TAKENS' LAST PROBLEM AND EXISTENCE OF NON-TRIVIAL WANDERING DOMAINS

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Abstract. In this paper, we give an answer to a C^r $(2 \leq r < \infty)$ version of the open problem of Takens in [41] which is related to historic behavior of dynamical systems. To obtain the answer, we show the existence of non-trivial wandering domains near a homoclinic tangency, which is conjectured by Colli-Vargas [6, §2]. Concretely speaking, it is proved that any Newhouse open set in the space of C^r-diffeomorphisms on a closed surface is contained in the closure of the set of diffeomorphisms which have non-trivial wandering domains whose forward orbits have historic behavior. Moreover, this result implies an answer in the C^r category to one of the open problems of van Strien [39] which is concerned with wandering domains for Hénon family.

1. Introduction

1.1. **Historic behavior and wandering domains.** Consider a dynamical system with a compact state space X, given by a continuous map $\varphi : X \to X$. We say that the forward orbit $\{x, \varphi(x), \varphi^2(x), \dots\}$ of $x \in X$ has *historic behavior* if the partial average

$$\frac{1}{m+1} \sum_{i=0}^{m} \delta_{\varphi^i(x)}$$

dose not converge as $m \to \infty$ in the weak topology, where $\delta_{\varphi^i(x)}$ is the Dirac measure on X supported at $\varphi^i(x)$. The terminology of historic behavior was given by Ruelle in [37]. The following is the last open problem presented by Takens (see [41]).

Takens' Last Problem. *Whether are there persistent classes of smooth dynamical systems such that the set of initial states which give rise to orbits with historic behavior has positive Lebesgue measure?*

The first example of historic behavior was given in [19], where it is shown that the logistic family contains elements for which almost all orbits have historic behavior. This was extended to generic full families of unimodal maps, see [8]. While Takens showed in [42] that Bowen's 2-dimensional flow with an attracting heteroclinic loop has a set of positive Lebesgue measure consisting of initial points of orbits with historic behavior, but the property is not preserved under arbitrarily small perturbations of the dynamics. Also, by using Dowker's result [10], Takens showed that the doubling map on the circle persistently has orbits with historic behavior, for which the collection of initial points is a residual subset on the circle, see [41].

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In this paper, we obtain an answer to Takens' Last Problem for non-hyperbolic diffeomorphisms having homoclinic tangencies by a different way from the previous works. To solve the problem we use a non-empty connected open set, called a wandering domain, whose images do not intersect each other but are wandering around non-trivial hyperbolic sets. Wandering domains have been studied from the beginning of 20th century. In fact, Bohl [3] in 1916 and Denjoy [9] in 1932 constructed examples of C^1 diffeomorphisms on a circle which have wandering domains whose ω-limit set is a Cantor set. However, it can not be extended to any C^2 as well as C^1 diffeomorphism whose derivative is a function of bounded variation, see in [8]. Subsequently, similar phenomena for high dimensional diffeomorphisms were studied by several authors, for example [22, 18, 30, 4, 31, 27]. Also, for unimodal as well as multimodal maps on an interval or a circle, the main difficulty in their classification in real analytic category was to show the absence of wandering domains, which were developed by many dynamicists [7, 28, 2, 8, 40], see the survey of van Strien [39].

On the other hand, a wide variety of investigations derived from Smale's works in 1960s yielded abundant developments, and provided a focal point for us to explore beyond hyperbolic phenomena. Thus, we here focus entirely on one of non-hyperbolic phenomena called homoclinic tangencies, which were pioneered by Newhouse, Palis, Takens and others. It is somewhat surprising that homoclinic tangencies and wandering domains were not studied together until Colli-Vargas' model in [6]. We furthermore discuss these two themes in more general situation to solve Takens' Last Problem.

1.2. **Main results.** To state our results we have to introduce some definitions. Let M be a closed two-dimensional C^∞ manifold and $\mathrm{Diff}^r(M)$, $r \geq 2$, the set of C^r diffeomorphisms on M endowed with C^r topology. We say that $f \in \mathrm{Diff}^r(M)$ has a *homoclinic tangency* of a saddle periodic point p if the stable manifold $W^s(p,f)$ and unstable manifold $W^u(p,f)$ have a non-empty and non-transversal intersection. Newhouse showed in [32] that, for any $f \in \mathrm{Diff}^r(M)$ with a homoclinic tangency of a dissipative saddle fixed point p, there is an open set $\mathcal{N} \subset \mathrm{Diff}^r(M)$ whose closure $\mathrm{Cl}(\mathcal{N})$ contains f and such that any element of $\mathcal{N}$ is arbitrarily C^r-approximated by a diffeomorphism g with a homoclinic tangency associated with a dissipative saddle fixed point p_g which is the continuation of p, and moreover g has a C^r-persistent tangency associated with some basic sets Λ_g containing p_g (i.e. there is a C^r neighborhood of g any element of which has a homoclinic tangency for the continuation of Λ_g). Such an open set $\mathcal{N}$ is called a *Newhouse open set* or a *Newhouse domain*. Various non-hyperbolic phenomena were observed in $\mathcal{N}$ but still far from being completely understood. For example, Newhouse also showed in [32] that generic elements of $\mathcal{N}$ have infinitely many sinks or sources. Kaloshin proved in [20] that the number of periodic points for diffeomorphisms in a residual subset of $\mathcal{N}$ grows super-exponentially. See [15, §1] and [24, §0] for detail descriptions of other results which are not mentioned here.

Our definition of a wandering domain is the same as one defined for one-dimensional dynamics in [8, 39]. In fact, we say that, for $f \in \mathrm{Diff}^r(M)$, a non-empty connected open set $D \subset M$ is a *wandering domain* if

- $f^i(D) \cap f^j(D) = \emptyset$ for any integer $i, j \geq 0$ with $i \neq j$;
- the union $\omega(D,f) = \bigcup_{x \in D} \omega(x,f)$ of ω-limit sets is not equal to a single periodic orbit.

A wandering domain D is called *contracting* if the diameter of $f^n(D)$ converges to zero as $n \to \infty$. Note that Denjoy's example is a contracting wandering domain, see [9, 39].

We now state the main result of this paper where a conjecture of Colli and Vargas is proved affirmatively in C^r topology and a solution to Takens' Last Problem will be obtained by using some non-trivial wandering domains.

Theorem A. *Let M be a closed surface and $\mathcal{N}$ any Newhouse open set in* $\mathrm{Diff}^r(M)$ *with $2 \le r < \infty$. Then there exists a dense subset of $\mathcal{N}$ each element f of which has a contracting wandering domain D such that*

(1) *$\omega(D, f)$ contains a hyperbolic set which is not just a periodic orbit;*
(2) *the forward orbit of every $x \in D$ under f has historic behavior.*

We note the following:

- (The absence of regularity) Any diffeomorphisms given in Theorem A as well as Colli-Vargas' examples in [6] are not necessarily guaranteed to be of class C^∞, see Remark 7.8 (2).
- (Variety of average measures) In Colli-Vargas' examples, the sequence of the average measures $\mu_x(m)$ can have various accumulation points, see [6, §9]. However, in our construction, one can not expect such a variety for some technical reasons, see Remark 2.1.
- (Positivity of Lebesgue measure and persistent property) Any wandering domain is an open set, and hence in particular D has positive Lebesgue measure, which is the condition required in Takens' Last Problem. While the property having a wandering domain obtained in Theorem A is not persistent. This is the reason why only dense subsets are obtained, which is exactly the same sort of restriction as in the paper by Colli [5] concerning the density of Hénon-like attractors.
- (Higher dimensions) We think that a similar result holds for codimension one dissipative homoclinic tangencies in dimensions higher than two, which is studied in [34].

Next we consider the *Hénon family* $f_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$f_{a,b}(x, y) = (1 - ax^2 + y, bx), \tag{1.1}$$

where a, b are real parameters. This family will play a significant role in the renormalization near the homoclinic tangency. As $b = 0$, the dynamics of $f_{a,0}$ is perfectly controlled by the family of quadratic maps $\varphi_a(x) = 1 - ax^2$. It is known that there is a parameter value of a such that there is a C^1 unimodal map which is semi-conjugated to φ_a and has a wandering interval [17]. Moreover, under the sufficient differentiability, some large class of multimodal maps including the quadratic maps cannot have non-trivial wandering intervals, see [40]. However, as $b \ne 0$, it has not been known whether or not $f_{a,b}$ has wandering domains, which is one of open problems in [39, 29]. We can answer such a problem in the C^r category with $2 \le r < \infty$ as in the following corollary of Theorem A together with the fact that there exists (a, b) arbitrarily close to $(2, 0)$ such that $f_{a,b}$ has a quadratic homoclinic tangency for some saddle fixed point which unfolds generically with respect to a, see for example in [23, 25].

Corollary B. *There is an open set $\mathcal{O}$ of the parameter space of Hénon family with* $\mathrm{Cl}(\mathcal{O}) \ni (2,0)$ *such that, for every* $(a,b) \in \mathcal{O}$, $f_{a,b}$ *is* C^r*-approximated by diffeomorphisms which have contracting non-trivial wandering domains.*

In the end of this section we recall that non-trivial wandering phenomena are observable in circle homeomorphisms in the C^∞ category by Hall [16] but not in the C^ω category by Yoccoz [43], which is the answer to one of problems by Poincaré [35]. Note that every discussion of the present paper unfolds in the C^r category with $2 \le r < \infty$, but some tools would not be applied directly to discussion in the C^∞ as well as C^ω category. See Remark 7.8-(2). Thus, the open problem for wandering domains of the Hénon family by van Strien et al. is unsolved yet in C^∞ and C^ω categories. So it is worth recalling the following:

Question. *Does there exist a parameter value* (a,b) *for the Hénon family (1.1) such that* $f_{a,b}$ *has a non-trivial wandering domain?*

Note that Astong et al. [1] study the existence of wandering Fatou components for polynomial skew-product maps and present an example which admits a wandering Fatou component intersecting $\mathbb{R}^2$. However, it does not contain any Hénon family.

2. Outline of the proof

In this section, we will sketch the proof of Theorem A, where several technical terminologies, e.g. *linked pair*, *linking property*, *critical chains*, etc, appear without definitions, all of which will be given in the following sections.

2.1. **Standard settings for general situations.** For the beginning, it could not be better than that assumptions are minimized. So we start our discussions not for specific models as in [6], but rather for any two-dimensional diffeomorphism f which has a homoclinic tangency for a dissipative saddle fixed point, say p. See Figure 2.1. For such a situation, it naturally reminds us of the renormalization scheme near homoclinic tangencies by Palis-Takens [33], see Theorem 3.2. In fact, we will take much advantage of the scheme as follows.

We may here assume that f is a linear map $f(x,y) = (\lambda x, \sigma y)$ in a small neighborhood $U(p)$ of p with $0 < \lambda < 1 < \sigma$, $\lambda\sigma < 1$ by performing an arbitrarily small perturbation for f and replacing f by f^2 if necessary. In our scheme, we have two main basic sets, where the *basic set* means a compact hyperbolic and locally maximal invariant set which is transitive and contains a dense subset of periodic orbits. One is a horseshoe Λ which is associated with a transverse homoclinic intersection of p but *not* affine in general. The other is a basic set, denoted by Γ_m, which is created by an Hénon-like return map φ_n of (3.2) in the renormalization near the homoclinic tangency, where m is the period of some periodic point for φ_n. Those ingredients and their cyclic interconnection by way of persistent heteroclinic tangencies are precisely described in Sections 3.

