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## ABSTRACT

In this study, weakly nonlinear pressure waves in quiescent compressible liquids comprising several uniformly-distributed spherical microbubbles, at moderately high-frequency and short-wavelength, are theoretically investigated. The energy equation at the bubble–liquid interface and the effective polytropic exponent are utilized to clarify thermal effects inside bubbles on wave dissipation. In addition, thermal conduction is investigated in detail using four temperature-gradient models. The following results are drawn: (i) Nonlinear Schrödinger equation is derived as an effective equation, wherein three types of dissipation factors, i.e., liquid viscosity, liquid compressibility, and thermal conduction, are unified into a linear combination as the dissipation coefficient. This is different from our previous result treating the low-frequency and long-wavelength case [Kamei *et al.*, Phys. Fluids **33**, 053302 (2021)], i.e., two types of dissipation terms appeared and did not unify into a linear combination. (ii) Dissipation due to thermal conduction is more than four times larger than that due to other dissipation factors. (iii) Dissipation due to thermal conduction at the bubble–liquid interface is considerably larger than that due to thermal conduction through the bubbly liquid. (iv) It is found that the dissipation effect in the short-wave case is smaller than that in the long-wave case.



## I. INTRODUCTION

This paper is a continuation of our recent paper[1] investigating theoretically nonlinear pressure waves in bubbly liquids from the macroscopic viewpoint based on averaged equations. Pressure wave propagation in a liquid containing microbubbles is one of the most fundamental and important physical phenomena in the field of multiphase flow.[2–11] Bubble dynamics[12–16] is also a key point (see also the reviews in Refs. 17 and 18) as oscillations of bubbles in a liquid induce the effects of dispersion[19] and dissipation.[19,20] Long-range propagation of pressure waves in bubbly liquids, accompanied by a weak nonlinearity, leads to the formation of shock and solitary waves in the cases of the dissipation–nonlinearity and dispersion–nonlinearity competitions, respectively. Estimating the effect of nonlinearity, dissipation, and dispersion on the pressure wave is important in determining the wave evolution into the shock or solitary wave. However, it is a challenging task from the viewpoint of experiments and direct numerical simulations. For weakly nonlinear pressure waves,[21,22] we can theoretically derive a nonlinear wave equation (i.e., a linear combination of the nonlinear, dissipation, and dispersion terms) from basic equations. This equation is an approximate description of the spatio–temporal development of the waveform obtained owing to the balance between the wave nonlinearity, dissipation, and dispersion.[21,22] We can then discuss the relative sizes of the dissipation and dispersion contributing to the nonlinearity. Numerous types of nonlinear wave equations describing weakly nonlinear propagation of pressure waves in bubbly liquids have been developed.[19,20,23–33]

Recently, we have clarified that low-frequency long waves are described by the Korteweg–de Vries–Burgers (KdVB) equation and that high-frequency short waves are described by a nonlinear Schrödinger (NLS) equation in the linear dispersion relation in quiescent bubbly liquids, as shown the KdVB band in Fig. 1.[34,35] For low-frequency long waves, the waveform obtained as a solution of the

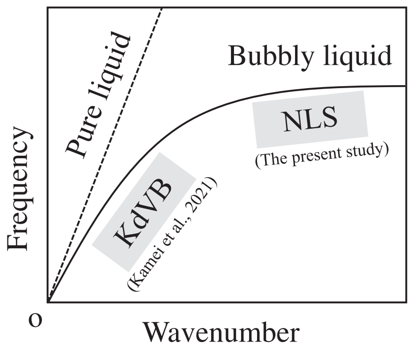


FIG. 1. Conceptual diagram of linear dispersion relationship for pressure waves in bubbly liquids, where the liquid compressibility is neglected; van Wijngaarden[19,20] originally drawn and Kanagawa *et al.*[34,40] decompose into two bands, i.e., KdVB and NLS equations for low-frequency long wave and moderately high-frequency short wave, respectively. After many derivations of KdVB equations,[20,23,24,36] Kanagawa *et al.*[34,40] derived KdVB and NLS equations in a unified way. Noting that there exists a quite high-frequency curve above the present curve induced by the consideration of liquid compressibility[31,39] and another NLS equation for quite high-frequency (not moderately high-frequency) band was reported.[27,31]

well-known KdVB equation corresponds to the waveform observed by experiments.[36–38] Furthermore, in our previous paper,[1] we re-derived the KdVB equation by incorporating the thermal conduction at the bubble–liquid interface and thermodynamics inside the bubble as an update model. Therein, we highlighted that the solution of the KdVB equation is similar to the waveform observed in an experimental study.[23] However, there are no reports on the propagation of moderately high-frequency [around the eigenfrequency of single bubble oscillations signified by (35) in Sec. II D] short waves (i.e., the NLS band in Fig. 1) as the solution of the NLS equation. Therefore, theoretical predictions for the NLS band have long been strongly desired. On the other hand, from the theoretical viewpoint, Khismatullin and Akhatov[31] derived another NLS equation focusing on a quite high (not moderately high) frequency curve, which is above the curve in Fig. 1 and is induced by the consideration of liquid compressibility.[31,39] Although Russian co-workers derived NLS and Ginzburg–Landau (GL) equations, they also focused on the quite high-frequency band.[27–29] Hence, so far, our original paper[34,40] is the only study on the derivation of the NLS equation in the case of a moderately high-frequency band; it should be noted that Refs. 34 and 40 utilized the two-fluid and mixture models as basic equations, respectively.

By focusing on the dissipation effect on high-frequency short waves, our group considered liquid viscosity at the bubble–liquid interface and liquid compressibility (i.e., acoustic radiation).[40] In addition, we considered the recently incorporated viscosity of bubbly liquids and thermal conduction through them.[41,42] However, we found that the effect of thermal conduction through a bubbly liquid is small.[41,42] Therefore, in our model, a consideration of thermal conduction was strongly desired. Table I summarizes four dissipation factors used in the present and previous studies.[27,31,40,41] The purpose of this study is to derive an NLS equation incorporating four dissipation effects: viscosity of a bubbly liquid, viscosity at the bubble–liquid interface, thermal conduction at the bubble–liquid interface, and acoustic radiation due to liquid compressibility. Based on the derived NLS equation, we clarified the effects of thermal conduction at the interface and thermodynamics inside the bubble on nonlinear propagation of moderately high-frequency short waves. We also utilized four widely used models for evaluating the temperature gradient at the bubble–liquid interface in (11) and determined the differences between these models.

This paper is organized as follows: Section II introduces basic equations and perturbation expansions based on the multiple scale method.[22] Especially, the energy equation at the bubble–liquid interface and the effective polytropic exponent[24] are incorporated into our model to investigate the thermal effect. Section III describes the derivation of the NLS equation. Section IV focuses on the coefficients of the NLS equation and clarifies the differences between the present and previous coefficients. We show that the thermal effect contributes to not only wave dissipation but also to wave dispersion and that thermal conduction strongly contributes to wave dissipation. We also compare the present NLS equation with the previous KdVB equation.[1] Section V concludes the study.

## II. PROBLEM FORMULATION

### A. Problem

Focusing on a moderately high-frequency and short-wavelength band, as shown in Fig. 1, we consider one-dimensional (i.e., plane) weakly nonlinear pressure waves in a liquid uniformly containing numerous spherical gas bubbles. We then clarify the effect of thermal conduction at the bubble–liquid interface on wave propagation.

For simplifying the formulation of the problem, the following assumptions[1] are used: (i) the viscous coefficients of the gas phase, bulk viscosity, phase change, and mass transport across the bubble–liquid interface are ignored. (ii) The temperature of the liquid phase is constant. (iii) The effect of initial flow[43] is neglected; hence, the bubbly liquid is initially quiescent. (iv) The bubbles do not coalesce, break, appear, and disappear. (v) The forces exerted on bubbles, such as drag force,[44,45] are neglected. (vi) The polydispersity,[46] multi-dimensionality,[47] and effect of the encapsulating shell[48] are not considered.

### B. Basic equations

We use the following conservation equations of mass and momentum for bubbly liquids:[49,50]

$$\frac{\partial \rho^*}{\partial t^*} + \frac{\partial \rho^* u^*}{\partial x^*} = 0, \tag{1}$$

$$\frac{\partial \rho^* u^*}{\partial t^*} + \frac{\partial \rho^* u^{*2}}{\partial x^*} + \frac{\partial p_{\mathrm{L}}^*}{\partial x^*} - \frac{4}{3}\mu^* \frac{\partial^2 u^*}{\partial x^{*2}} = 0, \tag{2}$$

where $t^*$ is time, $x^*$ space coordinate, $\rho^*$ density, $u^*$ fluid velocity, $p^*$ pressure, and $\mu^*$ viscosity; the subscript L denotes the liquid phase and the superscript $*$ denotes the dimensional quantity. Here, we assume that the pressure of the bubbly liquid is equivalent to the averaged pressure of the liquid.[29,40]

The bubbly liquid is assumed to be a Newtonian fluid, and the bulk viscosity is ignored based on the Stokes assumption. As derived in several works,[51,52] the viscosity of the bubbly liquid as a physical property, $\mu^*$, is expressed as follows:

$$\mu^* = (1 + \alpha_0)\mu_{\mathrm{L}}^*, \tag{3}$$

where $\alpha_0$ is the initial void fraction. Here, $\mu^*$ is higher than $\mu_{\mathrm{L}}^*$ because the mechanical work acting on the water and the velocity of the flow field are reduced owing to the containing bubbles.[53] Note that (3) is

**TABLE I.** Summary of four dissipation factors considered in the present and previous studies.[27,31,40–42] We emphasize that Khismatullin and Akhatov[31] focused on a quite high-frequency band with a clear statement and Gumerov[27] focused on a quite high-frequency band. In contrast, Kanagawa *et al.*,[40] Kamei and Kanagawa,[41,42] and the present study focus on a moderately high (not quite high) frequency band. These studies are compared in Sec. IV.

