon^2\frac{\partial}{\partial x_2}. \tag{26}$$

Note that far field II was not used in the previous study for the low-frequency long wave.[38]

The dependent variables are now regarded as functions of the extended independent variables in (22)–(24). They are converted to be dimensionless and expanded in the power series of ε,

$$R^*/R_0^* = 1 + \varepsilon R_1 + \varepsilon^2 R_2 + O(\varepsilon^3), \tag{27}$$

$$u_G^*/U^* = u_{G0} + \varepsilon u_{G1} + \varepsilon^2 u_{G2} + O(\varepsilon^3), \tag{28}$$

$$u_L^*/U^* = u_{L0} + \varepsilon u_{L1} + \varepsilon^2 u_{L2} + O(\varepsilon^3), \tag{29}$$

$$\alpha/\alpha_0 = 1 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + O(\varepsilon^3), \tag{30}$$

$$\rho_L^*/\rho_{L0}^* = 1 + \varepsilon^5\rho_{L2} + \varepsilon^6\rho_{L3} + O(\varepsilon^7), \tag{31}$$

$$p_L^*/(\rho_{L0}^* U^{*2}) = p_{L0} + \varepsilon p_{L1} + \varepsilon^2 p_{L2} + O(\varepsilon^3), \tag{32}$$

$$T_G^*/T_0^* = 1 + \varepsilon T_{G1} + \varepsilon^2 T_{G2} + O(\varepsilon^3), \tag{33}$$

where U^* ($\equiv L^*/T^*$) is a typical propagation speed, and the initial nondimensional pressures p_{L0} and p_{G0} are defined as

$$p_{L0} \equiv \frac{p_{L0}^*}{\rho_{L0}^* U^{*2}} \equiv O(1), \quad p_{G0} \equiv \frac{p_{G0}^*}{\rho_{L0}^* U^{*2}} \equiv O(1). \tag{34}$$

It should be noted that the expansion of the liquid density starts with $O(\varepsilon^5)$[26,28–30,33,34] and started with $O(\varepsilon^2)$ in the previous study on long wave.[38] The ratio of the initial densities of the gas and liquid phases is

$$\frac{\rho_{G0}^*}{\rho_{L0}^*} \equiv O(\varepsilon^3), \tag{35}$$

and hence, the density ratio can be excluded.[26–34,38]

In contrast to our previous study, wherein we treat the low-frequency long wave,[38] herein, we focus on the moderately high-frequency short wave. The set of sizes of the three nondimensional ratios is then determined using ε,[26,28–30]

$$\left(\frac{U^*}{c_{L0}^*}, \frac{R_0^*}{L^*}, \frac{\omega^*}{\omega_B^*}\right) = (O(\varepsilon^2), O(1), O(1)) = \left(V\varepsilon^2, \Delta, \Omega\right), \tag{36}$$

where V, Δ, and Ω are the constants of $O(1)$; the right-hand side was chosen as $\left(V\sqrt{\varepsilon}, \Delta\sqrt{\varepsilon}, \Omega\sqrt{\varepsilon}\right)$ in the previous study on long wave.[38] That is, the speed of sound in bubbly flows is considerably smaller than that in pure water, the initial radius of a bubble is comparable to the typical wavelength, and the incident frequency of waves is also comparable to the eigenfrequency of a single bubble. Although the method of averaged equations should not be ideally applied to such short waves, the plane wave problem can be excluded from the restriction because the average volume can be sufficiently large along the plane parallel to the wave front.[26] Nevertheless, the assumption of spherical symmetry of bubble oscillations should be validated.[26] We will address this problem in a future work.

Equation (9) is nondimensionalized as[31]

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[T_G R^{3(\kappa-1)}\right] = R^{3\kappa-1}\frac{3(\kappa-1)\lambda_G^*}{p_{G0}^*\omega_B^* R_0^*}\left.\frac{\partial T^*}{\partial r^*}\right|_{r^*=R^*}, \tag{37}$$

where $T = T_G^*/T_0^*$ and $R = R^*/R_0^*$. The nondimensionalized expression on the right-hand side of (37) is

$$(\text{RHS}) = \frac{3(\kappa-1)\lambda_G^*}{p_{G0}^*\omega^* R_0^*}R^{3\kappa-1}\left[\frac{\mathrm{Re}(\widetilde{L_P^*})T_0^*}{|\widetilde{L_P^*}|^2}(1-T_G) + \frac{\omega^*\mathrm{Im}(\widetilde{L_P^*})T_0^*}{\omega_B^*|\widetilde{L_P^*}|^2}\frac{\mathrm{D}T_G}{\mathrm{D}t}\right]. \tag{38}$$

Then, we determine the sizes of the nondimensional number using ε,

$$\frac{3(\kappa-1)\lambda_G^*}{p_{G0}^*\omega^* R_0^*}\frac{\mathrm{Re}(\widetilde{L_P^*})T_0^*}{|\widetilde{L_P^*}|^2} \equiv \zeta_1\varepsilon^2, \tag{39}$$

$$\frac{3(\kappa-1)\lambda_G^*}{p_{G0}^*\omega^* R_0^*}\frac{\omega^*\mathrm{Im}(\widetilde{L_P^*})T_0^*}{\omega_B^*|\widetilde{L_P^*}|^2} \equiv \zeta_2\varepsilon^3, \tag{40}$$

where ζ_1 and ζ_2 are constants of $O(1)$; ζ_2 does not affect the present result.

D. Drag force

Liquid viscosity μ_L^* and effective viscosity μ_{e0}^* are nondimensionalized as

$$\frac{\mu_L^*}{\rho_{L0}^* U^* L^*} \equiv O(\varepsilon^2) \equiv \mu_L\varepsilon^2, \tag{41}$$

$$\frac{\mu_{e0}^*}{\rho_{L0}^* U^* L^*} \equiv O(\varepsilon^2) \equiv \mu_{e0}\varepsilon^2, \tag{42}$$

where μ_L and μ_{e0} are constants of $O(1)$.

Drag coefficient C_D is then defined as

$$C_D \equiv \frac{8\mu_L^*}{|u_G^* - u_L^*|\rho_L^* R^*}, \tag{43}$$

where C_D depends on Reynolds number Re ($C_D = 16/\mathrm{Re}$).[2] The gas viscosity and Reynolds stress induced by volume averaging are ignored. However, for simplicity, we use a Stokes-type drag force. We intend to validate these assumptions in our forthcoming work.

III. RESULTS

A. Linear propagation of carrier wave at near field

Substituting (25)–(43) into (1)–(4), (8), and (9), the set of linear equations is derived with the aid of (17)–(20) from the leading order of approximation,

$$\frac{\mathrm{D}\alpha_1}{\mathrm{D}t_0} - 3\frac{\mathrm{D}R_1}{\mathrm{D}t_0} + \frac{\partial u_{\mathrm{G1}}}{\partial x_0} = 0, \tag{44}$$

$$\alpha_0 \frac{\mathrm{D}\alpha_1}{\mathrm{D}t_0} - (1-\alpha_0)\frac{\partial u_{\mathrm{L1}}}{\partial x_0} = 0, \tag{45}$$

$$\beta_1\left(\frac{\mathrm{D}u_{\mathrm{G1}}}{\mathrm{D}t_0} - \frac{\mathrm{D}u_{\mathrm{L1}}}{\mathrm{D}t_0}\right) + p_{\mathrm{G0}}\left(\frac{\partial T_{\mathrm{G1}}}{\partial x_0} - 3\frac{\partial R_1}{\partial x_0}\right) = 0, \tag{46}$$

$$(1-\alpha_0)\frac{\mathrm{D}u_{\mathrm{L1}}}{\mathrm{D}t_0} - \alpha_0\beta_1\left(\frac{\mathrm{D}u_{\mathrm{G1}}}{\mathrm{D}t_0} - \frac{\mathrm{D}u_{\mathrm{L1}}}{\mathrm{D}t_0}\right) - \alpha_0 u_0 \frac{\mathrm{D}\alpha_1}{\mathrm{D}t_0} + u_0(1-\alpha_0)\frac{\partial u_{\mathrm{L1}}}{\partial x_0} + (1-\alpha_0)\frac{\partial p_{\mathrm{L1}}}{\partial x_0} = 0, \tag{47}$$

$$\frac{\mathrm{D}^2 R_1}{\mathrm{D}t_0^2} - \left[\frac{3(\gamma_{\mathrm{e}}-1)p_{\mathrm{G0}}}{\Delta^2} - 1\right]R_1 - \frac{p_{\mathrm{G0}}}{\Delta^2}T_{\mathrm{G1}} + \frac{p_{\mathrm{L1}}}{\Delta^2} = 0, \tag{48}$$

$$3(\kappa-1)\frac{\mathrm{D}R_1}{\mathrm{D}t_0} + \frac{\mathrm{D}T_{\mathrm{G1}}}{\mathrm{D}t_0} = 0. \tag{49}$$