2.2. **Main Cantor sets and bridges.** From the basic sets Λ and Γ_m one can obtain several dynamically defined Cantor sets, among which the following three are especially important in this paper:

$$K^s_\Lambda := \pi_{\mathcal{F}^u_{\mathrm{loc}}(\Lambda)}(\Lambda), \quad K^u_\Lambda := \pi_{\mathcal{F}^s_{\mathrm{loc}}(\Lambda)}(\Lambda), \quad K^u_m := \pi_m(\Gamma_m),$$

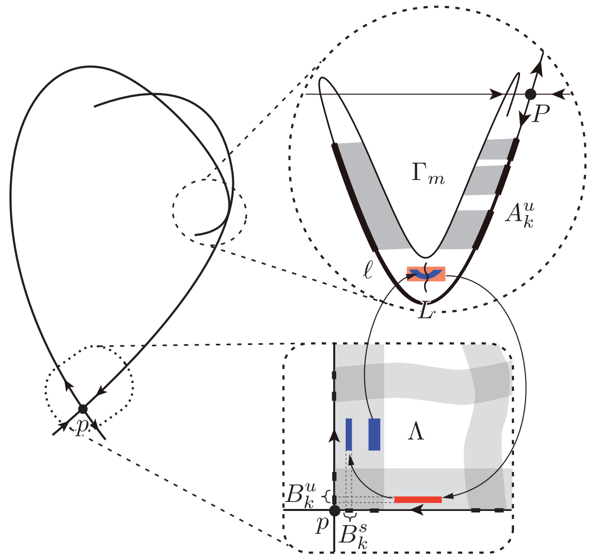

FIGURE 2.1. Transition of the wandering domains

where $\pi_{\mathcal{F}^u_{\mathrm{loc}}(\Lambda)}$, respectively $\pi_{\mathcal{F}^s_{\mathrm{loc}}(\Lambda)}$, is the projection on $W^s_{\mathrm{loc}}(p)$, respectively $W^u_{\mathrm{loc}}(p)$, along the leaves of an unstable foliation $\mathcal{F}^u_{\mathrm{loc}}(\Lambda)$, respectively stable foliation $\mathcal{F}^s_{\mathrm{loc}}(\Lambda)$, and π_m is the projection on an arc $\ell \subset W^u_{\mathrm{loc}}(P)$ along the leaves of a stable foliation $\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m)$ compatible with $W^s_{\mathrm{loc}}(\Gamma_m)$. Here P stands for the saddle fixed point for φ_n as illustrated in Figure 2.1, which is not contained in Γ_m. See (4.2) and (4.8) respectively. Moreover, one has the sequences of s, u-bridges related to the three Cantor sets, respectively denoted by

$$\{B^s_k\}, \quad \{B^u_k\}, \quad \{A^u_k\},$$

where the former two are defined in Subsection 4.2, see Figure 4.1, and the latter is in Subsections 4.3–5.1, see Figure 4.3. In Section 4, we give descriptions of bounded distortion properties of these bridges.

2.3. **Generalized uniformly linking properties for $\{B^s_k(\Delta)\}$ and $\{A^u_k(\Delta)\}$.** In the case of the Colli-Vargas model in [6, §3], non-trivial wandering domains were detected in intersections between stable gaps and unstable gaps of sufficiently thick affine Cantor sets. Obviously, we cannot directly unfold the same story into our case, because there is no promise in general such that the product of thickness of K^s_Λ and K^u_Λ is larger than one.

However, a bypass of this problem is already given in the renormalization scheme, see Theorem 3.2. In fact, since the thickness of the third Cantor set K^u_m has an arbitrarily large value by taking m large enough, one can obtain a C^2-persistent heteroclinic tangency associated with $W^u(\Lambda_g)$ and $W^s(\Gamma_{m,g})$, where Λ_g and $\Gamma_{m,g}$ are the continuations of Λ and Γ_m, respectively, see (S-v) in Section 3. For simplicity, we still denote such continuations by Λ and Γ_m, respectively, in this outline. Therefore, in Section 5, we can discuss linking properties between the images of continuations for bridges $\{B^s_k\}$ and $\{A^u_k\}$ on the arc L of tangencies between

$f^{N_2}(\mathcal{F}^u_{\mathrm{loc}}(\Lambda))$ and $f^{-N_0}(\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m))$ for some integers $N_0, N_2 > 0$. See Figure 5.3. Actually, these linking situations will be found by the projected images (5.5) on L.

To obtain a linked pair of the continuations for bridges, we have to add the first perturbation in $f^{-1}(\mathcal{B}_{\delta_0})$ where $\mathcal{B}_{\delta_0}$ is a small δ_0-disk which meets the inverse image $\tilde{L}$ of L, see Figure 5.5. Precisely, this perturbation is the horizontal δ-shift with $|\delta| \ll \delta_0$, and hence the perturbed map is given as

$$f_\delta(\boldsymbol{x}) = f(\boldsymbol{x}) + (\delta, 0)$$

for any $\boldsymbol{x} \in f^{-1}(\mathcal{B}_{\delta_0/2})$. Using this perturbation, we present Lemma 5.2 which is a generalization of *Linking Lemma* in [6, p. 1669]. Moreover, in Lemma 6.1, we show the existence of uniformly linking subsequence of $\{A^u_k(\Delta)\}$ and $\{B^s_k(\Delta)\}$ where $\Delta = \sum_{k=1}^\infty \delta_k$ for the sequence of local δ_k-shifts given in Lemma 6.2. This is a generalization of *Linear Growth Lemma* in [6, p. 1670]. The proofs of the results are supported by Lemma 4.3 and Lemma 4.5 in which bounded distortion properties for s-bridges $\{B^s_k\}$ and u-bridges $\{A^u_k\}$ are presented.

2.4. **Critical chains in $\{B^s_k(\Delta)\}$ and $\{B^u_k(\Delta)\}$.** Note that the uniformly linking subsequence of $\{B^s_k(\Delta)\}$ and $\{A^u_k(\Delta)\}$ are constructed on the arc L including *heteroclinic* persistent tangencies of $W^u(\Lambda)$ and $W^s(\Gamma_m)$. At this stage, since we use only a one-way from Λ to Γ_m, even if one takes any domain constructed from the linking subsequence by some perturbations, there is no certification that the orbit of domain comes back to and wanders around Λ non-trivially.

However, we can construct a return route by using the fact that Γ_m and Λ are homoclinically related to each other, see the condition (S-v) of Section 3. It follows that the stable foliation $\mathcal{F}^s(\Lambda)$ and some gap of $K^u_{m,L}$ have a transverse intersection. Hence, in Section 7, one obtains some gap $\widehat{G}^u_{L,k+1}$ of $A^u_{L,k+1}$ which contains some u-bridge $B^u_{L,k+1} := \pi_{\hat{A}^u_k}(B^u_{k+1})$ as illustrated in Figure 7.1, where $\pi_{\hat{A}^u_k} : B(0) \to L$ is the projection defined as (5.4) and B^u_{k+1} is the u-bridge whose itinerary satisfies (7.1).

Moreover, by taking the itinerary of the s-bridge B^{s*}_{k+1} as in (7.1), one can obtain linking situations which are desired in Lemma 7.1 (*Critical Chain Lemma*): there is an interval J^*_{k+1} such that $t \in J^*_{k+1}$ if and only if

$$(B^{s*}_{L,k+1} + t) \cap B^u_{L,k+1} \neq \emptyset,$$

where $B^{s*}_{L,k+1}$ is the image of the s-bridge B^{s*}_{k+1} of K^u_Λ by the projection $\pi^s : S \to L$ of (5.6). Note that the itineraries given in (7.1) will be crucial to control the orbit of any point in the wandering domain obtained in the later sections.

2.5. **Multidirectional perturbations and critical chains of rectangles.** We may consider the inverse image $\widetilde{L}$ of L which is contained in the neighborhood of Λ, see Figure 7.3. Lemma 7.2 implies that, for all sufficiently large k, there exists almost horizontal line L_k such that L_k meets L transversely at a single point $\boldsymbol{x}_k$.

Moreover we take a point $\widetilde{\boldsymbol{x}}_k = f^{N_1} \circ f^{\widehat{i_k}} \circ f^{N_0}(\boldsymbol{x}_k)$ in $\widetilde{L}_k$, and define a sequence $\{\widehat{\boldsymbol{x}}_k\}$ with $\widehat{\boldsymbol{x}}_k = f^{z_k k^2 + \langle k \rangle}(\widetilde{\boldsymbol{x}}_k)$, where z_k is either z_0 or $z_0 + 1$ for a fixed positive integer z_0 satisfying the conditions (8.4), (8.5) and $\langle k \rangle$ is the integer of (7.2) with $\lim_{k\to\infty} \langle k \rangle / k^2 = 1$. For $\{\boldsymbol{x}_k\}$ and $\{\widehat{\boldsymbol{x}}_k\}$, one has the sequence

$$\underline{\boldsymbol{t}} = (\boldsymbol{t}_2, \boldsymbol{t}_3, \ldots, \boldsymbol{t}_k, \ldots),$$

where each $\boldsymbol{t}_k$ is the vector given by $f^{N_2}(\widehat{\boldsymbol{x}}_k) + \boldsymbol{t}_{k+1} = \boldsymbol{x}_{k+1}$, see Figure 7.6, which is the second perturbation corresponding to the *perturbation vector* in [6, p.1673]. Note that entry vectors of the perturbation not necessarily have the same direction.

For $\underline{\boldsymbol{t}}$, we now define the diffeomorphism $f_{\underline{\boldsymbol{t}}}$ by

$$f_{\underline{\boldsymbol{t}}} := f \circ \psi_{\underline{\boldsymbol{t}}}$$

so that $f_{\underline{\boldsymbol{t}}}(\widehat{\boldsymbol{x}}_k) = \boldsymbol{x}_{k+1}$, where $\psi_{\underline{\boldsymbol{t}}}$ is the C^r-map defined as (7.17). Note that Lemma 7.6 claims that, if T is sufficiently large, then $\psi_{\underline{\boldsymbol{t}}}$ is arbitrarily C^r-close to the identify and hence $f_{\underline{\boldsymbol{t}}}$ is a C^r- diffeomorphism arbitrarily C^r-close to f.

2.6. **Non-trivial wandering domains.** Around each $\boldsymbol{x}_k = (x_k, y_k)$, we define a rectangle as

$$R_k = \big[x_k - b_k^{\frac{1}{2}}, x_k + b_k^{\frac{1}{2}}\big] \times [y_k - b_k, y_k + b_k],$$

where b_k is the positive number given in (8.2), and show in Lemma 8.2 (*Rectangle Lemma*) that there is an integer $k_0 > 0$ such that, for any $k \geq k_0$,

$$f_{\underline{\boldsymbol{t}}}^{m_k}(R_k) \subset \mathrm{Int} R_{k+1},$$

where m_k is the positive integer given in Remark 8.4. See Figure 8.1. This implies that the interior D of R_{k_0} is a wandering domain. It follows immediately from our construction of D that the wandering domain is contracting.

We will consider a sufficiently large positive integer z_0 independent of k and satisfying the conditions (8.4) and (8.5), and consider a sequence $\boldsymbol{z} = \{z_k\}_{k=1}^{\infty}$ of integers such that each entry z_k is either z_0 or $z_0 + 1$. This implies that the linear map $f_{\underline{\boldsymbol{t}}}^{z_k k^2}(x, y) = (\lambda^{z_k k^2} x, \sigma^{z_k k^2} y)$ in a small neighborhood of p occupies a major factor of $f_{\underline{\boldsymbol{t}}}^{m_k}$ and absorbs fluctuations caused by non-linear factors. See also Remark 2.1 for the sequence. Moreover, extra words $\underline{v}_{k+1} \in \{1, 2\}^k$ will be added to the itineraries of (7.1) so that the ω-limit set of any point in the wandering domain contains Λ, which is possible by choosing entries of v_k suitably in the proof of Proposition 8.3.

