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Contribution of initial bubble radius distribution to weakly nonlinear waves with a long wavelength in bubbly liquids

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ABSTRACT

In this study, the weakly nonlinear propagation of plane progressive pressure waves in an initially quiescent liquid was theoretically investigated. This liquid contains several small uniformly distributed spherical polydisperse gas bubbles. The polydispersity considered here represents various types of initial bubble radii, and the liquid contains multiple bubbles, each with an initial radius. Using the method of multiple scales, we first derived the Korteweg–de Vries–Burgers (KdVB) equation with a correction term as a nonlinear wave equation. This equation describes the long-range wave propagation with weak nonlinearity, low frequency, and long wavelength in the polydisperse bubbly liquid using the basic equations in a two-fluid model. The utilization of the two-fluid model incorporates the dependence of an initial void fraction on each coefficient in the nonlinear, dissipation, and dispersion terms in the KdVB equation. Furthermore, unlike previous studies on waves in polydisperse bubbly liquids, we achieved the formulation without assuming an explicit form of the polydispersity function. Consequently, we discovered the contribution of polydispersity to the various effects of wave propagation, that is, the nonlinear, dissipation, and dispersion effects. In particular, the dispersion effect of the waves was found to be strongly influenced by polydispersity.

I. INTRODUCTION

Pressure waves in liquid containing several microbubbles (i.e., bubbly liquid) convert either into shock waves or into an acoustic soliton[1] through competition between the nonlinear effects of waves and the dispersion effect owing to bubble oscillations in the medium. Because shock waves and solitons have different properties than waves, it is necessary to evaluate them (i.e., nonlinearity, dissipation, and dispersion) to predict their development. Several studies have been conducted on pressure waves in bubbly liquids by both experimental measurements[2–12] and numerical simulations.[5,10,13–52] Although it is difficult to directly evaluate the magnitude of the three properties with these methods, few theoretical approaches have facilitated it. Several theoretical studies on pressure wave propagation in bubbly liquids have been conducted to investigate waveforms by deriving a nonlinear wave equation.[53–60] For weakly nonlinear pressure waves, we can theoretically derive a weakly nonlinear wave equation, which describes the spatiotemporal evolution through a competition of the three properties. Using this equation, we can evaluate the balance between the three properties. Korteweg–de Vries–Burgers (KdVB) equation, in particular, is one of the most prominent weakly nonlinear wave equations that describes low-frequency long-wave pressure waves in bubbly liquids. It has been derived in several theoretical studies[1–3,9,61–65] and agrees with a few experimental results.[3]

Recently, our group has been improving and re-deriving the KdVB equation, and evaluating the effects of previously ignored viewpoints on wave propagation and the three properties. Some additional wave propagation properties being considered are the initial nonuniform distribution of velocities in gas and liquid phases,[66] drag force acting on the bubbles, and bubble translation,[67,68] and quasi-one-dimensional propagation as a focused ultrasound.[69]

Moreover, we re-considered previously incorporated viewpoints such as thermal conduction under monodispersity[70] and polydispersity[71] assumptions. In particular, our group[72,73] clarified the incorporation of the initial void fraction dependence on the three properties for the utilization of the two-fluid model equations in the case of monodisperse bubbly liquids.

The KdVB equation describing pressure wave propagation in bubbly liquids was first derived by van Wijngaarden.[65] Since then, several theoretical studies have proposed new and improved KdVB equations.[57,61–64] Although the KdVB equation predicts the theoretical evolution of pressure waves in bubbly liquids well, few papers have reported its disagreement with experimental results. Watanabe and Prosperetti[74] compared the numerical solution of the model by Caflisch *et al.*[75] based on Wijngaarden's model with the experimental results[76] and found significant differences in some results. They concluded that a few flaws existed in Wijngaarden's model and other models (i.e., basic equations and approximation methods). However, they could not clarify the physical reasons for these differences. Beylich and Gülhan[76] claimed the validity of the theoretical model, but this result challenges it. Kameda *et al.*[5,77] found that the cause of this discrepancy was the initial nonuniformity of bubble size (i.e., initial polydispersity). They compared the results of shock wave tube experiments using bubbly liquids with the numerical solutions of a mathematical model[65] and found close quantitative agreement. These results suggest that polydispersity is a non-negligible factor in the accurate prediction of pressure wave propagation in bubbly liquid.

In the last few decades, pressure wave propagation in bubbly liquids has been actively studied with a focus on polydispersity.[8,15,78–91] In particular, Ando *et al.*[15,86] numerically analyzed shock waves propagating in a polydisperse bubbly liquid and clarified the phenomenon of phase cancellation, which is caused by the independent oscillation of bubbles with different initial sizes and does not occur in the monodisperse case. It is understood that most bubbly liquids have polydispersity, and it is therefore a factor that cannot be ignored when considering pressure waves in bubbly liquids. Nevertheless, few studies[53,71] have considered polydispersity in the weakly nonlinear wave equation, which enables qualitative evaluation of pressure waves in bubbly liquids. However, the three properties have not been considered in the evaluation. This is a significant problem for the global understanding of pressure waves in bubbly liquids.

Gumerov[53] first derived the KdVB equation considering initial polydispersity by using mixture model equations.[62] However, the dependence of the three properties on the initial void fraction could not be reflected in the resultant KdVB equation. Later, using two-fluid model equations, our group[71] also derived the KdVB equation considering initial polydispersity, but the size of the polydispersity was significantly small. Surprisingly, the previous studies[53,71] assumed that bubbles obey only one type of eigenfrequency for a representative bubble with multiple bubbles.

In this paper, we propose introducing multiple eigenfrequencies for each bubble in polydisperse bubbly liquids with a discrete distribution of initial bubble radii. This assumption results in the incorporation of acoustic properties, such as phase cancellation, caused by interactions between the various bubble oscillations of each initial size. The purpose of this study was to theoretically elucidate the effects of a discrete distribution of the initial bubble radius on weakly nonlinear propagation of pressure waves in polydisperse bubbly liquids. This was based on deriving the KdVB equation, where the polydispersity is non-negligible, and the dependence of the initial void fraction is incorporated using basic equations by the two-fluid model.

The remainder of the paper is organized as follows. In Sec. II, we describe a problem in which several bubbles with a discrete distribution of initial radii are represented as N types of bubbles oscillating independently. To incorporate the effect of the independent motion of bubbles, for the polydisperse case, the oscillations of several bubbles in bubbly liquids are described by the Keller equation (7), for multiple, instead of single, initial radii. In this study, "polydispersity" represents a discrete distribution of bubble radii. In Sec. III, we derive the KdVB equation with polydispersity using the method of multiple scales for basic equations in the two-fluid model, and discuss the effects of polydispersity on the three properties (i.e., nonlinearity, dissipation, and dispersion) under the assumption that the bubble radius follows a lognormal distribution, which is often adopted as the distribution of polydisperse bubbly liquids. Finally, Sec. IV presents the conclusions of this study.

II. PROBLEM FORMULATION

A. Assumption

In this study, the weakly nonlinear (i.e., finite but small amplitude) propagation of a plane (one-dimensional) progressive pressure wave in a compressible liquid that contains a large number of uniformly distributed small spherical gas bubbles was theoretically considered, as shown in Fig. 1. We focused on examining the effects of polydispersity of the initial radius of bubble on wave propagation. The bubbly liquid was initially assumed to be quiescent. The coalescence, breakup, appearance, and extinction of bubbles were not considered. The following factors were not considered for simplicity: gas viscosity, thermal conductivities of the liquid,[61,70,74,92,93] bubble interaction,[24,94–99] phase change, and mass transport across the bubble wall (i.e., gas–liquid interface).[100] As clarified by our previous papers,[69,73] initial nonuniformity of bubble distribution (i.e., number density of bubbles) contributes to an advection effect of waves (i.e., propagation speed of waves), but we here assume the initially uniform bubble distribution for simplicity. The pressure and density of gas were assumed to be uniform inside each bubble.

Polydispersity was considered; that is, it was assumed that the initial bubble radius has N types[80] at any position, as shown in Fig. 1. These bubble groups move independently from each other. It should be noted that the content rate of each bubbly liquid bubble type is initially uniform.

B. Basic equations

The utilization of basic equations based on a two-fluid model[101] enables us to describe the dependence of the initial void fraction on the weakly nonlinear wave equations. Therefore, mass and momentum conservation equations for the gas and liquid phases in the two-fluid model were initially used

$$\frac{\partial}{\partial t^*}(\alpha\rho_{\mathrm{G}}^*) + \frac{\partial}{\partial x^*}(\alpha\rho_{\mathrm{G}}^* u_{\mathrm{G}}^*) = 0, \tag{1}$$

$$\frac{\partial}{\partial t^*}\left[(1-\alpha)\rho_{\mathrm{L}}^*\right] + \frac{\partial}{\partial x^*}\left[(1-\alpha)\rho_{\mathrm{L}}^* u_{\mathrm{L}}^*\right] = 0, \tag{2}$$

$$\frac{\partial}{\partial t^*}(\alpha\rho_{\mathrm{G}}^* u_{\mathrm{G}}^*) + \frac{\partial}{\partial x^*}\left(\alpha\rho_{\mathrm{G}}^* {u_{\mathrm{G}}^*}^2\right) + \alpha\frac{\partial p_{\mathrm{G}}^*}{\partial x^*} = F^*, \tag{3}$$

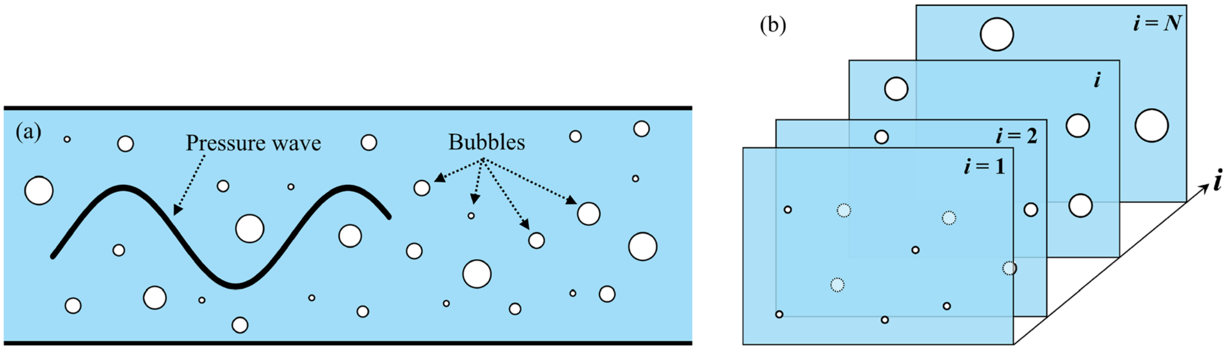

FIG. 1. Conceptual diagram of problem statement: (a) propagation of pressure waves in a polydisperse bubbly liquid and (b) the model for the elements of a polydisperse bubbly liquid.

$$\frac{\partial}{\partial t^*}\left[(1-\alpha)\rho_L^* u_L^*\right] + \frac{\partial}{\partial x^*}\left[(1-\alpha)\rho_L^* u_L^{*2}\right] + (1-\alpha)\frac{\partial p_L^*}{\partial x^*} + P^*\frac{\partial \alpha}{\partial x^*} = -F^*, \tag{4}$$

where t^* is time, x^* is the space coordinate, α is the void fraction (volume fraction of the gas phase), u^* is the fluid velocity, ρ^* is the volume-averaged density, p^* is the volume-averaged pressure, and P^* is the liquid pressure averaged at the bubble–liquid interface. The subscripts G and L represent volume-averaged variables in the gas and liquid phases, respectively, and the superscript * represents a dimensional quantity. As an interfacial momentum transportation model, the model for virtual mass force F^* was employed[102,103]

$$F^* = -\beta_1 \alpha \rho_L^* \left(\frac{D_G u_G^*}{Dt^*} - \frac{D_L u_L^*}{Dt^*}\right) - \beta_2 \rho_L^* (u_G^* - u_L^*)\frac{D_G \alpha}{Dt^*} - \beta_3 \alpha (u_G^* - u_L^*)\frac{D_G \rho_L^*}{Dt^*}, \tag{5}$$

where coefficients β_j $(j = 1, 2, 3)$ are set to 1/2 for the case of spherical bubbles. Lagrange derivatives D_G/Dt^* and D_L/Dt^* are defined as follows:

$$\frac{D_G}{Dt^*} \equiv \frac{\partial}{\partial t^*} + u_G^* \frac{\partial}{\partial x^*}, \quad \frac{D_L}{Dt^*} \equiv \frac{\partial}{\partial t^*} + u_L^* \frac{\partial}{\partial x^*}. \tag{6}$$

The spherically symmetric bubble oscillations in a compressible liquid are governed by the Keller equation[104] $(i = 1, 2, \ldots, N)$

$$\left(1 - \frac{1}{c_{L0}^*}\frac{D_G R_i^*}{Dt^*}\right) R_i^* \frac{D_G^2 R_i^*}{Dt^{*2}} + \frac{3}{2}\left(1 - \frac{1}{3c_{L0}^*}\frac{D_G R_i^*}{Dt^*}\right)\left(\frac{D_G R_i^*}{Dt^*}\right)^2 = \left(1 + \frac{1}{c_{L0}^*}\frac{D_G R_i^*}{Dt^*}\right)\frac{P_i^*}{\rho_{L0}^*} + \frac{R_i^*}{\rho_{L0}^* c_{L0}^*}\frac{D_G}{Dt^*}(p_L^* + P_i^*), \tag{7}$$

where c_{L0}^* is the speed of sound in pure water and $R_i^*(x^*, t^*)$ is the bubble radius when the initial type is ith. We introduce polydispersity with a different initial condition of oscillations for each i type.