Here, all the partial derivatives with respect to t_0 (i.e., operator $\partial/\partial t_0$) in our original paper[26] varied with Lagrange derivative $\mathrm{D}/\mathrm{D}t_0$ owing to consideration of the initial constant flow velocity, u_0,

$$\frac{\mathrm{D}}{\mathrm{D}t_0} \equiv \frac{\partial}{\partial t_0} + u_0\frac{\partial}{\partial x_0}. \tag{50}$$

It should be noted that the initial velocities of both phases are assumed to be the same ($u_{\mathrm{G0}} = u_{\mathrm{L0}} \equiv u_0$) for simplicity; however, the perturbations of each velocity are not the same ($u_{\mathrm{G1}} \neq u_{\mathrm{L1}}$ and $u_{\mathrm{G2}} \neq u_{\mathrm{L2}}$).

Combining (44)–(48) into a single equation for the first-order variation of bubble radius, R_1, we obtain the fourth-order linear partial differential equation,

$$\mathscr{L}_1[R_1] \equiv \frac{\mathrm{D}^2 R_1}{\mathrm{D}t_0^2} - \frac{3p_{\mathrm{G0}}\{[\kappa(1-\alpha_0)+\beta_1]\alpha_0 + \beta_1(\kappa-\gamma_{\mathrm{e}})\} + \beta_1\Delta^2(1-\alpha_0)}{3\beta_1\alpha_0(1-\alpha_0)} \times \frac{\partial^2 R_1}{\partial x_0^2} - \frac{\Delta^2}{3\alpha_0}\frac{\partial^2}{\partial x_0^2}\left(\frac{\mathrm{D}^2 R_1}{\mathrm{D}t_0^2}\right) = 0, \tag{51}$$

where $\mathscr{L}_1$ denotes the linear differential operator. Equation (51) is the linear wave equation with the dispersion term (i.e., third term) owing to bubble oscillations. In the previous study on long wave,[38] (i) the fourth order derivative in (51) [i.e., the second order derivative in (48)] did not appear as the dispersion effect was weak and (ii) the thermal conduction was discarded for simplicity.

Owing to a strong dispersion effect, the wave profile is broken down into each component with its own propagation speed if an initial wave is a superposition of different harmonic components. Hence, we consider a solution of (51) in the form of a quasi-monochromatic wave train evolving into a slowly modulated wave packet,[10,26]

$$R_1 = A(t_1, x_1; t_2, x_2)\mathrm{e}^{\mathrm{i}\theta} + \mathrm{c.c.}, \quad \theta \equiv kx_0 - \Omega t_0, \tag{52}$$

where A is the slowly varying complex amplitude depending on only the slow scales and is a constant in the near field, characterized by t_0 and x_0; $k\,(\equiv k^* L^*)$ is the nondimensional wavenumber (k^* is the dimensional wavenumber); and c.c. denotes the complex conjugate. Then, $\mathrm{e}^{\mathrm{i}\theta}$ corresponds to the carrier wave and A to the envelope wave.[10] Here, we focus on only the right-running carrier wave. By substituting (52) into (44)–(48) and integrating them with respect to t_0 and x_0, the other first-order variations, i.e., α_1, u_{G1}, u_{L1}, p_{L1}, and T_{G1}, can be expressed in terms of R_1,

$$\alpha_1 = b_1 R_1, \quad u_{\mathrm{G1}} = b_2 R_1, \quad u_{\mathrm{L1}} = b_3 R_1, \quad p_{\mathrm{L1}} = b_4 R_1, \quad T_{\mathrm{G1}} = b_5 R_1 \tag{53}$$

with

$$b_1 = \frac{1-\alpha_0}{\alpha_0(1-\alpha_0+\beta_1)}\left[3\alpha_0\beta_1 - \frac{(1-\alpha_0)b_4}{v_{\mathrm{p}}^2}\right],$$
$$b_2 = v_{\mathrm{p}}(b_1 - 3), \quad b_3 = -v_{\mathrm{p}}\frac{\alpha_0 b_1}{1-\alpha_0}, \tag{54}$$
$$b_4 = \Delta^2(v_{\mathrm{p}}^2 k^2 - 1) - 3(\kappa-\gamma_{\mathrm{e}})p_{\mathrm{G0}},$$
$$b_5 = -3(\kappa-1),$$

where nondimensional phase velocity v_{p} is obtained in (56).

Let us discuss the linear dispersion relation of the implicit form, $D(k,\Omega)$, including the calculation of phase velocity v_{p} and group velocity v_{g}. Substitution of (52) into (51) yields

$$D(k,\Omega) \equiv \frac{\Delta^2 k^2}{3\alpha_0}\left[1 - (\Omega - u_0 k)^2\right] - (\Omega - u_0 k)^2 + \frac{[\beta_1 + \kappa(1-\alpha_0)]\alpha_0 + \beta_1(\kappa-\gamma_{\mathrm{e}})}{\beta_1\alpha_0(1-\alpha_0)}p_{\mathrm{G0}}k^2 = 0, \tag{55}$$

or

$$\Omega = u_0 k + k\sqrt{\frac{\Delta^2}{3\alpha_0 + \Delta^2 k^2} + \frac{3p_{\mathrm{G0}}\{[\kappa(1-\alpha_0)+\beta_1]\alpha_0 + \beta_1(\kappa-\gamma_{\mathrm{e}})\}}{\beta_1(1-\alpha_0)(3\alpha_0+\Delta^2 k^2)}}. \tag{56}$$

The phase and group velocities are calculated as

$$V_{\mathrm{p}} \equiv \frac{\Omega}{k} = v_{\mathrm{p}} + u_0,$$
$$v_{\mathrm{p}} \equiv \sqrt{\frac{\Delta^2}{3\alpha_0 + \Delta^2 k^2} + \frac{3p_{\mathrm{G0}}\{[\beta_1 + \kappa(1-\alpha_0)]\alpha_0 + \beta_1(\kappa-\gamma_{\mathrm{e}})\}}{\beta_1\alpha_0(1-\alpha_0)(3\alpha_0+\Delta^2 k^2)}}, \tag{57}$$

$$V_{\mathrm{g}} \equiv \frac{\mathrm{d}\Omega}{\mathrm{d}k} = -\frac{\partial D/\partial k}{\partial D/\partial\Omega} = v_{\mathrm{g}} + u_0, \quad v_{\mathrm{g}} \equiv \frac{3\alpha_0 v_{\mathrm{p}}}{3\alpha_0 + \Delta^2 k^2}. \tag{58}$$

The explicit form of typical phase velocity U^* should be determined to obtain the coefficients depicted in Figs. 2–5 in Secs. III D and III E. As the nondimensional phase velocity v_{p} is defined by U^*, the choice of v_{p} determines U^*. Then, we put $v_{\mathrm{p}} \equiv 1$ under $\Omega \equiv 1$ (i.e., $k \equiv 1$); this leads to

$$U^* \equiv \sqrt{\frac{\{[\beta_1 + \kappa(1-\alpha_0)]\alpha_0 + \beta_1(\kappa-\gamma_{\mathrm{e}})\}p_{\mathrm{G0}}^*}{\beta_1\alpha_0(1-\alpha_0)\rho_{\mathrm{L0}}^*}}. \tag{59}$$

Although the effect of the initial flow velocity can be observed, those of the translation and drag force do not appear in the near field. For the case without the initial flow velocity (i.e., the initially quiescent case, $u_0 = 0$), $\mathrm{D}/\mathrm{D}t_0$ changes $\partial/\partial t_0$, and the result in the present study coincides with that in our original study.[26]