2.7. **Historic behavior.** The diffeomorphism f_n obtained in Proposition 8.3 and the wandering domain $D = \mathrm{Int} R_{k_0}$ depend on the sequence $\boldsymbol{z}$. In the proof of Theorem A, we express the dependence by $f_{n,\boldsymbol{z}}$ and $D_{\boldsymbol{z}} = \mathrm{Int} R_{k_0,\boldsymbol{z}}$. From the setting of (7.1), the itinerary of any orbit starting from the wandering domain $D_{\boldsymbol{z}}$ contains $\underline{1}^{(z_k k^2)}$ and $\underline{2}^{(k^2)}$, and the remaining part of the itinerary is corresponding to at most order k iterations of $f_{n,\boldsymbol{z}}$. Using this, one can show that for any $x \in D_{\boldsymbol{z}}$ there is the subsequence $\{\mu_x(\widehat{m}_k)\}_{k=k_0}^{\infty}$ of partial averages

$$\mu_x(\widehat{m}_k) = \frac{1}{\widehat{m}_k + 1} \sum_{i=0}^{\widehat{m}_k} \delta_{f_{n,\boldsymbol{z}}^i(x)}$$

which tends to the mutually distinct two probability measures:

$$\nu_0 = \frac{1}{z_0 + 1}\big(z_0 \delta_p + \delta_{\widehat{p}}\big), \quad \nu_1 = \frac{1}{z_0 + 2}\big((z_0 + 1)\delta_p + \delta_{\widehat{p}}\big)$$

as $k \to \infty$, where $\widehat{p}$ is a saddle fixed point in the horseshoe Λ other than p. It implies that the orbit of any point in the wandering domain $D_{\boldsymbol{z}}$ has historic behavior. This finishes the outline of the proof of Theorem A.

Remark 2.1. Our diffeomorphism f is linear only in a small neighborhood $U(p)$ of p but not necessarily in neighborhoods of other points of M, which may yield some fluctuations to the dynamics. To reduce influences of such fluctuations relatively, we first rearranged f so that, for any $k \in \mathbb{N}$, the orbit of $x \in D$ under f spends k^2 times in the linearizing neighborhood $U(p)$ and $O(k)$ times in any other small neighborhoods. However, in such a construction, the sequence of $\mu_x(m)$ would converge to the Dirac measure δ_p and hence the forward orbit of x would not have historic behavior. So, we have rearranged f again so that the orbit spends $z_k k^2$ times in $U(p)$, k^2 times in $U(\widehat{p})$ and $O(k)$ times in any other small neighborhoods. In such an example, we can not expect that the sequence of $\mu_x(m)$ has various accumulation points, which is essentially different from the example in [6, Section 9]. Note that we have taken the integer z_0 large enough so that any fluctuation caused in $U(\widehat{p})$ is relatively small. On the other hand, since $k^2/(z_k k^2) \geq 1/(z_0+1) > 0$, the restriction of $\mu_x(m)$ on $U(\widehat{p})$ does not converge to zero as $m \to \infty$.

3. Preliminaries

In this section, we present standard notions concerning planer homoclinic bifurcations introduced by Palis-Takens [33], which are given in forms adaptable to our discussions.

Throughout the remainder of this paper, suppose that M is a closed surface and r an integer r with $2 \leq r < \infty$. Let $\mathcal{N}$ be any Newhouse open set in $\mathrm{Diff}^r(M)$. Any element of $\mathcal{N}$ is arbitrarily C^r-approximated a diffeomorphism f with a dissipative saddle periodic point p whose stable manifold $W^s(p)$ and unstable manifold $W^u(p)$ have a quadratic tangency, where a periodic point p of period $\mathrm{Per}(p)$ is said to be *dissipative* if $|\det(Df^{\mathrm{Per}(p)})_p| < 1$. Denoting $f^{\mathrm{Per}(p)}$ again by f for simplicity, one can suppose that p is a dissipative saddle fixed point of f.

The following lemma is an elementary but crucial fact for justifying our replacement of f by $f^{n\mathrm{Per}(p)}$ with $n \in \mathbb{N}$ here and later.

Lemma 3.1. *Suppose that a is a positive multiple of* $\mathrm{Per}(p)$*. If D is a contracting wandering domain for f^a with $\omega_{f^a}(\boldsymbol{x}) \ni p$ for some (or equivalently any) $\boldsymbol{x} \in D$, then D is a contracting wandering domain also for f.*

Proof. Since $C = \sup\{\|Df^i_{\boldsymbol{x}}\|\,;\, \boldsymbol{x} \in M,\ i = 0, 1, \dots, a-1\} < \infty$, $\mathrm{diam}(f^j(D)) \leq C\mathrm{diam}(f^{am}(D))$ for any $j \in \mathbb{N}$, where m is the greatest integer with $ma \leq j$. Thus our assumption $\lim_{m\to\infty} \mathrm{diam}(f^{am}(D)) = 0$ implies that $\lim_{j\to\infty} \mathrm{diam}(f^j(D)) = 0$.

If D were not a wandering domain for f, then there would exist a positive integer n with $D \cap f^n(D) \neq \emptyset$. Then we have the chain

$$D,\ f^n(D),\ f^{2n}(D),\ \dots,\ f^{an}(D)$$

with $f^{ni}(D) \cap f^{n(i+1)}(D) = f^{ni}(D \cap f^n(D)) \neq \emptyset$ for $i = 0, 1, \dots, a-1$. Since $\lim_{j\to\infty} \mathrm{diam}(f^j(D)) = 0$,

$$\lim_{m\to\infty} \mathrm{diam}\big(f^{am}\big(D \cup f^n(D) \cup \cdots \cup f^{na}(D)\big)\big) = 0. \tag{3.1}$$

Moreover we have $D \cap W^s(p) = \emptyset$. Otherwise, the sequence $\{f^{aj}(D)\}$ of open sets would converge to $W^u(p)$, which contradicts that D is a contracting wandering domain. Since $\omega_{f^a}(\boldsymbol{x}) \ni p$ and $\boldsymbol{x} \notin W^s(p)$ for any $\boldsymbol{x} \in D$, there exists a monotone increasing sequence $\{m_j\}$ of positive integers such that $f^{am_j}(\boldsymbol{x})$ converges to a point $\boldsymbol{z}$ in $W^u_{\mathrm{loc}}(p) \setminus \{p\}$ as $j \to \infty$. Then $\lim_{j\to\infty} f^{a(m_j+n)}(\boldsymbol{x}) = f^{an}(\boldsymbol{z}) \in W^u_{\mathrm{loc}}(p) \setminus \{p\}$.

On the other hand, (3.1) implies that $\lim_{j\to\infty} f^{a(m_j+n)}(\boldsymbol{x}) = \lim_{j\to\infty} f^{am_j}(\boldsymbol{x}) = \boldsymbol{z}$ and so $f^{an}(\boldsymbol{z}) = \boldsymbol{z}$. This contradicts that $\boldsymbol{z} \in W^u_{\mathrm{loc}}(p) \setminus \{p\}$ is not a fixed point of f^{an}. Thus D is a contracting wandering domain for f. □

By a small perturbation near the homoclinic tangency q, one can also suppose that the tangency is quadratic. Moreover, as in [33, Section 6.5], performing several arbitrarily small perturbations, we obtain a diffeomorphism which has both a transverse homoclinic intersection and a homoclinic tangency of the continuation of p simultaneously. Using the transverse homoclinic intersection, one obtains a *basic set* containing the continuation of p called a *horseshoe*, i.e. a compact invariant hyperbolic set which is transitive and contains a dense subset of periodic orbits and such that the restriction of the diffeomorphism on the set is conjugate to the 2-shift. In what follows, if no confusion can arise, we will denote a C^∞ diffeomorphism which is arbitrarily C^r close to f again by f. So we will work under the assumption that f is a diffeomorphism of C^∞-class and return to a diffeomorphism of $C^{\overline{r}}$-class with $3 \le \overline{r} < \infty$ at the stage of (7.19) in Subsection 7.2.

In accordance with the discussion as above, we may suppose that f has

(S-i) a horseshoe Λ containing a dissipative saddle fixed point p;
(S-ii) a non-degenerate homoclinic tangency q of p.

Moreover, if a C^∞ diffeomorphism C^r-close to f satisfies an open and dense Sternberg condition [38] concerning the eigenvalues at the continuation of p, then f is C^r-linearizable in a neighborhood U of the continuation. So, one can suppose that the above f has

(S-iii) a C^r-linearizing coordinate in a neighborhood of p such that $f(x, y) = (\lambda x, \sigma y)$ with $0 < \lambda < 1 < \sigma$ and $\lambda\sigma < 1$ (we replace f by f^2 if λ or σ is negative).

Note that one might proceed without (S-iii) by using techniques in Gonchenko et al. [11, 12, 13, 14], but (S-iii) is more appropriate here from a standpoint of simple descriptions.

Let $\{f_\mu\}_{\mu\in\mathbb{R}}$ be a one-parameter family of C^∞ diffeomorphisms on M with $f_0 = f$ and such that the homoclinic quadratic tangency q of p unfolds generically at $\mu = 0$. The *renormalization* of return maps near the tangency provides a better description as follows.

Theorem 3.2 (Renormalization Theorem, Palis-Takens [33])**.** *There exists an integer $N_* > 0$ such that, for any sufficiently large integer $n > 0$, there are a C^r parametrization $\Theta_n : \mathbb{R} \to \mathbb{R}$ and a $\bar{\mu}$-dependent C^r coordinate change $\Phi_n : \mathbb{R}^2 \to M$ satisfying the following:*

- $\frac{d\Theta_n}{d\mu}(\bar{\mu}) > 0$;
- *for any $(\bar{\mu}, \bar{x}, \bar{y}) \in \mathbb{R} \times \mathbb{R}^2$, $(\Theta_n(\bar{\mu}), \Phi_n(\bar{x}, \bar{y}))$ converges to $(0, q)$ as $n \to \infty$;*
- *for any $\bar{\mu} \in \mathbb{R}$, the diffeomorphisms φ_n on $\mathbb{R}^2$ defined by*

$$\varphi_n : (\bar{x}, \bar{y}) \longmapsto \Phi_n^{-1} \circ f_{\mu_n}^{N_*+n} \circ \Phi_n(\bar{x}, \bar{y}) \tag{3.2}$$

converge to

$$(\bar{x}, \bar{y}) \longmapsto (\bar{y}, \bar{y}^2 + \bar{\mu}). \tag{3.3}$$

as $n \to \infty$ in the C^r topology, where $\mu_n := \Theta_n(\bar{\mu})$.

Proof. See Palis-Takens [33, §3.4, Theorem 1] for the proof. □

Here the integer N_* is taken so that $f^{-N_*}(q)$ is a point in $W^u_{\mathrm{loc}}(p)$ sufficiently near p. In short this lemma ensures that the return maps $f^{N_*+n}_{\mu_n}$ near the homoclinic tangency can be approximated by diffeomorphisms so called *Hénon-like maps* which are close to the quadratic endomorphism (3.3).