Equation for thermal conduction at the gas–liquid interface[61] is described as

$$\frac{D_G p_{Gi}^*}{Dt^*} = \frac{3}{R_i^*}\left[(\kappa - 1)\lambda_G^* \left.\frac{\partial T_{Gi}^*}{\partial r^*}\right|_{r^*=R_i^*} - \kappa p_{Gi}^* \frac{D_G R_i^*}{Dt^*}\right], \tag{8}$$

where T_{Gi}^* is the temperature of the gas phase for the ith bubble, κ is the ratio of the specific heats, r^* is the radial distance from the center of the bubble, and λ_G is the thermal conductivity of the gas phase. For the temperature gradient expressed in the first term on the right-hand side of (8), we used the following model:[105]

$$\left.\frac{\partial T_{Gi}^*}{\partial r^*}\right|_{r^*=R_i^*} = \frac{\mathrm{Re}(\tilde{L}_P^*)(T_{G0}^* - T_{Gi}^*)}{|\tilde{L}_P^*|^2} + \frac{\mathrm{Im}(\tilde{L}_P^*)}{\omega_B^* |\tilde{L}_P^*|^2}\frac{D_G T_{Gi}^*}{Dt^*}, \tag{9}$$

where T_{G0}^* is the initial temperature (note isothermal as the initial condition) and ω_B^* is the natural angular frequency of the spherically symmetric oscillations of a single bubble, for which we use

$$\omega_B^* = \sqrt{\frac{3\gamma_e p_{G0}^* - 2\sigma^*/R_0^{\mathrm{ref}*}}{\rho_{L0}^* R_0^{\mathrm{ref}*2}} - \left(\frac{2\mu_{e0}^*}{\rho_{L0}^* R_0^{\mathrm{ref}*2}}\right)^2}, \tag{10}$$

$$\gamma_e = \mathrm{Re}\left(\frac{\Gamma_N}{3}\right), \tag{11}$$

$$\mu_{e0}^* = \mu^* + \mathrm{Im}\left(\frac{p_{G0}^* \Gamma_N}{4\omega_B^*}\right), \tag{12}$$

$$\Gamma_N = \frac{3\alpha_N^2 \kappa}{\alpha_N^2 + 3(\kappa - 1)(\alpha_N \coth \alpha_N - 1)}, \tag{13}$$

$$\alpha_N = \sqrt{\frac{\kappa \omega_B^* p_{G0}^* R_0^{\mathrm{ref}*2}}{2(\kappa - 1) T_{G0}^* \lambda_G^*}(1 + \mathrm{j})}, \tag{14}$$

where γ_e is the effective polytropic exponent, μ_{e0}^* is the initial effective viscosity, Γ_N and α_N are complex numbers, and j denotes the imaginary unit. Further, $\tilde{L}_P^*$ is the complex number with the length dimension

$$\tilde{L}_P^* = \frac{R_0^{\mathrm{ref}*}(\alpha_N^2 - 3\alpha_N \coth \alpha_N + 3)}{\alpha_N^2(\alpha_N \coth \alpha_N - 1)}. \tag{15}$$

To close the set of (1)–(8), we introduce (i) the equation of an ideal gas state; (ii) Tait, equation of liquid state; (iii) the conservation law of mass inside the bubble; and (iv) the balance of normal stresses across the gas–liquid interface

$$\frac{p_{Gi}^*}{p_{Gi0}^*} = \left(\frac{\rho_{Gi}^*}{\rho_{Gi0}^*}\right)\left(\frac{T_{Gi}^*}{T_{G0}^*}\right), \tag{16}$$

$$p_{\rm L}^* = p_{\rm L0}^* + \frac{\rho_{\rm L0}^* c_{\rm L0}^{*2}}{n}\left[\left(\frac{\rho_{\rm L}^*}{\rho_{\rm L0}^*}\right)^n - 1\right], \tag{17}$$

$$\frac{\rho_{{\rm G}i}^*}{\rho_{{\rm G}i0}^*} = \left(\frac{R_{i0}^*}{R_i^*}\right)^3, \tag{18}$$

$$p_{{\rm G}i}^* - (p_{\rm L}^* + P_i^*) = \frac{2\sigma^*}{R_i^*} + \frac{4\mu^*}{R_i^*}\frac{{\rm D_G}R_i^*}{{\rm D}t^*}, \tag{19}$$

where $\rho_{{\rm G}i}^*$ is the gas density inside the bubbles of the ith type, $p_{{\rm G}i}^*$ is the gas pressure inside the bubbles of the ith type, P_i^* is the pressure on the bubbles of the ith type, n is the material constant, σ^* is the surface tension, and μ^* is the liquid viscosity. The physical quantities in the initial unperturbed state are constants and are denoted by the subscript 0. Equations (16), (18), and (19) are used for $i = 1, \ldots, N$, as in (7). Note that our previous monodisperse cases[66–70,72,106,107] considered R_i as R, and our previous polydisperse case[71] also regarded R_i as R, because the previous study[71] defined the initial polydispersity in perturbation expansions. These extensions are justified by the assumption that the bubbles do not directly affect each other. Essentially, the motion of a single bubble is the same as that of a bubble in a bubbly liquid.

In addition, we can close these equations explicitly by introducing the relations between averaged quantities and quantities of each bubble type

$$p_{\rm G}^* = \sum_{i=1}^{N}(X_i R_i^{*3} p_{{\rm G}i}^*) \Big/ \sum_{i=1}^{N}(X_i R_i^{*3}), \tag{20}$$

$$\rho_{\rm G}^* = \sum_{i=1}^{N}(X_i R_i^{*3} \rho_{{\rm G}i}^*) \Big/ \sum_{i=1}^{N}(X_i R_i^{*3}), \tag{21}$$

$$P^* = \sum_{i=1}^{N}(X_i R_i^{*2} P_i^*) \Big/ \sum_{i=1}^{N}(X_i R_i^{*2}), \tag{22}$$

where X_i are the content rates of the ith bubble type and are constants.

We emphasize that the present problem incorporates the effect of gas–liquid interface (i.e., bubble wall); that is, the dynamics of bubble wall is described by (7), the balance of normal stresses is by (19), and the surface averaged liquid pressure P focuses on the gas–liquid interface.

C. Analysis by multiple scales

The relationship $U^* = L^*\omega^*$ exists, where U^*, L^*, and T^* are the typical propagation speed of waves, wavelength, and period of waves, respectively [ω^* ($\equiv 1/T^*$) is the typical incident frequency of waves]. We determine the magnitudes of the set of three ratios[72]

$$\frac{U^*}{c_{\rm L0}^*} \equiv O(\sqrt{\epsilon}) = V\sqrt{\epsilon}, \tag{23}$$

$$\frac{R_0^{\rm ref*}}{L^*} \equiv O(\sqrt{\epsilon}) = \Delta\sqrt{\epsilon}, \tag{24}$$

$$\frac{\omega^*}{\omega_{\rm B}^*} \equiv O(\sqrt{\epsilon}) = \Omega\sqrt{\epsilon}, \tag{25}$$

where ϵ (\ll1) is a typical nondimensional amplitude of weakly nonlinear waves and is used as a perturbation; the constants V, Δ, and Ω are of $O(1)$; and $R_0^{\rm ref*}$ is the representative initial bubble radius. The physical meaning of (23) is as follows: the speed of sound in bubbly liquids is significantly low compared with that in pure water, the representative bubble radius is considerably short compared with the wavelength, and the incident frequency of waves is significantly low compared with the natural frequency of bubble oscillations.

The magnitude of nondimensional ratio with regard to thermal conduction is also determined[70,71]

$$\frac{3(\kappa-1)\lambda_{\rm G}^*}{p_{\rm G0}^*\omega^* R_{i0}^*}\frac{{\rm Re}(\tilde{L}_{\rm P}^*)T_{\rm G0}^*}{|\tilde{L}_{\rm P}^*|^2} = \zeta_{i1}\epsilon, \tag{26}$$

$$\frac{3(\kappa-1)\lambda_{\rm G}^*}{p_{\rm G0}^*\omega^* R_{i0}^*}\frac{\omega^*{\rm Im}(\tilde{L}_{\rm P}^*)T_{\rm G0}^*}{\omega_{\rm B}^*|\tilde{L}_{\rm P}^*|^2} = \zeta_{i2}\epsilon^2, \tag{27}$$

where ζ_{i1} and ζ_{i2} are constants of $O(1)$.

Furthermore, we determine the magnitude of polydispersity (i.e., the distribution of initial size of each bubble) as follows:

$$\frac{R_{i0}^*}{R_0^{\rm ref*}} \equiv O(1) = R_{i0} \quad (i = 1, 2, \ldots, N). \tag{28}$$

Equation (28) indicates that the initial radius of each type is comparable to the representative radius. This equation expresses the magnitude of the polydispersity allowed in this study, and we considered and incorporated polydispersity that is not small (i.e., not on the order of ϵ).

The nondimensionalization results of time t^* and space coordinates x^* are $t \equiv t^*/T^*$ and $x \equiv x^*/L^*$, respectively. Subsequently, we introduce four independent variables for near field denoted by the subscript 0 and far field denoted by the subscript 1[108]

$$t_0 = t, \quad x_0 = x; \quad t_1 = \epsilon t, \quad x_1 = \epsilon x. \tag{29}$$

Dependent variables are expanded in power series of ϵ after nondimensionalization. For example, perturbation expansions of $p_{\rm L}^*$, α, R_i, and $\rho_{\rm L}^*$ are given by

$$p_{\rm L}^*/(\rho_{\rm L0}^* U^{*2}) = p_{\rm L0} + \epsilon p_{\rm L1} + \epsilon^2 p_{\rm L2} + O(\epsilon^3), \tag{30}$$

$$\alpha/\alpha_0 = 1 + \epsilon\alpha_1 + \epsilon^2\alpha_2 + O(\epsilon^3), \tag{31}$$

$$R_i^*/R_{i0}^* = 1 + \epsilon R_{i1} + \epsilon^2 R_{i2} + O(\epsilon^3), \tag{32}$$

$$\rho_{\rm L}^*/\rho_{\rm L0}^* = 1 + \epsilon^2\rho_{\rm L1} + O(\epsilon^3). \tag{33}$$

Notably, all the perturbation expansions begin with $O(\epsilon^1)$ except for $\rho_{\rm L}^*$.[72]

The initial nondimensional pressures for the gas and liquid phases are $p_{\rm G0}^*$ and $p_{\rm L0}^*$, respectively, and liquid viscosity μ^* is introduced as

$$\frac{p_{\rm G0}^*}{\rho_{\rm L0}^* U^{*2}} \equiv O(1) = p_{\rm G0}, \tag{34}$$

$$\frac{p_{\rm L0}^*}{\rho_{\rm L0}^* U^{*2}} \equiv O(1) = p_{\rm L0}, \tag{35}$$

$$\frac{\mu^*}{\rho_{\rm L0}^* U^* L^*} \equiv O(\epsilon) = \mu\epsilon, \tag{36}$$

where $p_{\rm G0}$, $p_{\rm L0}$, and μ are constants of the order of unity. In this study, an effect of liquid viscosity acting on only the gas–liquid interface was incorporated.