B. Linear propagation of envelope wave at far field I

As in the case of $O(\varepsilon)$, the following set of inhomogeneous equations of $O(\varepsilon^2)$ is derived:

$$\frac{\mathrm{D}\alpha_2}{\mathrm{D}t_0} - 3\frac{\mathrm{D}R_2}{\mathrm{D}t_0} + \frac{\partial u_{\mathrm{G}2}}{\partial x_0} = M_1, \tag{60}$$

$$\alpha_0\frac{\mathrm{D}\alpha_2}{\mathrm{D}t_0} - (1-\alpha_0)\frac{\partial u_{\mathrm{L}2}}{\partial x_0} = M_2, \tag{61}$$

$$\beta_1\left(\frac{\mathrm{D}u_{\mathrm{G}2}}{\mathrm{D}t_0} - \frac{\mathrm{D}u_{\mathrm{L}2}}{\mathrm{D}t_0}\right) - 3p_{\mathrm{G}0}\frac{\partial R_2}{\partial x_0} + p_{\mathrm{G}0}\frac{\partial T_{\mathrm{G}2}}{\partial x_0} = M_3, \tag{62}$$

$$\begin{aligned}(1-\alpha_0)\frac{\mathrm{D}u_{\mathrm{L}2}}{\mathrm{D}t_0} - \alpha_0\beta_1\left(\frac{\mathrm{D}u_{\mathrm{G}2}}{\mathrm{D}t_0} - \frac{\mathrm{D}u_{\mathrm{L}2}}{\mathrm{D}t_0}\right) - \alpha_0 u_0\frac{\mathrm{D}\alpha_2}{\mathrm{D}t_0}\\ + u_0(1-\alpha_0)\frac{\partial u_{\mathrm{L}2}}{\partial x_0} + (1-\alpha_0)\frac{\partial p_{\mathrm{L}2}}{\partial x_0} = M_4,\end{aligned} \tag{63}$$

$$\frac{\mathrm{D}^2R_2}{\mathrm{D}t_0^2} - \left[\frac{3(\gamma_{\mathrm{e}}-1)p_{\mathrm{G}0}}{\Delta^2} - 1\right]R_2 - \frac{p_{\mathrm{G}0}}{\Delta^2}T_{\mathrm{G}2} + \frac{p_{\mathrm{L}2}}{\Delta^2} = M_5, \tag{64}$$

$$3(\kappa-1)\frac{\mathrm{D}R_2}{\mathrm{D}t_0} + \frac{\mathrm{D}T_{\mathrm{G}2}}{\mathrm{D}t_0} = M_6. \tag{65}$$

Equations (60)–(64) are then combined into a single inhomogeneous equation,

$$\mathscr{L}_1[R_2] = M(R_1), \tag{66}$$

where

$$\begin{aligned}M = &-\frac{1}{3}\frac{\mathrm{D}M_1}{\mathrm{D}t_0} + \frac{1}{3\alpha_0}\frac{\mathrm{D}M_2}{\mathrm{D}t_0} + \frac{u_0}{3\alpha_0(1-\alpha_0)}\frac{\partial M_2}{\partial x_0}\\ &+ \frac{1-\alpha_0+\beta_1}{3(1-\alpha_0)\beta_1}\frac{\partial M_3}{\partial x_0} + \frac{1}{3\alpha_0(1-\alpha_0)}\frac{\partial M_4}{\partial x_0}\\ &- \frac{\Delta^2}{3\alpha_0}\frac{\partial^2 M_5}{\partial x_0^2} - \frac{[\beta_1+\alpha_0(1-\alpha_0)]p_{\mathrm{G}0}}{3\beta_1\alpha_0(1-\alpha_0)}\int\frac{\partial^2 M_6}{\partial x_0^2}\mathrm{d}t_0.\end{aligned} \tag{67}$$

Introducing (52) and (53) into (67) rewrites the single inhomogeneous term M,

$$M = \Gamma A^2 \mathrm{e}^{2\mathrm{i}\theta} + \mathrm{i}\left(-\frac{\partial D}{\partial\Omega}\right)\left[\frac{\partial A}{\partial t_1} + (v_{\mathrm{g}}+u_0)\frac{\partial A}{\partial x_1}\right]\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.}, \tag{68}$$

where real constant Γ is

$$\begin{aligned}\Gamma = -\frac{2}{3}\Bigg[&v_{\mathrm{p}}k m_1 - \frac{v_{\mathrm{p}}k m_2}{\alpha_0} + \frac{1-\alpha_0+\beta_1}{(1-\alpha_0)\beta_1}k m_3 + \frac{k m_4}{\alpha_0(1-\alpha_0)}\\ &-\frac{2\Delta^2k^2m_5}{\alpha_0} - \frac{\beta_1+\alpha_0(1-\alpha_0)}{\beta_1\alpha_0(1-\alpha_0)\Omega}p_{\mathrm{G}0}k^2m_6\Bigg]\end{aligned} \tag{69}$$

with real constants m_i $(i = 1, 2, 3, 4, 5, 6)$,

$$m_1 = -6(b_1-2)v_{\mathrm{p}}k - 2b_2(b_1-3)k, \tag{70}$$

$$m_2 = -2\alpha_0 b_1 b_3 k, \tag{71}$$

$$m_3 = \hat{m} + p_{\mathrm{G}0}(3b_1 - b_1b_5 + 6b_5 - 12)k, \tag{72}$$

$$\begin{aligned}m_4 = &-\alpha_0\hat{m} - 2(1-\alpha_0)b_3^2k + \alpha_0 b_1 b_4 k - 2\alpha_0 b_1 b_3 v_{\mathrm{p}}k\\ &+ \alpha_0 b_1\left[-3(\gamma_{\mathrm{e}}-\kappa)p_{\mathrm{G}0} + \Delta^2\right]k,\end{aligned} \tag{73}$$

$$\hat{m} = (\beta_1+\beta_2)(b_2-b_3)b_1v_{\mathrm{p}}k - \beta_1(b_2^2-b_3^2)k, \tag{74}$$

$$m_5 = 1 - 3b_2v_{\mathrm{p}}k^2 + \frac{3(2-\gamma_{\mathrm{e}}-b_5)}{2\Delta^2}p_{\mathrm{G}0} + \frac{5}{2}v_{\mathrm{p}}^2k^2 + \frac{(b_3-b_2)^2}{4\Delta^2}, \tag{75}$$

$$m_6 = 2v_{\mathrm{p}}b_5^2k + 6\left[3\kappa - 2 - \frac{\kappa(3\kappa-1)}{2}\right], \tag{76}$$

where m_5 increases from the original coefficient[26] owing to the appearance of the fifth term in the right-hand side of (75), induced by the effect of translation, and m_6 is obtained owing to the consideration of the thermal effect.

The solvability condition for (67) requires that the coefficient of $\mathrm{e}^{\mathrm{i}\theta}$ should be zero;[10,26] then, we obtain the linear wave equation for envelope wave A,[30]

$$\frac{\partial A}{\partial t_1} + (v_{\mathrm{g}}+u_0)\frac{\partial A}{\partial x_1} = 0. \tag{77}$$

Substituting (77) into (68) simplifies (67) into

$$\mathscr{L}_1[R_2] = \Gamma A^2\mathrm{e}^{2\mathrm{i}\theta} + \text{c.c.}, \tag{78}$$

and its solution is obtained as follows:[26]

$$R_2 = c_0 A^2 \mathrm{e}^{2\mathrm{i}\theta} + \text{c.c.} \tag{79}$$

with

$$c_0 \equiv \frac{\Gamma}{D_{22}}, \quad D_{22} \equiv D(2k, 2\Omega) = -\frac{4\Delta^2 {v_{\mathrm{p}}}^2 k^4}{\alpha_0}. \tag{80}$$