| | Viscosity of bubbly liquid | Viscosity at bubble–liquid interface | Liquid compressibility | Thermal conduction |
|---|---|---|---|---|
| Gumerov (1992)[27] | Not considered | Considered | Not considered | Considered (at bubble–liquid interface) |
| Khismatullin and Akhatov (2001)[31] | Not considered | Considered | Not considered | Considered |
| Kanagawa *et al.* (2011)[40] | Not considered | Considered | Considered | Not considered |
| Kamei and Kanagawa (2019)[41] | Considered | Considered | Considered | Considered (through bubbly liquid) |
| This study | Considered | Considered | Considered | Considered (at bubble–liquid interface) |

applicable if $\alpha_0 < 0.05$.[54] The volume-averaged density of the bubbly liquid, $\rho^*$, is defined as follows:

$$\rho^* = (1-\alpha)\rho_{\mathrm{L}}^*, \tag{4}$$

where $\alpha$ is the void fraction and the density of the gas is neglected. The void fraction, $\alpha$, is related to the number density of the bubbles, $N^*$, using the following equation:

$$\alpha = \frac{4}{3}\pi R^{*3} N^*, \tag{5}$$

$$\frac{\partial N^*}{\partial t^*} + \frac{\partial N^* u^*}{\partial x^*} = 0, \tag{6}$$

where $R^*$ is a representative bubble radius. Equation (5) defines the void fraction, $\alpha$, and (6) represents the conservation of the number density of the bubbles, $N^*$.

Substituting the following conservation equation of mass inside the bubble:

$$\frac{\rho_{\mathrm{G}}^*}{\rho_{\mathrm{G0}}^*} = \left(\frac{R_0^*}{R^*}\right)^3, \tag{7}$$

and (5) into (6), we obtain

$$\frac{\partial}{\partial t^*}(\alpha\rho_{\mathrm{G}}^*) + \frac{\partial}{\partial x^*}(\alpha\rho_{\mathrm{G}}^* u^*) = 0, \tag{8}$$

where the subscript G denotes the gas phase. Furthermore, substituting (4) into (1) and (2), we obtain

$$\frac{\partial}{\partial t^*}\left[(1-\alpha)\rho_{\mathrm{L}}^*\right] + \frac{\partial}{\partial x^*}\left[(1-\alpha)\rho_{\mathrm{L}}^* u^*\right] = 0, \tag{9}$$

$$\frac{\partial}{\partial t^*}\left[(1-\alpha)\rho_{\mathrm{L}}^* u^*\right] + \frac{\partial}{\partial x^*}\left[(1-\alpha)\rho_{\mathrm{L}}^* u^{*2}\right] + \frac{\partial p_{\mathrm{L}}^*}{\partial x^*} - \frac{4}{3}\mu^*\frac{\partial^2 u^*}{\partial x^{*2}} = 0. \tag{10}$$

To examine the thermal effect inside the bubble, we use the following relationship at the bubble–liquid interface proposed by Prosperetti:[24]

$$\frac{\mathrm{D}p_{\mathrm{G}}^*}{\mathrm{D}t^*} = \frac{3}{R^*}\left[(\kappa-1)\lambda_{\mathrm{G}}^*\frac{\partial T_{\mathrm{G}}^*}{\partial r^*}\bigg|_{r^*=R^*} - \kappa p_{\mathrm{G}}^*\frac{\mathrm{D}R^*}{\mathrm{D}t^*}\right], \tag{11}$$

where $\lambda_{\mathrm{G}}^*$ is the thermal conductivity of the gas inside the bubble and $\kappa$ is the ratio of specific heats. Note that $R^*$ is not $R^*(t^*)$ but $R(t^*, x^*)$; the bubble radius is regarded as a field variable defined in all $t^*$ and $x^*$.

Note that $p_{\mathrm{G}}^*$, $T_{\mathrm{G}}^*$, and $R^*$ do not depend on the time $t^*$ but also on the space $x^*$ [e.g., $p_{\mathrm{G}}^*(t^*, x^*)$].

The Keller equation for spherical oscillations of a bubble in a compressible liquid is given as follows:[55]

$$\left(1 - \frac{1}{c_{\mathrm{L0}}^*}\frac{\mathrm{D}R^*}{\mathrm{D}t^*}\right)R^*\frac{\mathrm{D}^2R^*}{\mathrm{D}t^{*2}} + \frac{3}{2}\left(1 - \frac{1}{3c_{\mathrm{L0}}^*}\frac{\mathrm{D}R^*}{\mathrm{D}t^*}\right)\left(\frac{\mathrm{D}R^*}{\mathrm{D}t^*}\right)^2$$
$$= \left(1 + \frac{1}{c_{\mathrm{L0}}^*}\frac{\mathrm{D}R^*}{\mathrm{D}t^*}\right)\frac{P^*}{\rho_{\mathrm{L0}}^*} + \frac{R^*}{\rho_{\mathrm{L0}}^* c_{\mathrm{L0}}^*}\frac{\mathrm{D}}{\mathrm{D}t^*}(p_{\mathrm{L}}^* + P^*), \tag{12}$$

where $P^*$ is the liquid pressure averaged on the bubble–liquid interface,[39] $c_{\mathrm{L0}}^*$ is the sound speed in initial unperturbed pure water, subscript 0 denotes the initial unperturbed state, and the material differential operator, $\mathrm{D}/\mathrm{D}t^*$ represents the following differential operator:

$$\frac{\mathrm{D}}{\mathrm{D}t^*} = \frac{\partial}{\partial t^*} + u^*\frac{\partial}{\partial x^*}. \tag{13}$$

To close the system represented in (8)–(12), we introduce the following Tait's equation of state for liquids, equation of state for ideal gas, and balance of normal stresses across the bubble–liquid interface:

$$p_{\mathrm{L}}^* = p_{\mathrm{L0}}^* + \frac{\rho_{\mathrm{L0}}^* c_{\mathrm{L0}}^{*2}}{n}\left[\left(\frac{\rho_{\mathrm{L}}^*}{\rho_{\mathrm{L0}}^*}\right)^n - 1\right], \tag{14}$$

$$\frac{p_{\mathrm{G}}^*}{p_{\mathrm{G0}}^*} = \frac{\rho_{\mathrm{G}}^*}{\rho_{\mathrm{G0}}^*}\frac{T_{\mathrm{G}}^*}{T_0^*}, \tag{15}$$

$$p_{\mathrm{G}}^* - (p_{\mathrm{L}}^* + P^*) = \frac{2\sigma^*}{R^*} + \frac{4\mu_{\mathrm{L}}^*}{R^*}\frac{\mathrm{D}R^*}{\mathrm{D}t^*}, \tag{16}$$

where $n$ is the material constant and $\sigma^*$ is the surface tension. Note that the polytropic equation of state used in our original study[40] is changed to (15).

### C. Temperature-gradient model

Similar to the long-wave case,[1] we incorporated the thermal conductivity at the bubble–liquid interface by introducing (11). Further, we examined the effect of thermal conductivity caused by the differences between the temperature-gradient models $\partial T_{\mathrm{G}}^*/\partial r^*|_{r^*=R^*}$ on the first term on the right-hand side of (11). We used the following four models:[56–59]

(i) Shimada *et al.* (SMK) model,[56]

$$\left.\frac{\partial T_{\rm G}^*}{\partial r^*}\right|_{r^*=R^*} = \frac{5}{4}\frac{T_0^* - T_{\rm G}^*}{R^*}, \tag{17}$$

(ii) Lertnuwat *et al.* (LSM) model,[57]

$$\left.\frac{\partial T_{\rm G}^*}{\partial r^*}\right|_{r^*=R^*} = \frac{T_0^* - T_{\rm G}^*}{\sqrt{2\pi D^*/\omega_{\rm B}^*}}, \tag{18}$$

(iii) Preston *et al.* (PCB) model,[58]

$$\left.\frac{\partial T_{\rm G}^*}{\partial r^*}\right|_{r^*=R^*} = \frac{T_0^* - T_{\rm G}^*}{|\widetilde{L}_{\rm P}^*|}, \tag{19}$$

(iv) Sugiyama *et al.* (STM) model,[59]

$$\left.\frac{\partial T_{\rm G}^*}{\partial r^*}\right|_{r^*=R^*} = \frac{{\rm Re}(\widetilde{L}_{\rm P}^*)(T_0^* - T_{\rm G}^*)}{|\widetilde{L}_{\rm P}^*|^2} + \frac{{\rm Im}(\widetilde{L}_{\rm P}^*)}{\omega_{\rm B}^*|\widetilde{L}_{\rm P}^*|^2}\frac{{\rm D}T_{\rm G}^*}{{\rm D}t^*}, \tag{20}$$

where $T_0^*$ is the initial temperature, $D^*$ is the thermal diffusivity of the gas inside the bubble, and $\omega_{\rm B}^*$ is the natural frequency of a single bubble and is given as follows:[59]

$$\omega_{\rm B}^* = \sqrt{\frac{3\gamma_{\rm e}p_{\rm G0}^* - 2\sigma^*/R_0^*}{\rho_{\rm L0}^* R_0^{*2}} - \left(\frac{2\mu_{\rm e0}^*}{\rho_{\rm L0}^* R_0^{*2}}\right)^2}, \tag{21}$$

$$\gamma_{\rm e} = {\rm Re}\left(\frac{\Gamma_{\rm N}}{3}\right), \tag{22}$$

$$\mu_{\rm e0}^* = \mu_{\rm L}^* + {\rm Im}\left(\frac{p_{\rm G0}^*\Gamma_{\rm N}}{4\omega_{\rm B}^*}\right), \tag{23}$$

where $\gamma_{\rm e}$ is the effective polytropic exponent and we do not explicitly assume $\gamma_{\rm e}$ except for Sec. IV; and $\mu_{\rm e0}^*$ is the initial effective viscosity; and Re and Im denote the real and imaginary parts, respectively. The explicit form of (21) is different from that used in our previous studies.[34,40–48]

