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Nonlinear Pressure Waves in Bubbly Flows with Drag Force: Theoretical and Numerical Comparison of Acoustic and Thermal and Drag Force Dissipations

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Weakly nonlinear propagation of plane pressure waves in flowing compressible water containing many spherical microbubbles is theoretically investigated. Special focus is placed on the thermal conduction inside the bubble and drag force acting on translational bubbles. From the method of multiple scales, the Korteweg–de Vries–Burgers equation is derived and two types of dissipation terms appear: a term with a second-order partial derivative owing to the liquid compressibility and a term without differentiation owing to the drag force and thermal conduction. Finally, we numerically found that the dissipation effect due to the thermal conduction is the largest, followed by that due to the acoustic radiation and drag force.

In nonlinear acoustics or nonlinear wave theory, a pressure wave evolves into a shock wave owing to the competition between the nonlinear and the dissipation effects, or into a stable wave [a so-called (acoustic) soliton] owing to the competition between the nonlinear and the dispersion effects. It is essential to evaluate the nonlinear, dissipation, and dispersion effects, because the shock wave and the soliton have considerably different properties. In the field of weakly nonlinear (i.e., finite but small amplitude) waves in bubbly liquids, the derivation of nonlinear wave equations such as the Korteweg–de Vries–Burgers (KdVB) equation is effective for obtaining the sizes of the nonlinear, dissipation, and dispersion effects as constant coefficients.[1)] Kuznetsov et al.[2)] showed that the theoretically predicted waveform obtained using the KdVB equation agreed with the experimental result.

Although the KdVB equation has long been used to predict weakly nonlinear waves in bubbly flows,[2–4)] previous theoretical studies have ignored some factors. In particular, the translation of the bubble and drag force acting on bubbles are practically significant factors for high-speed cavitating flows in hydraulic machinery. Our previous study[1)] was the first to consider the translation and drag forces; in this study, the KdVB equation was derived from a two-fluid model,[5)] and it was found that the translation increased the nonlinearity and the drag force increased the dissipation. However, for simplicity, our previous study[1)] ignored the effect of thermal conduction at the bubble–liquid interface on the dissipation of waves; instead, only the drag force and acoustic radiation were compared. Nevertheless, the importance of the effect of thermal conduction on wave dissipation has been well known.[3)] Therefore, the present study aims to derive the KdVB equation by considering the thermal effect and to compare the dissipation effects due to the drag force, acoustic radiation, and thermal conduction.

In this study, we theoretically investigate the weakly nonlinear (i.e., finite but small amplitude) propagation of one-dimensional (plane) progressive pressure waves in flowing compressible water that uniformly contains many spherical oscillating bubbles, as shown in Fig. 1.[1)] As in our previous studies,[1,6)] we newly introduce the drag force and translation as bubble dynamics. The main assumptions are as follows: (i) the bubble motion is spherically symmetric; (ii) the bubbles do not coalesce, break up, disappear, and appear; (iii) in the initial state, the gas and liquid phases flow with constant velocity; (iv) the temperature of the liquid phase is constant; and (v) for simplicity, the lift, gravitation, direct interactions between bubbles, gas phase viscosity, Reynolds stress, and phase change and mass transport across the bubble–liquid interface are omitted. Although these assumptions are the same as those in our previous study,[1)] in the present study, the equation of the thermal conduction at the bubble–liquid interface[3)] with a temperature gradient model[7)] is used to express thermal effect inside bubbles.

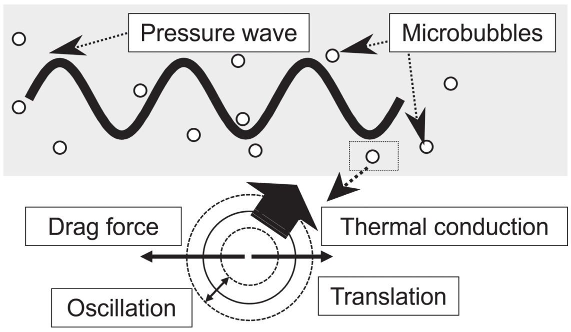

Fig. 1. Concept of problem: pressure wave propagation in bubbly flows.