Substituting (79) into (60)–(64) yields the explicit forms of the second-order perturbations,

$$\begin{pmatrix}\alpha_2\\ u_{\mathrm{G}2}\\ u_{\mathrm{L}2}\\ p_{\mathrm{L}2}\\ T_{\mathrm{G}2}\end{pmatrix} = \begin{pmatrix}c_1 & d_1 & 0\\ c_2 & d_2 & 0\\ c_3 & d_3 & 0\\ c_4 & d_4 & c_{\mathrm{s}}\\ c_5 & 0 & 0\end{pmatrix}\begin{pmatrix}A^2\mathrm{e}^{2\mathrm{i}\theta} + \text{c.c.}\\ \mathrm{i}(\partial A/\partial t_1)\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.}\\ |A|^2\end{pmatrix} \tag{81}$$

with real constants c_i $(i = 1, 2, 3, 4, 5)$, d_i $(i = 1, 2, 3, 4)$, and c_{s},

$$c_1 = -\frac{1}{\alpha_0 v_{\mathrm{p}}k}\left[(1-\alpha_0)c_3k + \frac{m_2}{2}\right], \tag{82}$$

$$c_2 = (c_1 - 3c_0)v_{\mathrm{p}} + \frac{m_1}{2k}, \tag{83}$$

$$c_3 = \frac{1}{v_{\mathrm{p}}}\left[c_4 - \frac{u_0m_2 + \alpha_0m_3 + m_4}{2(1-\alpha_0)k} - \frac{\alpha_0p_{\mathrm{G}0}(3c_0-c_5)}{1-\alpha_0}\right], \tag{84}$$

$$c_4 = [4v_{\mathrm{p}}^2\Delta^2k^2 + 3(\gamma_{\mathrm{e}}-1)p_{\mathrm{G}0} + c_5p_{\mathrm{G}0} - \Delta^2]c_0 + m_5\Delta^2, \tag{85}$$

$$c_5 = \frac{m_6}{2v_{\mathrm{p}}k}, \tag{86}$$

$$d_1 = -\frac{1}{\alpha_0v_{\mathrm{p}}k}\left[(1-\alpha_0)d_3k + \alpha_0b_1\frac{v_{\mathrm{g}}}{v_{\mathrm{g}}+u_0} + \frac{1-\alpha_0}{v_{\mathrm{g}}+u_0}b_3\right], \tag{87}$$

$$d_2 = v_{\mathrm{p}}d_1 - \frac{(3-b_1)v_{\mathrm{g}}}{(v_{\mathrm{g}}+u_0)k} - \frac{b_2}{(v_{\mathrm{g}}+u_0)k}, \tag{88}$$

$$d_3 = \frac{1}{v_{\mathrm{p}}}\left[d_4 - \frac{b_3v_{\mathrm{g}}}{(v_{\mathrm{g}}+u_0)k} + \frac{b_4}{(v_{\mathrm{g}}+u_0)k} - \frac{p_{\mathrm{G}0}\alpha_0(3-b_5)}{(1-\alpha_0)(v_{\mathrm{g}}+u_0)}\right], \tag{89}$$

$$d_4 = 2\nu_{\rm p}\Delta^2 \frac{v_{\rm g}}{v_{\rm g}+u_0}k, \tag{90}$$

$$c_s = 6(2-\gamma_{\rm e}-b_5)p_{\rm G0} + (2-v_{\rm p}^2k^2 - 2v_{\rm p}b_2k^2)\Delta^2 + \frac{(b_2-b_3)^2}{2}. \tag{91}$$

Here, c_1 and c_2 are the same as our original coefficients;[26] d_1, d_2, and d_3 are the same as our original and previous coefficients[26,30] without the translation and drag force; note that our original paper[26] and previous paper[30] treated without the initial flow velocities and with the initial flow velocities, respectively.

Before proceeding to the approximation of $O(\varepsilon^3)$, we note that the effect of the drag force did not appear in the approximation of $O(\varepsilon^2)$, even though that of the translation appeared.

C. Nonlinear propagation of envelope wave at far field II and resultant NLS equation

In contrast to the low-frequency long wave case,[38] the present moderately-high-frequency short wave case requires third order approximation. The set of inhomogeneous equations of $O(\varepsilon^3)$ is also derived,

$$\frac{{\rm D}\alpha_3}{{\rm D}t_0} - 3\frac{{\rm D}R_3}{{\rm D}t_0} + \frac{\partial u_{\rm G3}}{\partial x_0} = N_1, \tag{92}$$

$$\alpha_0\frac{{\rm D}\alpha_3}{{\rm D}t_0} - (1-\alpha_0)\frac{\partial u_{\rm L3}}{\partial x_0} = N_2, \tag{93}$$

$$\beta_1\left(\frac{{\rm D}u_{\rm G3}}{{\rm D}t_0} - \frac{{\rm D}u_{\rm L3}}{{\rm D}t_0}\right) - 3p_{\rm G0}\frac{\partial R_3}{\partial x_0} + p_{\rm G0}\frac{\partial T_{\rm G3}}{\partial x_0} = N_3, \tag{94}$$

$$\begin{aligned}(1-\alpha_0)\frac{{\rm D}u_{\rm L3}}{{\rm D}t_0} &- \alpha_0\beta_1\left(\frac{{\rm D}u_{\rm G3}}{{\rm D}t_0} - \frac{{\rm D}u_{\rm L3}}{{\rm D}t_0}\right) - \alpha_0u_0\frac{{\rm D_L}\alpha_3}{{\rm D}t_0}\\ &+ u_0(1-\alpha_0)\frac{\partial u_{\rm L3}}{\partial x_0} + (1-\alpha_0)\frac{\partial p_{\rm L3}}{\partial x_0} = N_4,\end{aligned} \tag{95}$$

$$\frac{{\rm D}^2R_3}{{\rm D}t_0^2} - \left[\frac{3(\gamma_{\rm e}-1)p_{\rm G0}}{\Delta^2} - 1\right]R_3 - \frac{p_{\rm G0}}{\Delta^2}T_{\rm G3} + \frac{p_{\rm L3}}{\Delta^2} = N_5, \tag{96}$$

$$3(\kappa-1)\frac{{\rm D}R_3}{{\rm D}t_0} + \frac{{\rm D}T_{\rm G3}}{{\rm D}t_0} = N_6. \tag{97}$$

Equations (92)–(96) are combined into

$$\begin{aligned}\mathscr{L}_1[R_3] = &-\frac{1}{3}\frac{{\rm D}N_1}{{\rm D}t_0} + \frac{1}{3\alpha_0}\frac{{\rm D}N_2}{{\rm D}t_0} + \frac{u_0}{3\alpha_0(1-\alpha_0)}\frac{\partial N_2}{\partial x_0}\\ &+ \frac{1-\alpha_0+\beta_1}{3(1-\alpha_0)\beta_1}\frac{\partial N_3}{\partial x_0} + \frac{1}{3\alpha_0(1-\alpha_0)}\frac{\partial N_4}{\partial x_0} - \frac{\Delta^2}{3\alpha_0}\frac{\partial^2 N_5}{\partial x_0^2}\\ &- \frac{[\beta_1+\alpha_0(1-\alpha_0)]p_{\rm G0}}{3\beta_1\alpha_0(1-\alpha_0)}\int\frac{\partial^2 N_6}{\partial x_0^2}{\rm d}t_0 = N.\end{aligned} \tag{98}$$

The inhomogeneous term N is rewritten as

$$N = \Lambda_1 e^{3{\rm i}\theta} + \Lambda_2 e^{2{\rm i}\theta} + \Lambda_3 e^{{\rm i}\theta} + \text{c.c.}, \tag{99}$$

where $\Lambda_i\ (i=1,2,3)$ are the complex variables including A,

$$\begin{aligned}\Lambda_1 &= \lambda_1 A^3, \quad \Lambda_2 = {\rm i}\lambda_2 A\frac{\partial A}{\partial x_1},\\ \Lambda_3 &= \left(-\frac{\partial D}{\partial\Omega}\right)\left\{{\rm i}\left[\frac{\partial A}{\partial t_2} + (v_{\rm g}+u_0)\frac{\partial A}{\partial x_2}\right]\right.\\ &\quad\left. + \nu_1|A|^2A + {\rm i}\nu_2 A + \nu_3\frac{\partial^2 A}{\partial x_1^2}\right\} = 0,\end{aligned} \tag{100}$$

where the explicit forms of real constants λ_1 and λ_2 are not presented as since they are not essential for the following discussion. Imposing the solvability condition for (98) and (99),[10,26] i.e., $\Lambda_3 = 0$, yields

$${\rm i}\left[\frac{\partial A}{\partial t_2} + (v_{\rm g}+u_0)\frac{\partial A}{\partial x_2}\right] + \nu_1|A|^2A + {\rm i}\nu_2A + \nu_3\frac{\partial^2 A}{\partial x_1^2} = 0. \tag{101}$$

Combining (69) and (102) with the help of (25) and (26) yields

$${\rm i}\left[\frac{\partial A}{\partial t} + (v_{\rm g}+u_0)\frac{\partial A}{\partial x}\right] + \varepsilon^2(\nu_1|A|^2A + {\rm i}\nu_2A) + \nu_3\frac{\partial^2 A}{\partial x^2} = 0. \tag{102}$$