III. THEORETICAL ANALYSIS

A. First-order approximation

By substituting (23)–(34) into (1)–(22) and equating each coefficient of the like powers of ϵ in the resultant equations, the set of linearized equations is obtained as follows:

(i) Mass conservation in the gas phase

$$\frac{\partial \alpha_1}{\partial t_0} - 3\sum_{i=1}^{N}\left(X_{\mathrm{V}i}\frac{\partial R_{i1}}{\partial t_0}\right) + \frac{\partial u_{\mathrm{G}1}}{\partial x_0} = 0, \tag{37}$$

(ii) Mass conservation in the liquid phase

$$\alpha_0\frac{\partial \alpha_1}{\partial t_0} - (1-\alpha_0)\frac{\partial u_{\mathrm{L}1}}{\partial x_0} = 0, \tag{38}$$

(iii) Momentum conservation in the gas phase

$$\beta_1\frac{\partial u_{\mathrm{G}1}}{\partial t_0} - \beta_1\frac{\partial u_{\mathrm{L}1}}{\partial t_0} - 3p_{\mathrm{G}0}\sum_{i=1}^{N}\left(X_{\mathrm{V}i}\frac{\partial R_{i1}}{\partial x_0}\right) + p_{\mathrm{G}0}\sum_{i=1}^{N}\left(X_{\mathrm{M}i}\frac{\partial T_{\mathrm{G}i1}}{\partial x_0}\right) = 0, \tag{39}$$

(iv) Momentum conservation in the liquid phase

$$(1-\alpha_0+\beta_1\alpha_0)\frac{\partial u_{\mathrm{L}1}}{\partial t_0} - \beta_1\alpha_0\frac{\partial u_{\mathrm{G}1}}{\partial t_0} + (1-\alpha_0)\frac{\partial p_{\mathrm{L}1}}{\partial x_0} = 0, \tag{40}$$

(v) Keller equation for bubbles of the ith type ($i = 1, 2, \ldots, N$)

$$R_{i1} + \frac{3(\gamma_{\mathrm{e}}-1)R_{i0}}{2\Delta^2/\Omega^2 + 3[(3\gamma_{\mathrm{e}}-1)R_{i0} - 2\gamma_{\mathrm{e}}]p_{\mathrm{L}0}}p_{\mathrm{L}1} - \frac{\Delta^2/\Omega^2 + [(3\gamma_{\mathrm{e}}-1)R_{i0} - 3\gamma_{\mathrm{e}}]p_{\mathrm{L}0}}{2\Delta^2/\Omega^2 + 3[(3\gamma_{\mathrm{e}}-1)R_{i0} - 2\gamma_{\mathrm{e}}]p_{\mathrm{L}0}}T_{\mathrm{G}i1} = 0, \tag{41}$$

(vi) Equation of thermal conduction for bubbles of the ith type

$$\frac{\partial T_{\mathrm{G}i1}}{\partial t_0} + 3(\kappa-1)\frac{\partial R_{i1}}{\partial t_0} = 0. \tag{42}$$

Note that the N Keller equation (41) includes R_{i0}. This implies that N types of bubbles with different initial bubble radii follow N different equations.

By eliminating α_1, $u_{\mathrm{G}1}$, $u_{\mathrm{L}1}$, and $p_{\mathrm{L}1}$ from (37)–(42), as a single equation, the linear wave equation for a first-order variation of the liquid pressure $p_{\mathrm{L}1}$ is derived as follows:

$$\frac{\partial^2 p_{\mathrm{L}1}}{\partial t_0^2} - v_{\mathrm{p}}^2\frac{\partial^2 p_{\mathrm{L}1}}{\partial x_0^2} = 0, \tag{43}$$

with the phase velocity v_{p},

$$v_{\mathrm{p}} = \sqrt{\frac{3\alpha_0(1-\alpha+\beta_1)p_{\mathrm{G}0}D + \beta_1(1-\alpha_0)}{3\beta_1\alpha_0(1-\alpha_0)C}}, \tag{44}$$

where

$$C = -\sum_{i=1}^{N}(X_{\mathrm{V}i}s_{Ri}), \quad D = -\sum_{i=1}^{N}\{[X_{\mathrm{V}i} + (\kappa-1)X_{\mathrm{M}i}]s_{Ri}\},$$
$$X_{\mathrm{V}i} = X_iR_{i0}^3\Big/\sum_{i=1}^{N}(X_iR_{i0}^3), \quad X_{\mathrm{M}i} = X_ip_{\mathrm{G}i0}R_{i0}^3\Big/\sum_{i=1}^{N}(X_ip_{\mathrm{G}i0}R_{i0}^3), \tag{45}$$
$$s_{Ri} = -\frac{(3\gamma_{\mathrm{e}}-1)R_{i0}}{(3\kappa-1)\Delta^2/\Omega^2 + 3p_{\mathrm{L}0}[\kappa(3\gamma_{\mathrm{e}}-1)R_{i0} - \gamma_{\mathrm{e}}(3\kappa-1)]}.$$

In the first order of approximation, the second- and first-order derivatives of the original Keller equation (7) are dropped, and the bubble dynamics are regarded as static (not dynamic) in (44). Furthermore, because v_{p} includes α_0, only static bubbles were observed. This result is correctly reflected by the long-wave approximation in (44).

By focusing only on the right-running wave $p_{\mathrm{L}1}$, a phase function φ_0 is introduced[72] as

$$\varphi_0 = x_0 - t_0. \tag{46}$$

On rewriting (37)–(41) using φ_0, we have the expressions of the other variations $u_{\mathrm{G}1}$, $u_{\mathrm{L}1}$, α_1, R_{i1}, and $T_{\mathrm{G}i1}$, in terms of the liquid pressure variation $f(=p_{\mathrm{L}1})$,

$$u_{\mathrm{G}1} = s_1p_{\mathrm{L}1}, \quad u_{\mathrm{L}1} = s_2p_{\mathrm{L}1}, \quad \alpha_1 = s_3p_{\mathrm{L}1}, \quad R_{i1} = s_{Ri}p_{\mathrm{L}1}, \quad T_{\mathrm{G}i1} = s_{Ti}p_{\mathrm{L}1} \tag{47}$$

with

$$s_1 = 1 + 3p_{\mathrm{G}0}\frac{(1-\alpha_0+\beta_1\alpha_0)}{\beta_1(1-\alpha_0)}D, \quad s_2 = 1 + 3p_{\mathrm{G}0}\frac{\alpha_0}{1-\alpha_0}D,$$
$$s_3 = -\frac{1-\alpha_0}{\alpha_0}s_2, \quad s_{Ti} = -3(\kappa-1)s_{Ri}. \tag{48}$$

Notably, from the boundary conditions at $x_0 \to \infty$ where the bubbly liquid is quiescent, constants of integration are dropped. That is, wave propagation does not have the three properties in the first-order approximation, as in our monodisperse case.[72]

B. Second-order approximation and resultant KdVB equation

In this subsection, the approximation of second order is done and the system of equations is derived

(i) Mass conservation law in gas phase

$$\frac{\partial \alpha_2}{\partial t_0} - 3\sum_{i=1}^{N}\left(X_{\mathrm{V}i}\frac{\partial R_{i2}}{\partial t_0}\right) + \frac{\partial u_{\mathrm{G}2}}{\partial x_0} = K_1, \tag{49}$$

(ii) Mass conservation law in liquid phase

$$\alpha_0\frac{\partial \alpha_2}{\partial t_0} - (1-\alpha_0)\frac{\partial u_{\mathrm{L}2}}{\partial x_0} = K_2, \tag{50}$$

(iii) Momentum conservation law in gas phase

$$\beta_1\frac{\partial u_{\mathrm{G}2}}{\partial t_0} - \beta_1\frac{\partial u_{\mathrm{L}2}}{\partial t_0} - 3p_{\mathrm{G}0}\sum_{i=1}^{N}\left\{[X_{\mathrm{V}i} + (\kappa-1)X_{\mathrm{M}i}]\frac{\partial R_{i2}}{\partial x_0}\right\} = K_3, \tag{51}$$

(iv) Momentum conservation law in liquid phase

$$(1-\alpha_0+\beta_1\alpha_0)\frac{\partial u_{\mathrm{L}2}}{\partial t_0} - \beta_1\alpha_0\frac{\partial u_{\mathrm{G}2}}{\partial t_0} + (1-\alpha_0)\frac{\partial p_{\mathrm{L}2}}{\partial x_0} = K_4, \tag{52}$$

(v) Keller equation for bubbles of the ith type

$$R_{i2} + \frac{3(\gamma_{\mathrm{e}}-1)R_{i0}}{2\Delta^2/\Omega^2 + 3[(3\gamma_{\mathrm{e}}-1)R_{i0} - 2\gamma_{\mathrm{e}}]p_{\mathrm{L}0}}p_{\mathrm{L}2} - \frac{\Delta^2/\Omega^2 + [(3\gamma_{\mathrm{e}}-1)R_{i0} - 3\gamma_{\mathrm{e}}]p_{\mathrm{L}0}}{2\Delta^2/\Omega^2 + 3[(3\gamma_{\mathrm{e}}-1)R_{i0} - 2\gamma_{\mathrm{e}}]p_{\mathrm{L}0}}T_{\mathrm{G}i2} = K_{Ri}, \tag{53}$$

(vi) Equation of thermal conduction for bubbles of the ith type

$$\frac{\partial T_{\mathrm{G}i2}}{\partial t_0} + 3(\kappa - 1)\frac{\partial R_{i2}}{\partial t_0} = K_{Ti}, \tag{54}$$

with inhomogeneous terms K_m ($m = 1, 2, 3, 4, Ri, Ti$); they are explicitly presented in Appendix A. In the second-order approximation, the second-order and first-order derivatives in the original Keller equation (7) remain unlike the first-order approximation. Therefore, N kinds of bubbles with different initial bubble radii follow N oscillation equations describing different dynamics (e.g., inertia, damping, and eigenfrequency). Equations (49)–(54) are combined into

$$\begin{aligned}
&- C\frac{\partial^2 p_{\mathrm{L}2}}{\partial t_0^2} + \left[p_{\mathrm{G}0}\frac{1-\alpha_0+\beta_1}{\beta_1(1-\alpha_0)}D + \frac{1}{3\alpha_0}\right]\frac{\partial^2 p_{\mathrm{L}2}}{\partial x_0^2} \\
&= K(p_{\mathrm{L}1}; \varphi_0, t_1, x_1) \\
&= \frac{1}{3}\frac{\partial K_1}{\partial \varphi_0} - \frac{1}{3\alpha_0}\frac{\partial K_2}{\partial \varphi_0} + \frac{1-\alpha_0+\beta_1}{3\beta_1(1-\alpha_0)}\frac{\partial K_3}{\partial \varphi_0} + \frac{1}{3\alpha_0(1-\alpha_0)}\frac{\partial K_4}{\partial \varphi_0} \\
&\quad - \sum_{i=1}^{N}\left(\frac{2\Delta^2/\Omega^2 + 3[(3\gamma_{\mathrm{e}}-1)R_{i0} - 2\gamma_{\mathrm{e}}]p_{\mathrm{L}0}}{(3\kappa-1)\Delta^2/\Omega^2 + 3[\kappa(3\gamma_{\mathrm{e}}-1)R_{i0} - \gamma_{\mathrm{e}}(3\kappa-1)]p_{\mathrm{L}0}}\right. \\
&\quad \left.\times\left\{X_{\mathrm{V}i} - \frac{p_{\mathrm{G}0}(1-\alpha_0+\beta_1)}{\beta_1(1-\alpha_0)}[X_{\mathrm{V}i} + (\kappa-1)X_{\mathrm{M}i}]\right\}\frac{\partial^2 K_{Ri}}{\partial \varphi_0^2}\right) \\
&\quad + \sum_{i=1}^{N}\left(\frac{\Delta^2/\Omega^2 + [(3\gamma_{\mathrm{e}}-1)R_{i0} - 3\gamma_{\mathrm{e}}]p_{\mathrm{L}0}}{(3\kappa-1)\Delta^2/\Omega^2 + 3[\kappa(3\gamma_{\mathrm{e}}-1)R_{i0} - \gamma_{\mathrm{e}}(3\kappa-1)]p_{\mathrm{L}0}}\right. \\
&\quad \left.\times\left\{X_{\mathrm{V}i} - \frac{p_{\mathrm{G}0}(1-\alpha_0+\beta_1)}{\beta_1(1-\alpha_0)}[X_{\mathrm{V}i} + (\kappa-1)X_{\mathrm{M}i}]\right\}\frac{\partial K_{Ti}}{\partial \varphi_0}\right) \\
&\quad + \sum_{i=1}^{N}\left[\frac{p_{\mathrm{G}0}(1-\alpha_0+\beta_1)}{3\beta_1(1-\alpha_0)}X_{\mathrm{M}i}\frac{\partial K_{Ti}}{\partial \varphi_0}\right].
\end{aligned} \tag{55}$$