Moreover, the complex number $\Gamma_{\rm N}$ is given as[59]

$$\Gamma_{\rm N} = \frac{3\alpha_{\rm N}^2\kappa}{\alpha_{\rm N}^2 + 3(\kappa - 1)(\alpha_{\rm N}\coth\alpha_{\rm N} - 1)}, \tag{24}$$

$$\alpha_{\rm N} = \sqrt{\frac{\kappa\omega_{\rm B}^* p_{\rm G0}^* R_0^{*2}}{2(\kappa - 1)T_0^*\lambda_{\rm G}^*}}(1 + {\rm i}), \tag{25}$$

where $\alpha_{\rm N}$ is a complex number and i denotes the imaginary unit. Therefore, $\widetilde{L}_{\rm P}^*$ in (19) and (20) is the complex number with the dimensions of length and is given by

$$\widetilde{L}_{\rm P}^* = \frac{R_0^*(\alpha_{\rm N}^2 - 3\alpha_{\rm N}\coth\alpha_{\rm N} + 3)}{\alpha_{\rm N}^2(\alpha_{\rm N}\coth\alpha_{\rm N} - 1)}. \tag{26}$$

## D. Multiple scale analysis

The time $t^*$ and space coordinate $x^*$ are nondimensionalized using $t = \omega^* t^*$ and $x = x^*/L^*$, respectively, where $\omega^*$ is the typical angular frequency of a wave and $L^*$ is the typical wavelength. Next, we introduce new independent variables based on the typical dimensionless amplitude of a wave, $\varepsilon$ ($\ll 1$), for near field [i.e., the temporal and spatial scales of $O(1)$], far field I [i.e., the temporal and spatial scales of $O(1/\varepsilon)$], and far field II [i.e., the temporal and spatial scales of $O(1/\varepsilon^2)$],[22]

$$\begin{cases} t_0 = t, & x_0 = x, \\ t_1 = \varepsilon t, & x_1 = \varepsilon x, \\ t_2 = \varepsilon^2 t, & x_2 = \varepsilon^2 x. \end{cases} \tag{27}$$

Note that far field II was not used in the long-wave case.[1]

The dependent variables are nondimensionalized and expanded to powers of $\varepsilon$,

$$\frac{R^*}{R_0^*} - 1 = \varepsilon R_1 + \varepsilon^2 R_2 + \varepsilon^3 R_3 + O(\varepsilon^4), \tag{28}$$

$$\frac{\alpha}{\alpha_0} - 1 = \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \varepsilon^3\alpha_3 + O(\varepsilon^4), \tag{29}$$

$$\frac{T_{\rm G}^*}{T_0^*} - 1 = \varepsilon T_{\rm G1} + \varepsilon^2 T_{\rm G2} + \varepsilon^3 T_{\rm G3} + O(\varepsilon^4), \tag{30}$$

$$\frac{u^*}{U^*} = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + O(\varepsilon^4), \tag{31}$$

where $U^*$ is the typical phase velocity of the wave. Next, the expansion of the liquid density in $\varepsilon$ is given as follows:[34,40]

$$\frac{\rho_{\rm L}^*}{\rho_{\rm L0}^*} = 1 + \varepsilon^5\rho_{\rm L1} + \varepsilon^6\rho_{\rm L2} + O(\varepsilon^7), \tag{32}$$

which is determined from (14). The expansion was started from $O(\varepsilon^2)$ in the long-wave case.[1] Furthermore, the pressures are nondimensionalized as

$$p_{\rm L} = \frac{p_{\rm L}^*}{\rho_{\rm L0}^* U^{*2}}, \quad p_{\rm L0} = \frac{p_{\rm L0}^*}{\rho_{\rm L0}^* U^{*2}}, \quad p_{\rm G0} = \frac{p_{\rm G0}^*}{\rho_{\rm L0}^* U^{*2}}, \tag{33}$$

where $p_{\rm L}$, $p_{\rm L0}$, and $p_{\rm G0}$ are $O(1)$; $p_{\rm L}$ is expanded as

$$p_{\rm L} = p_{\rm L0} + \varepsilon p_{\rm L1} + \varepsilon^2 p_{\rm L2} + \varepsilon^3 p_{\rm L3} + O(\varepsilon^4). \tag{34}$$

As order estimation of constants, there exists a relationship $U^* = L^*\omega^*$ among $U^*$, $L^*$, and $\omega^*$; we determine the values of the three nondimensionalized parameters as follows:[34,40,42,45]

$$\left(\frac{U^*}{c_{\rm L0}^*}, \frac{R_0^*}{L^*}, \frac{\omega^*}{\omega_{\rm B}^*}\right) = (V\varepsilon^2, \Delta, \Omega), \tag{35}$$

where $V$, $\Delta$, and $\Omega$ are constants of $O(1)$. The following important points should be noted: (i) The incident frequency $\omega^* \approx 17$ kHz and the phase velocity $U^* \approx 9$ m/s for milimeter bubble case (i.e., $R_0^* = 1$ mm), and $\omega^* \approx 24$ MHz and $U^* \approx 14$ m/s for micrometer bubble case (i.e., $R_0^* = 1\,\mu$m) are examples of the order estimation of each scale. (ii) Three ratios in the set (35) were chosen as $O(\sqrt{\varepsilon})$ in the long-wave case.[1] (iii) Although the method of averaged equations should not be ideally applied to such short waves, the plane wave problem can be excluded from the restriction because the average volume along the plane-parallel to the wave front can be sufficiently large;[34] nevertheless, the assumption of spherical symmetry of bubble oscillations should be validated;[34] we will address this problem in future work. Finally, the nondimensional liquid viscosity and the initial effective viscosity are defined using $\varepsilon$,

$$\frac{\mu_{\rm L}^*}{\rho_{\rm L0}^* U^* L^*} = \mu_{\rm L}\varepsilon^2, \tag{36}$$

$$\frac{\mu_{e0}^*}{\rho_{L0}^* U^* L^*} = \mu_{e0}\varepsilon^2, \tag{37}$$

where $\mu_L$ and $\mu_{e0}$ are constants of $O(1)$.

### E. Nondimensionalization of energy equation (11)

Equation (11) is nondimensionalized as

$$\frac{D}{Dt}\left[T_G R^{3(\kappa-1)}\right] = R^{3\kappa-1}\frac{3(\kappa-1)\lambda_G^*}{p_{G0}^*\omega_B^* R_0^*}\frac{\partial T^*}{\partial r^*}\bigg|_{r^*=R^*}, \tag{38}$$

where $T_G = T_G^*/T_{G0}^*$ and $R = R^*/R_0^*$. The nondimensionalized expression on the right-hand side of (38) depends on the selected temperature-gradient model. First, in the case of (17),

$$(\mathrm{RHS}) = \frac{3(\kappa-1)\lambda_G^*}{p_{G0}^*\omega^* R_0^*}\frac{5}{4}\frac{T_0^*}{R_0^*}R^{3\kappa-2}(1-T_G); \tag{39}$$

then, we determine the sizes of the nondimensional number in (39) using $\varepsilon$ as follows:

$$\frac{3(\kappa-1)\lambda_G^*}{p_{G0}^*\omega^* R_0^*}\frac{5}{4}\frac{T_0^*}{R_0^*} \equiv \zeta_{SMK}\varepsilon^2. \tag{40}$$

In the same manner, we determine the sizes of the nondimensional numbers in other cases: in (18),

$$(\mathrm{RHS}) = \frac{3(\kappa-1)\lambda_G^*}{p_{G0}^*\omega^* R_0^*}\frac{T_0^*}{\sqrt{2\pi D^*/\omega_B^*}}R^{3\kappa-1}(1-T_G), \tag{41}$$

$$\frac{3(\kappa-1)\lambda_G^*}{p_{G0}^*\omega^* R_0^*}\frac{T_0^*}{\sqrt{2\pi D^*/\omega_B^*}} \equiv \zeta_{LSM}\varepsilon^2; \tag{42}$$

in (19),

$$(\mathrm{RHS}) = \frac{3(\kappa-1)\lambda_G^*}{p_{G0}^*\omega^* R_0^*}\frac{T_0^*}{|\widetilde{L}_P^*|}R^{3\kappa-1}(1-T_G), \tag{43}$$

$$\frac{3(\kappa-1)\lambda_G^*}{p_{G0}^*\omega^* R_0^*}\frac{T_0^*}{|\widetilde{L}_P^*|} \equiv \zeta_{PCB}\varepsilon^2; \tag{44}$$

and in (20),

$$(\mathrm{RHS}) = \frac{3(\kappa-1)\lambda_G^*}{p_{G0}^*\omega^* R_0^*}R^{3\kappa-1}\left[\frac{\mathrm{Re}(\widetilde{L}_P^*)T_0^*}{|\widetilde{L}_P^*|^2}(1-T_G) + \frac{\omega^*\mathrm{Im}(\widetilde{L}_P^*)T_0^*}{\omega_B^*|\widetilde{L}_P^*|^2}\frac{DT_G}{Dt}\right], \tag{45}$$

$$\frac{3(\kappa-1)\lambda_G^*}{p_{G0}^*\omega^* R_0^*}\frac{\mathrm{Re}(\widetilde{L}_P^*)T_0^*}{|\widetilde{L}_P^*|^2} \equiv \zeta_{STM1}\varepsilon^2, \tag{46}$$

$$\frac{3(\kappa-1)\lambda_G^*}{p_{G0}^*\omega^* R_0^*}\frac{\omega^*\mathrm{Im}(\widetilde{L}_P^*)T_0^*}{\omega_B^*|\widetilde{L}_P^*|^2} \equiv \zeta_{STM2}\varepsilon^2, \tag{47}$$

where $\zeta_{SMK}$, $\zeta_{LSM}$, $\zeta_{PCB}$, $\zeta_{STM1}$, and $\zeta_{STM2}$ are constants of $O(1)$. Note that (42), (44), (46), and (47) determine the sizes of nondimensional ratios.