Finally, we obtain the following NLS equation (precisely speaking, the NLS equation with a correction dissipation term, i.e., the third term on the left-hand side):

$${\rm i}\frac{\partial A}{\partial\tau} + \nu_1|A|^2A + {\rm i}\nu_2A + \nu_3\frac{\partial^2 A}{\partial\xi^2} = 0, \tag{103}$$

via a variable transform

$$\tau = \varepsilon^2 t, \quad \xi = \varepsilon\left[x - (v_{\rm g}+u_0)t\right]. \tag{104}$$

The second, third, and fourth terms on the left-hand side of (103) represent the nonlinear, dissipation, and dispersion effects, respectively. Real constants ν_1, ν_2, and ν_3 represent the sizes of the nonlinearity, dissipation, and dispersion, respectively. Dispersion coefficient ν_3 is given by

$$\nu_3 = -\frac{9\alpha_0 k v_{\rm p}\Delta^2}{2(3\alpha_0+\Delta^2k^2)^2}\left(=\frac{1}{2}\frac{{\rm d}v_{\rm g}}{{\rm d}k} = \frac{1}{2}\frac{{\rm d}V_{\rm g}}{{\rm d}k}\right) < 0, \tag{105}$$

and this coincides with the original dispersion coefficient.[26] Nonlinear coefficient ν_1 is

$$\begin{aligned}\nu_1 = \frac{1}{3}\frac{1}{\partial D/\partial\Omega}&\left[kv_{\rm p}n_1 - \frac{kv_{\rm p}}{\alpha_0}n_2 + \frac{(1-\alpha_0+\beta_1)}{(1-\alpha_0)\beta_1}kn_3 + \frac{kn_4}{\alpha_0(1-\alpha_0)}\right.\\ &\left. - \frac{\Delta^2k^2n_5}{\alpha_0} - \frac{\beta_1+\alpha_0(1-\alpha_0)}{\beta_1\alpha_0(1-\alpha_0)\Omega}p_{\rm G0}k^2n_6\right] < 0,\end{aligned} \tag{106}$$

with real constants $n_i\ (i = 1, 2, 3, 4, 5, 6)$,

$$\begin{aligned}n_1 = &-3v_{\rm p}[c_0(4-b_1) - c_1 + 6b_1 - 10]k\\ &+ [c_2(3-b_1) + b_2(3c_0 - c_1 + 9b_1 - 18)]k,\end{aligned} \tag{107}$$

$$n_2 = -\alpha_0(b_1c_3 + b_3c_1)k, \tag{108}$$

$$\begin{aligned}n_3 = \hat{n} + p_{\rm G0}[&(b_5-3)c_1 + 6c_0b_1 - 2b_1b_5\\ &+ 6b_1(b_5-2) + 3c_0b_5 + 3c_5 + 12(5-3b_1)]k,\end{aligned} \tag{109}$$

$$\begin{aligned}n_4 = &-\alpha_0\hat{n} - \alpha_0v_{\rm p}(b_3c_1 + b_1c_3)k - 2(1-\alpha_0)b_3c_3k\\ &+ \alpha_0\left[u_0b_3c_1 + 6b_1b_3^2 - b_4c_3 + b_1c_4\right.\\ &\left. - b_1(c_4 - c_s) + u_0b_1c_3 + 2c_1b_4\right]k\\ &+ \alpha_0p_{\rm G0}[b_1c_5 - 6c_1(\gamma_{\rm e}-\kappa) + 3b_1c_0(\gamma_{\rm e}-1)\\ &- 6b_1(2-\gamma_{\rm e}-b_5)]k + \alpha_0\Delta^2(2c_1 - b_1c_0 - 2b_1)k,\end{aligned} \tag{110}$$

$$\begin{aligned}\hat{n} = &(2\beta_1-\beta_2)b_1(c_2-c_3)v_{\rm p}k - (\beta_1-2\beta_2)(b_2-b_3)c_1v_{\rm p}k\\ &- b_1(b_2-b_3)[\beta_1(b_2+b_3) + \beta_2b_2]k - \beta_1(b_2c_2 - b_3c_3)k,\end{aligned} \tag{111}$$

$$\begin{aligned}n_5 = &-3 + c_0(-v_{\rm p}^2k^2 + 2) + 2b_2\left[b_2k - v_{\rm p}(1+3c_0)k\right]k\\ &+ 3p_{\rm G0}\frac{c_0(2-\gamma_{\rm e}-b_5) + 3\gamma_{\rm e} + 6b_5 - c_5 - 10}{\Delta^2}\\ &+ \frac{(b_2-b_3)(c_2-c_3)}{2\Delta^2},\end{aligned} \tag{112}$$

$$n_6 = -3v_p\left[(\kappa+3)c_0 + 9\kappa^2(\kappa-3) + 2(13\kappa-4) - (\kappa-1)c_5\right]k$$
$$\quad - 2b_2[3(\kappa-1)c_0 + c_5 - 3(\kappa-2)(3\kappa+1)]k. \qquad (113)$$

Here, the last term of n_5 is owing to the effect of translation which exhibits increase of nonlinearity compared with the original nonlinear coefficient,[26] and n_6 is obtained because the thermal effect is considered, as in m_6. Then, the absolute value of nonlinear coefficient ν_1 increases by considering the translation of bubbles; however, an analytical explanation of why this occurs is difficult (see also Sec. III D).

Dissipation coefficient ν_2 is decomposed into a linear combination,

$$\nu_2 = \nu_{2\mathrm{vis}} + \nu_{2\mathrm{ac}} + \nu_{2\mathrm{th}} + \nu_{2\mathrm{dr}} > 0, \qquad (114)$$

where the dissipation component owing to the liquid viscosity, $\nu_{2\mathrm{vis}}$, the acoustic radiation (i.e., liquid compressibility), $\nu_{2\mathrm{ac}}$, the thermal conduction, $\nu_{2\mathrm{th}}$, and drag force, $\nu_{2\mathrm{dr}}$, are given by

$$\nu_{2\mathrm{vis}} = \frac{2k^2}{3\alpha_0 + \Delta^2 k^2}\mu > 0, \qquad (115)$$

$$\nu_{2\mathrm{ac}} = \frac{\Delta^2 - 3(\gamma_e - \kappa)p_{G0}}{2(3\alpha_0 + \Delta^2 k^2)}\Delta k^2 V > 0, \qquad (116)$$

$$\nu_{2\mathrm{th}} = -\frac{1}{\partial D/\partial\Omega}\frac{\beta_1 + \alpha_0(1-\alpha_0)}{\beta_1\alpha_0(1-\alpha_0)}(\kappa-1)p_{G0}\frac{k^2}{\Omega}\zeta_1 > 0, \qquad (117)$$

$$\nu_{2\mathrm{dr}} = \frac{1}{\partial D/\partial\Omega}\frac{k}{\beta_1\Delta^2}(b_2 - b_3)\mu > 0. \qquad (118)$$

Here, $\nu_{2\mathrm{vis}}$ is the same as that in our original paper,[26] $\nu_{2\mathrm{ac}}$ is different from that in Ref. 26, and $\nu_{2\mathrm{th}}$ and $\nu_{2\mathrm{dr}}$ are introduced. It should be noted that $\nu_{2\mathrm{dr}}$ is positive, because $\partial D/\partial\Omega$ and $(b_2 - b_3)$ are negative, wherein an analytical explanation of the latter may be difficult. Therefore, the dissipation coefficient ν_2 increases by considering the drag force acting on the bubbles (see also Sec. III E).

As in the case of a long wave,[38] the translation of bubbles only affects the nonlinear coefficient, and the drag force acting on the bubbles only affects the dissipation coefficient; the dispersion coefficient is not affected by the translation and the drag force. However, the tendencies of the long and short waves are quite different. From now on, the tendencies are described in detail.

D. Nonlinearity

Figures 2 and 3 illustrate the dependence of nonlinear coefficient ν_1 on initial void fraction α_0 and wavenumber k, respectively. The nonlinearity (i.e., absolute value of the nonlinear coefficient) significantly increased with the translation case compared to that without the translation case; as the explicit form of ν_1 is quite complex, an analytical explanation of why this occurs is difficult. Further, the difference between the two cases increased for a large k and a small α_0. It should be noted that the dependence of ν_1 on the initial bubble radius, R_0^*, is quite small.

It implies that the increase in nonlinearity owing to the translation for a short wave is quite large compared to that for a long wave.[38] This may be justified by the fact that a slip between the gas and liquid phases is prominent in short waves.