As basic equations, to introduce the drag force[8)] in interfacial momentum transport, we use the mass and momentum conservation laws for the gas and liquid phases based on a two-fluid model:[5)]

$$\frac{\partial}{\partial t^*}(\alpha\rho_{\mathrm{G}}^*) + \frac{\partial}{\partial x^*}(\alpha\rho_{\mathrm{G}}^* u_{\mathrm{G}}^*) = 0, \tag{1}$$

$$\frac{\partial}{\partial t^*}[(1-\alpha)\rho_{\mathrm{L}}^*] + \frac{\partial}{\partial x^*}[(1-\alpha)\rho_{\mathrm{L}}^* u_{\mathrm{L}}^*] = 0, \tag{2}$$

$$\frac{\partial}{\partial t^*}(\alpha\rho_{\mathrm{G}}^* u_{\mathrm{G}}^*) + \frac{\partial}{\partial x^*}(\alpha\rho_{\mathrm{G}}^* u_{\mathrm{G}}^{*2}) + \alpha\frac{\partial p_{\mathrm{G}}^*}{\partial x^*} = F^* + D^*, \tag{3}$$

$$\frac{\partial}{\partial t^*}[(1-\alpha)\rho_L^* u_L^*] + \frac{\partial}{\partial x^*}[(1-\alpha)\rho_L^* {u_L^*}^2] + (1-\alpha)\frac{\partial p_L^*}{\partial x^*} + P^*\frac{\partial \alpha}{\partial x^*} = -F^* - D^*, \tag{4}$$

where t^* is the time, x^* is the space coordinate normal to the wave front, α is the void fraction ($0 < \alpha < 1$), ρ^* is the density, u^* is the velocity, p^* is the pressure, P^* is the liquid pressure averaged on the bubble–liquid interface,[5] and F^* is the virtual mass force utilizing the model;[9] the subscripts G and L denote the gas and liquid phases, respectively, and the superscript $*$ denotes a dimensional quantity. The following model for the drag force term D^* for spherical bubbles is introduced:[1,6,10]

$$D^* = -\frac{3}{8R^*}C_D\alpha\rho_L^*(u_G^* - u_L^*)|u_G^* - u_L^*|, \tag{5}$$

where R^* is the bubble radius and C_D is the drag coefficient for a single spherical bubble. In addition, we use the equation of motion for bubbles, which is represented as a linear combination of the volumetric oscillations of bubbles[11] and the translation of bubbles:[12]

$$\left(1 - \frac{1}{c_{L0}^*}\frac{D_G R^*}{Dt^*}\right)R^*\frac{D_G^2 R^*}{D{t^*}^2} + \frac{3}{2}\left(1 - \frac{1}{3c_{L0}^*}\frac{D_G R^*}{Dt^*}\right)\left(\frac{D_G R^*}{Dt^*}\right)^2 = \left(1 + \frac{1}{c_{L0}^*}\frac{D_G R^*}{Dt^*}\right)\frac{P^*}{\rho_{L0}^*} + \frac{R^*}{\rho_{L0}^* c_{L0}^*}\frac{D_G}{Dt^*}(p_L^* + P^*) + \frac{(u_G^* - u_L^*)^2}{4}, \tag{6}$$

where c_{L0}^* is the speed of sound in pure water, the subscript 0 denotes the initial unperturbed state, and the Lagrange derivative D_G/Dt^* is defined as

$$\frac{D_G}{Dt^*} = \frac{\partial}{\partial t^*} + u_G^*\frac{\partial}{\partial x^*}. \tag{7}$$

The novelty of this paper is the introduction of equation for thermal conduction at the bubble–liquid interface[3] to express the thermal effect inside a bubble:

$$\frac{D_G p_G^*}{Dt^*} = \frac{3}{R^*}\left[(\gamma - 1)\lambda_G^*\frac{\partial T_G^*}{\partial r^*}\bigg|_{r^*=R^*} - \gamma p_G^*\frac{D_G R^*}{Dt^*}\right], \tag{8}$$

where λ_G^* is the thermal conductivity of the gas inside the bubble, γ is the ratio of specific heats, T_G^* is the temperature of the gas phase, and r^* is the radial distance from the center of the bubble. Although some models[13–17] have been proposed for the temperature gradient as an approximation of the first term on the right-hand side of (8), in this study, we use the following model by Shimada et al.:[7]

$$\frac{\partial T_G^*}{\partial r^*}\bigg|_{r^*=R^*} = \frac{5}{4}\frac{T_{G0}^* - T_G^*}{R^*}. \tag{9}$$