## III. DERIVATION

This section focuses on the derivation of the NLS equation and a discussion will be presented in Sec. IV.

### A. Leading order of approximation

By equating the coefficients of the like powers of $\varepsilon$ in the governing equations (8)–(12), a set of linearized first-order equations can be derived

$$\frac{\partial \alpha_1}{\partial t_0} - 3\frac{\partial R_1}{\partial t_0} + \frac{\partial u_1}{\partial x_0} = 0, \tag{48}$$

$$\alpha_0\frac{\partial \alpha_1}{\partial t_0} - (1-\alpha_0)\frac{\partial u_1}{\partial x_0} = 0, \tag{49}$$

$$(1-\alpha_0)\frac{\partial u_1}{\partial t_0} + \frac{\partial p_{L1}}{\partial x_0} = 0, \tag{50}$$

$$\frac{\partial T_{G1}}{\partial t_0} + 3(\kappa-1)\frac{\partial R_1}{\partial t_0} = 0, \tag{51}$$

$$\left[3(\gamma_e-1)p_{G0} - \Delta^2\right]R_1 + p_{G0}T_1 - p_{L1} - \Delta^2\frac{\partial^2 R_1}{\partial t_0^2} = 0. \tag{52}$$

Subsequently, we obtain a single partial differential equation with a dispersion term for the first-order perturbation of the bubble radius, $R_1$,

$$\mathscr{L}[R_1] = 0,$$
$$\mathscr{L} = \frac{\partial^2}{\partial t_0^2} + \frac{\Delta^2 - 3(\gamma_e-\kappa)p_{G0}}{3\alpha_0(1-\alpha_0)}\frac{\partial^2}{\partial x_0^2} - \frac{\Delta^2}{3\alpha_0(1-\alpha_0)}\frac{\partial^4}{\partial t_0^2\partial x_0^2}. \tag{53}$$

Because of the dispersion effect, the wave profile is split into each component with its own propagation speed if the initial wave is a superposition of different harmonic components. We, therefore, consider the solution of (53) in the form of a quasimonochromatic wave train that evolves into a slowly modulated wave packet,[34]

$$R_1 = A(t_1, x_1; t_2, x_2)e^{i\theta} + \text{c.c.}, \tag{54}$$

$$\theta = kx_0 - \Omega(k)\,t_0, \tag{55}$$

where $A$ is the complex amplitude, $k = k^*L^*$ is the nondimensional wavenumber ($k^*$ is the dimensional wavenumber), $\theta$ is the phase function, and c.c. denotes complex conjugate. Therefore, the slow variation of the carrier wave $e^{i\theta}$ is described using envelope wave $A$.

After substituting (54) into (53), we obtain a linear dispersion relation,

$$D(k,\Omega) = -\Omega^2 + \frac{\Delta^2 + 3(\kappa-\gamma_e)p_{G0}}{3\alpha_0(1-\alpha_0)}k^2 - \frac{\Delta^2\Omega^2k^2}{3\alpha_0(1-\alpha_0)} = 0, \tag{56}$$

or

$$\Omega = k\sqrt{\frac{\Delta^2 + 3(\kappa-\gamma_e)p_{G0}}{3\alpha_0(1-\alpha_0) + \Delta^2k^2}}. \tag{57}$$

From (57), the nondimensional phase velocity $v_p$ and group velocity $v_g$ are obtained as

$$v_p = \frac{\Omega}{k} = \sqrt{\frac{\Delta^2 + 3(\kappa-\gamma_e)p_{G0}}{3\alpha_0(1-\alpha_0) + \Delta^2k^2}}, \tag{58}$$

$$v_g = \frac{d\Omega}{dk} = \frac{3\alpha_0(1-\alpha_0)\Omega}{k\left[3\alpha_0(1-\alpha_0) + \Delta^2k^2\right]}. \tag{59}$$

Note that both $v_p$ and $v_g$ include $\gamma_e$, which did not appear in our previous studies.[34,40–44,46–48] This change is the same as in the long-wave case.[1]

After substituting (54) into (48)–(52) and integrating them with respect to $t_0$ and $x_0$ under the boundary condition at $x_0 \to \infty$, where the bubbly liquid is uniform and at rest, we obtain

$$\alpha_1 = b_1 R_1, \quad u_1 = b_2 R_1, \quad T_1 = b_3 R_1, \quad p_{L1} = b_4 R_1 \tag{60}$$

with

$$\begin{aligned} b_1 &= 3(1-\alpha_0), \quad b_2 = -3\alpha_0 v_p, \\ b_3 &= -3(\kappa - 1), \quad b_4 = b_2(1-\alpha_0)v_p. \end{aligned} \tag{61}$$

## B. Second order of approximation

The set of second-order equations is as follows:

$$\frac{\partial \alpha_2}{\partial t_0} - 3\frac{\partial R_2}{\partial t_0} + \frac{\partial u_2}{\partial x_0} = M_1, \tag{62}$$

$$\alpha_0 \frac{\partial \alpha_2}{\partial t_0} - (1-\alpha_0)\frac{\partial u_2}{\partial x_0} = M_2, \tag{63}$$

$$(1-\alpha_0)\frac{\partial u_2}{\partial t_0} + \frac{\partial p_{L2}}{\partial x_0} = M_3, \tag{64}$$

$$\frac{\partial T_{G2}}{\partial t_0} + 3(\kappa-1)\frac{\partial R_2}{\partial t_0} = M_4, \tag{65}$$

$$\left[3(\gamma_e - 1)p_{G0} - \Delta^2\right]R_2 + p_{G0}T_2 - p_{L2} - \Delta^2\frac{\partial^2 R_2}{\partial t_0^2} = M_5, \tag{66}$$

where the explicit forms of the inhomogeneous terms $M_j$ $(j = 1, 2, 3, 4, 5)$ are shown in Appendix A. Equations (62)–(66) can be reduced to a single inhomogeneous equation for $R_2$,

$$\mathscr{L}[R_2] = \Gamma A^2 \mathrm{e}^{2\mathrm{i}\theta} + \mathrm{i}\left(-\frac{\partial D}{\partial \Omega}\right)\left(\frac{\partial A}{\partial t_1} + v_g\frac{\partial A}{\partial x_1}\right)\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.}, \tag{67}$$

where the real constant $\Gamma$ is given by

$$\Gamma = -\frac{2}{3}\left[\Omega m_1 - \frac{\Omega m_2}{\alpha_0} - \frac{k m_3}{\alpha_0(1-\alpha_0)} + \frac{p_{G0} m_4}{\alpha_0(1-\alpha_0)} + \frac{2\Delta^2 k^2 m_5}{\alpha_0(1-\alpha_0)}\right] \tag{68}$$

with

$$\begin{aligned} m_1 &= 6(2-b_1)\Omega + 2b_2(3-b_1)k, \\ m_2 &= -2\alpha_0 b_1 b_2 k, \quad m_3 = 2\alpha_0 b_1 b_2 \Omega + 2(1-\alpha_0)b_2^2 k, \\ m_4 &= 6(\kappa-1)\left(\frac{3\kappa-4}{2!} + b_3\right)k^2, \\ m_5 &= 3b_2\Omega k - \frac{5}{2}\Omega^2 + \frac{1}{\Delta^2}\left[3(b_3 + \gamma_e - 2)p_{G0} - \Delta^2\right]. \end{aligned} \tag{69}$$

From the solvability condition for (67), the coefficient of $\mathrm{e}^{\mathrm{i}\theta}$ on the right-hand side of (67) should be zero.[22,34] Hence, we obtain

$$\frac{\partial A}{\partial t_1} + v_g\frac{\partial A}{\partial x_1} = 0. \tag{70}$$

Applying (70) to (67), the solution of (67) uniformly valid up to the concerned far field is given by[34]

$$\begin{aligned} R_2 &= c_0 A^2 \mathrm{e}^{2\mathrm{i}\theta} + \text{c.c.}, \\ c_0 &= \frac{\Gamma}{D_{22}}, \quad D_{22} = -\frac{4\Delta^2 k^2 \Omega^2}{\alpha_0(1-\alpha_0)}. \end{aligned} \tag{71}$$

After substituting (71) into (62)–(66), we obtain

$$\begin{pmatrix} \alpha_2 \\ u_2 \\ T_2 \\ p_{L2} \end{pmatrix} = \begin{pmatrix} c_1 & d_1 & 0 \\ c_2 & d_2 & 0 \\ c_3 & d_3 & 0 \\ c_4 & d_4 & c_s \end{pmatrix} \begin{pmatrix} A^2\mathrm{e}^{2\mathrm{i}\theta} + \text{c.c.} \\ \mathrm{i}\partial A/\partial t_1 \mathrm{e}^{\mathrm{i}\theta} + \text{c.c.} \\ |A|^2 \end{pmatrix} \tag{72}$$

with

$$\begin{aligned} c_1 &= -\frac{1}{2\alpha_0\Omega}\left[2(1-\alpha_0)c_2 k + m_2\right], \\ c_2 &= \frac{1}{2k}(m_1 - m_2 - 6\alpha_0 c_0 \Omega), \quad c_3 = -3(\kappa-1)c_0 - \frac{m_4}{2k^2}, \\ c_4 &= \left[3(\gamma_e - 1)p_{G0} - (1-4\Omega^2)\Delta^2\right]c_0 + p_{G0}c_3 - \Delta^2 m_5, \\ d_1 &= -\frac{1}{\alpha_0\Omega}\left[(1-\alpha_0)d_2 k + \alpha_0 b_1 + \frac{1-\alpha_0}{v_g}b_2\right], \\ d_2 &= \frac{1}{k}\left(2\alpha_0 b_1 - 3\alpha_0 - \frac{2\alpha_0 - 1}{v_g}b_2\right), \\ d_3 &= 0, \quad d_4 = p_{G0}d_3 - 2\Delta^2\Omega, \\ c_s &= \Delta^2(2b_2 k + \Omega)\Omega + 2\left[3(\gamma_e + b_3 - 2)p_{G0} - \Delta^2\right]. \end{aligned} \tag{73}$$