For simplicity, we consider the map

$$\varphi_{\mu,\nu}(x,y) = (y, \mu + \nu x + y^2),$$

which is equivalent to the original Hénon map given in (1.1) via appropriate parameter and coordinate changes, see [23]. More general Hénon-like families are obtained by adding small higher-order terms to the above form. From [33, §6.3, Proposition 1], for a given $m \geq 3$, there exist a neighborhood $\mathcal{U}(-2,0)$ of the point $(-2,0)$ in the parameter space and continuous maps P, Q_m and Γ_m which map $(\mu,\nu) \in \mathcal{U}(-2,0)$ to the fixed point $P_{\mu,\nu}$, the periodic point $Q_{m;\mu,\nu}$, and the non-trivial invariant set $\Gamma_{m;\mu,\nu}$ for $\varphi_{\mu,\nu}$, respectively, and furthermore satisfy the following properties:

- $P_{-2,0} = (2,2)$ is a saddle fixed point and $Q_{m;-2,0}$ is a saddle periodic point of period m both of which are contained in a parabolic arc which is convex downward between $P_{-2,0}$ and $\tilde{P}_{-2,0} := (-2,2)$, see Figure 4.4.
- For any $(\mu,\nu) \in \mathcal{U}(-2,0)$ with $\nu \neq 0$, $\Gamma_{m;\mu,\nu}$ is a basic set containing the orbit of $Q_{m;\mu,\nu}$.

More detailed information on these ingredients will be given in the next section. Note that the same properties hold for any Hénon-like map which is sufficiently close to $\varphi_{-2,0}$.

By the above results, there exists a positive integer $n(m) > 0$ such that, for any $n \geq n(m)$, f_{ν_n} is a diffeomorphism arbitrarily C^r close to the original f and satisfying not only (S-i)–(S-iii) but also

(S-iv) the restriction φ_n of $f^{N_*+n}_{\mu_n}$ near the tangency q is a return map C^r-approximated by an Hénon-like map and has the continuation $P_{\bar{\mu}}$ of the saddle fixed point P, the continuation $\Gamma_{m;\bar{\mu}}$ of the basic set Γ_m which contains the continuation $Q_{m;\bar{\mu}}$ of the saddle periodic point Q_m of period m, where $\bar{\mu} = \Theta_n^{-1}(\mu_n)$.

We here recall two important relations on a pair of basic sets. We say that disjoint basic sets Λ and Γ are *homoclinically related* if both $W^u(\Lambda) \cap W^s(\Gamma)$ and $W^s(\Lambda) \cap W^u(\Gamma)$ contain non-trivial transverse intersections. Basic sets Λ and Γ for f have a *C^2-robust tangency* if there exists a C^2 neighborhood $\mathcal{U}(f)$ of f satisfying the following condition: for every $g \in \mathcal{U}(f)$, either $W^u(\Lambda_g) \cap W^s(\Gamma_g)$ or $W^s(\Lambda_g) \cap W^u(\Gamma_g)$ contains a tangency, where Λ_g and Γ_g are the continuations of Λ and Γ, respectively.

By [33, Section 6.4], we may also suppose that

(S-v) the continuation $\Lambda_n := \Lambda(f_{\mu_n})$ of the horseshoe Λ in (S-i) and the basic set $\Gamma_{m,n} := \Gamma_m(\varphi_n)$ in (S-iv) are homoclinically related, and they have a C^2-robust tangency. To be more precise, $W^u(\Lambda_g) \cap W^s(\Gamma_{m,g})$ contains a tangency a for any diffeomorphism g C^2-near f, see Figure 3.1.

Now we recall the construction of the return map φ_n by Palis-Takens. There exists a small rectangle D_n near q such that $\Gamma_{m,n} = \bigcap_{k=-\infty}^{\infty} f^{k(N_*+n)}_{\mu_n}(D_n)$. According to [33, Section 6.4], there is a transverse intersection point b of $W^u(p)$ and $W^s(P)$ such that

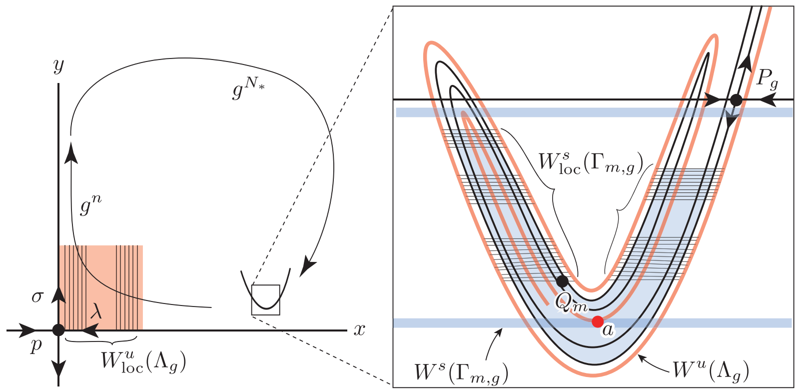

FIGURE 3.1. A homoclinical relation between Λ_g and $\Gamma_{m,g}$ and a quadratic tangency between $W^u(\Lambda_g)$ and $W^s(\Gamma_{m,g})$.

(S-vi) the sub-arc α^u in $W^u(p)$ connecting p with b is disjoint from the union $X_n = D_n \cup f_{\mu_n}(D_n) \cup \cdots \cup f_{\mu_n}^{N_*+n}(D_n)$, and

(S-vii) $f^i_{\mu_n}(D_n) \cap f^j_{\mu_n}(D_n) = \emptyset$ for any $i, j \in \{1, 2, \ldots, N_* + n\}$ with $i \neq j$, see Figure 3.2.

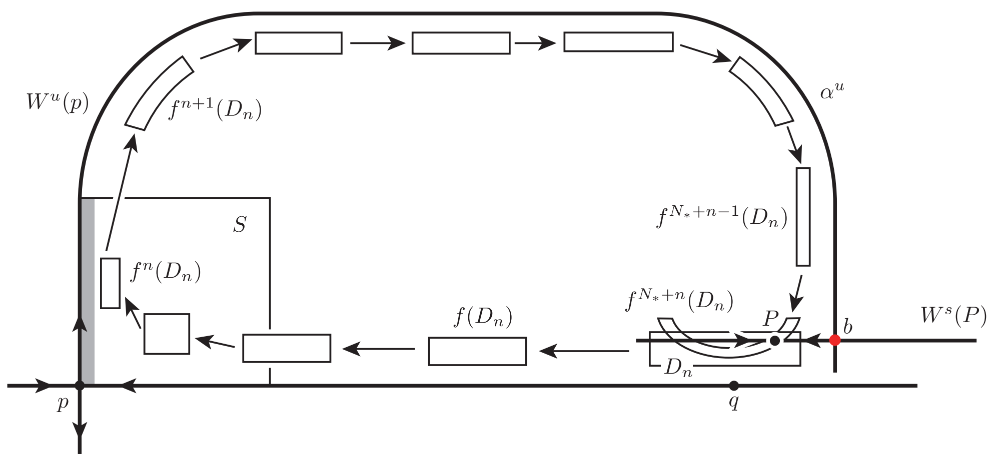

FIGURE 3.2. f_{μ_n}, $P(\varphi_n)$ are represented shortly by f and P respectively. The shaded region disjoint from X_n contains the supports of the perturbations of f_{μ_n} considered in Subsections 5.2 and 7.2 respectively.

Note that the condition (S-vii) does not necessarily hold for $i, j \notin \{1, 2, \ldots, N_*+n\}$. In fact, for any integer N sufficiently larger than N_*, $f^N_{\mu_n}(\alpha^u)$ meets all leaves of $W^s_{\rm loc}(\Gamma_m)$ transversely, which is suggested in Figure 3.1, and hence in particular $f^N_{\mu_n}(\alpha^u) \cap D_n \neq \emptyset$.

A nonempty compact subset K of an interval I is called a *Cantor set* if K has neither interior points nor isolated points. A *gap* of the Cantor set K is the closure of a connected component of $I \setminus K$. Let G be a gap and p a boundary point of

G. A closed interval $B \subset I$ is called the *bridge* at p if B is maximal among all closed intervals B' in I with $G \cap B' = \{p\}$ and such that B' does not intersect any gap whose length is at least that of G. The *thickness* for the Cantor set K at p is defined by $\tau(K,p) = |B|/|G|$, where B and G are a bridge and a gap satisfying $G \cap B = \{p\}$. The *thickness* $\tau(K)$ of K is the infimum over these $\tau(K,p)$ for all boundary points p of gaps of K. Two Cantor sets K_1 and K_2 are said to be *linked* if neither K_1 is contained in the interior of any gap of K_2 nor K_2 is contained in the interior of any gap of K_1. *Gap Lemma* (see [32, §4], [33, §4.2], [26]) shows that, for any linked Cantor sets K_1 and K_2 with $\tau(K_1)\tau(K_2) > 1$, $K_1 \cap K_2 \neq \emptyset$ holds.

We say that a bridge B of K is *adjacent* to a gap G if $B \cap G \neq \emptyset$ and $\mathrm{Int}B \cap G = \emptyset$. If two bridges B, B' are adjacent to a common gap G, then G is called the *connecting gap* for B and B' and denoted by $\mathrm{Gap}(B,B')$.

4. Bounded distortions

4.1. **Classical bounded distortion lemma.** A Cantor set K in an interval I is said to be *dynamically defined* if the following conditions hold: there are mutually disjoint closed sub-intervals $B_1, B_2, \dots, B_r \subset I$ and a differentiable map Ψ defined in a neighborhood U of $B_1 \sqcup \dots \sqcup B_r$ in I such that

- Ψ is uniformly hyperbolic on K, that is, there are constants $C > 0$ and $\sigma > 1$ such that $|(\Psi^n)'(x)| \geq C\sigma^n$ for every $x \in K$ and $n \geq 1$, and
- $\{B_1, \dots, B_r\}$ is a Markov partition satisfying

$$K = \bigcap_{n \in \mathbb{N}} \Psi^{-n}(B_1 \sqcup \dots \sqcup B_r).$$

The next classical result, called *Bounded Distortion Lemma*, will play an important role in this paper.

Lemma 4.1 (Palis-Takens [33])**.** *Let K be a dynamically defined Cantor set as above associated with a uniformly hyperbolic map Ψ and a_0 the minimum positive integer with $C\sigma^{a_0} > 1$. If Ψ satisfies the $C^{1+\alpha}$ Hölder condition for some $0 < \alpha \leq 1$, then, for every $\delta > 0$, there exists a constant $c(\delta) > 0$ satisfying*

$$e^{-c(\delta)} \leq |(\Psi^{na_0})'(q)||(\Psi^{na_0})'(\tilde{q})|^{-1} \leq e^{c(\delta)} \tag{4.1}$$

for any $q, \tilde{q} \in I$ and integer $n \geq 1$ such that (i) $|\Psi^{na_0}(q) - \Psi^{na_0}(\tilde{q})| \leq \delta$; (ii) the interval between $\Psi^i(q)$ and $\Psi^i(\tilde{q})$ is contained in $B_1 \sqcup \dots \sqcup B_r$ for all $0 \leq i \leq (n-1)a_0$. Moreover, $c(\delta)$ is of order δ^α. In particular, $c(\delta)$ converges to zero as $\delta \to 0$.

Proof. Since Ψ is uniformly hyperbolic on K, $|(\Psi^n)'(x)| \geq C\sigma^n$ for any $x \in K$ and $n \geq 1$. Then there exists a constant $\gamma > 1$ and an open neighborhood V of K in U such that $|(\Psi^{a_0})'(x)| \geq \gamma$ for any $x \in V$ and $n \geq 1$. For all sufficiently small $\delta > 0$, any points q, $\tilde{q}$ of I satisfying the conditions (i) and (ii) bound an interval contained in V. Hence, one can apply [33, §4.1, Theorem 1] directly to the expanding map Ψ^{a_0}, and obtain a constant

$$c(\delta) = \widetilde{C}\delta^\alpha \frac{(C\sigma^{a_0})^{-\alpha}}{1 - (C\sigma^{a_0})^{-\alpha}}$$

satisfying the condition (4.1), where $\widetilde{C}$ is a positive constant independent of δ. This completes the proof. $\square$

4.2. **Horseshoes and s-bridges.** Let us recall a simple example of a dynamically defined Cantor set and give the bounded distortion property for the set.