E. Dissipation

Figures 4 and 5 illustrate the dependence of each component of the dissipation coefficient [i.e., the liquid viscosity $\nu_{2\mathrm{vis}}$, acoustic

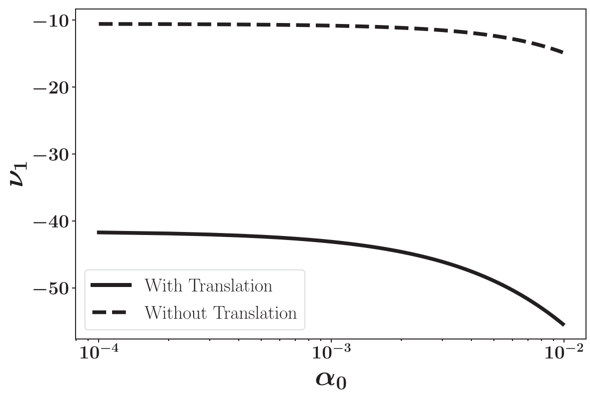

FIG. 2. Nonlinear coefficient ν_1 vs initial void fraction α_0 under $k = 2$ for the case of $\varepsilon = 0.07$, $R_0^* = 10\,\mu$m, $\rho_{L0}^* = 10^3$ kg/m^3, $p_{L0}^* = 101\,325$ Pa, $\beta_1 = \beta_2 = 1/2$, $c_{L0}^* = 1500$ m/s, $\sigma^* = 0.0728$ N/m, $\mu_L^* = 10^{-3}$ Pa · s, $\lambda^* = 0.0257$ W/(m·K), $\kappa = 1.41$, $T_0^* = 290$ K, and $u_0^* = 1$ m/s. The same condition is used in Figs. 3–5.

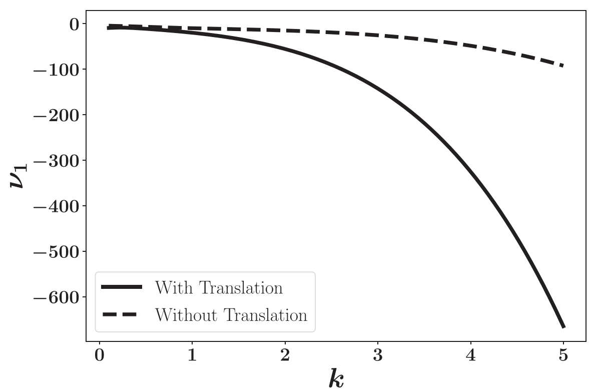

FIG. 3. Nonlinear coefficient ν_1 vs wavenumber k under $\alpha_0 = 0.01$.

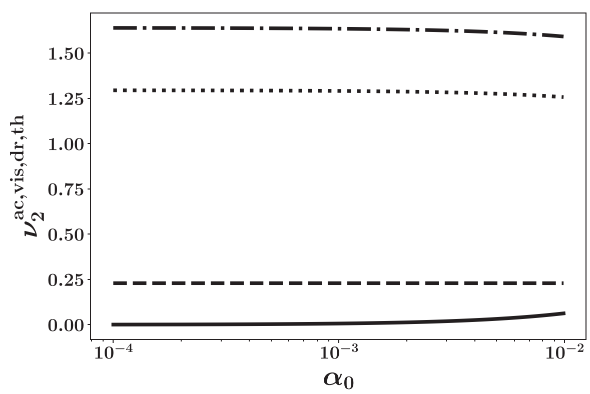

FIG. 4. Each component of dissipation coefficient ν_2 vs α_0 in (114)–(118) under $k = 1$: dissipation owing to liquid viscosity $\nu_{2\mathrm{vis}}$ is denoted by the dashed-dotted curve, that owing to acoustic radiation $\nu_{2\mathrm{ac}}$ by the dotted curve, that owing to thermal conduction $\nu_{2\mathrm{th}}$ by the dashed curve, and that owing to drag force $\nu_{2\mathrm{dr}}$ by the solid curve (same as Fig. 5).

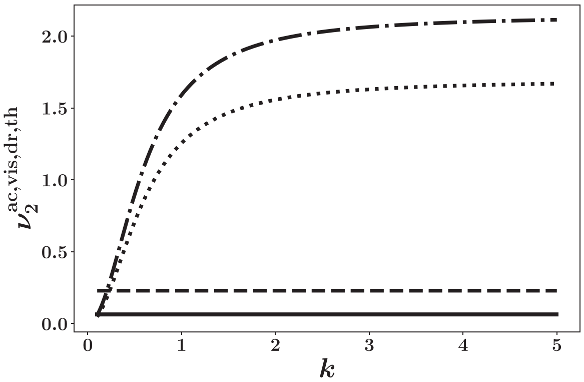

FIG. 5. Each component of dissipation coefficient ν_2 vs k under $\alpha_0 = 0.01$.

radiation $\nu_{2\mathrm{ac}}$, thermal dissipation $\nu_{2\mathrm{th}}$, and drag force $\nu_{2\mathrm{dr}}$; see also (114)–(118)] on α_0 and k, respectively. These results are different from case A of the long wave case (see Sec. III B in our previous paper[38]). In addition to the dissipation owing to the liquid viscosity and the acoustic radiation, that owing to the thermal conduction and the drag force have been introduced, and the four dissipation factors coexist in the form of a linear combination.

The dissipation owing to the drag force, $\nu_{2\mathrm{dr}}$, is the smallest. Here, $\nu_{2\mathrm{vis}}$ and $\nu_{2\mathrm{ac}}$ are almost of a similar order, and $\nu_{2\mathrm{vis}} > \nu_{2\mathrm{ac}}$; moreover, $\nu_{2\mathrm{th}}$ and $\nu_{2\mathrm{dr}}$ are almost of a similar order, and $\nu_{2\mathrm{th}} > \nu_{2\mathrm{dr}}$. Further, $\nu_{2\mathrm{vis}}$ and $\nu_{2\mathrm{ac}}$ are dependent on α_0 and k, whereas $\nu_{2\mathrm{th}}$ is independent of α_0 and k; $\nu_{2\mathrm{dr}}$ is dependent on α_0 and independent of k.

IV. CONCLUSIONS

We have theoretically examined the weakly nonlinear propagation of plane progressive pressure waves with thermal conduction in water flows that uniformly contain many translational microbubbles, with a special focus on the effects of a translation of bubbles and a drag force acting on the bubbles. Further, the result obtained in our previous study on low-frequency long wave[38] has successfully been extended to moderately-high-frequency short wave. The main conclusions are summarized as follows:

(i) Using the method of multiple scales, we derived the NLS equation that could describe the weakly nonlinear propagation of the envelope wave of a short carrier wave.

(ii) The translation of bubbles contributed to the nonlinearity and increased the absolute value of the nonlinear coefficient, and the drag force acting on the bubbles contributed to the dissipation and increased the value of the dissipation coefficient, as in the case of long wave.[38]

(iii) Owing to the incorporation of bubble translation, the nonlinearity (i.e., absolute value of the nonlinear coefficient) significantly increased compared to the case of nonlinearity without translation. Compared to the case of long wave,[38] although this trend was qualitatively the same, quantitatively, the difference might be considerable. This implied that the effect of a slip between the gas and liquid phases was dominant for the case of short wave.

(iv) The dissipation coefficient was expressed as the linear combination of the liquid viscosity, acoustic radiation, thermal dissipation, and drag force. The dissipation owing to the drag force was most smallest.

ACKNOWLEDGMENTS

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APPENDIX A: INHOMOGENEOUS TERMS IN (60)–(65)

Although the present inhomogeneous terms M_1 and M_2 are essentially the same as the previous inhomogeneous terms, i.e., S_1 and S_2 in Appendix B in Ref. 30, respectively; the present terms assume $\mathrm{D_G}/\mathrm{D}t = \mathrm{D_L}/\mathrm{D}t = \mathrm{D}/\mathrm{D}t$.