## C. Third order of approximation and resultant NLS equation

The set of third-order equations is as follows:

$$\frac{\partial \alpha_3}{\partial t_0} - 3\frac{\partial R_3}{\partial t_0} + \frac{\partial u_3}{\partial x_0} = N_1, \tag{74}$$

$$\alpha_0 \frac{\partial \alpha_3}{\partial t_0} - (1-\alpha_0)\frac{\partial u_3}{\partial x_0} = N_2, \tag{75}$$

$$(1-\alpha_0)\frac{\partial u_3}{\partial t_0} + \frac{\partial p_{L3}}{\partial x_0} = N_3, \tag{76}$$

$$\frac{\partial T_{G3}}{\partial t_0} + 3(\kappa-1)\frac{\partial R_3}{\partial t_0} = N_4, \tag{77}$$

$$\left[3(\gamma_e - 1)p_{G0} - \Delta^2\right]R_3 + p_{G0}T_3 - p_{L3} - \Delta^2\frac{\partial^2 R_3}{\partial t_0^2} = N_5, \tag{78}$$

where the explicit forms of the inhomogeneous terms $N_j$ $(j = 1, 2, 3, 4, 5)$ are shown in Appendix B. Equations (74)–(78) are reduced to the following single inhomogeneous equation for $R_3$:

$$\mathscr{L}[R_3] = \Lambda_1 \mathrm{e}^{3\mathrm{i}\theta} + \Lambda_2 \mathrm{e}^{2\mathrm{i}\theta} + \Lambda_3 \mathrm{e}^{\mathrm{i}\theta} + \text{c.c.}, \tag{79}$$

where $\Lambda_1$ and $\Lambda_2$ are not shown because they are not essential for the derivation of the NLS equation. By imposing the nonsecular condition[22,34] to (79), we obtain

$$\Lambda_3 = \left(-\frac{\partial D}{\partial \Omega}\right)\left[\mathrm{i}\left(\frac{\partial A}{\partial t_2} + v_g\frac{\partial A}{\partial x_2}\right) + \nu_1|A|^2 A + \mathrm{i}\nu_2 A + \nu_3\frac{\partial^2 A}{\partial x_1^2}\right] = 0. \tag{80}$$

Combining (70) and (80) with the help of the derivative expansion method[22] based on multiple-scales (27), we have

$$\mathrm{i}\left(\frac{\partial A}{\partial t_2} + v_g\frac{\partial A}{\partial x_2}\right) + \varepsilon^2(\nu_1|A|^2 A + \mathrm{i}\nu_2 A) + \nu_3\frac{\partial^2 A}{\partial x_1^2} = 0. \tag{81}$$

Finally, we obtain the NLS equation,

$$i\frac{\partial A}{\partial \tau}+\nu_1|A|^2A+i\nu_2A+\nu_3\frac{\partial^2 A}{\partial \xi^2}=0, \tag{82}$$

$$\tau=\varepsilon^2 t, \quad \xi=\varepsilon(x-v_{\mathrm{g}}t), \tag{83}$$

where the real constant $\nu_1$ is the nonlinear coefficient,

$$\nu_1=\frac{1}{3}\frac{1}{\partial D/\partial\Omega}\left[\Omega n_1-\frac{\Omega}{\alpha_0}n_2+\frac{kn_3}{\alpha_0(1-\alpha_0)}+\frac{p_{\mathrm{G0}}n_4}{\alpha_0(1-\alpha_0)}-\frac{\Delta^2k^2n_5}{\alpha_0(1-\alpha_0)}\right] \tag{84}$$

with

$$\begin{aligned} n_1&=3[c_0(4-b_1)-c_1+6b_1-10]\Omega\\ &\quad+[c_2(3-b_1)+b_2(3c_0-c_1+9b_1-18)]k,\\ n_2&=-\alpha_0(b_1c_2+b_2c_1)k,\\ n_3&=n_2v_{\mathrm{p}}+b_2[3\alpha_0b_1b_2-2(1-\alpha_0)c_2]k,\\ n_4&=\left\{3(\kappa-1)\left[c_3+b_3c_0+(3\kappa-4)\left(c_0+\frac{3b_3}{2}+\frac{3\kappa-5}{2}\right)\right]\right.\\ &\quad-[2b_2c_3+2b_2c_0+b_3c_2+3(\kappa-1)c_2\\ &\quad\left.+(3\kappa-3)(3\kappa-4)b_2+6(\kappa-1)b_2b_3]\frac{k}{\Omega}\right\}k^2,\\ n_5&=-3+c_0(2-\Omega^2)+2b_2[b_2k-(1+3c_0)\Omega]k\\ &\quad-\frac{3(10-4c_0+c_3+b_3c_0-6b_3+2\gamma_{\mathrm{e}}c_0-3\gamma_{\mathrm{e}})}{\Delta^2}p_{\mathrm{G0}}. \end{aligned} \tag{85}$$

Furthermore, the real constants $\nu_2$ and $\nu_3$ are the dissipation and dispersion coefficients, respectively,

$$\begin{aligned}\nu_2=-\frac{1}{\partial D/\partial\Omega}\frac{\Omega k^2}{3\alpha_0(1-\alpha_0)}\Bigg(&-\frac{4(1+\alpha_0)b_2\mu_{\mathrm{L}}k}{3\Omega}+4\mu_{\mathrm{L}}\\ &+V\Delta\left\{\Delta^2-[3(\gamma_{\mathrm{e}}-1)+b_3]p_{\mathrm{G0}}\right\}-\frac{p_{\mathrm{G0}}b_3\zeta}{\Omega^2}\Bigg),\end{aligned} \tag{86}$$

$$\begin{aligned}\nu_3=-\frac{1}{\partial D/\partial\Omega}\frac{v_{\mathrm{g}}}{3\alpha_0(1-\alpha_0)}\Bigg\{&\left[(1-\alpha_0)d_2k+\frac{p_{\mathrm{G0}}d_3k^2}{\Omega}-\Delta^2k^2\right]v_{\mathrm{g}}\\ &-[(1-\alpha_0)d_2\Omega+d_4k]\Bigg\},\end{aligned} \tag{87}$$

where $\zeta$ represents $\zeta_{\mathrm{SMK}}$, $\zeta_{\mathrm{LSM}}$, $\zeta_{\mathrm{PCB}}$, and $\zeta_{\mathrm{STM1}}$.

## IV. DISCUSSION

In this section, we discuss the differences between the present study and two previous studies (Kanagawa *et al.*[40] and Kamei and Kanagawa[41,42]). It should be noted that the existing studies, which derived NLS equations, focused on a quite-high (not moderately high) frequency band.[27,31] Herein, first, we introduce the NLS equation derived in the previous studies.[40–42] The equation reported in Ref. 40 is as follows:

$$i\frac{\partial A}{\partial \tau}+\widetilde{\nu_1}|A|^2A+i\widetilde{\nu_2}A+\widetilde{\nu_3}\frac{\partial^2 A}{\partial \xi^2}=0, \tag{88}$$

$$\tau=\varepsilon^2 t, \quad \xi=\varepsilon(x-v_{\mathrm{g}}t) \tag{89}$$

with

$$\widetilde{\nu_1}=\frac{1}{\partial D/\partial\Omega}\left[\frac{\Omega\widetilde{n_1}}{3}-\frac{\Omega\widetilde{n_2}}{3\alpha_0}+\frac{k\widetilde{n_3}}{3\alpha_0(1-\alpha_0)}-k^2\widetilde{n_4}\right], \tag{90}$$

$$\widetilde{\nu_2}=\frac{(4\mu_{\mathrm{L}}+\Delta^3V)\Omega^2}{2\Delta^2}, \tag{91}$$

$$\widetilde{\nu_3}=-\frac{3(1-\Omega^2)\Omega^3}{2k^2}. \tag{92}$$

The constants $\widetilde{n_j}$ ($j=1,2,3,4,5$) are defined in Eq. (63) in our original paper[40] (see the detailed definitions and explanations in Ref. 40). In our original study,[40] we used $\gamma$ rather than $\gamma_{\mathrm{e}}$; the thermal conduction was not considered. In Sec. IV A, we express the quantity obtained in our previous study[40] using $\widetilde{\ }$.