From the solvability condition[72] of (55), which is equivalent to the nonsecular condition for expansion in (30)–(33), we have

$$\begin{aligned}
K = &- 2C\frac{\partial}{\partial \varphi_0}\left(\frac{\partial p_{\mathrm{L}1}}{\partial t_1} + \frac{\partial p_{\mathrm{L}1}}{\partial x_1} + \Pi_0\frac{\partial p_{\mathrm{L}1}}{\partial \varphi_0} + \Pi_1 p_{\mathrm{L}1}\frac{\partial p_{\mathrm{L}1}}{\partial \varphi_0}\right. \\
&\left. - \Pi_{21}\frac{\partial^2 p_{\mathrm{L}1}}{\partial \varphi_0^2} + \Pi_3\frac{\partial^3 p_{\mathrm{L}1}}{\partial \varphi_0^3} + \Pi_{22} p_{\mathrm{L}1}\right) = 0.
\end{aligned} \tag{56}$$

Subsequently, we get the KdVB equation

$$\frac{\partial p_{\mathrm{L}1}}{\partial \tau} + \Pi_1 p_{\mathrm{L}1}\frac{\partial p_{\mathrm{L}1}}{\partial \xi} - \Pi_{21}\frac{\partial^2 p_{\mathrm{L}1}}{\partial \xi^2} + \Pi_3\frac{\partial^3 p_{\mathrm{L}1}}{\partial \xi^3} + \Pi_{22} p_{\mathrm{L}1} = 0, \tag{57}$$

via a variable transformation

$$\tau \equiv \epsilon t, \quad \xi \equiv x - (1 + \epsilon\Pi_0)t, \tag{58}$$

where Π_1 is the nonlinear coefficient (constant coefficient of nonlinear term), Π_{21} and Π_{22} are the dissipation coefficients (constant coefficients of dissipation terms), Π_3 is the dispersion coefficient (constant coefficient of dispersion term), and Π_0 is the advection coefficient related to phase velocity (advection). It is a linear sum of terms that represent each of the three properties. They are constant coefficients given by

$$\Pi_0 = -\frac{(1-\alpha_0)V^2}{6\alpha_0 C}, \tag{59}$$

$$\begin{aligned}
\Pi_{21} = &\frac{1}{2C}\sum_{i=1}^{N}\left(\left\{X_{\mathrm{V}i} - p_{\mathrm{G}0}\frac{1-\alpha_0+\beta_1}{\beta_1(1-\alpha_0)}[X_{\mathrm{V}i} + (\kappa-1)X_{\mathrm{M}i}]\right\}\right. \\
&\left.\times\Big(\underbrace{4\mu s_{Ri}^2}_{\text{viscosity}} - \underbrace{V\Delta R_{i0} s_{Ri}}_{\text{acoustic radiation}}\Big)\right),
\end{aligned} \tag{60}$$

$$\begin{aligned}
\Pi_{22} = &-3(\kappa-1)\sum_{i=1}^{N}\left(\left\{X_{\mathrm{V}i} - \frac{p_{\mathrm{G}0}(1-\alpha_0+\beta_1)}{\beta_1(1-\alpha_0)}\right.\right. \\
&\left.\left.\times[X_{\mathrm{V}i} + (\kappa-1)X_{\mathrm{M}i}]\right\}p_{\mathrm{G}i0}\zeta_{i1}s_{Ri}^2\right) \\
&+ 3(\kappa-1)\sum_{i=1}^{N}\left[\frac{p_{\mathrm{G}0}(1-\alpha_0+\beta_1)}{3\beta_1(1-\alpha_0)}X_{\mathrm{M}i}\zeta_{i1}s_{Ri}\right],
\end{aligned} \tag{61}$$

$$\Pi_3 = \frac{1}{2C}\sum_{i=1}^{N}\left(\left\{X_{\mathrm{V}i} - p_{\mathrm{G}0}\frac{1-\alpha_0+\beta_1}{\beta_1(1-\alpha_0)}[X_{\mathrm{V}i} + (\kappa-1)X_{\mathrm{M}i}]\right\}(\Delta^2 R_{i0}^2 s_{Ri}^2)\right). \tag{62}$$

The form of Π_1 is explicitly shown in Appendix B because its explicit form is significantly complex.

As dissipation coefficients, Π_{21} is attributed to the viscosity and acoustic radiation owing to the compressibility of liquids, and Π_{22} is attributed to thermal conduction.[70] Two types of coefficients representing dissipation exist, which implies that the dissipation mechanism owing to viscosity and acoustic radiation is different from that owing to thermal conduction. The effects of polydispersity are observed in all nonlinear coefficients Π_1, dissipation coefficients Π_{21} and Π_{22}, and dispersion coefficients Π_3.

C. Effect of polydispersity on three properties

In this discussion, as an example of the bubble radius distribution for bubbly liquid, we incorporate the commonly used lognormal distribution.[86,109] The lognormal distribution for bubble radius is given by

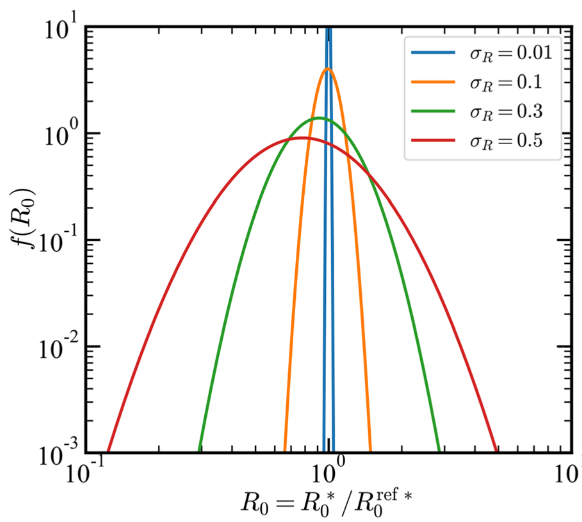

FIG. 2. Lognormal distribution (63) for dimensionless equilibrium bubble radius $R_0 = R_0^*/R_0^{\mathrm{ref}^*}$, where R_0^* is the equilibrium bubble radius and $R_0^{\mathrm{ref}^*}$ is the representative initial bubble radius for four cases of $\sigma_R = 0.01, 0.1, 0.3$, and 0.5.

$$f(R_0) = \frac{1}{\sqrt{2\pi}\sigma_R R_0} \exp\left[\frac{-(\log R_0)^2}{\sigma_R^2}\right], \tag{63}$$

where $R_0 \equiv R_0^*/R_0^{\mathrm{ref}^*}$ is the dimensionless equilibrium bubble radius, σ_R represents the magnitude of polydispersity, and $\sigma_R \to 0$ leads $f(R_0)$ to the Dirac delta function (i.e., the monodisperse case) as shown in Fig. 2.

Figures 3 and 4 show the nonlinear coefficient Π_1, dissipation coefficients Π_{21} and Π_{22}, and dispersion coefficient Π_3 vs the initial void fraction α_0 for X_i, satisfying $f(R_0)\mathrm{d}R_0$ with $N = 1000$ for micrometer bubble scale and millimeter bubble scale, respectively.

The dissipation coefficient owing to viscosity and acoustic radiation Π_{21} and the dispersion coefficient Π_3 in all polydisperse cases are larger than those in the monodisperse case. However, the dissipation coefficient owing to thermal conductivity Π_{22} is reduced by polydispersity from the monodisperse case. Π_{21}, Π_{22}, and Π_3 were significantly affected when σ_R was large. Since the effect of polydispersity on the two types of dissipation coefficients is opposite, the effect on the total dissipation is discussed later with regard to the waveform. Figure 3 shows that Π_1 is reduced by polydispersity, but the change is relatively small. In Figs. 3 and 4, we observe that the effect of polydispersity on each coefficient is similar even if the scale of $R_0^{\mathrm{ref}^*}$ is different. The focus of these results is that the change in Π_3 is significantly larger than the change in other coefficients. Originally, wave dispersion was caused by bubble oscillations[1] and was affected by bubble size (i.e., polydispersity). This result implies a strong connection between polydispersity and wave dispersion, which is strongly affected by bubble size.

D. Numerical example

As an example of the waveform, we numerically solve the KdVB equation (57) using the split-step Fourier–Galerkin method (see also the detailed scheme in Refs. 110 and 111). We assume the initial waveform as