The inhomogeneous terms in the momentum equations, M_3 and M_4, are given by

$$M_3 = p_{\mathrm{G0}}\left[\left(\frac{\partial}{\partial x_1} + \alpha_1\frac{\partial}{\partial x_0}\right)(3R_1 - T_{\mathrm{G1}}) + 3\frac{\partial}{\partial x_0}(R_1 T_{\mathrm{G1}} - 2R_1^2)\right] + M_{\mathrm{F}}, \tag{A1}$$

$$\begin{aligned} M_4 = {} & \frac{\mathrm{D}}{\mathrm{D}t_1}[u_0\alpha_0\alpha_1 - (1-\alpha_0)u_{\mathrm{L1}}] - (1-\alpha_0)\frac{\partial}{\partial x_1}(p_{\mathrm{L1}} + u_0 u_{\mathrm{L1}}) \\ & + \alpha_0\frac{\mathrm{D}\alpha_1 u_{\mathrm{L1}}}{\mathrm{D}t_0} + u_0\alpha_0\frac{\partial \alpha_1 u_{\mathrm{L1}}}{\partial x_0} - 2(1-\alpha_0)u_{\mathrm{L1}}\frac{\partial u_{\mathrm{L1}}}{\partial x_0} + \alpha_0\alpha_1\frac{\partial p_{\mathrm{L1}}}{\partial x_0} \\ & - \alpha_0\left[3(\gamma_e - 1)p_{\mathrm{G0}}R_1 + p_{\mathrm{G0}}T_{\mathrm{G1}} - p_{\mathrm{L1}} - \Delta^2 R_1\right]\frac{\partial \alpha_1}{\partial x_0} - \alpha_0 M_{\mathrm{F}} \end{aligned} \tag{A2}$$

with

$$\begin{aligned} M_{\mathrm{F}} = {} & -\beta_1\frac{\mathrm{D}}{\mathrm{D}t_1}(u_{\mathrm{G1}} - u_{\mathrm{L1}}) - \beta_1\alpha_1\frac{\mathrm{D}}{\mathrm{D}t_0}(u_{\mathrm{G1}} - u_{\mathrm{L1}}) \\ & - \frac{\beta_1}{2}\frac{\partial}{\partial x_0}(u_{\mathrm{G1}}^2 - u_{\mathrm{L1}}^2) - \beta_2(u_{\mathrm{G1}} - u_{\mathrm{L1}})\frac{\mathrm{D}\alpha_1}{\mathrm{D}t_0}. \end{aligned} \tag{A3}$$

In Keller's equation, owing to the consideration of translation, the present inhomogeneous term, M_5, is increased from the previous inhomogeneous term,[30]

$$\begin{aligned} M_5 = {} & -2\frac{\mathrm{D}^2 R_1}{\mathrm{D}t_0\mathrm{D}t_1} - R_1\frac{\mathrm{D}^2 R_1}{\mathrm{D}t_0^2} - 2u_{\mathrm{G1}}\frac{\mathrm{D}}{\mathrm{D}t_0}\left(\frac{\partial R_1}{\partial x_0}\right) - \frac{\mathrm{D}u_{\mathrm{G1}}}{\mathrm{D}t_0}\frac{\partial R_1}{\partial x_0} - \frac{3}{2}\left(\frac{\mathrm{D}R_1}{\mathrm{D}t_0}\right)^2 \\ & + \left[1 + \frac{3(2-\gamma_e)p_{\mathrm{G0}}}{\Delta^2}\right]R_1^2 - \frac{3p_{\mathrm{G0}}}{\Delta^2}R_1 T_{\mathrm{G1}} + \frac{(u_{\mathrm{G1}} - u_{\mathrm{L1}})^2}{\Delta^2}. \end{aligned} \tag{A4}$$

Owing to the incorporation of thermal effect, M_6 can be expressed as,

$$M_6 = -\frac{\mathrm{D}}{\mathrm{D}t_1}[T_{\mathrm{G1}} + 3(\kappa-1)R_1] - 3\frac{\mathrm{D}}{\mathrm{D}t_0}\left\{(\kappa-1)T_{\mathrm{G1}}R_1 - \left[3\kappa - \frac{\kappa(\kappa-1)}{2} - 2\right]R_1^2\right\} - u_{\mathrm{G1}}\frac{\partial}{\partial x_0}[T_{\mathrm{G1}} + 3(\kappa-1)R_1] + \zeta_2\frac{\mathrm{D}T_{\mathrm{G1}}}{\mathrm{D}t_0}. \tag{A5}$$

APPENDIX B: INHOMOGENEOUS TERMS IN (92)–(97)

Although the present inhomogeneous terms, N_1 and N_2, are essentially the same as the previous inhomogeneous terms, i.e., W_1 and W_2 in Appendix B in Ref. 30, respectively; the present terms assume $\mathrm{D_G}/\mathrm{D}t = \mathrm{D_L}/\mathrm{D}t = \mathrm{D}/\mathrm{D}t$.

The inhomogeneous terms in the momentum equations, N_3 and N_4, are given by

$$\begin{aligned} N_3 = p_{\mathrm{G0}}\Bigg[&\left(\frac{\partial}{\partial x_1} + \alpha_1\frac{\partial}{\partial x_0}\right)(3R_2 - T_{\mathrm{G2}}) + \left(\frac{\partial}{\partial x_2} + \alpha_1\frac{\partial}{\partial x_1}\right)(3R_1 - T_{\mathrm{G1}}) + 3\frac{\partial}{\partial x_1}(R_1T_{\mathrm{G1}} - 2R_1^2) + 3\frac{\partial R_2T_{\mathrm{G1}}}{\partial x_0} \\ &+ \alpha_2\frac{\partial}{\partial x_0}(3R_1 - T_{\mathrm{G1}}) + \alpha_1\frac{\partial}{\partial x_0}(3R_2 - T_{\mathrm{G2}} - 6R_1^2 + 3R_1T_{\mathrm{G1}}) + 10\frac{\partial R_1^3}{\partial x_0} + 3\frac{\partial}{\partial x_0}(R_2T_{\mathrm{G1}} + R_1T_{\mathrm{G2}} - 2R_1^2T_{\mathrm{G1}})\Bigg] + N_{\mathrm{F}}, \end{aligned} \tag{B1}$$

$$\begin{aligned} N_4 = &\frac{\mathrm{D}}{\mathrm{D}t_2}[u_0\alpha_0\alpha_1 - (1-\alpha_0)u_{\mathrm{L1}}] + \frac{\mathrm{D}}{\mathrm{D}t_1}[u_0\alpha_0\alpha_2 - (1-\alpha_0)u_{\mathrm{L2}}] - (1-\alpha_0)\left[\frac{\partial}{\partial x_2}(p_{\mathrm{L1}} + u_0u_{\mathrm{L1}}) + \frac{\partial}{\partial x_1}(p_{\mathrm{L2}} + u_0u_{\mathrm{L2}})\right] \\ &+ \alpha_0\left[\frac{\mathrm{D}\alpha_1u_{\mathrm{L1}}}{\mathrm{D}t_1} + \frac{\mathrm{D}}{\mathrm{D}t_0}(\alpha_2u_{\mathrm{L1}} + \alpha_1u_{\mathrm{L2}})\right] + u_0\alpha_0\frac{\partial}{\partial x_0}(\alpha_2u_{\mathrm{L1}} + \alpha_1u_{\mathrm{L2}}) + \alpha_0\frac{\partial\alpha_1u_{\mathrm{L1}}^2}{\partial x_0} - 2(1-\alpha_0)\left(u_{\mathrm{L1}}\frac{\partial u_{\mathrm{L1}}}{\partial x_1} + \frac{\partial u_{\mathrm{L1}}u_{\mathrm{L2}}}{\partial x_0}\right) \\ &+ \alpha_0\left(\alpha_1\frac{\partial p_{\mathrm{L1}}}{\partial x_1} + \alpha_2\frac{\partial p_{\mathrm{L1}}}{\partial x_0} + \alpha_1\frac{\partial p_{\mathrm{L2}}}{\partial x_0}\right) - \alpha_0[3(\gamma_{\mathrm{e}}-1)p_{\mathrm{G0}}R_1 + p_{\mathrm{G0}}T_{\mathrm{G1}} - p_{\mathrm{L1}} - \Delta^2R_1]\left(\frac{\partial\alpha_2}{\partial x_0} + \frac{\partial\alpha_1}{\partial x_1}\right) \\ &- \alpha_0\left\{[3(\gamma_{\mathrm{e}}-1)p_{\mathrm{G0}} - \Delta^2]R_2 + p_{\mathrm{G0}}(T_{\mathrm{G2}} - 3R_1T_{\mathrm{G1}}) - p_{\mathrm{L2}} + [3(2-\gamma_{\mathrm{e}})p_{\mathrm{G0}} + \Delta^2]R_1^2\right\}\frac{\partial\alpha_1}{\partial x_0} - \alpha_0N_{\mathrm{F}} \end{aligned} \tag{B2}$$