Further, the other NLS equation[41,42] is given by

$$i\frac{\partial A}{\partial \tau}+\widehat{\nu_1}|A|^2A+i\widehat{\nu_2}A+\widehat{\nu_3}\frac{\partial^2 A}{\partial \xi^2}=0, \tag{93}$$

$$\tau=\varepsilon^2 t, \quad \xi=\varepsilon(x-v_{\mathrm{g}}t) \tag{94}$$

with

$$\widehat{\nu_1}=-\frac{1}{3}\frac{1}{\partial D/\partial\Omega}\left[-\Omega\widehat{n_1}+\frac{\Omega\widehat{n_2}}{\alpha_0}-\frac{k\widehat{n_3}}{\alpha_0(1-\alpha_0)}-\frac{p_{\mathrm{G0}}\widehat{n_4}}{\alpha_0(1-\alpha_0)(1-\alpha_0-\delta p_{\mathrm{G0}})}+\frac{\Delta^2k^2\widehat{n_5}}{\alpha_0(1-\alpha_0-\delta p_{\mathrm{G0}})}\right], \tag{95}$$

$$\begin{aligned}\widehat{\nu_2}=\frac{k^2}{2[3\alpha_0(1-\alpha_0-\delta p_{\mathrm{G0}})+\Delta^2k^2]}&\left\{4\mu_{\mathrm{L}}\alpha_0(1+\alpha_0)\frac{1-\alpha_0-\delta p_{\mathrm{G0}}}{1-\alpha_0}+\lambda\frac{3\alpha_0\delta p_{\mathrm{G0}}}{1-\alpha_0}+4\mu_{\mathrm{L}}\right.\\ &\left.+V\Delta\left[\Delta^2-3\left(\gamma-1-\alpha_0\delta\frac{\Omega^2}{k^2}\right)p_{\mathrm{G0}}\right]\right\},\end{aligned} \tag{96}$$

$$\widehat{\nu_3}=-\frac{1}{\partial D/\partial\Omega}\left[\frac{-3\Omega+kv_{\mathrm{g}}}{6\alpha_0(1-\alpha_0)}d_4-\frac{\Delta^2k^2v_{\mathrm{g}}}{3\alpha_0(1-\alpha_0-\delta p_{\mathrm{G0}})}\right]v_{\mathrm{g}}. \tag{97}$$

The constants $\widehat{n_j}$ ($j=1,2,3,4,5$) are defined in Eqs. (53)–(55) in the previous paper[42] (see the detailed definitions and explanations in Ref. 42). In the previous studies,[41,42] we used $\gamma$ rather than $\gamma_{\mathrm{e}}$ and considered the thermal conduction through the bubbly liquid instead of the bubble–liquid interface. In Subsections IV A–IV E, we express the quantity obtained in the previous studies[41,42] using $\widehat{\ }$.

### A. Effective polytropic exponent

Herein, we discuss the dependences of some coefficients on the initial bubble radius $R_0^*$ and the nondimensional wavenumber $k$. First, the effective polytropic exponent, $\gamma_{\mathrm{e}}$, depends on $R_0^*$ but not on $k$. Next, for a small value of $R_0^*$, the thermodynamic processes occurring inside the bubble become isothermal (i.e., $\gamma_{\mathrm{e}}\to 1$), and for a large value of $R_0^*$, the processes become adiabatic (i.e., $\gamma_{\mathrm{e}}\to\kappa$).

### B. Dispersion coefficient

Dispersion coefficient in this study, $\nu_3$, increases with increasing $R_0^*$, while $\widehat{\nu_3}$ decreases with increasing $R_0^*$; $\widetilde{\nu_3}$ does not depend on $R_0^*$. For the dependence of $k$, $\nu_3$, $\widetilde{\nu_3}$, and $\widehat{\nu_3}$ have an extremum at a small wavenumber. With regard to the size of dispersion (i.e., the absolute value of dispersion coefficient), we obtain $|\nu_3| < |\widetilde{\nu}_3| < |\widehat{\nu}_3|$.

## C. Dissipation coeffcient: Overall trend

Herein, we discuss the most important aspect: dissipation coefficients. In this study, we obtained only one type of dissipation term $\mathrm{i}\nu_2 A$, subsequently, all the dissipation factors were included as linear combination in the dissipation coefficient $\nu_2$, while the dissipation effects for the case of KdVB equation (i.e., low frequency long wave)[1] were included in two types of dissipation terms—the well-known second-order derivative with respect to the space coordinate and the term without differentiation. Therefore, we can discuss the dissipation effect by focusing on only $\nu_2$. In the following, we present the dependence of $\nu_2$ on the initial bubble radius $R_0^*$ and nondimensional wavenumber $k$ since the dependence of $\nu_2$ on the initial void fraction $\alpha_0$ is quite small.

Figure 2(a) shows the dependence of the dissipation coefficients, $\nu_2$, $\widetilde{\nu}_2$, and $\widehat{\nu}_2$ on $R_0^*$, where the SMK model (17) is used as the temperature-gradient model in $\nu_2$. All coefficients decrease with increasing $R_0^*$. The present coefficient $\nu_2$ is the largest among all coefficients because the thermal conduction at the bubble–liquid interface is incorporated, and the present dissipation effect becomes more than four times larger than the cases without thermal conduction. Figure 2(b) shows the dependence of $\nu_2$, $\widetilde{\nu}_2$, and $\widehat{\nu}_2$ on $k$. All

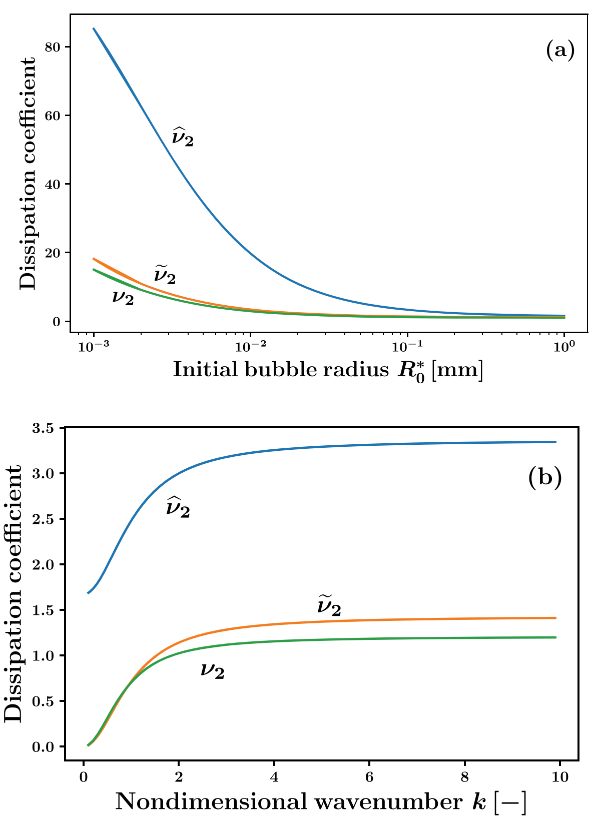


**FIG. 2.** Dissipation coefficients, $\nu_2$ (green), $\widetilde{\nu}_2$ (orange), and $\widehat{\nu}_2$ (green): dependence of (a) initial bubble radius $R_0^*$ and (b) nondimensional wavenumber $k$ for $\varepsilon = 0.07$, $\alpha_0 = 0.01$, $k = 10$, $c_{L0}^* = 1500$ m/s, and $\mu_L^* = 1 \times 10^{-3}$ Pa s; the same condition is used in Figs. 3–5.

coefficients increase with increasing $k$. As in the case of the dependence on $R_0^*$ [Fig. 2(a)], the present coefficient $\nu_2$ has the largest value among all coefficients because the thermal conduction at the bubble–liquid interface is incorporated.

## D. Dissipation coefficient: Three factors

We next discuss each dissipation factor. The three dissipation coefficients, $\nu_2$, $\widetilde{\nu}_2$, and $\widehat{\nu}_2$, are decomposed into

$$\begin{aligned} \nu_2 &= \nu_{\mathrm{vis}} + \nu_{\mathrm{ac}} + \nu_{\mathrm{th}}, \\ \nu_{\mathrm{vis}} &= -\frac{1}{\partial D/\partial\Omega}\frac{\Omega k^2}{3\alpha_0(1-\alpha_0)}\left[4\mu_{\mathrm{L}} + \frac{4(1+\alpha_0)b_2\mu_{\mathrm{L}}k}{3\Omega}\right], \\ \nu_{\mathrm{ac}} &= -\frac{1}{\partial D/\partial\Omega}\frac{\Omega k^2}{3\alpha_0(1-\alpha_0)}V\Delta\left\{\Delta^2 - [3(\gamma_{\mathrm{e}}-1)+b_3]p_{\mathrm{G0}}\right\}, \\ \nu_{\mathrm{th}} &= \frac{1}{\partial D/\partial\Omega}\frac{\Omega k^2}{3\alpha_0(1-\alpha_0)}\frac{p_{\mathrm{G0}}b_3\zeta}{\Omega^2}, \end{aligned} \tag{98}$$

$$\begin{aligned} \widetilde{\nu}_2 &= \widetilde{\nu}_{\mathrm{vis}} + \widetilde{\nu}_{\mathrm{ac}}, \\ \widetilde{\nu}_{\mathrm{vis}} &= \frac{4\mu_{\mathrm{L}}\Omega^2}{2\Delta^2}, \quad \widetilde{\nu}_{\mathrm{ac}} = \frac{V\Delta\Omega^2}{2}, \end{aligned} \tag{99}$$

$$\begin{aligned} \widehat{\nu}_2 &= \widehat{\nu}_{\mathrm{vis}} + \widehat{\nu}_{\mathrm{ac}} + \widehat{\nu}_{\mathrm{th}}, \\ \widehat{\nu}_{\mathrm{vis}} &= \frac{k^2}{2[3\alpha_0(1-\alpha_0-\delta p_{\mathrm{G0}})+\Delta^2k^2]} \\ &\quad\times\left[4\mu_{\mathrm{L}}\alpha_0(1+\alpha_0)\frac{1-\alpha_0-\delta p_{\mathrm{G0}}}{1-\alpha_0} + 4\mu_{\mathrm{L}}\right], \\ \widehat{\nu}_{\mathrm{ac}} &= \frac{k^2}{2[3\alpha_0(1-\alpha_0-\delta p_{\mathrm{G0}})+\Delta^2k^2]} \\ &\quad\times V\Delta\left[\Delta^2 - 3\left(\gamma - 1 - \alpha_0\delta\frac{\Omega^2}{k^2}\right)p_{\mathrm{G0}}\right], \\ \widehat{\nu}_{\mathrm{th}} &= \frac{k^2}{2[3\alpha_0(1-\alpha_0-\delta p_{\mathrm{G0}})+\Delta^2k^2]}\lambda\frac{3\alpha_0\delta p_{\mathrm{G0}}}{1-\alpha_0}, \end{aligned} \tag{100}$$

where $\zeta$ represents $\zeta_{\mathrm{SKM}}$, $\zeta_{\mathrm{LSM}}$, $\zeta_{\mathrm{PCB}}$, and $\zeta_{\mathrm{STM1}}$, and the subscripts "vis," "ac," and "th" represent the viscosity, acoustic radiation due to the liquid compressibility, and thermal conduction, respectively.