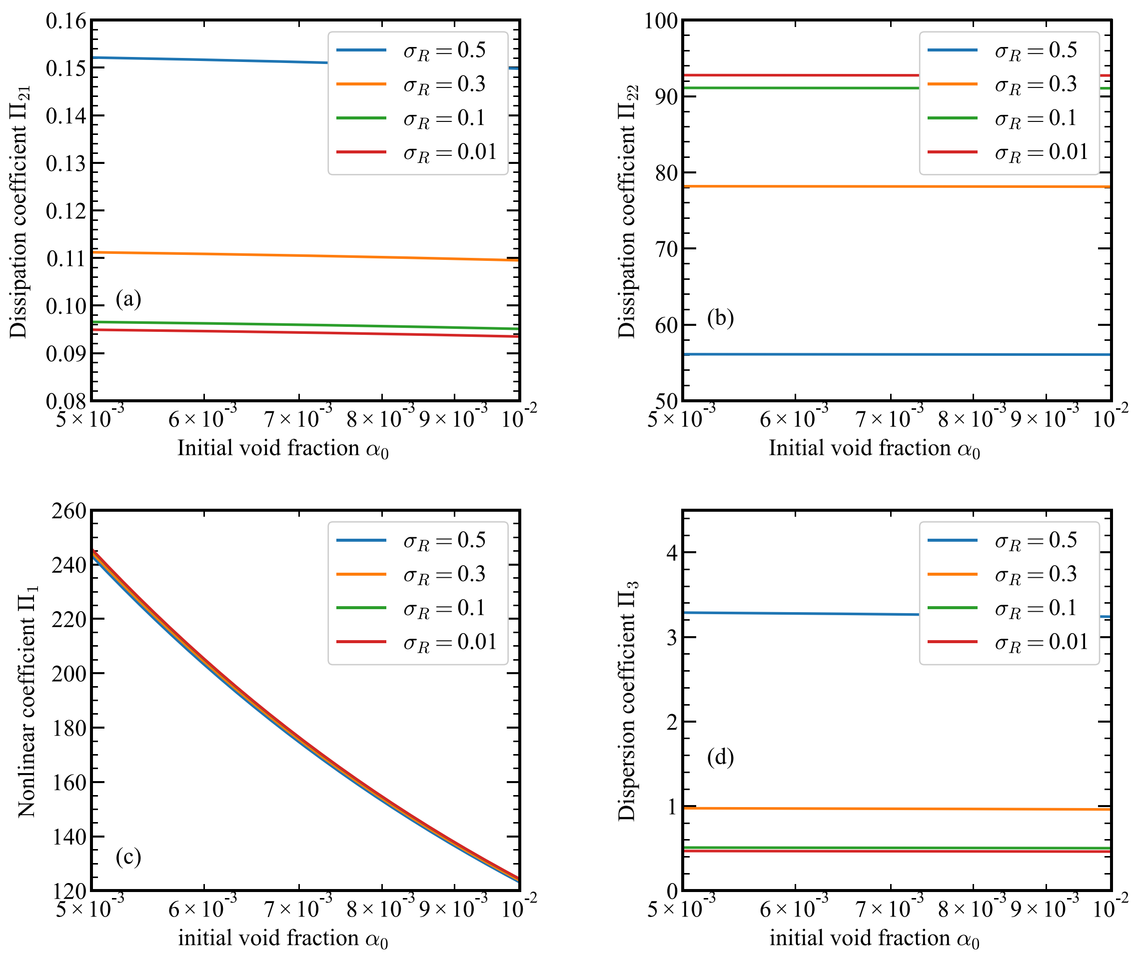

FIG. 3. (a) Dissipation coefficient owing to viscosity and acoustic radiation Π_{21}, (b) dissipation coefficient owing to thermal conductivity Π_{22},[70] (c) nonlinear coefficient Π_1, and (d) dispersion coefficient Π_3 as a function of the initial void fraction α_0 when X_i follows lognormal distribution (63). A representative initial bubble radius $R_0^{\mathrm{ref}^*} = 10\mu$m, $\sqrt{\epsilon} = 0.15$, $p_{L0}^* = 101\,325$ Pa, $\rho_{L0}^* = 1000\,\mathrm{kg/m^3}$, $\sigma^* = 0.0728\,\mathrm{N/m}$, $c_{L0}^* = 1500\,\mathrm{m/s}$, $\mu^* = 1 \times 10^3\,\mathrm{Pa \cdot s}$. $\sigma_R = 0.01$ corresponds to the monodisperse case. Note that Π_{21}, Π_{22}, and Π_3 depend weakly on α_0. Π_1 decreases slightly to the extent that it is difficult to see it in the graph.

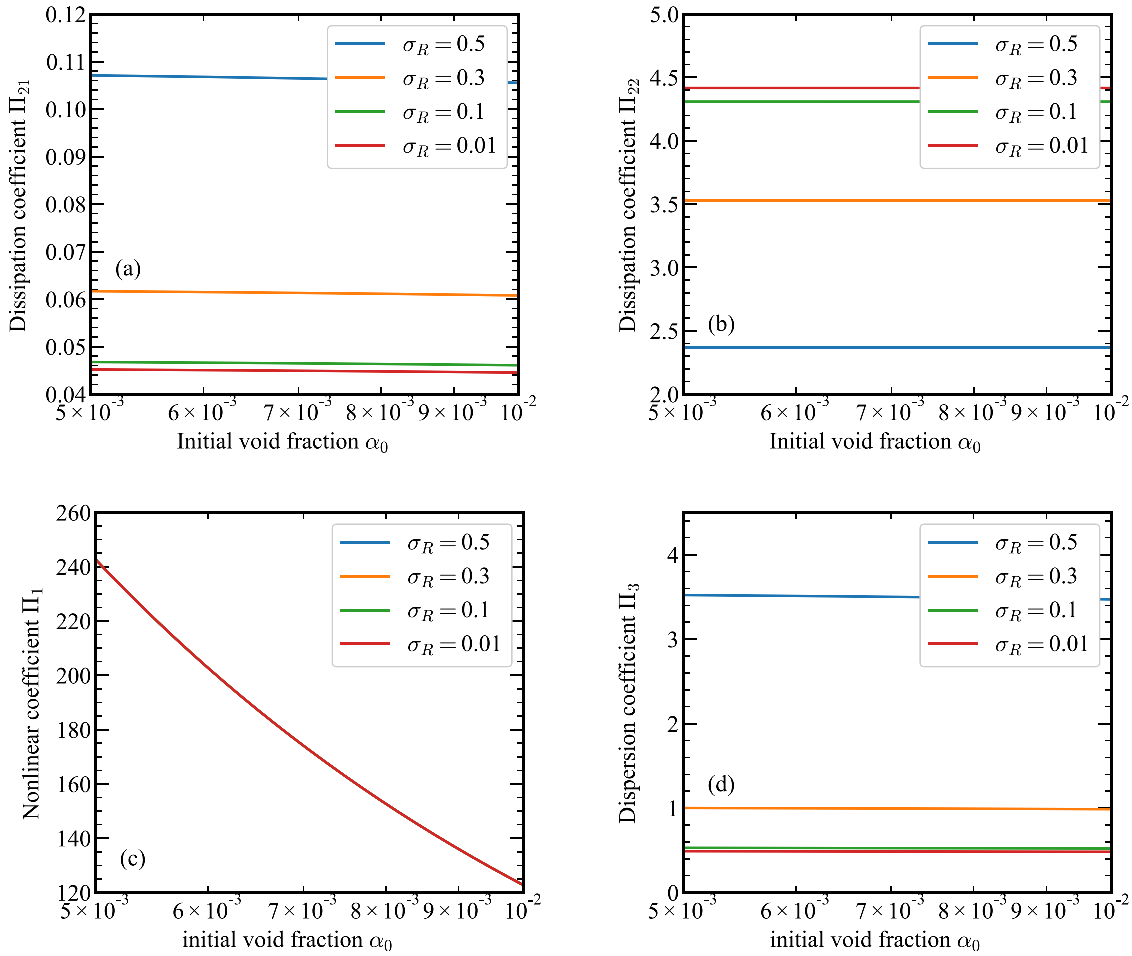

FIG. 4. (a) Π_{21}, (b) Π_{22}, (c) Π_1, and (d) Π_3 for millimeter case (i.e., $R_0^{\mathrm{ref}^*} = 1$ mm) as counterpart of Fig. 3 (i.e., the micrometer case). These values, except for $R_0^{\mathrm{ref}^*}$, are used in Fig. 3.

$$p_{\mathrm{L1}} = \frac{1}{4}\exp\left[\frac{-2(x-10)^2}{9}\right], \quad (64)$$

where p_{L1} is the first-order perturbation of nondimensional liquid pressure. The temporal evolution of the wave (64) is investigated. Figures 5 and 6 show the cases of $\sigma_R = 0.01$ (i.e., monodispersity) and 0.5 (i.e., polydisperse), respectively. There is a clear difference between monodisperse and polydisperse cases.

IV. CONCLUSIONS

The weakly nonlinear propagation process of plane progressive pressure waves in liquids containing several spherical gas bubbles was

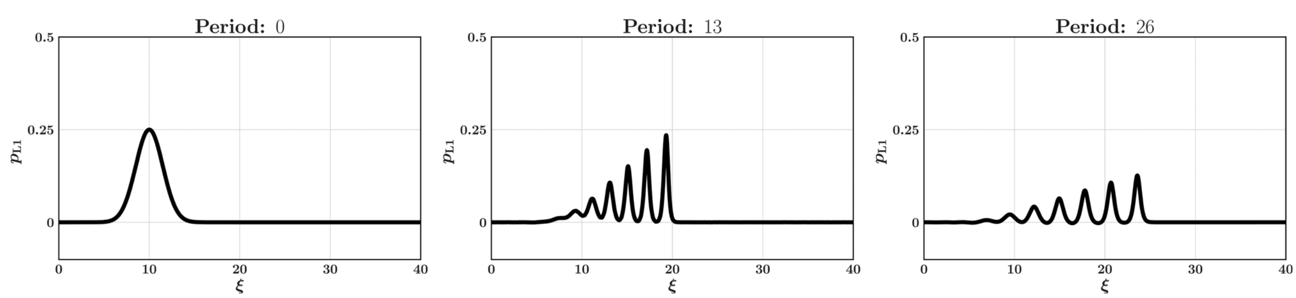

FIG. 5. Polydisperse case of temporal evolution of the initial waveform (64) by KdVB equation (57) for $\alpha_0 = 0.005$, $R_0^{\mathrm{ref}^*} = 10\,\mu$m, gridsteps $N_{\mathrm{grid}} = 1024$, duration of numerical integration $\Delta\tau = 0.001$, and size of the computational domain $W = 40$ and $\sigma_R = 0.01$.

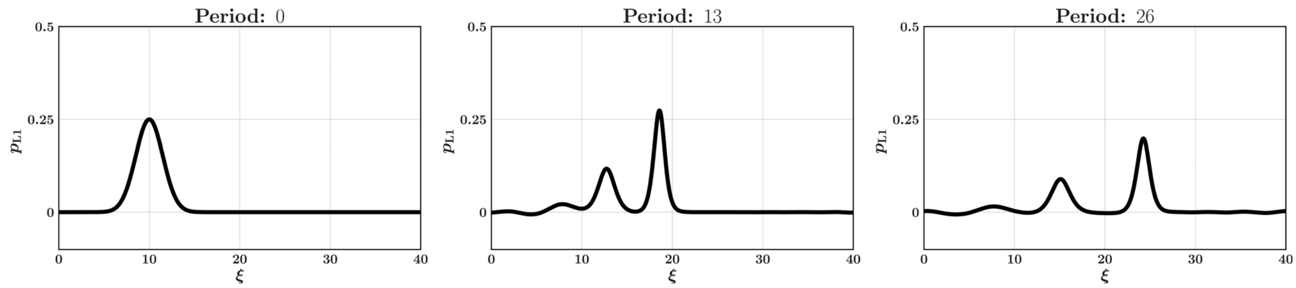

FIG. 6. Polydisperse case of temporal evolution of the initial waveform (64) by KdVB equation (57) for $\alpha_0 = 0.005$, $R_0^{\mathrm{ref}^*} = 10\,\mu$m, gridsteps $N_{\mathrm{grid}} = 1024$, duration of numerical integration $\Delta\tau = 0.001$, and size of the computational domain $W = 40$ and $\sigma_R = 0.5$.

analytically investigated. Our special focus was polydispersity (i.e., discrete distribution of initial bubble radius), where a bubbly liquid contains multiple bubbles of different sizes and the content rate is uniform and optional.

Subsequently, we derived the KdVB equation with a correction term, describing pressure waves in polydisperse bubbly liquids, where all the coefficients of the nonlinear, dissipation, and dispersion terms are influenced by polydispersity. In particular, the dispersion effect in the present polydisperse case increased significantly compared with that in our previous monodisperse case. Furthermore, in the case of the dispersion coefficient, polydispersity contributes more significantly compared with the cases of the nonlinear and dissipation coefficients. This implies a strong relationship between wave dispersion and the polydispersity of bubble size.