Consider a two-dimensional C^r diffeomorphism f admitting a horseshoe Λ as in (S-i) of Section 3, which contains a saddle fixed point p. Let $\mathcal{F}^u_{\mathrm{loc}}(\Lambda)$ and $\mathcal{F}^s_{\mathrm{loc}}(\Lambda)$ be local unstable and stable foliations on $S = [0,2]\times[0,2]$ compatible with $W^u_{\mathrm{loc}}(\Lambda)$ and $W^s_{\mathrm{loc}}(\Lambda)$ respectively. Here we may assume that

- $W^s_{\mathrm{loc}}(p) = [-2,2]\times\{0\}$, $W^u_{\mathrm{loc}}(p) = \{0\}\times[-2,2]$,
- $[0,2]\times\{2\}$ is a leaf of $\mathcal{F}^s_{\mathrm{loc}}(\Lambda)$ disjoint from $W^s_{\mathrm{loc}}(\Lambda)$ and $\{2\}\times[0,2]$ is a leaf of $\mathcal{F}^u_{\mathrm{loc}}(\Lambda)$ disjoint from $W^u_{\mathrm{loc}}(\Lambda)$.

Let $\pi_{\mathcal{F}^u_{\mathrm{loc}}(\Lambda)} : S \to W^s_{\mathrm{loc}}(p)$ be the projection along the leaves of $\mathcal{F}^u_{\mathrm{loc}}(\Lambda)$, and $\pi_{\mathcal{F}^s_{\mathrm{loc}}(\Lambda)} : S \to W^u_{\mathrm{loc}}(p)$ the projection along the leaves of $\mathcal{F}^s_{\mathrm{loc}}(\Lambda)$. Consider the Cantor sets

$$K^s_\Lambda := \pi_{\mathcal{F}^u_{\mathrm{loc}}(\Lambda)}(\Lambda), \quad K^u_\Lambda := \pi_{\mathcal{F}^s_{\mathrm{loc}}(\Lambda)}(\Lambda), \tag{4.2}$$

associated with Λ dynamically defined by $\Psi_s := \pi_{\mathcal{F}^u_{\mathrm{loc}}(\Lambda)}\circ f^{-1}$ and $\Psi_u := \pi_{\mathcal{F}^s_{\mathrm{loc}}(\Lambda)}\circ f$, respectively. Since f is a C^r map and $\pi_{\mathcal{F}^u_{\mathrm{loc}}}$, $\pi_{\mathcal{F}^s_{\mathrm{loc}}}$ are $C^{1+\alpha}$ maps with $0<\alpha<1$ (see [33, §4.1]), it follows that both Ψ_s and Ψ_u are of $C^{1+\alpha}$ class. Note that both Ψ_s and Ψ_u are expanding maps.

Remark 4.2. (1) In this paper, we use three local 'stable' foliation on S, one of which is the above $\mathcal{F}^s_{\mathrm{loc}}(\Lambda)$. The other two are $\mathcal{F}_S$ in Subsection 5.1 and $\mathcal{G}^s_{\mathrm{loc}}(\Lambda)$ in Subsection 7.2.
(2) The unstable foliation $\mathcal{F}^u_{\mathrm{loc}}(\Lambda)$ will be chosen carefully in Subsection 5.1 so as to be suitable to our purpose. However the Cantor set K^s_Λ and hence the bridges and gaps associated with K^s_Λ are independent of the choice.

Let $B^s(0)$ (respectively $B^u(0)$) be the smallest interval in $W^s_{\mathrm{loc}}(p)$ (respectively $W^u_{\mathrm{loc}}(p)$) containing K^s (respectively K^u). We give here descriptions associated only with K^s_Λ. Similar arguments work also in the case of K^u_Λ. There exists a Markov partition of K^s_Λ in $B^s(0)$ which consists of two components, and denote one of them by $B^s(1;1)$ and the other by $B^s(1;2)$. For each integer $k\ge 1$ and $w_i\in\{1,2\}$ $(i=1,\dots,k)$, we define the interval $B^s(k;w_1\dots w_k)$, called an *s-bridge of generation k*, as

$$B^s(k;w_1\dots w_k) = \left\{x\in I\ ;\ \Psi_s^{i-1}(x)\in B^s(1;w_i),\ i=1,\dots,k\right\},$$

where the sequence $w_1w_2\dots w_k$ is the *itinerary* for the s-bridge. If one writes $\underline{w} = w_1w_2\dots w_k$, then $\underline{w}^{-1}$ stands for the reverse sequence $w_kw_{k-1}\dots w_1$. The *u-bridges* $B^u(k;w_1\dots w_k)$ of *generation* k associated with K^u_Λ can be defined similarly by using Ψ_u. For our convenience, we regard that $B^s(0)$ and $B^u(0)$ are the bridges of generation 0 with empty itinerary.

For any integer $k\ge 0$, let $\mathcal{B}^s_k$ be the collection of all $B^s(k;w_1\dots w_k)$, see Figure 4.1. Note that $\mathcal{B}^s_k$ consists of mutually disjoint 2^k s-bridges. The union $\mathcal{B}^s = \bigcup_{k=0}^\infty \mathcal{B}^s_k$ is the set of all u-bridges of K^s_Λ. The set $\mathcal{B}^u = \bigcup_{k=0}^\infty \mathcal{B}^u_k$ of all u-bridges of K^u_Λ is defined similarly.

If necessary replacing the original f by f^{a_0} with a large integer a_0, S by a thiner sub-rectangle S' with $\partial S'\supset[0,2]\times\{0\}$ and such that $\Lambda' = \bigcap_{n=-\infty}^\infty f^{na_0}(S')$ is an f^{a_0}-invariant basic set, we may suppose that

$$\max\left\{|B^s(1;1)|,\ |B^s(1;2)|\right\} < \frac{9}{21}|B^s(0)|, \tag{4.3}$$

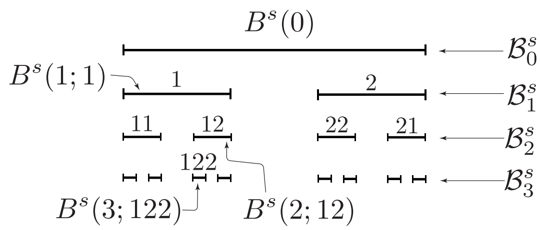

FIGURE 4.1. A collection of nested stable bridges. For any $w_1, w_2 \in \{1,2\}$, $\Psi_s(B^s(2; w_1w_2)) = B^s(1; w_2)$.

where $|\cdot|$ stands for the length of the corresponding interval. See Figure 4.2. Note that the new bridges $B^s(1;1)$, $B^s(1;2)$ for f^{a_0} coincide with the original

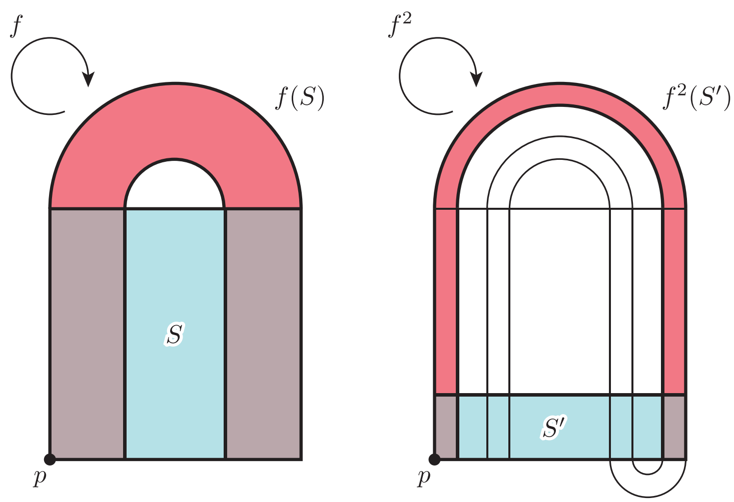

FIGURE 4.2. Replacement of a horseshoe by a more slender one. The case of $a_0 = 2$.

bridges $B^s(a_0; 11\cdots 1)$ and $B^s(a_0; 21\cdots 1)$ for f respectively. Since moreover one can choose a_0 so that $\max\{|B^s(1;1)|,\ |B^s(1;2)|\}$ is arbitrarily small, we may also assume by Lemma 4.1 that

$$\frac{9}{10} \le |(\Psi_s^{a_0 n})'(q)||(\Psi_s^{a_0 n})'(\tilde{q})|^{-1} \le \frac{11}{10} \tag{4.4}$$

for any $q, \tilde{q} \in I$ such that the intervals between $\Psi_s^{a_0 i}(q)$ and $\Psi_s^{a_0 i}(\tilde{q})$ $(i = 0, 1, \ldots, n-1)$ are contained in $B(1;1) \sqcup B(1;2)$. We set f^{a_0}, S', Λ' again by f, S and Λ respectively, and retake the local coordinate on M near p so that the new S equals $[0,2] \times [0,2]$.

Lemma 4.3 (Bounded distortions for s-bridges). *The exist constants $r_{s+} \ge r_{s-} > 2$ satisfying the following conditions. For any integer $k > 0$, let B_k^s and B_{k+1}^s be s-bridges for K_Λ^s of generation k and $k+1$ such that the first k entries in the itinerary of B_{k+1}^s are identical to the entries in the itinerary of B_k^s, that is,*

$$B_k^s := B^s(k; w_1 \ldots w_k), \quad B_{k+1}^s := B^s(k+1; w_1 \ldots w_k w_{k+1}).$$

Then the inequality

$$r_{s-} \le |B^s_k||B^s_{k+1}|^{-1} \le r_{s+}.$$

holds.

Proof. By the mean-value theorem, for any bridges B^s_k and B^s_{k+1} as above, there are points $q \in B^s_{k+1}$, $\tilde{q} \in B^s_k$ such that

$$|B^s(0)| = |B^s_k||(\Psi^k_s)'(\tilde{q})|, \quad |B^s(1;w_{k+1})| = |B^s_{k+1}||(\Psi^k_s)'(q)|.$$

By (4.3) and (4.4),

$$\begin{aligned}|B^s_k||B^s_{k+1}|^{-1} &= |B^s(0)||B^s(1;w_{k+1})|^{-1}|(\Psi^k_s)'(q)||(\Psi^k_s)'(\tilde{q})|^{-1}\\ &> \frac{21}{9}\cdot\frac{9}{10} = \frac{21}{10} =: r_{s-}.\end{aligned}$$

Moreover, we have

$$|B^s_k||B^s_{k+1}|^{-1} \le \frac{|B^s(0)|}{\min\{|B^s(1;1)|,\,|B^s(1;2)|\}}\cdot\frac{11}{10} =: r_{s+}.$$

This completes the proof. □

4.3. **Quadratic maps and u-bridges.** We recall another Cantor set for one-dimensional maps fundamental properties of which are succeeded by Hénon-like maps in Section 4.4.