Regarding the viscosity, Fig. 3(a) shows the dependence of the dissipation coefficients: $\nu_{\mathrm{vis}}$, $\widetilde{\nu}_{\mathrm{vis}}$, and $\widehat{\nu}_{\mathrm{vis}}$ on $R_0^*$. Clearly, $\nu_{\mathrm{vis}}$ is larger than $\widehat{\nu}_{\mathrm{vis}}$ but is comparable to $\widetilde{\nu}_{\mathrm{vis}}$. Figure 3(b) shows the dependence of $\nu_{\mathrm{vis}}$, $\widetilde{\nu}_{\mathrm{vis}}$, and $\widehat{\nu}_{\mathrm{vis}}$ on $k$. Whereas $\nu_2$ has the smallest value among all coefficients for small $k$, $\nu_{\mathrm{vis}}$ is larger than $\widehat{\nu}_{\mathrm{vis}}$ and is smaller than $\widetilde{\nu}_{\mathrm{vis}}$ with increasing $k$.

Regarding the acoustic radiation, Fig. 4(a) shows the dependence of the dissipation coefficients: $\nu_{\mathrm{ac}}$, $\widetilde{\nu}_{\mathrm{ac}}$, and $\widehat{\nu}_{\mathrm{ac}}$ on $R_0^*$. All the coefficients decrease with increasing $R_0^*$, $\nu_{\mathrm{ac}}$ has the largest value among all coefficients. Figure 4(b) shows the dependence of $\nu_{\mathrm{ac}}$, $\widetilde{\nu}_{\mathrm{ac}}$, and $\widehat{\nu}_{\mathrm{ac}}$ on $k$. All the coefficients increase with increasing $R_0^*$. Similar to the case of the dependence on the bubble radius, $\nu_{\mathrm{ac}}$ has the largest value among all coefficients.

Regarding the thermal conduction, Fig. 5(a) shows the dependence of $\nu_{\mathrm{th}}$ and $\widehat{\nu}_{\mathrm{th}}$ on $R_0^*$. All the temperature-gradient models [i.e., SMK model (17), LSM model (18), PCB model (19), and STM model (20)] are used in $\nu_{\mathrm{th}}$. Note that our original study[40] neglected the effect of thermal conduction. From the comparison, the effect of thermal conduction at the bubble–liquid interface (green, red, blue, and orange curves) is larger than that of thermal conduction through the bubbly

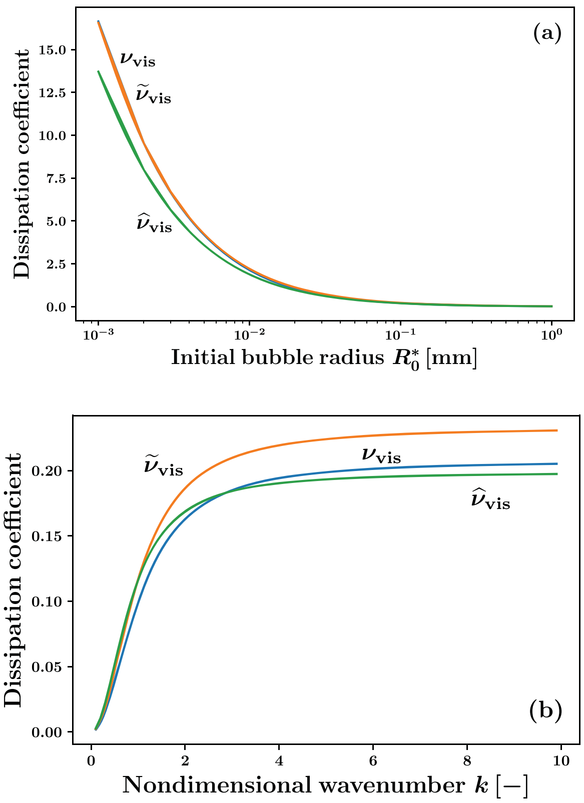


FIG. 3. Dissipation coefficients due to the viscosity, $\nu_{\mathrm{vis}}$ (blue), $\widetilde{\nu_{\mathrm{vis}}}$ (orange), and $\widehat{\nu_{\mathrm{vis}}}$ (green): dependence of (a) $R_0^*$ and (b) $k$. Note that $\nu_{\mathrm{vis}}$ is quite close to $\widetilde{\nu_{\mathrm{vis}}}$ in (a).

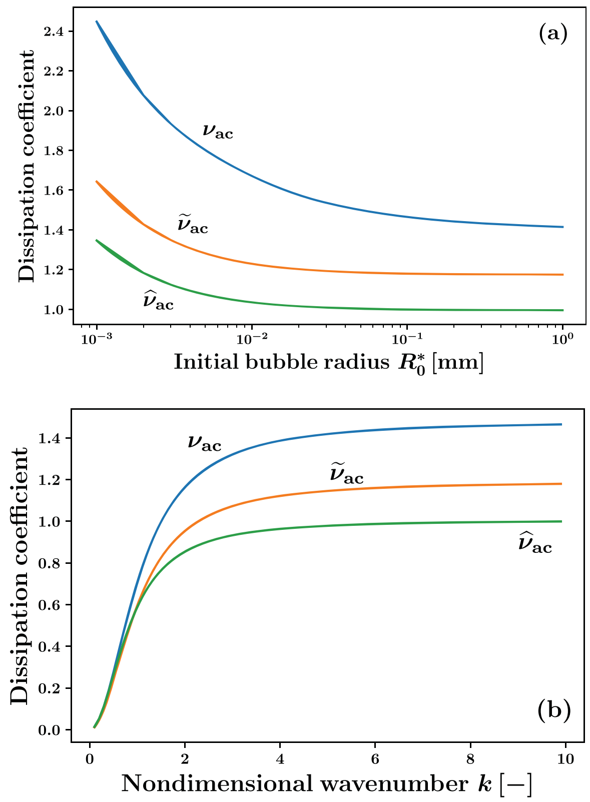


FIG. 4. Dissipation coefficients due to the acoustic radiation, $\nu_{\mathrm{ac}}$ (blue), $\widetilde{\nu_{\mathrm{ac}}}$ (orange), and $\widehat{\nu_{\mathrm{ac}}}$ (green): dependence of (a) $R_0^*$ and (b) $k$.

liquid (purple curve). The order of $\nu_{\mathrm{th}}$ is more than $O(10)$, whereas that of $\widehat{\nu_{\mathrm{th}}}$ is less than $O(10^{-6})$.[41] This indicates that, in this study, the effect of thermal conduction is evaluated correctly by focusing on the heat entering and leaving individual bubbles by incorporating the energy equation (11). Although the difference among the four models (17)–(20) is small for milliscale bubbles, the values in the PCB and STM models are large compared with those in the LSM and SMK models for microscale bubbles. The PCB model is similar to the STM model, and the LSM model is similar to the SMK model. Figure 5(b) shows the dependence of $\nu_{\mathrm{th}}$ and $\widehat{\nu_{\mathrm{th}}}$ on $k$; $\nu_{\mathrm{th}}$ is a constant and has no dependence. The order of $\nu_{\mathrm{th}}$ is $O(10)$, whereas that of $\widehat{\nu_{\mathrm{th}}}$ is less than $O(10^{-7})$.[41] Hence, similar to the discussion of $R_0^*$ dependence [Fig. 5(a)], we conclude that the effect of thermal conduction is evaluated correctly.

## E. Comparison of present NLS case with previous KdVB case

Finally, we compare the results of the present NLS equation (82) with those of the previous KdVB equation.[1] The similarities between the NLS and KdVB equations are summarized as follows: (i) The dissipation effect due to viscosity is decreased; (ii) the dissipation effect due to acoustic radiation is increased; and (iii) the thermal conduction at the bubble–liquid interface has the largest contribution to the dissipation effect.

The differences between the NLS and KdVB equations are summarized as follows: (iv) The dissipation factors are consolidated into one type of dissipation coefficient $\nu_2$ as a linear combination in the NLS equation (82), in the KdVB equation, the dissipation factors are divided into two types of dissipation coefficients; (v) the dissipation effect is mostly due to thermal conduction in the KdVB equation, whereas dissipation due to thermal conduction is approximately half of all dissipation effects in the NLS equation (82).

## V. CONCLUSIONS

Weakly nonlinear propagation of pressure waves with a moderately high (not quite-high) frequency and a short wavelength (Fig. 1 and Table I) was theoretically studied in an initially quiescent compressible liquid comprising several uniformly-distributed spherical microbubbles. The investigation was based on the derivation of the NLS equation (82). In particular, the energy equation at the bubble–liquid interface (11) and the effective polytropic exponent (22)[24] were introduced.

The main results of this paper are summarized as follows:

(i) The effect of thermal conduction at the bubble–liquid interface on dissipation was the largest among all dissipation effects (i.e., the viscosity, acoustic radiation, and thermal conduction).