Furthermore, the equation of state for ideal gas, the Tait's equation of state for liquid, conservation equation of mass inside the bubble, and the balance of normal stresses across the bubble–liquid interface are introduced:

$$\frac{p_G^*}{p_{G0}^*} = \frac{\rho_G^*}{\rho_{G0}^*}\frac{T_G^*}{T_{G0}^*}, \tag{10}$$

$$p_L^* = p_{L0}^* + \frac{\rho_{L0}^* {c_{L0}^*}^2}{n}\left[\left(\frac{\rho_L^*}{\rho_{L0}^*}\right)^n - 1\right], \tag{11}$$

$$\frac{\rho_G^*}{\rho_{G0}^*} = \left(\frac{R_0^*}{R^*}\right)^3, \tag{12}$$

$$p_G^* - (p_L^* + P^*) = \frac{2\sigma^*}{R^*} + \frac{4\mu_L^*}{R^*}\frac{D_G R^*}{Dt^*}, \tag{13}$$

where n is a material constant (e.g., $n = 7.15$ for water), σ^* is the surface tension, and μ_L^* is the liquid viscosity.

The temperature of the gas phase is nondimensionalized and expanded in a power series of ε:

$$T_G^*/T_{G0}^* = 1 + \varepsilon T_{G1} + \varepsilon^2 T_{G2} + O(\varepsilon^3), \tag{14}$$

where ε ($\ll 1$) is a finite but small nondimensional wave amplitude. The expansions of the other dependent variables are the same as those in our previous paper.[1] As the scaling relations of nondimensional ratios derived by using ε, low-frequency long waves are described by introducing the following ratios:[1]

$$\frac{U^*}{c_{L0}^*} \equiv O(\sqrt{\varepsilon}) \equiv V\sqrt{\varepsilon}, \tag{15}$$

$$\frac{R_0^*}{L^*} \equiv O(\sqrt{\varepsilon}) \equiv \Delta\sqrt{\varepsilon}, \tag{16}$$

$$\frac{\omega^*}{\omega_B^*} \equiv O(\sqrt{\varepsilon}) \equiv \Omega\sqrt{\varepsilon}, \tag{17}$$

where V, Δ, and Ω are constants of $O(1)$, U^* is the typical propagation speed, L^* is the typical wavelength, ω^* is the typical frequency of waves, and ω_B^* is the natural frequency of a single bubble.[1] The liquid viscosity μ_L^* is also nondimensionalized as

$$\frac{\mu_L^*}{\rho_{L0}^* U^* L^*} \equiv O(\varepsilon^2) \equiv \mu\varepsilon^2, \tag{18}$$

where μ is a constant of $O(1)$. Furthermore, the size of the nondimensional ratio for thermal conduction is introduced:[13]

$$\frac{3(\gamma - 1)\lambda_G^*}{p_{G0}^*\omega^* R_0^*}\frac{5}{4}\frac{T_{G0}^*}{R_0^*} = \zeta\varepsilon, \tag{19}$$

where ζ is a constant of $O(1)$.

As a result, the leading order of approximation in (8) is derived from the method of multiple scales:

$$\frac{DT_{G1}}{Dt_0} + 3(\gamma - 1)\frac{DR_1}{Dt_0} = 0. \tag{20}$$

Further, the second order of approximation in (8) is

$$\frac{DT_{G2}}{Dt_0} + 3(\gamma - 1)\frac{DR_2}{Dt_0} = -\zeta T_{G1} - 3(\gamma - 1)\frac{D}{Dt_0}\left[T_{G1}R_1 + \frac{1}{2}(3\gamma - 4)R_1^2\right], \tag{21}$$

where R_1 and R_2 are first and second order perturbations of bubble radius, respectively,[1] and the definition of the linear Lagrange derivative D/Dt_0 is

$$\frac{D}{Dt_0} = \frac{\partial}{\partial t_0} + u_0\frac{\partial}{\partial x_0}, \tag{22}$$

where t_0 ($= t$) and x_0 ($= x$) are extended independent variables; and t_1 ($= \varepsilon t$) and x_1 ($= \varepsilon x$) represent slow scales appearing in the second order of approximation, as in Appendix in Ref. 18.