Let F_μ be the family of one-dimensional quadratic maps on $\mathbb{R}$ defined as

$$F_\mu(x) = x^2 + \mu, \tag{4.5}$$

where μ is a real parameter. For any integer $m \ge 3$, F_μ has a periodic orbit of period m if μ is sufficiently close to -2. Denote by q_1 and q_2, respectively, the minimum and maximum points of the m-periodic orbit, which satisfies $F_\mu(q_1) = q_2$. We denote by $A^u(0)$ the interval $[q_1, q_2]$, and define the mutually disjoint $m-1$ intervals as

$$A^u(1;z_1) := \begin{cases} F^{-1}_\mu(A^u(0)) \cap \{x<0\} & \text{if } z_1 = 1;\\ F^{-z_1+1}_\mu(F^{-1}_\mu(A^u(0)) \cap \{x<0\}) \cap \{x>0\} & \text{if } z_1 = 2,\dots,m-1.\end{cases}$$

Moreover, for every integer $k \ge 2$, we inductively define the $(m-1)^k$ intervals $A^u(k;z_1z_2\dots z_k)$ as

$$A^u(k;z_1z_2\dots z_k) := A^u(k-1;z_1\dots z_{k-1}) \cap F^{-z_1}_\mu(A^u(k-1;z_2\dots z_k)),$$

where each entry z_i is an element of $\{1,2,\dots,m-1\}$. See Figure 4.3 for the case of $m=4$. The interval $A^u(k;z_1z_2\dots z_k)$ is called a *u-bridge* for F_μ of *generation* k, and the sequence $(z_1\dots z_k)$ is the *itinerary* of the u-bridge. A unique point of $\partial A^u(k;z_1z_2\dots z_k) \cap \partial A^u(k+1;z_1z_2\dots z_k1)$ (resp. $\partial A^u(k;z_1z_2\dots z_k) \cap \partial A^u(k+1;z_1z_2\dots z_km-1)$) is called the *leading point* (resp. *bottom point*) of $A^u(k;z_1z_2\dots z_k)$. Consider the Cantor set

$$K^u_m(\mu) := A^u(0) \cap \bigcap_{k=1}^{\infty} \bigsqcup_{\substack{(z_1,\dots,z_k)\\ \in\{1,\dots,m-1\}^k}} A^u(k;z_1\dots z_k)$$

dynamically defined by F_μ and associated with the m-periodic orbit. For our convenience, we regard that $A^u(0)$ is a u-bridge of generation 0 with empty itinerary. The *leading gap* of $A^u(k;z_1\dots z_k)$ is the gap in $A^u(k;z_1\dots z_k)$ bounded by $A^u(k+$

$1; z_1 \dots z_k 1)$ and $A^u(k+1; z_1 \dots z_k 2)$. For each integer $k \geq 0$, let $\mathcal{A}^u_k$ be the collection of all k-bridges $A^u(k; z_1 \dots z_k)$, see Figure 4.3.

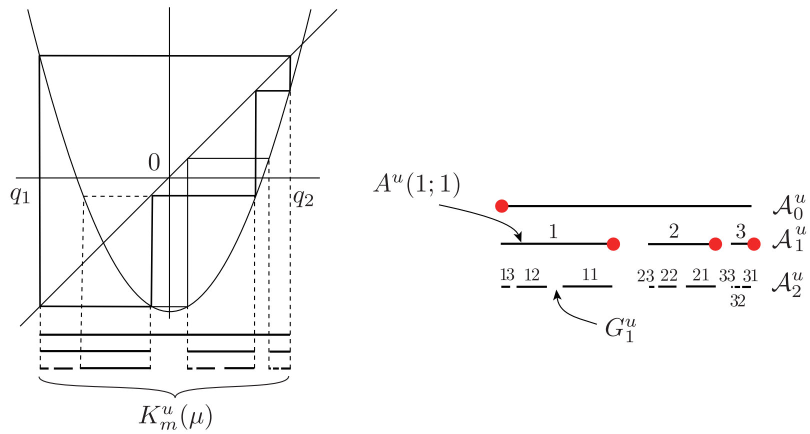

FIGURE 4.3. A nested sequence of unstable bridges. For any $z \in \{1,2,3\}$, $F_\mu(A^u(2;3z)) = A^u(2;2z)$, $F_\mu(A^u(2;2z)) = A^u(2;1z)$, and $F_\mu(A^u(2;1z)) = A^u(1;z)$. The red dots represent the leading points of u-bridges of generation 0 and 1. G^u_1 is the leading gap of $A^u(1;1)$.

Remark 4.4. As $\mu = -2$, F_{-2} is topologically conjugate to the tent map $T : x \mapsto |2x-1|+1$ via $g : x \mapsto 2 - 4\sin^2(\pi/2)x$. It implies that, if μ is contained in a small neighborhood I of -2 and $m \geq 3$, $K^u_m(\mu)$ is a uniformly hyperbolic set for F_μ, see [33, §6.2].

Lemma 4.5 (Bounded distortions for u-bridges)**.** *For any integer $m \geq 3$, there exist $\eta(m) > 0$ and an integer $\kappa(m) \geq 1$ such that, for any $k \geq \kappa(m)$ and $\mu \in (-2-\eta(m), -2+\eta(m))$, the following conditions (1)–(3) hold, where A^u_k and A^u_{k+1} are u-bridges for $K^u_m(\mu)$ of generation k and $k+1$ respectively such that the first k entries in the itinerary of A^u_{k+1} are the same as the entries in the itinerary of A^u_k, that is,*

$$A^u_k = A^u(k; z_1 \dots z_k),\ A^u_{k+1} = A^u(k+1; z_1 \dots z_k z_{k+1}).$$

(1) *If $z_{k+1} = j \in \{1, \dots, m-1\}$, then*

$$3 \cdot 2^{j-2} \leq |A^u_k||A^u_{k+1}|^{-1} \leq 5 \cdot 2^{j-2}.$$

In particular, we have

$$\frac{3}{2} \leq |A^u_k||A^u_{k+1}|^{-1} \leq 5 \cdot 2^{m-3}$$

for any $z_{k+1} \in \{1, \dots, m-1\}$. Moreover, $|A^u_k||A^u_{k+1}|^{-1} \leq 5$ if z_{k+1} is either 1 or 2.

(2) *Let I^u_k be the minimum sub-interval of A^u_k containing A^u_{k+1} and the bottom point of A^u_k. Then*

$$|A^u_{k+1}|\,|I^u_k|^{-1} \geq \frac{1}{3}.$$

(3) *Suppose that* $z_{k+1} \le m-2$ *and* $\widetilde{A}^u_{k+1} = A^u(k+1; z_1 \dots z_k\ z_{k+1}+1)$. *Let* G^u_{k+1} *be the connecting gap for* A^u_{k+1} *and* $\widetilde{A}^u_{k+1}$, *i.e.* $G_{k+1} = \mathrm{Gap}(A^u_{k+1}, \widetilde{A}^u_{k+1})$. *Then*

$$|\widetilde{A}^u_{k+1}|\,|A^u_{k+1}|^{-1} \ge \frac{1}{3} \quad \text{and} \quad |G^u_{k+1}|\,|A^u_{k+1}|^{-1} \ge \frac{1}{2^{m+1}}.$$

Proof. (1) There are constants $C > 0$ and $\sigma > 1$ such that $|(F^k_\mu)'(x)| \ge C\sigma^k$ for any $x \in K^u_m(\mu)$ and $k \ge 1$. Let $k_0 = k_0(\mu, m)$ be the minimum non-negative integer with $C\sigma^{k_0} > 1$. As in the proof of Lemma 4.1, we have a constant $\gamma > 1$ and an open neighborhood V of $K^u_m(\mu)$ such that $|(F^{k_0}_\mu)'(x)| \ge \gamma$ for any $x \in V$ and $n \ge 1$.

Take a positive integer κ. For any integer $k \ge k_0 + \kappa$, there exist integers $n \ge 0$ and $h \in \{\kappa, \dots, \kappa + k_0 - 1\}$ such that $F^{nk_0}_\mu(A^u_k) = A^u_h$, $F^{nk_0}_\mu(A^u_{k+1}) = A^u_{h+1}$, where

$$A^u_h = A^u(h; z_{k-h+1} \dots z_k),\ A^u_{h+1} = A^u(h+1; z_{k-h+1} \dots z_k z_{k+1}).$$

By the mean-value theorem, there are $q \in A^u_k$ and $\tilde{q} \in A^u_{k+1}$ such that

$$|A^u_h| = |A^u_k||(F^{nk_0}_\mu)'(q)|, \quad |A^u_{h+1}| = |A^u_{k+1}||(F^{nk_0}_\mu)'(\tilde{q})|. \tag{4.6}$$

Let $\delta(k)$ be the maximum width of elements in $\mathcal{A}(k)$. Applying Lemma 4.1 to $F^{nk_0}_\mu$, there exists a constant $\tilde{c} = \tilde{c}(\delta(\kappa))$ of order $\delta(\kappa)^\alpha$ independent of n such that

$$e^{-\tilde{c}} \le |(F^{nk_0}_\mu)'(\tilde{q})||(F^{nk_0}_\mu)'(q)|^{-1} \le e^{\tilde{c}}.$$

Thus we have

$$e^{-\tilde{c}} r_{u-}(m,j) \le |A^u_k||A^u_{k+1}|^{-1} \le e^{\tilde{c}} r_{u+}(m,j),$$

where

$$\begin{aligned} r_{u-}(m,j) = \min\bigl\{&|A^u_h|\,|A^u_{h+1}|^{-1}\ ;\ h \in \{\kappa, \dots, \kappa+k_0-1\}, \\ &z_{k-h+1}, \dots, z_k \in \{1, \dots, m-1\},\ z_{k+1} = j\bigr\}; \\ r_{u+}(m,j) = \max\bigl\{&|A^u_h|\,|A^u_{h+1}|^{-1}\ ;\ h \in \{\kappa, \dots, \kappa+k_0-1\}, \\ &z_{k-h+1}, \dots, z_k \in \{1, \dots, m-1\},\ z_{k+1} = j\bigr\}. \end{aligned}$$

One can suppose that $\delta(\kappa)$ is arbitrarily small by taking κ large enough, and hence $\tilde{c}(\delta(\kappa))$ is arbitrarily close to 0.

First we consider the case of $\mu = -2$. Set $A^{u\prime}_h = g^{-1}(A^u_h)$ for the conjugation map g given in Remark 4.4. By [33, §6.2], $A^{u\prime}(0) = [2\delta, 1-\delta]$ and $A^{u\prime}(1;j) = [\frac{1}{2^j} + \frac{1}{2^j}\delta, \frac{1}{2^{j-1}} - \frac{1}{2^{j-1}}\delta]$ for $j = 1, \dots, m-1$. $A^{u\prime}(1;2) = [\frac{1}{4} + \frac{1}{4}\delta, \frac{1}{2} - \frac{1}{2}\delta]$, where $\delta = \frac{1}{2^m - 1}$. Thus we have $|A^{u\prime}(0)| = 1 - 3\delta$ and $|A^{u\prime}(1;j)| = \frac{1}{2^j}(1-3\delta)$. This implies that

$$|A^{u\prime}(0)|\,|A^{u\prime}(1;j)|^{-1} = 2^j.$$

Since the tent map T is a piecewise linear map with $|DT_x| = 2$ for any $x \ne \frac{1}{2}$, $|A^{u\prime}_h|\,|{A^u_{h+1}}'|^{-1} = 2^j$ for any $h \in \{\kappa, \dots, \kappa+k_0-1\}$, $z_{k-h+1}, \dots, z_k \in \{1, \dots, m-1\}$ and $z_{k+1} = j$. The width of the interval $A^{u\prime}_h$ can be arbitrarily small if we take κ sufficiently large. Since the conjugation map g is almost affine on such a short interval, one can suppose that $r_{u-}(m,j) > \frac{3}{4} \cdot 2^j = 3 \cdot 2^{j-2}$ and $r_{u+}(m,j) < \frac{5}{4} 2^j = 5 \cdot 2^{j-2}$ for $\mu = -2$. Since F^t_μ uniformly C^1 converges to F^t_{-2} on $[-3,3]$ as $\mu \to -2$ for $t = 1, \dots, k_0$, there exist $\eta(m) > 0$ and an integer $\kappa(m) \ge 1$ such that $e^{-\tilde{c}(k)} r_{u-}(m,j) > 3 \cdot 2^{j-2}$ and $e^{\tilde{c}(k)} r_{u+}(m,j) < 5 \cdot 2^{j-2}$ if $k \ge \kappa(m)$ and $\mu \in (-2-\eta(m), -2+\eta(m))$. This shows (1).