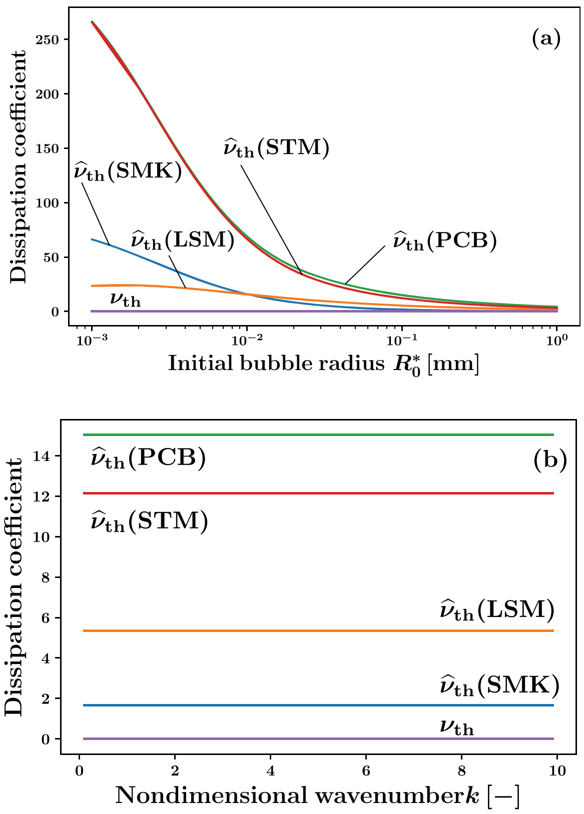


**FIG. 5.** Dissipation coefficients due to the thermal conduction, $\nu_{\rm th}$ (purple) and $\widehat{\nu_{\rm th}}$ (blue, orange, red, and green): dependence of (a) $R_0^*$ and (b) $k$. Blue, orange, red, and green curves represent SMK, LSM, STM, and PCB models, respectively.

(ii) Four models (17)–(20)[56–59] were utilized to evaluate the thermal dissipation effect. The difference between the four models was significant for microscale bubbles and insignificant for milliscale bubbles.

(iii) All dissipation effects were summarized in one type of dissipation term $i\nu_2 A$ as a linear combination. However, in the case of the KdVB equation,[1] those were summarized in two types of dissipation terms: the well-known second-order derivative term with respect to the space coordinate and the newly discovered undifferentiated term.

(iv) The presented thermal effect contributed not only to the dissipation coefficient $\nu_2$ but also to the dispersion coefficient $\nu_3$.

(v) Dissipation due to thermal conduction was approximately half of all dissipation effects in the present short-wave case. In the long-wave case, dissipation due to thermal conduction was close to all dissipation effects.[1]

In summary, we achieved a detailed understanding of the thermal effect inside bubbles on the pressure wave in bubbly liquids in the framework of weakly nonlinear theory for the case of the NLS equation. In a forthcoming paper, we will introduce the temperature in the liquid phase as an unknown variable.

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## APPENDIX A: INHOMOGENEOUS TERMS IN (62)–(66)

Explicit forms of the inhomogeneous terms $M_j$ $(j = 1, 2, 3, 4, 5)$ are given by

$$M_1 = -\left(\frac{\partial \alpha_1}{\partial t_1} - 3\frac{\partial R_1}{\partial t_1} + \frac{\partial u_1}{\partial x_1}\right) + 3\frac{\partial}{\partial t_0}\left(\alpha_1 R_1 - 2R_1^2\right) + \frac{\partial}{\partial x_0}\left(3u_1 R_1 - \alpha_1 u_1\right), \tag{A1}$$

$$M_2 = -\left[\alpha_0 \frac{\partial \alpha_1}{\partial t_1} - (1-\alpha_0)\frac{\partial u_1}{\partial x_1}\right] - \alpha_0 \frac{\partial \alpha_1 u_1}{\partial x_0}, \tag{A2}$$

$$M_3 = -\left[(1-\alpha_0)\frac{\partial u_1}{\partial t_1} + \frac{\partial p_{L1}}{\partial x_1}\right] + \alpha_0 \frac{\partial \alpha_1 u_1}{\partial t_0} - (1-\alpha_0)\frac{\partial u_1^2}{\partial x_0}, \tag{A3}$$

$$M_4 = -\left[\frac{\partial T_{G1}}{\partial t_1} + 3(\kappa - 1)\frac{\partial R_1}{\partial t_1}\right] - u_1 \frac{\partial T_{G1}}{\partial x_0} - 3(\kappa-1)\left[u_1 \frac{\partial R_1}{\partial x_0} + \frac{\partial T_{G1} R_1}{\partial t_0} + \frac{3(\kappa-1)-1}{2}\frac{\partial R_1^2}{\partial t_0}\right], \tag{A4}$$

$$M_5 = 2\Delta^2 \frac{\partial^2 R_1}{\partial t_0 \partial t_1} + \Delta^2 \frac{\partial u_1}{\partial t_0}\frac{\partial R_1}{\partial x_0} + 2\Delta^2 u_1 \frac{\partial^2 R_1}{\partial t_0 \partial x_0} + \Delta^2 \frac{\partial^2 R_1}{\partial t_0^2} + \frac{3\Delta^2}{2}\left(\frac{\partial R_1}{\partial t_0}\right)^2 + 3p_{G0} R_1 T_1 + \left[3(\gamma_e - 2)p_{G0} - \Delta^2\right]. \tag{A5}$$

## APPENDIX B: INHOMOGENEOUS TERMS IN (73)–(77)

Explicit forms of the inhomogeneous terms $N_j$ $(j = 1, 2, 3, 4, 5)$ are given by

$$\begin{aligned} N_1 = &-\left(\frac{\partial \alpha_1}{\partial t_2} - 3\frac{\partial R_1}{\partial t_2} + \frac{\partial u_1}{\partial x_2}\right) - \left(\frac{\partial \alpha_2}{\partial t_1} - 3\frac{\partial R_2}{\partial t_1} + \frac{\partial u_2}{\partial x_1}\right) \\ &+ 3\frac{\partial}{\partial t_1}\left(\alpha_1 R_1 - 2R_1^2\right) + \frac{\partial}{\partial x_1}\left(3u_1 R_1 - \alpha_1 u_1\right) \\ &- 12\frac{\partial R_1 R_2}{\partial t_0} + 10\frac{\partial R_1^3}{\partial t_0} + 3\frac{\partial}{\partial t_0}\left[\alpha_1\left(R_2 - 2R_1^2\right)\right] + 3\frac{\partial R_1 \alpha_2}{\partial t_0} \\ &+ 3\frac{\partial \alpha_1 R_1 u_1}{\partial x_0} + 3\frac{\partial R_1 u_2}{\partial x_0} + 3\frac{\partial}{\partial x_0}\left[u_1\left(R_2 - 2R_1^2\right)\right] \\ &- \frac{\partial \alpha_2 u_1}{\partial x_0} - \frac{\partial \alpha_1 u_2}{\partial x_0}, \end{aligned} \tag{B1}$$

$$\begin{aligned} N_2 = &-\left[\alpha_0 \frac{\partial \alpha_1}{\partial t_2} - (1-\alpha_0)\frac{\partial u_1}{\partial x_2}\right] - \left[\alpha_0 \frac{\partial \alpha_2}{\partial t_1} - (1-\alpha_0)\frac{\partial u_2}{\partial x_1}\right] \\ &- \alpha_0 \frac{\partial \alpha_1 u_1}{\partial x_1} - \alpha_0 \frac{\partial \alpha_1 u_2}{\partial x_0} - \alpha_0 \frac{\partial \alpha_2 u_1}{\partial x_0}, \end{aligned} \tag{B2}$$

$$
\begin{aligned}
N_3 = &-\left[(1-\alpha_0)\frac{\partial u_1}{\partial t_2}+\frac{\partial p_{\mathrm{L}1}}{\partial x_2}\right]-\left[(1-\alpha_0)\frac{\partial u_2}{\partial t_1}+\frac{\partial p_{\mathrm{L}2}}{\partial x_1}\right] \\
&+\alpha_0\frac{\partial \alpha_1 u_1}{\partial t_1}-(1-\alpha_0)\frac{\partial u_1^2}{\partial x_1}+\alpha_0\frac{\partial \alpha_1 u_2}{\partial t_0}+\alpha_0\frac{\partial \alpha_2 u_1}{\partial t_0} \\
&-2(1-\alpha_0)\frac{\partial u_1 u_2}{\partial x_0}+\alpha_0\frac{\partial \alpha_1 u_1^2}{\partial x_0}+\frac{4\mu_{\mathrm{L}}}{3}\frac{\partial^2 u_1}{\partial x_0^2},
\end{aligned}
\tag{B3}
$$

$$
\begin{aligned}
N_4 = &-\left[\frac{\partial T_{\mathrm{G}2}}{\partial t_1}+3(\kappa-1)\frac{\partial R_2}{\partial t_1}\right]-\left[\frac{\partial T_{\mathrm{G}1}}{\partial t_2}+3(\kappa-1)\frac{\partial R_1}{\partial t_2}\right] \\
&-u_1\frac{\partial T_{\mathrm{G}1}}{\partial x_1}-3(\kappa-1)\left[u_1\frac{\partial R_1}{\partial x_1}+\frac{\partial T_{\mathrm{G}1}R_1}{\partial t_1}+\frac{3(\kappa-1)-1}{2}\frac{\partial R_1^2}{\partial t_1}\right] \\
&-3(\kappa-1)\left(\frac{\partial T_{\mathrm{G}2}R_1}{\partial t_0}+\frac{\partial T_{\mathrm{G}1}R_2}{\partial t_0}+u_1\frac{\partial R_2}{\partial x_0}+u_2\frac{\partial R_1}{\partial x_0}+u_1\frac{\partial T_{\mathrm{G}1}R_1}{\partial x_0}\right) \\
&-\frac{3(\kappa-1)[3(\kappa-1)-1]}{2}\left[2\frac{\partial R_1 R_2}{\partial t_0}+\frac{\partial T_{\mathrm{G}1}R_1^2}{\partial t_0}+u_1\frac{\partial R_1^2}{\partial x_0}\right. \\
&\left.+\frac{3(\kappa-1)-2}{3}\frac{\partial R_1^3}{\partial t_0}\right]-u_1\frac{\partial T_{\mathrm{G}2}}{\partial x_0}-u_2\frac{\partial T_{\mathrm{G}1}}{\partial x_0}.
\end{aligned}
\tag{B4}
$$

## DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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