Although the present results discussed dissipation as a linear combination, we will incorporate a nonlinear dissipation effect[112–114] in a forthcoming paper.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Tetsuya Kanagawa: Conceptualization (lead); Data curation (equal); Formal analysis (equal); Funding acquisition (lead); Investigation (equal); Methodology (lead); Project administration (lead); Software (equal); Supervision (lead); Validation (equal); Visualization (equal); Writing – original draft (lead); Writing – review and editing (lead). **Reona Ishitsuka:** Conceptualization (supporting); Data curation (equal); Formal analysis (equal); Investigation (equal); Methodology (supporting); Writing – original draft (supporting). **Shuya Arai:** Data curation (equal); Formal analysis (supporting); Investigation (equal); Validation (equal); Writing – original draft (supporting); Writing – review and editing (supporting). **Takahiro Ayukai:** Data curation (equal); Formal analysis (supporting); Investigation (supporting); Software (equal); Validation (supporting); Visualization (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX A: INHOMOGENEOUS TERMS

The inhomogeneous terms K_m ($m = 1, 2, 3, 4, Ri, Ti$) are given by

$$\begin{aligned} K_1 = {} & 3\sum_{i=1}^{N} X_{\mathrm{V}i}\frac{\partial(\alpha_1 R_{i1})}{\partial t_0} - \frac{\partial \alpha_1}{\partial t_1} + 3\sum_{i=1}^{N} X_{\mathrm{V}i}\frac{\partial R_{i1}}{\partial t_1} - \frac{\partial(\alpha_1 u_{\mathrm{G1}})}{\partial x_0} \\ & + 3\sum_{i=1}^{N} X_{\mathrm{V}i}\frac{\partial(u_{\mathrm{G1}} R_{i1})}{\partial t_0} - \frac{\partial u_{\mathrm{G1}}}{\partial x_1} + 6\sum_{i=1}^{N}\left(X_{\mathrm{V}i} R_{i1}\frac{\partial R_{i1}}{\partial t_0}\right) \\ & - 18\sum_{i=1}^{N}(X_{\mathrm{V}i} R_{i1})\sum_{i=1}^{N}\left(X_{\mathrm{V}i}\frac{\partial R_{i1}}{\partial t_0}\right), \end{aligned} \tag{A1}$$

$$K_2 = (1-\alpha_0)V^2\frac{\partial p_{\mathrm{L1}}}{\partial t_0} - \alpha_0\frac{\partial \alpha_1}{\partial t_1} - \alpha_0\frac{\partial(\alpha_1 u_{\mathrm{L1}})}{\partial t_0} + (1-\alpha_0)\frac{\partial u_{\mathrm{L1}}}{\partial x_1}, \tag{A2}$$

$$\begin{aligned} K_3 = {} & -\beta_1\alpha_1\left(\frac{\partial u_{\mathrm{G1}}}{\partial t_0} - \frac{\partial u_{\mathrm{L1}}}{\partial t_0}\right) - \beta_1\left(\frac{\partial u_{\mathrm{G1}}}{\partial t_1} - \frac{\partial u_{\mathrm{L1}}}{\partial t_1}\right) \\ & - \beta_1\left(u_{\mathrm{G1}}\frac{\partial u_{\mathrm{G1}}}{\partial t_0} - u_{\mathrm{L1}}\frac{\partial u_{\mathrm{L1}}}{\partial t_0}\right) - \beta_2(u_{\mathrm{G1}} - u_{\mathrm{L1}})\frac{\partial \alpha_1}{\partial t_0} \\ & + 3p_{\mathrm{G0}}\sum_{i=1}^{N}\left(X_{\mathrm{V}i}\frac{\partial R_{i1}^2}{\partial x_0}\right) - 18p_{\mathrm{G0}}\sum_{i=1}^{N}(X_{\mathrm{V}i} R_{i1})\sum_{i=1}^{N}\left(X_{\mathrm{V}i}\frac{\partial R_{i1}}{\partial x_0}\right) \\ & + p_{\mathrm{G0}}\alpha_1\sum_{i=1}^{N}\left(3X_{\mathrm{V}i}\frac{\partial R_{i1}}{\partial x_0} - X_{\mathrm{M}i}\frac{\partial T_{\mathrm{G}i1}}{\partial x_0}\right) + p_{\mathrm{G0}}\sum_{i=1}^{N}\left(3X_{\mathrm{V}i}\frac{\partial R_{i1}}{\partial x_1}\right. \\ & \left. - X_{\mathrm{M}i}\frac{\partial T_{\mathrm{G}i1}}{\partial x_1}\right) + 3p_{\mathrm{G0}}\sum_{i=1}^{N}(X_{\mathrm{V}i} R_{i1})\sum_{i=1}^{N}\left(X_{\mathrm{M}i}\frac{\partial T_{\mathrm{G}i1}}{\partial x_0}\right) \\ & + 3p_{\mathrm{G0}}\sum_{i=1}^{N}(X_{\mathrm{M}i} T_{\mathrm{G}i1})\sum_{i=1}^{N}\left(X_{\mathrm{V}i}\frac{\partial R_{i1}}{\partial x_0}\right), \end{aligned} \tag{A3}$$

$$
\begin{aligned}
K_4 =& \beta_1\alpha_1\alpha_0\left(\frac{\partial u_{\mathrm{G}1}}{\partial t_0}-\frac{\partial u_{\mathrm{L}1}}{\partial t_0}\right)+\beta_1\alpha_0\left(\frac{\partial u_{\mathrm{G}1}}{\partial t_1}-\frac{\partial u_{\mathrm{L}1}}{\partial t_1}\right)\\
&+\beta_1\alpha_0\left(u_{\mathrm{G}1}\frac{\partial u_{\mathrm{G}1}}{\partial t_0}-u_{\mathrm{L}1}\frac{\partial u_{\mathrm{L}1}}{\partial t_0}\right)+\beta_2\alpha_0(u_{\mathrm{G}1}-u_{\mathrm{L}1})\frac{\partial \alpha_1}{\partial t_0}\\
&+\alpha_0\frac{\partial(\alpha_1 u_{\mathrm{L}1})}{\partial t_0}-(1-\alpha_0)\frac{\partial u_{\mathrm{L}1}}{\partial t_1}-(1-\alpha_0)\frac{\partial u_{\mathrm{L}1}^2}{\partial x_0}\\
&+\alpha_0\alpha_1\frac{\partial p_{\mathrm{L}1}}{\partial x_0}-(1-\alpha_0)\frac{\partial p_{\mathrm{L}1}}{\partial x_1},
\end{aligned}
\tag{A4}
$$

$$
\begin{aligned}
K_{Ri} =& \frac{5\Delta^2/\Omega^2+3[2(3\gamma_{\mathrm{e}}-1)R_{i0}-5\gamma_{\mathrm{e}}]p_{\mathrm{L}0}}{2\Delta^2/\Omega^2+3[(3\gamma_{\mathrm{e}}-1)R_{i0}-2\gamma_{\mathrm{e}}]p_{\mathrm{L}0}}R_{i1}^2\\
&-\frac{(3\gamma_{\mathrm{e}}-1)R_{i0}}{2\Delta^2/\Omega^2+3[(3\gamma_{\mathrm{e}}-1)R_{i0}-2\gamma_{\mathrm{e}}]p_{\mathrm{L}0}}\left(4\mu\frac{\partial R_{i1}}{\partial t_0}-V\Delta R_{i0}\frac{\partial p_{\mathrm{L}1}}{\partial t_0}\right)\\
&-\frac{(3\gamma_{\mathrm{e}}-1)\Delta^2 R_{i0}^3}{2\Delta^2/\Omega^2+3[(3\gamma_{\mathrm{e}}-1)R_{i0}-2\gamma_{\mathrm{e}}]p_{\mathrm{L}0}}\frac{\partial^2 R_{i1}}{\partial t_0^2}\\
&-\frac{3\Delta^2/\Omega^2+3[(3\gamma_{\mathrm{e}}-1)R_{i0}-3\gamma_{\mathrm{e}}]p_{\mathrm{L}0}}{2\Delta^2/\Omega^2+3[(3\gamma_{\mathrm{e}}-1)R_{i0}-2\gamma_{\mathrm{e}}]p_{\mathrm{L}0}}R_{i1}T_{\mathrm{G}i1},
\end{aligned}
\tag{A5}
$$

$$
\begin{aligned}
K_{Ti} =& -3(\kappa-1)\frac{\partial R_{i1}}{\partial t_1}-\frac{\partial T_{\mathrm{G}i1}}{\partial t_1}-3(\kappa-1)(3\kappa-4)R_{i1}\frac{\partial R_{i1}}{\partial t_0}\\
&-3(\kappa-1)\frac{\partial(T_{\mathrm{G}i1}R_{i1})}{\partial t_0}-3(\kappa-1)u_{\mathrm{G}1}\frac{\partial R_{i1}}{\partial x_0}-u_{\mathrm{G}1}\frac{\partial T_{\mathrm{G}i1}}{\partial x_0}-\zeta_{i1}T_{\mathrm{G}i1}.
\end{aligned}
\tag{A6}
$$

APPENDIX B: THE NONLINEAR COEFFICIENT

We shall present the explicit forms of the nonlinear coefficient of the KdVB equation, Π_1,

t

$$
\begin{aligned}
\Pi_1 \equiv& -\frac{1}{6C}k_1+\frac{1}{6\alpha_0 C}k_2-\frac{1-\alpha_0+\beta_1}{6\beta_1(1-\alpha_0)C}k_3-\frac{1}{6\alpha_0(1-\alpha_0)C}k_4\\
&+\frac{1}{2C}\sum_{i=1}^{N}\left(\frac{2\Delta^2/\Omega^2+3[(3\gamma_{\mathrm{e}}-1)R_{i0}-2\gamma_{\mathrm{e}}]p_{\mathrm{L}0}}{(3\kappa-1)\Delta^2/\Omega^2+3[\kappa(3\gamma_{\mathrm{e}}-1)R_{i0}-\gamma_{\mathrm{e}}(3\kappa-1)]p_{\mathrm{L}0}}\right.\\
&\left.\times\left\{X_{\mathrm{V}i}-\frac{p_{\mathrm{G}0}(1-\alpha_0+\beta_1)}{\beta_1(1-\alpha_0)}[X_{\mathrm{V}i}+(\kappa-1)X_{\mathrm{M}i}]\right\}k_{Ri}\right)\\
&-\frac{1}{2C}\sum_{i=1}^{N}\left(\frac{\Delta^2/\Omega^2+[(3\gamma_{\mathrm{e}}-1)R_{i0}-3\gamma_{\mathrm{e}}]p_{\mathrm{L}0}}{(3\kappa-1)\Delta^2/\Omega^2+3[\kappa(3\gamma_{\mathrm{e}}-1)R_{i0}-\gamma_{\mathrm{e}}(3\kappa-1)]p_{\mathrm{L}0}}\right.\\
&\left.\times\left\{X_{\mathrm{V}i}-\frac{p_{\mathrm{G}0}(1-\alpha_0+\beta_1)}{\beta_1(1-\alpha_0)}[X_{\mathrm{V}i}+(\kappa-1)X_{\mathrm{M}i}]\right\}k_{Ti}\right)\\
&-\frac{1}{2C}\sum_{i=1}^{N}\left[\frac{p_{\mathrm{G}0}(1-\alpha_0+\beta_1)}{3\beta_1(1-\alpha_0)}X_{\mathrm{M}i}k_{Ti}\right],
\end{aligned}
\tag{B1}
$$

where

$$
\begin{aligned}
k_1 =& -2s_1s_3-6(s_1-s_3)C-6\sum_{i=1}^{N}(X_{\mathrm{V}i}s_{Ri}^2)+18C^2,\\
k_2 =& -2\alpha_0 s_2 s_3,\\
k_3 =& [(\beta_1+\beta_2)s_3-\beta_1(s_1+s_2)](s_1-s_2)-3p_{\mathrm{G}0}s_3D\\
&+6p_{\mathrm{G}0}\sum_{i=1}^{N}(X_{\mathrm{V}i}s_{Ri}^2)-18p_{\mathrm{G}0}CD,
\end{aligned}
\tag{B2}
$$

$$
\begin{aligned}
k_4 =& -\alpha_0[(\beta_1+\beta_2)s_3-\beta_1(s_1+s_2)](s_1-s_2)-2\alpha_0 s_2 s_3\\
&-2(1-\alpha_0)s_2^2+\alpha_0 s_3,\\
k_{Ri} =& 2\frac{(9\kappa-4)\Delta^2/\Omega^2+3[(3\kappa-1)(3\gamma_{\mathrm{e}}-1)R_{i0}-\gamma_{\mathrm{e}}(9\kappa-4)]p_{\mathrm{L}0}}{2\Delta^2/\Omega^2+3[(3\gamma_{\mathrm{e}}-1)R_{i0}-2\gamma_{\mathrm{e}}]p_{\mathrm{L}0}}s_{Ri}^2,\\
k_{Ti} =& -3(3\kappa-2)(\kappa-1)s_{Ri}^2.
\end{aligned}
$$

REFERENCES

[1] L. van Wijngaarden, "On the equations of motion for mixtures of liquid and gas bubbles," J. Fluid Mech. **33**, 465–474 (1968).