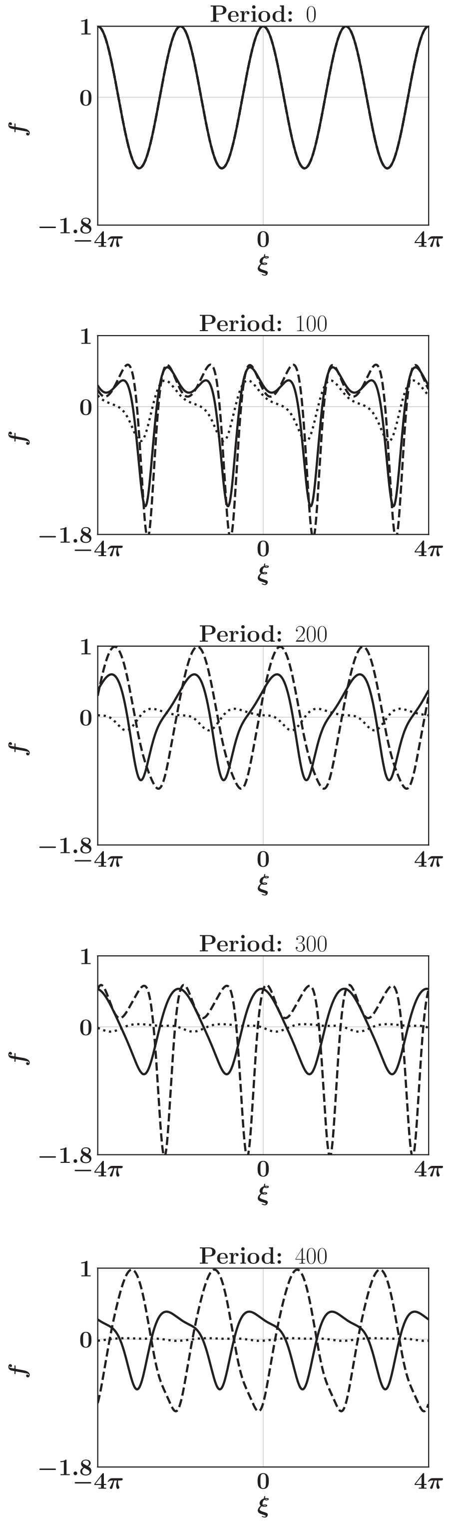

Fig. 2. Temporal evolution of the numerical solutions of the KdVB equation for $\alpha_0 = 0.01$, $R_0^* = 500\,\mu$m, $\sqrt{\varepsilon} = 0.15$, $\gamma = 1.4$, $v_p = 1$, $\rho_{L0}^* = 1000\,$kg/m^3, $\mu_L^* = 10^{-3}\,$Pa·s, $T_{G0}^* = 298\,$K, $\lambda_G^* = 0.025\,$W/(m·K), $p_{L0}^* = 101325\,$Pa, $\sigma^* = 0.0728\,$N/m, grid steps = 1024, duration of numerical integration of time = 0.001, and size of computational domain = 8π. The solid, dashed, and dotted curves represent the waveforms with only the acoustic radiation, only the drag force, and only the thermal conduction, respectively.