(2) Suppose that $\mu = -2$. Then $A^{u\prime}(0) = [2\delta, 1-\delta]$ and $A^{u\prime}(1;l) = \bigl[\frac{1}{2^l} + \frac{1}{2^l}\delta, \frac{1}{2^{l-1}} - \frac{1}{2^{l-1}}\delta\bigr]$ for $l = 1,\dots,m-1$. Let $I_0^{u\prime}$ be the minimum sub-interval of $A^{u\prime}(0)$ containing $A^{u\prime}(1;l)$ and 2δ, that is, $I_0^{u\prime} = \bigl[2\delta, \frac{1}{2^{l-1}} - \frac{1}{2^{l-1}}\delta\bigr]$. Since $|A^{u\prime}(1;l)| = \frac{1}{2^l}(1-3\delta)$ and $|I_0^{u\prime}| = \bigl(\frac{1}{2^{l-1}} - \frac{1}{2^{l-2}}\delta\bigr) - 2\delta \le \frac{1}{2^{l-1}}(1-3\delta)$, $|A^{u\prime}(1;l)|\,|I_0^{u\prime}|^{-1} \ge \frac{1}{2}$. By using the argument as in (1), one can show that

$$|A_{k+1}^u|\,|I_k^u|^{-1} \ge \frac{1}{3}$$

if necessary retaking $\kappa(m)$ by a larger integer and $\mu(m)$ by a smaller positive number. This shows (2).

(3) The proof is quite similar to that of (2). Since $|A^{u\prime}(1;l)| = \frac{1}{2^l}(1-3\delta)$ and $|A^{u\prime}(1;l+1)| = \frac{1}{2^{l+1}}(1-3\delta)$ for any $l \in \{1,\dots,m-2\}$, we have $|A^{u\prime}(1;l+1)|\,|A^{u\prime}(1;l)|^{-1} = \frac{1}{2}$. The connecting gap $G_l^{u\prime}$ for $A^{u\prime}(1;l)$ and $A^{u\prime}(1;l+1)$ has the length $|G_l^{u\prime}| = \frac{1}{2^l}\delta$. Thus $|A^{u\prime}(1;l)|\,|G_l^{u\prime}|^{-1} \le (1-3\delta)\delta^{-1} = 2^m - 4 < 2^m$. Then one can retake $\kappa(m)$ and $\eta(m)$ again so that the inequalities of (3) hold. This completes the proof. □

As μ is close to -2, $K_m^u(\mu)$ is contained in the interior $\mathrm{Int}(I)$ of $I = [-3,3]$. In such a situation, observe that, for every $p \in K_m^u(\mu)$, there exist $k \ge 0$, $A^u \in \mathcal{A}_k^u$ and the closure G a component of $I \setminus \bigsqcup_{A^u \in \mathcal{A}_k^u} A^u$ such that A^u and G are the bridge and gap satisfying $A^u \cap G = \{p\}$. Note that $K_m^u(\mu)$ depends on the initially given m-periodic orbit containing q_1 and q_2. This fact together with Remark 4.4 implies the following:

Remark 4.6. $\tau(K_m^u(\mu))$ can be arbitrarily large if we take m sufficiently large, see [33, §6.2].

4.4. **Translation into Hénon-like maps.** Hénon map introduced in Section 3 is written as

$$\varphi_{\mu,\nu}(x,y) = (y,\ \nu x + F_\mu(y)),$$

where ν is a real parameter and F_μ is the quadratic map of (4.5). Fortunately all the properties given in Section 4.3 are inherited by Hénon maps with $\nu \approx 0$.

Using the Cantor set $K_m^u(\mu)$ for F_μ with $\mu \approx -2$, one can define the subset $\{(x, F_\mu(x))\ ;\ x \in K_m^u(\mu)\}$ of $\mathbb{R}^2$, which is a Cantor set on the parabolic curve $\mathrm{Im}(\varphi_{\mu,0}) = \{(x, \mu + x^2); -\infty < x < \infty\}$. For simplicity, we denote the Cantor set $\varphi_{\mu,0}(K_m^u(\mu))$ in $\mathrm{Im}(\varphi_{\mu,0})$ again by $K_m^u(\mu)$.

Let $\mathcal{U}(-2,0)$ be a small neighborhood of $(-2,0)$ in the parameter space, and let $P_{\mu,\nu}$, $Q_{m;\mu,\nu}$, $\Gamma_{m;\mu,\nu}$ be respectively the fixed point, periodic point and basic set for $\varphi_{\mu,\nu}$ with $(\mu,\nu) \in \mathcal{U}(-2,0)$ defined in Section 3. Now we consider a continuous map $\widetilde{P} : \mathcal{U}(-2,0) \to \mathbb{R}^2$ with $\widetilde{P}_{\mu,\nu} := \widetilde{P}(\mu,\nu) \in W^u(P_{\mu,\nu})$ and $\widetilde{P}_{-2,0} = (-2,2)$, see Figure 4.4. For any $(\mu,\nu) \in \mathcal{U}(-2,0)$, let $\widehat{\ell}_{\mu,\nu}^u$ be the arc in $W^u(P_{\mu,\nu})$ connecting $P_{\mu,\nu}$ with $\widetilde{P}_{\mu,\nu}$. By the stable manifold theorem (for example see [36], Chapter 5, Theorem 10.1U)), $\widehat{\ell}_{\mu,\nu}^u$ C^r-converges to $\widehat{\ell}_{-2,0}^u$ as $(\mu,\nu) \to (-2,0)$, see Figure 4.4.

For any $(\mu,\nu) \approx (-2,0)$ with $\nu \ne 0$, consider arcs h_i $(i = 1,2)$ in $W_{\mathrm{loc}}^s(\varphi_{\mu,\nu}^i(Q_{m;\mu,\nu}))$ as illustrated in Figure 4.5. Let l_j $(j = 1,2)$ be parabolic curves in $\mathbb{R}^2$ with $\partial l_1 \cup \partial l_2 = \partial h_1 \cup \partial h_2$ and such that the union $l_1 \cup h_1 \cup l_2 \cup h_2$ is a simple closed curve in $\mathbb{R}^2$ bounding a compact region $E_{\mu,\nu}$ which contains the basic set $\Gamma_{m;\mu,\nu}$ and such that $\ell_{\mu,\nu}^u = \widehat{\ell}_{\mu,\nu}^u \cap E_{\mu,\nu}$ is an arc connecting h_1 with h_2. Consider a local stable foliation $\mathcal{F}_{\mu,\nu}^s = \mathcal{F}_{\mathrm{loc}}^s(\Gamma_{m;\mu,\nu})$ on $E_{\mu,\nu}$ *compatible with* $W_{\mathrm{loc}}^s(\Gamma_{m;\mu,\nu})$, that is,

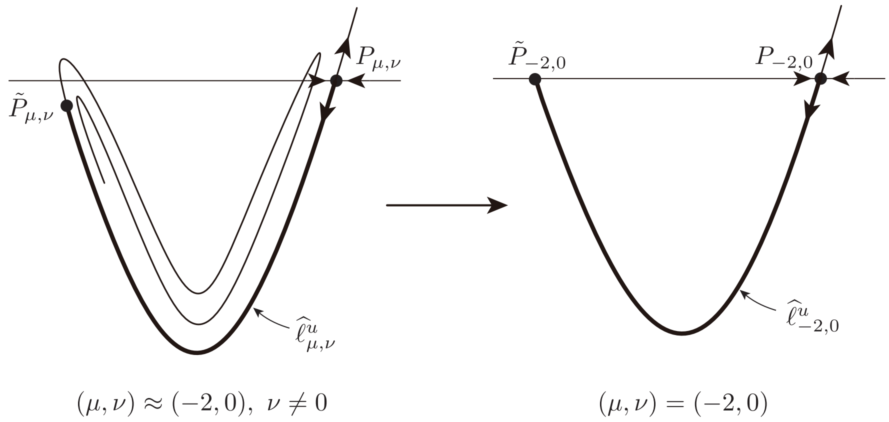

FIGURE 4.4. Folding of $\widehat{\ell}^u_{\mu,\nu}$ as $(\mu,\nu) \to (-2,0)$.

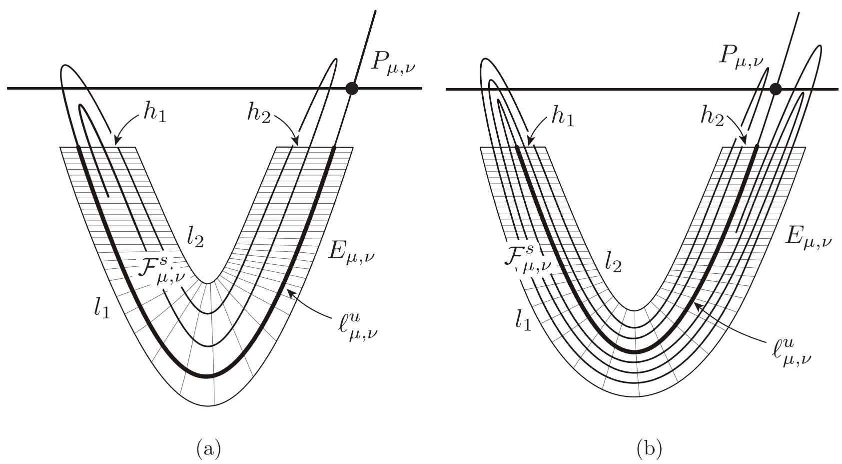

FIGURE 4.5. Local stable foliations compatible with $W^s_{\text{loc}}(\Gamma_{m;\mu,\nu})$.
(a) The case of $\nu > 0$. (b) The case of $\nu < 0$.

(F-i) each component of $W^s_{\text{loc}}(\Gamma_{m;\mu,\nu}) \cap E_{\mu,\nu}$ is a leaf of $\mathcal{F}^s_{\mu,\nu}$;
(F-ii) $\ell^u_{\mu,\nu}$ crosses $\mathcal{F}^s_{\mu,\nu}$ *exactly*, that is, each leaf of $\mathcal{F}^s_{\mu,\nu}$ intersects $\ell^u_{\mu,\nu}$ transversely in a single point and any point of $\ell^u_{\mu,\nu}$ is passed through by a leaf of $\mathcal{F}^s_{\mu,\nu}$;
(F-iii) leaves of $\mathcal{F}^s_{\mu,\nu}$ are C^3 curves such that themselves, their directions, and their curvatures vary C^1 with respect to any transverse direction and (μ,ν).

See [21, Lemma 4.1] and [23, §2.3] for details.

Here we take the curves l_1, l_2 so that they are sufficiently C^r-close to the outermost components of $W^u_{\text{loc}}(\Gamma_{m;\mu,\nu}) \cap E_{\mu,\nu}$. Then there exists a local stable foliation $\mathcal{F}^s_{\mu,\nu} = \mathcal{F}^s_{\text{loc}}(\Gamma_{m;\mu,\nu})$ on $E_{\mu,\nu}$ compatible with $W^s_{\text{loc}}(\Gamma_{m;\mu,\nu})$ which contains h_1, h_2 as leaves.

Let $\pi^s_{\mu,\nu} : E_{\mu,\nu} \to \ell^u_{\mu,\nu}$ be the projection along the leaves of $\mathcal{F}^s_{\mu,\nu}$. Define

$$K^u_{m;\mu,\nu} := \pi^s_{\mu,\nu}(\Gamma_{m;\mu,\nu}),$$

which is a Cantor set dynamically defined by $\pi^s_{\mu,\nu} \circ \varphi_{\mu,\nu}$. Here we note that the set $K^u_{m;\mu,\nu}$ does not depend on the choice of the local stable foliation $\mathcal{F}^s_{\mu,\nu}$ on $E_{\mu,\nu}$ compatible with $W^s_{\mathrm{loc}}(\Gamma_{m;\mu,\nu})$.