with

$$\begin{aligned} N_{\mathrm{F}} = &-\beta_1\frac{\mathrm{D}}{\mathrm{D}t_2}(u_{\mathrm{G1}} - u_{\mathrm{L1}}) - \beta_1\alpha_1\frac{\mathrm{D}}{\mathrm{D}t_1}(u_{\mathrm{G1}} - u_{\mathrm{L1}}) - \frac{\beta_1}{2}\frac{\partial}{\partial x_1}(u_{\mathrm{G1}}^2 - u_{\mathrm{L1}}^2) - \beta_2(u_{\mathrm{G1}} - u_{\mathrm{L1}})\frac{\mathrm{D}\alpha_1}{\mathrm{D}t_1} \\ &- \beta_1\frac{\mathrm{D}}{\mathrm{D}t_1}(u_{\mathrm{G2}} - u_{\mathrm{L2}}) - \beta_1\alpha_1\frac{\mathrm{D}}{\mathrm{D}t_0}(u_{\mathrm{G2}} - u_{\mathrm{L2}}) - \frac{\beta_1\alpha_1}{2}\frac{\partial}{\partial x_0}(u_{\mathrm{G1}}^2 - u_{\mathrm{L1}}^2) - \beta_2(u_{\mathrm{G2}} - u_{\mathrm{L2}})\frac{\mathrm{D}\alpha_1}{\mathrm{D}t_0} \\ &- \beta_1\alpha_2\frac{\mathrm{D}}{\mathrm{D}t_0}(u_{\mathrm{G1}} - u_{\mathrm{L1}}) - \beta_1\frac{\partial}{\partial x_0}(u_{\mathrm{G1}}u_{\mathrm{G2}} - u_{\mathrm{L1}}u_{\mathrm{L2}}) - \beta_2(u_{\mathrm{G1}} - u_{\mathrm{L1}})\left(\frac{\mathrm{D}\alpha_2}{\mathrm{D}t_1} + u_{\mathrm{G1}}\frac{\partial\alpha_1}{\partial x_0}\right) - \frac{3\mu}{\Delta^2}(u_{\mathrm{G1}} - u_{\mathrm{L1}}). \end{aligned} \tag{B3}$$

In Keller's equation, owing to the consideration of translation, the present inhomogeneous term, N_5, is increased from the previous inhomogeneous term,[30]

$$\begin{aligned} N_5 = &-2\frac{\mathrm{D}^2R_1}{\mathrm{D}t_0\mathrm{D}t_2} - \frac{\mathrm{D}^2R_1}{\mathrm{D}t_1^2} - 2\frac{\mathrm{D}^2R_2}{\mathrm{D}t_0\mathrm{D}t_1} - 2R_1\frac{\mathrm{D}^2R_1}{\mathrm{D}t_0\mathrm{D}t_1} - 2u_{\mathrm{G1}}\left(\frac{\mathrm{D}}{\mathrm{D}t_0}\frac{\partial R_1}{\partial x_1} + \frac{\mathrm{D}}{\mathrm{D}t_1}\frac{\partial R_1}{\partial x_0}\right) - 3\frac{\mathrm{D}R_1}{\mathrm{D}t_0}\frac{\mathrm{D}R_1}{\mathrm{D}t_1} - \frac{\mathrm{D}u_{\mathrm{G1}}}{\mathrm{D}t_0}\frac{\partial R_1}{\partial x_1} - \frac{\mathrm{D}u_{\mathrm{G1}}}{\mathrm{D}t_1}\frac{\partial R_1}{\partial x_0} \\ &- R_1\frac{\mathrm{D}^2R_2}{\mathrm{D}t_0^2} - R_2\frac{\mathrm{D}^2R_1}{\mathrm{D}t_0^2} - 2u_{\mathrm{G1}}\left(R_1\frac{\mathrm{D}}{\mathrm{D}t_0}\frac{\partial R_1}{\partial x_0} + \frac{\mathrm{D}}{\mathrm{D}t_0}\frac{\partial R_2}{\partial x_0}\right) - \frac{\mathrm{D}u_{\mathrm{G1}}}{\mathrm{D}t_0}\frac{\partial R_2}{\partial x_0} - 2u_{\mathrm{G2}}\frac{\mathrm{D}}{\mathrm{D}t_0}\frac{\partial R_1}{\partial x_0} - u_{\mathrm{G1}}^2\frac{\partial^2R_1}{\partial x_0^2} \\ &- \frac{\partial R_1}{\partial x_0}\left(u_{\mathrm{G1}}\frac{\partial u_{\mathrm{G1}}}{\partial x_0} + \frac{\mathrm{D}u_{\mathrm{G2}}}{\mathrm{D}t_0} + R_1\frac{\mathrm{D}u_{\mathrm{G1}}}{\mathrm{D}t_0} + 3u_{\mathrm{G1}}\frac{\mathrm{D}R_1}{\mathrm{D}t_0}\right) - 3\frac{\mathrm{D}R_1}{\mathrm{D}t_0}\frac{\mathrm{D}R_2}{\mathrm{D}t_0} + 2\left[\frac{3(2-\gamma_{\mathrm{e}})p_{\mathrm{G0}}}{\Delta^2} + 1\right]R_1R_2 - \left[\frac{(10-3\gamma_{\mathrm{e}})p_{\mathrm{G0}}}{\Delta^2} + 1\right]R_1^3 - \frac{3p_{\mathrm{G0}}}{\Delta^2}(R_1T_{\mathrm{G2}} \\ &+ R_2T_{\mathrm{G1}} - 2R_1^2T_{\mathrm{G1}}) + \frac{(u_{\mathrm{G1}} - u_{\mathrm{L1}})(u_{\mathrm{G2}} - u_{\mathrm{L2}})}{2\Delta^2} + \frac{V}{\Delta}\left\{p_{\mathrm{G0}}\frac{\mathrm{D}T_{\mathrm{G1}}}{\mathrm{D}t_0} + [3(\gamma_{\mathrm{e}}-1)p_{\mathrm{G0}} - \Delta^2]\frac{\mathrm{D}R_1}{\mathrm{D}t_0}\right\} - \frac{4\mu}{\Delta^2}\frac{\mathrm{D}R_1}{\mathrm{D}t_0}. \end{aligned} \tag{B4}$$

Owing to the incorporation of thermal effect, N_6 is expressed as,

$$\begin{aligned} N_6 = &-\frac{\mathrm{D}}{\mathrm{D}t_2}[T_{\mathrm{G1}} + 3(\kappa-1)R_1] - \frac{\mathrm{D}}{\mathrm{D}t_1}[T_{\mathrm{G2}} + 3(\kappa-1)R_2] - 3\frac{\mathrm{D}}{\mathrm{D}t_1}\left\{(\kappa-1)T_{\mathrm{G1}}R_1 - \left[3\kappa - \frac{\kappa(\kappa-1)}{2} - 2\right]R_1^2\right\} \\ &- \left(u_{\mathrm{G1}}\frac{\partial}{\partial x_1} + u_{\mathrm{G2}}\frac{\partial}{\partial x_0}\right)[T_{\mathrm{G1}} + 3(\kappa-1)R_1] + \frac{\mathrm{D}}{\mathrm{D}t_0}(3R_2T_{\mathrm{G1}} + 3R_1T_{\mathrm{G2}} + 10R_1^3 - 6R_1^2T_{\mathrm{G1}}) \\ &+ 3\kappa\frac{\mathrm{D}}{\mathrm{D}t_0}\left[(7-3\kappa)R_1R_2 + \frac{7-3\kappa}{2}R_1^2T_{\mathrm{G1}} - T_{\mathrm{G2}}R_1 - 6R_1^3 - T_{\mathrm{G1}}R_2 + \frac{3(3\kappa-1)}{2}R_1^3 - \frac{(3\kappa-1)(3\kappa-2)}{6}R_1^3\right] \\ &- 3u_{\mathrm{G1}}\frac{\partial}{\partial x_0}\left\{(\kappa-1)R_1T_{\mathrm{G1}} - \left[3\kappa - \frac{\kappa(3\kappa-1)}{2} - 2\right]R_1^2 + (\kappa-1)R_2 + \frac{T_{\mathrm{G2}}}{3}\right\} - \zeta_1T_{\mathrm{G1}}. \end{aligned} \tag{B5}$$

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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