Finally, we obtain the KdVB equation:

$$\frac{\partial f}{\partial \tau} + \Pi_1 f \frac{\partial f}{\partial \xi} + \Pi_2 \frac{\partial^2 f}{\partial \xi^2} + \Pi_3 \frac{\partial^3 f}{\partial \xi^3} + (\Pi_{4A} + \Pi_{4B}) f = 0, \quad (23)$$

through the variable transform

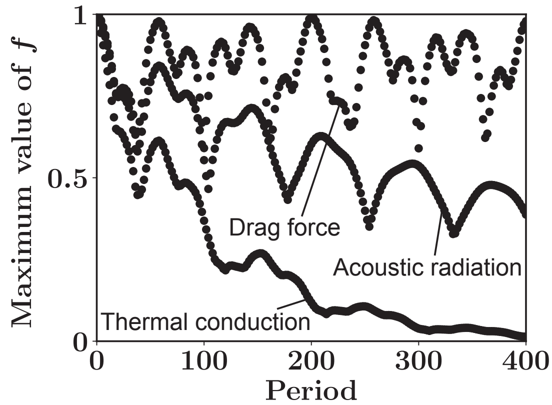

Fig. 3. Comparison of the three dissipation effects. The calculation condition is the same as that in Fig. 2.

$$\tau \equiv \varepsilon t, \quad \xi \equiv x - (v_p + u_0 + \varepsilon \Pi_0) t, \quad (24)$$

where f is the first order perturbation of the nondimensional bubble radius[1] and v_p is the phase velocity. Here, Π_0, Π_1, and Π_3 are the advection, nonlinear, and dispersion coefficients, respectively, and Π_2, Π_{4A}, and Π_{4B} are the dissipation coefficients due to acoustic radiation, drag force, and thermal conduction, respectively. The forms of Π_0 and Π_3 are given by

$$\Pi_0 = -\frac{(1-\alpha_0) v_p \Delta^2 V^2}{6\alpha_0 \Omega^2} < 0, \quad (25)$$

$$\Pi_3 = \frac{v_p \Delta^2}{6\alpha_0} > 0, \quad (26)$$

and are not affected by drag force, translation, and thermal conduction; i.e., they are the same as in Ref. 4. The explicit form of Π_1 is affected by translation and is quite complex [see Eq. (57) in Ref. 1]. The dissipation coefficients in (23) are given by

$$\Pi_2 = -\frac{1}{6\alpha_0} \frac{V\Delta^3}{\Omega^2} < 0, \quad (27)$$

$$\Pi_{4A} = \frac{\mu}{\Delta^2} \left\{ 3 - \frac{1}{\alpha_0 (3 - 2\alpha_0)} \left[3\alpha_0 + 2(1-\alpha_0) \frac{\Delta^2}{\Omega^2} \right] \right\} > 0, \quad (28)$$

$$\Pi_{4B} = \frac{p_{G0}[2\alpha_0(1-\alpha_0) + 1]}{2\alpha_0(1-\alpha_0)} (\gamma - 1)\zeta > 0. \quad (29)$$

In the case in which the thermal effect inside the bubble is ignored (i.e., $\Pi_{4B} = 0$), the present result agrees with the result of our previous study.[1]

Because the dissipation term due to the acoustic radiation ($\Pi_2 \partial^2 f / \partial \xi^2$) has a different mechanism from that due to the drag force and thermal conduction ($\Pi_{4A} f$ and $\Pi_{4B} f$) with regard to the unknown variable, they cannot be compared using the coefficients (Π_2, Π_{4A}, and Π_{4B}). Therefore, we conducted a numerical analysis employing the split-step Fourier method used in our previous studies.[1,13,19–21] Figure 2 shows the temporal evolution of the numerical solutions for the KdVB equation. The solid, dashed, and dotted curves represent the waveforms with only the acoustic radiation, only the drag force, and only the thermal conduction, respectively. We assume that the initial waveform of the solution is a cosine wave. Figure 3 shows the maximum value of the waveform as time passes. The dissipation effect of thermal conduction is the largest, followed by that of the acoustic radiation and drag force. This order is effective in the range from $R_0^* = 50\,\mu\text{m}$ to 1 mm.

In summary, we theoretically investigated the weakly nonlinear propagation of one-dimensional progressive pressure waves in flowing compressible water that uniformly contained many spherical oscillating bubbles by deriving the KdVB equation, and identified three dissipation effects: acoustic radiation, drag force, and thermal conduction. From the numerical analysis, we clarified that the dissipation effect of the thermal conduction was the largest, followed by that of the acoustic radiation and drag force.

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