[2] L. Noordzij and L. van Wijngaarden, "Relaxation effects, caused by relative motion, on shock waves in gas–bubble/liquid mixtures," J. Fluid Mech. **66**, 115–143 (1974).

[3] V. V. Kuznetsov, V. A. Nakoryakov, B. G. Pokusaev, and I. R. Shreiber, "Propagation of perturbations in a gas–liquid mixture," J. Fluid Mech. **85**, 85–96 (1978).

[4] D. S. Drumheller, M. E. Kipp, and A. Bedford, "Transient wave propagation in bubbly liquids," J. Fluid Mech. **119**, 347–365 (1982).

[5] M. Kameda and Y. Matsumoto, "Shock waves in a liquid containing small gas bubbles," Phys. Fluids **8**, 322 (1996).

[6] G. E. Reisman, Y.-C. Wang, and C. E. Brennen, "Observations of shock waves in cloud cavitation," J. Fluid Mech. **355**, 255–283 (1998).

[7] V. Leroy, S. Strybulevych, J. H. Page, and M. G. Scanlon, "Sound velocity and attenuation in bubbly gels measured by transmission experiments," J. Acoust. Soc. Am. **123**, 1931–1940 (2008).

[8] S. M. Frolov, K. A. Avdeev, V. S. Aksenov, A. A. Borisov, F. S. Frolov, I. O. Shamshin, R. R. Tukhvatullina, B. Basara, W. Edelbauer, and K. Pachler, "Experimental and computational studies of shock wave-to-bubbly water momentum transfer," Int. J. Multiphase Flow **92**, 20–38 (2017).

[9] Y. Chukkol and M. Muminov, "Shock-wave propagation in an isothermal slightly compressible bubbly viscoelastic fluid flow," Mod. Phys. Lett. B **34**, 2050042 (2020).

[10] N. N. Liu, W. B. Wu, A. M. Zhang, and Y. L. Liu, "Experimental and numerical investigation on bubble dynamics near a free surface and a circular opening of plate," Phys. Fluids **29**, 107102 (2017).

[11] E. Kadivar, M. V. Timoshevskiy, M. Yu Nichik, O. Moctar, T. E. Schellin, and K. S. Pervunin, "Control of unsteady partial cavitation and cloud cavitation in marine engineering and hydraulic systems," Phys. Fluids **32**, 052108 (2020).

[12] A. Kiyama, R. Rabbi, Z. Pan, S. Dutta, J. S. Allen, and T. T. Truscott, "Morphology of bubble dynamics and sound in heated oil," Phys. Fluids **34**, 062107 (2022).

[13] C. F. Delale, S. Nas, and G. Tryggvason, "Direct numerical simulation of shock propagation in bubbly liquids," Phys. Fluids **17**, 121705 (2005).

[14] N. Kudryashov and N. Teterev, "Numerical simulation of acoustic wave propagation via a liquid with gas bubbles," J. Acoust. Soc. Am. **123**, 3841 (2008).

[15] K. Ando, T. Colonius, and C. E. Brennen, "Numerical simulation of shock propagation in a polydisperse bubbly liquid Int. J. Multiphase Flow **37**, 596–608 (2011).

[16] D. Fuster and T. Colonius, "Modelling bubble clusters in compressible liquids," J. Fluid Mech. **688**, 352–389 (2011).

[17] H. Grandjean, N. Jacques, and S. Zaleski, "Shock propagation in liquids containing bubbly clusters: A continuum approach," J. Fluid Mech. **701**, 304–332 (2012).

[18] R. Jamshidi, B. Pohl, and U. A. Peuker, "Numerical investigation of sonochemical reactors considering the effect of inhomogeneous bubble clouds on ultrasonic wave propagation," Chem. Eng. J. **189**, 364–375 (2012).

[19] R. Jamshidi and G. Brenner, "Dissipation of ultrasonic wave propagation in bubbly liquids considering the effect of compressibility to the first order of acoustical Mach number," Ultrasonics **53**, 842–848 (2013).

[20] M. R. Betney, B. Tully, N. A. Hawker, and Y. Ventikos, "Computational modelling of the interaction of shock waves with multiple gas-filled bubbles in a liquid," Phys. Fluids **27**, 036101 (2015).

[21]S. H. Bryngelson, K. Schmidmayer, and T. Colonius, "A quantitative comparison of phase-averaged models for bubbly, cavitating flows," Int. J. Multiphase Flow **115**, 137–143 (2019).
[22]C. Vanhille, "Numerical simulations of stable cavitation bubble generation and primary Bjerknes forces in a three-dimensional nonlinear phased array focused ultrasound field," Ultrason. Sonochem. **63**, 104972 (2020).
[23]C. Vanhille, "A fourth-order approximation Rayleigh–Plesset equation written in volume variation for an adiabatic-gas bubble in an ultrasonic field: Derivation and numerical solution," Results Phys. **25**, 104193 (2021).
[24]D. Qin, Q. Zou, S. Lei, W. Wang, and Z. Li, "Nonlinear dynamics and acoustic emissions of interacting cavitation bubbles in viscoelastic tissues," Ultrason. Sonochem. **78**, 105712 (2021).
[25]Y. Fan, H. Li, and D. Fuster, "Time-delayed interactions on acoustically driven bubbly screens," J. Acoust. Soc. Am. **150**, 4219 (2021).
[26]K. Pham, J. Mercier, D. Fuster, J. Marigo, and A. Maurel, "Scattering of acoustic waves by a nonlinear resonant bubbly screen," J. Fluid Mech. **906**, A19 (2021).
[27]A. Modarreszadeh, E. Timofeev, A. Merlen, and P. Pernod, "Numerical simulation of the interaction of wave phase conjugation with bubble clouds," Int. J. Multiphase Flow **141**, 103638 (2021).
[28]S. H. Bryngelson, R. O. Fox, and T. Colonius, "Conditional moment methods for polydisperse cavitating flows," arXiv:2112.14172 (2021).
[29]Z. Wang, W. Zhou, T. Shu, Q. Xue, R. Zhang, and M. Wiercigroch, "Modelling of low-frequency acoustic wave propagation in dilute gas-bubbly liquids," Int. J. Mech. Sci. **216**, 106979 (2022).
[30]Z. Li, Y. Zhou, and L. Xu, "Sinking bubbles in a fluid under vertical vibration," Phys. Fluids **33**, 037130 (2021).
[31]S. R. Gonzalez-Avila, F. Denner, and C. D. Ohl, "The acoustic pressure generated by the cavitation bubble expansion and collapse near a rigid wall," Phys. Fluids **33**, 032118 (2021).
[32]A. Saha and A. K. Das, "Bubble dynamics in concentric multi-orifice column under normal and reduced gravity," Phys. Fluids **34**, 042113 (2022).
[33]A. Zhang, D. Su, C. Li, Y. Zhang, B. Jiang, and F. Pan, "Investigation of bubble dynamics in a micro-channel with obstacles using a conservative phase-field lattice Boltzmann method," Phys. Fluids **34**, 043312 (2022).
[34]H. Yan, H. Gong, Z. Huang, P. Zhou, and L. Liu, "Euler–Euler modeling of reactive bubbly flow in a bubble column," Phys. Fluids **34**, 053306 (2022).
[35]B. Tian, J. Chen, X. Zhao, M. Zhang, and B. Huang, "Numerical analysis of interaction between turbulent structures and transient sheet/cloud cavitation," Phys. Fluids **34**, 047116 (2022).
[36]M. H. Bappy, P. M. Carrica, J. Li, J. E. Martin, A. Vela-Martin, L. S. Freire, and G. C. Buscaglia, "A sub-grid scale cavitation inception model," Phys. Fluids **34**, 033308 (2022).
[37]A. M. Zhang, W. B. Wu, Y. L. Liu, and Q. X. Wang, "Nonlinear interaction between underwater explosion bubble and structure based on fully coupled model," Phys. Fluids **29**, 082111 (2017).
[38]A. Dolev, M. Kaynak, and M. S. Sakar, "Dynamics of entrapped microbubbles with multiple openings," Phys. Fluids **34**, 012012 (2022).
[39]E. Mahravan and D. Kim, "Bubble collapse and jet formation inside a liquid film," Phys. Fluids **33**, 112102 (2021).
[40]E. Ezzatneshan and H. Vaseghnia, "Dynamics of an acoustically driven cavitation bubble cluster in the vicinity of a solid surface," Phys. Fluids **33**, 123311 (2021).
[41]Q. Wang, W. Liu, C. Corbett, and W. R. Smith, "Microbubble dynamics in a viscous compressible liquid subject to ultrasound," Phys. Fluids **34**, 012105 (2022).
[42]L. A. Teran, S. A. Rodríguez, S. Lain, and S. Jung, "Interaction of particles with a cavitation bubble near a solid wall," Phys. Fluids **30**, 123304 (2018).
[43]E. Kadivar, T.-H. Phan, W.-G. Park, and O. el Moctar, "Dynamics of a single cavitation bubble near a cylindrical rod," Phys. Fluids **33**, 113315 (2021).
[44]M. A. Maiga, O. Coutier-Delgosha, and D. Buisine, "A new cavitation model based on bubble-bubble interactions," Phys. Fluids **30**, 123301 (2018).
[45]J. Mifsud, D. A. Lockerby, Y. M. Chung, and G. Jones, "Numerical simulation of a confined cavitating gas bubble driven by ultrasound," Phys. Fluids **33**, 122114 (2021).
[46]S. Wang, Q. Gui, J. Zhang, Y. Gao, J. Xu, and X. Jia, "Theoretical and experimental study of bubble dynamics in underwater explosions," Phys. Fluids **33**, 126113 (2021).
[47]Y. Qin, Z. Wang, L. Zou, and Z. Zong, "Analytical study on the dynamic characteristics of multiple gas-filled spherical bubbles in typical spatial locations," Phys. Fluids **34**, 022004 (2022).
[48]J. Zhang, S. Wang, X. Jia, Y. Gao, and F. Ma, "An engineering application of Prosperetti and Lezzi equation to solve underwater explosion bubbles," Phys. Fluids **33**, 017118 (2021).
[49]T. Maruishi and A. Toramaru, "Effect of bubble deformation on the coalescence of two ascending bubbles in a viscous liquid," Phys. Fluids **34**, 043302 (2022).
[50]Z. Wang, H. Cheng, and B. Ji, "Numerical prediction of cavitation erosion risk in an axisymmetric nozzle using a multi-scale approach," Phys. Fluids **34**, 062112 (2022).
[51]C. Wang, N. Li, X. Huang, W. Liu, and C. Weng, "Shock wave and bubble pulsation characteristics in a field generated by single underwater detonation," Phys. Fluids **34**, 066108 (2022).
[52]A. Clausse, K. Chetty, J. Buchanan, R. Ram, and M. L. de Bertodano, "Kinematic stability and simulations of the variational two-fluid model for slug flow," Phys. Fluids **34**, 043301 (2022).
[53]N. A. Gumerov, "On the propagation of finite-amplitude long waves in polydisperse mixtures of liquid and gas bubbles," J. Appl. Mech. Tech. Phys. **1**, 75–82 (1992).
[54]I. Akhatov, U. Parlitz, and W. Lauterborn, "Pattern formation in acoustic cavitation," J. Acoust. Soc. Am. **96**, 3627–3635 (1994).
[55]I. Akhatov, U. Parlitz, and W. Lauterborn, "Towards a theory of self-organization phenomena in bubble–liquid mixtures," Phys. Rev. E **54**, 4990–5003 (1996).
[56]V. Q. Vuong and A. J. Szeri, "Sonoluminescence and diffusive transport," Phys. Fluids **8**, 2354–2364 (1996).
[57]D. B. Khismatullin and I. S. Akhatov, "Sound-ultrasound interaction in bubbly fluids: Theory and possible applications," Phys. Fluids **13**, 3582–3598 (2001).
[58]N. A. Kudryashov and D. I. Sinelshchikov, "An extended equation for the description of nonlinear waves in a liquid with gas bubbles," Wave Motion **50**, 351–362 (2013).
[59]N. A. Kudryashov and D. I. Sinelshchikov, "Extended models of nonlinear waves in liquid with gas bubbles," Int. J. Nonlinear Mech. **63**, 31–38 (2014).
[60]Y. Haiyang, W. Haiyan, Z. Zhichen, X. Yong, and J. Kurths, "A stochastic nonlinear differential propagation model for underwater acoustic propagation: Theory and solution," Chaos Solitons Fractals **150**, 111105 (2021).
[61]A. Prosperetti, "The thermal behaviour of oscillating gas bubbles," J. Fluid Mech. **222**, 587–616 (1991).
[62]R. I. Nigmatulin, *Dynamics of Multiphase Media, Part 2* (CRC Press, Hemisphere/New York, 1991).
[63]V. E. Nakoryakov, B. G. Pokusaev, and I. R. Shreiber, *Wave Propagation in Gas–Liquid Media* (CRC Press, Boca Raton, 1993).
[64]R. J. Thiessen and A. F. Cheviakov, "Nonlinear dynamics of a viscous bubbly fluid," Commun. Nonlinear Sci. Numer. Simul. **73**, 244–264 (2019).
[65]L. van Wijngaarden, "One-dimensional flow of liquids containing small gas bubbles," Annu. Rev. Fluid Mech. **4**, 369–396 (1972).
[66]T. Maeda and T. Kanagawa, "Derivation of weakly nonlinear wave equations for pressure waves in bubbly flows with different types of nonuniform distribution of initial flow velocities of gas and liquid phases," J. Phys. Soc. Jpn. **89**, 114403 (2020).
[67]T. Yatabe, T. Kangawa, and T. Atykai, "Theoretical elucidation of effect of drag force and translation of bubble on weakly nonlinear pressure waves in bubbly flows," Phys. Fluids. **33**, 033315 (2021).
[68]T. Kanagawa, T. Ayukai, T. Maeda, and T. Yatabe, "Effect of drag force and translation of bubbles on nonlinear pressure waves with a short wavelength in bubbly flows," Phys. Fluids **33**, 053314 (2021).
[69]T. Kanagawa, "Two types of nonlinear wave equations for diffractive beams in bubbly liquids with nonuniform bubble number density," J. Acoust. Soc. Am. **137**, 2642–2654 (2015).
[70]T. Kamei, T. Kanagawa, and T. Ayukai, "An exhaustive theoretical analysis of thermal effect inside bubbles for weakly nonlinear pressure waves in bubbly liquids," Phys. Fluids **33**, 053302 (2021).
[71]T. Kanagawa, T. Ayukai, T. Kawame, and R. Ishitsuka, "Weakly nonlinear theory on pressure waves in bubbly liquids with a weak polydispersity Int. J. Multiphase Flow **142**, 103622 (2021).

[72]T. Kanagawa, T. Yano, M. Watanabe, and S. Fujikawa, "Unified theory based on parameter scaling for derivation of nonlinear wave equations in bubbly liquids," J. Fluid Sci. Technol. **5**, 351–369 (2010).
[73]T. Kanagawa, M. Watanabe, and S. Fujikawa, "Nonlinear wave equations for pressure wave propagation in liquids containing gas bubbles," J. Fluid Sci. Technol. **6**, 838–850 (2011).
[74]M. Watanabe and A. Prosperetti, "Shock waves in dilute bubbly liquids," J. Fluid Mech. **274**, 349–381 (1994).
[75]R. E. Caflisch, M. J. Miksis, G. C. Papanicolaou, and L. Ting, "Effective equations for wave propagation in bubbly liquids," J. Fluid Mech. **153**, 259–273 (1985).
[76]A. E. Beylich and A. Gülhan, "On the structure of nonlinear waves in liquids with gas bubbles," Phys. Fluids A: Fluid Dyn. **2**, 1412–1428 (1990).
[77]M. Kameda, N. Shimamura, and F. Hagashino, "Shock waves in a uniform bubbly flow," Phys. Fluids **10**, 2661 (1998).
[78]Z. D. Zhang and A. Prosperetti, "Ensemble-averaged equations for bubbly flows," Phys. Fluids **6**, 2956–2970 (1994).
[79]Y. Zhang, Z. Jiang, J. Yuan, T. Chen, Y. Zhang, N. Ning, and X. Du, "Influences of bubble size distribution on propagation of acoustic waves in dilute polydisperse bubbly liquids," J. Hydrodyn. B **31**, 50–57 (2019).
[80]S. L. Gavrilyuk and S. A. Fil'ko, "Shock waves in polydisperse bubbly media with dissipation," J. Appl. Mech. Tech. Phys. **32**, 669–677 (1991).
[81]D. A. Gubaidullin, D. D. Gubaidullina, and Y. V. Fedorov, "Acoustic waves in polydispersed bubbly liquids," J. Phys.: Conf. Ser. **567**, 012011 (2014).
[82]V. G. Gasenko and V. E. Nakoryakov, "Nonlinear three-wave equation for a polydisperse gas–liquid mixture," J. Eng. Thermophys. **17**, 158–165 (2008).
[83]D. A. Gubaidullin and Y. V. Fedorov, "Sound waves in two-fraction polydispersed bubbly media," J. Appl. Math. Mech. **77**, 532–540 (2013).
[84]D. A. Gubaidullin and Y. V. Fedorov, "Sound waves in liquids with polydisperse vapour–gas and gas bubbles," Fluid Dyn. **50**, 61–70 (2015).
[85]D. A. Gubaidullin and Y. V. Fedorov, "Sound waves in a liquid with polydisperse vapour–gas bubbles," Acoust. Phys. **62**, 179–186 (2016).
[86]K. Ando, T. Colonius, and C. E. Brennen, *Shock Propagation in Polydisperse Bubbly Liquids Bubble Dynamics and Shock Waves* (Springer, 2013), pp. 141–175.
[87]T. Colonius, R. Hagmeijer, K. Ando, and C. E. Brennen, "Statistical equilibrium of bubble oscillations in dilute bubbly flows," Phys. Fluids **20**, 040902 (2008).
[88]Y. Fan, H. Li, C. Xu, and T. Zhou, "Influence of bubble distributions on the propagation of linear waves in polydisperse bubbly liquids," J. Acoust. Soc. Am. **145**, 16 (2019).
[89]N. A. Gumerov and I. S. Akhatov, "Modes of self-organization of diluted bubbly liquids in acoustic fields: One-dimensional theory," J. Acoust. Soc. Am. **141**, 1190 (2017).
[90]J. C. Heylmun, B. Kong, A. Passalacqua, and R. Fox, "A quadrature-based moment method for polydisperse bubbly flows," Comput. Phys. Commun. **244**, 187–204 (2019).
[91]R. Mettin and W. Lauterborn, "Secondary acoustic waves in a polydisperse bubbly medium," J. Appl. Mech. Tech. Phys. **44**, 17–26 (2003).
[92]G. Zhou and A. Prosperetti, "Modelling the thermal behaviour of gas bubbles," J. Fluid Mech. **901**, R3 (2020).
[93]G. Zhou, "Modeling the thermal behavior of an acoustically driven gas bubble," J. Acoust. Soc. Am. **149**, 923–933 (2021).
[94]D. Fuster, J. M. Conoir, and T. Colonius, "Effect of direct bubble–bubble interactions on linear-wave propagation in bubbly liquids," Phys. Rev. E **90**, 063010 (2014).
[95]M. Guedra, C. Cornu, and C. Inserra, "A derivation of the stable cavitation threshold accounting for bubble–bubble interactions," Ultrason. Sonochem. **38**, 168–173 (2017).
[96]Y. Shen, L. Zhang, Y. Wu, and W. Chen, "The role of the bubble–bubble interaction on radial pulsations of bubbles," Ultrason. Sonochem. **73**, 105535 (2021).
[97]A. J. Sojahrood, Q. Li, H. Haghi, R. Karshafian, T. M. Porter, and M. C. Kolios, "Investigation of the nonlinear propagation of ultrasound through a bubbly medium including multiple scattering and bubble–bubble interaction: Theory and experiment," in *IEEE International Ultrasonics Symposium (IUS)* (IEEE, 2017), pp. 1–4.
[98]A. J. Sojahrood, R. Earl, H. Haghi, Q. Li, T. M. Porter, M. C. Kolios, and R. Karshafian, "Nonlinear dynamics of acoustic bubbles excited by their pressure-dependent subharmonic resonance frequency: influence of the pressure amplitude, frequency, encapsulation and multiple bubble interactions on oversaturation and enhancement of the subharmonic signal," Nonlinear Dyn. **103**, 429–466 (2021).
[99]K. Yasui, T. Tuziuti, J. Lee, T. Kozuka, A. Towata, and Y. Iida, "Numerical simulations of acoustic cavitation noise with the temporal fluctuation in the number of bubbles," Ultrason. Sonochem. **17**, 460–472 (2010).
[100]D. Fuster and F. Montel, "Mass transfer effects on linear wave propagation in diluted bubbly liquids," J. Fluid Mech. **779**, 598–621 (2015).
[101]R. Egashira, T. Yano, and S. Fujikawa, "Linear wave propagation of fast and slow modes in mixtures of liquid and gas bubbles," Fluid Dyn. Res. **34**, 317–334 (2004).
[102]T. Yano, R. Egashira, and S. Fujikawa, "Linear analysis of dispersive waves in bubbly flows based on averaged equations," J. Phys. Soc. Jpn. **75**, 104401 (2006).
[103]I. Eames and J. C. R. Hunt, "Forces on bodies moving unsteadily in rapidly compressed flows," J. Fluid Mech. **505**, 349–364 (2004).
[104]J. B. Keller and I. I. Kolodner, "Damping of underwater explosion bubble oscillations," J. Appl. Phys. **10**, 1152–1161 (1956).
[105]K. Sugiyama, S. Takagi, and Y. Matsumoto, "A new reduced-order model for the thermal damping effect on radial motion of a bubble (1st report, perturbation analysis)," Trans. JSME, Ser. B **71**, 1011–1019 (2005).
[106]T. Kanagawa and T. Kamei, "Thermal effect inside bubbles for weakly nonlinear pressure waves in bubbly liquids: Theory on short waves," Phys. Fluids **33**, 063319 (2021).
[107]Y. Kikuchi and T. Kanagawa, "Weakly nonlinear theory on ultrasound propagation in liquids containing many microbubbles encapsulated by visco-elastic shell," Jpn. J. Appl. Phys. **60**(SD), SDDD14 (2021).
[108]A. Jeffrey and T. Kawahara, *Asymptotic Methods in Nonlinear Wave Theory* (Pitman, London, 1982), Vol. 3.
[109]E. M. A. Frederix, T. L. W. Cox, J. G. M. Kuerten, and E. M. J. Komen, "Polydispersed modeling of bubbly flow using the log-normal size distribution," Chem. Eng. Sci. **201**, 237–246 (2019).
[110]T. Ayukai and T. Kanagawa, "Numerical study on formation of an acoustic soliton in bubbly liquids based on weakly nonlinear wave equation," J. Acoust. Soc. Am. **146**, 3077 (2019).
[111]T. Ayukai and T. Kanagawa, "Numerical analysis on nonlinear evolution of pressure waves in bubbly liquids based on KdV-Burgers equation," Jpn. J. Multiphase Flow **34**, 158–165 (2020).
[112]A. J. Sojahrood, H. Haghi, Q. Li, T. M. Porter, R. Karshafian, and M. C. Kolios, "Nonlinear power loss in the oscillations of coated and uncoated bubbles: Role of thermal, radiation and encapsulating shell damping at various excitation pressures," Ultrason. Sonochem. **66**, 105070 (2020).
[113]A. J. Sojahrood, H. Haghi, R. Karshafian, and M. C. Kolios, "Critical corrections to models of nonlinear power dissipation of ultrasonically excited bubbles," Ultrason. Sonochem. **66**, 105089 (2020).
[114]A. J. Sojahrood, H. Haghi, R. Karshafian, and M. C. Kolios, "Classification of the major nonlinear regimes of oscillations, oscillation properties, and mechanisms of wave energy dissipation in the nonlinear oscillations of coated and uncoated bubbles," Phys. Fluids **33**, 016105 (2021).