If (μ,ν) is close to $(-2,0)$, then one can define the presentation involved with bridges and gaps for the Cantor set $K^u_{m;\mu,\nu}$ in a manner quite similar to that for K^u_m. Then Lemma 4.5 and Remark 4.6 are translated as follows if necessary replacing $\mathcal{U}(-2,0)$ by a smaller neighborhood of $(-2,0)$:

Remark 4.7. (1) For every $(\mu,\nu)\in\mathcal{U}(-2,0)$, the bounded distortion property in Lemma 4.5 holds for $K^u_{m;\mu,\nu}$.
(2) The thickness $\tau(K^u_{m;\mu,\nu})$ of $K^u_{m;\mu,\nu}$ converges to $\tau(K^u_{m;\mu,0})$ as $\nu\to 0$, and hence it can have an arbitrarily large value if we take m large enough and $(\mu,\nu)\in\mathcal{U}(-2,0)$ sufficiently close to $(-2,0)$, see [33, §6.3, Proposition 1].

Note that the return map

$$\varphi_n := \Phi_n^{-1}\circ f_{\mu_n}^{N_*+n}\circ\Phi_n(\bar{x},\bar{y})$$

of Theorem 3.2 is arbitrarily C^r-close to $\varphi_{-2,0}$ if n is sufficiently large. Locally identifying the coordinate on $\mathbb{R}^2$ with that on a small neighborhood $U(q)$ of q in M, we may set

$$\varphi_n = f_{\mu_n}^{N_*+n}. \tag{4.7}$$

The φ_n is an Hénon-like map with the saddle fixed point $P(\varphi_n)$ and the basic set $\Gamma_m = \Gamma_m(\varphi_n)$ corresponding to $P_{\mu,\nu}$ and $\Gamma_{m;\mu,\nu}$. Let E be a compact region corresponding to $E_{\mu,\nu}$ and $\pi_m : E \to \ell^u(\varphi_n)$ the projection along the leaves of a stable foliation $\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m)$ on E compatible with $W^s_{\mathrm{loc}}(\Gamma_m)$, where $\ell^u(\varphi_n)$ is the curve in $W^u(P(\varphi_n))\cap E$ corresponding to $l^u(\varphi_{\mu,\nu})$. Then one obtains the dynamically defined Cantor set

$$K^u_m := \pi_m(\Gamma_m). \tag{4.8}$$

Remark 4.8. The distortion and thickness for K^u_m have the same properties as those in Remarks 4.7 if φ_n is sufficiently close to $\varphi_{-2,0}$.

5. Linking property for bridges

Recall that $\{f_\mu\}_{\mu\in\mathbb{R}}$ is the one-parameter family of two-dimensional C^r diffeomorphisms given in Section 3. In this section, we will present Linking Lemma, which is crucial in the proof of Theorem A.

5.1. **Heteroclinic tangencies.** Let $h:[0,1]\times[0,1]\to M$ be an embedding. Set $R = h([0,1]\times[0,1])$, $\sharp R = h([0,1]\times\{0,1\})$ and $\flat R = h(\{0,1\}\times[0,1])$. Note that $\sharp R\cup\flat R = \partial R$. We say that the pair $(R,\sharp R)$ (for short R) is a *strip* with the *edge* $\sharp R$. Similarly, the pair $(R,\flat R)$ is a *strip* with the *edge* $\flat R$. A strip $(R',\sharp R')$ is called a *sub-strip* of $(R,\sharp R)$ if $R'\subset R$ and each component of $\sharp R$ contains a component of $\sharp R'$.

One can take a coordinate neighborhood of the fixed point p in M such that $p=(0,0)$, $f_0^{-N_*}(q)=(0,1)$ and f_μ is linear on $S=[0,2]\times[0,2]$ and satisfying the condition (S-iii) in Section 3, where N_* is the positive integer given in Theorem 3.2. Consider the foliation $\mathcal{F}_S$ on S consisting of horizontal leaves, that is, any leaf of $\mathcal{F}_S$ has form $[0,2]\times\{y\}$ for some $0\le y\le 2$. Though $\mathcal{F}_S$ is not in general an f_μ-invariant foliation, the f_μ^{-n}-image of the restriction of $\mathcal{F}_S$ on $[0,2\lambda^n]\times[0,2]$ is a sub-lamination of $\mathcal{F}_S$ for any positive integer n. Such a partial invariance for $\mathcal{F}_S$ is sufficient in our arguments below.

For any $\bar{\mu}$ near -2, let $\mu_n = \Theta_n(\bar{\mu})$ be the parameter value used in (3.3). We set $f_{\mu_n} = f_n$ for short throughout the remainder of this subsection. Take an integer integer $m \geq 3$. From the condition (S-v), for any sufficiently large integer n, we have the horseshoe $\Lambda := \Lambda(f_n)$ containing p for f_n and the basic set $\Gamma_m := \Gamma_m(\varphi_n)$ for the return map $\varphi_n := f_n^{N_*+n}$ defined as (4.7) near the homoclinic tangency q. Recall that Γ_m contains the saddle fixed point $P := P(\varphi_n)$ and the m-periodic orbit $Q_m^{(i)} := \varphi_n^i(Q_m)$ for $i = 0, 1, \dots, m-1$. Let E be the compact region and $\ell^u(\varphi_n)$ the arc given in Subsection 4.4. Again by the condition (S-v), there is an integer $N_1 > 0$ such that each leaf of $f_n^{-N_1}(\mathcal{F}_S)$ meets leaves of $W^u_{\mathrm{loc}}(\Gamma_m)$ transversely in E. See Figure 5.1 for the case of $m = 4$. For any integer $i \geq 0$, let $\mathcal{A}^u(i)$ be the set

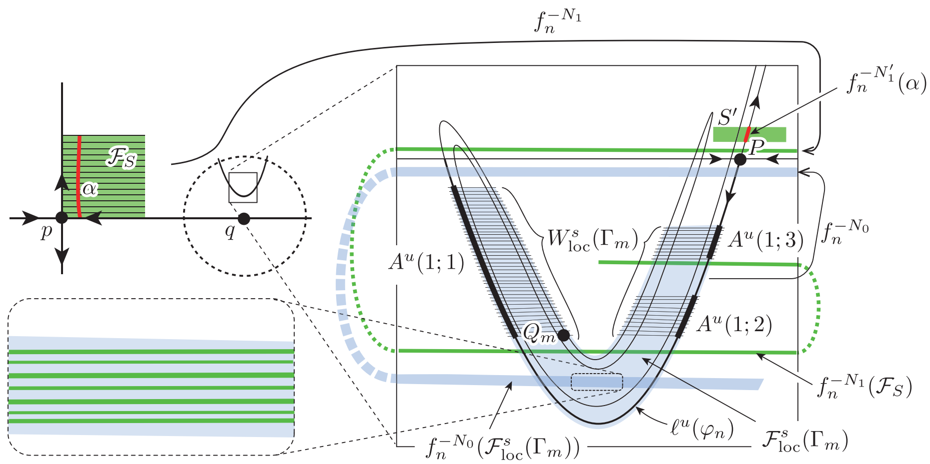

FIGURE 5.1. Pull backs of the foliation $\mathcal{F}_S$.

of u-bridges $A^u(i, \underline{z})$ in $\ell^u(\varphi_n)$ of generation i with $A^u(0) = \ell^u(\varphi_n)$ with respect to the Cantor set K^u_m of (4.8), and let $\mathcal{A}^u = \bigcup_{i=0}^{\infty} \mathcal{A}^u(i)$. Note that the itinerary $\underline{z} = (z_1 \dots z_i)$ of $A^u(i, \underline{z})$ is an element of $\{1, \dots, m-1\}^i$. Let $G^u(0)$ be the leading gap of $A^u(0)$. The closure $\mathbb{G}^u(0)$ of the component of $E \setminus W^s_{\mathrm{loc}}(\Gamma_m)$ containing $G^u(0)$ is a strip with $W^s_{\mathrm{loc}}(\Gamma_m) \cap \mathbb{G}^u(0) = \flat\mathbb{G}^u(0)$. For any $A^u \in \mathcal{A}^u$, the strip $\mathbb{G}(A^u)$ containing the leading gap $G(A^u)$ of A^u is defined similarly, see Figure 5.2.

For the square $S = [0,2] \times [0,2]$, set $\sharp S = [0,2] \times \{0,2\}$ and $\flat S = \{0,2\} \times [0,2]$. For our choice of N_1, there exists a almost vertical sub-strip $(S(0), \sharp S(0))$ of $(S, \sharp S)$ such that $\flat S(0)$ consists of two arcs in S meeting $\mathcal{F}_S$ transversely and $(f_n^{-N_1}(S(0)), f_n^{-N_1}(\flat S(0)))$ is a sub-strip of the strip $(\mathbb{G}^u(0), \sharp\mathbb{G}^u(0))$. We show that one can choose the strip so that

$$S(0) \cap W^u_{\mathrm{loc}}(\Lambda) = \emptyset. \tag{5.1}$$

Since Γ_m is homoclinically related to Λ, $W^u(P)$ contains a almost vertical arc α in S connecting the components of $\sharp S$. If we take an integer N_1' sufficiently large, the arc $f^{-N_1'}(\alpha)$ is contained in a small neighborhood of P, see Figure 5.1. Since $\alpha \cap W^u_{\mathrm{loc}}(\Lambda) = \emptyset$, we have a sub-strip S' of $f_n^{-N_1'}(S)$ with $f_n^{-N_1'}(\alpha)$ as a core and such that $f_n^{N_1'}(S') \cap W^u_{\mathrm{loc}}(\Lambda) = \emptyset$. If we take an integer N_1 sufficiently larger than N_1',

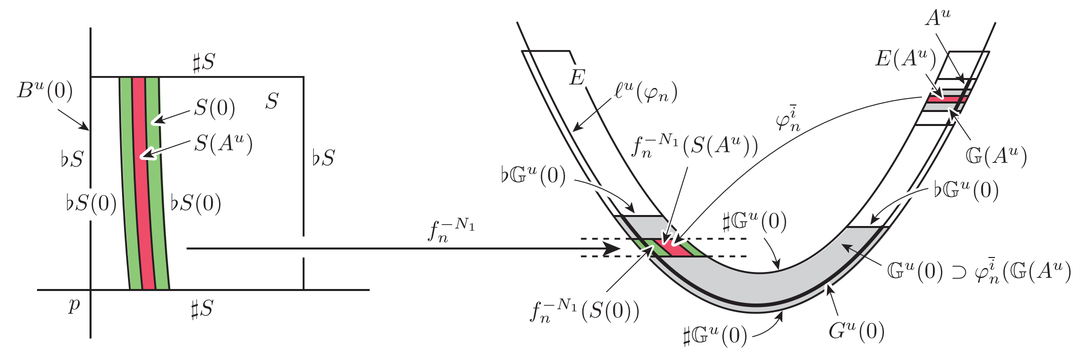

FIGURE 5.2. Pull back of the strip $S(A^u)$.

then $f_n^{-(N_1-N_1')}(S')$ is a sub-strip of $f_n^{-N_1}(S)$ containing the sub-strip $f^{-N_1}(S(0))$ of $\mathbb{G}^u(0)$ as above. Then $S(0)$ is contained in $f^{N_1'}(S')$ and hence in particular it satisfies (5.1).

If the generation of a u-bridge A^u is i, then there exists a unique integer $\bar{i}$ with

$$i \le \bar{i} \le (m-1)i \tag{5.2}$$

such that $(\varphi_n^{\bar{i}}(\mathbb{G}(A^u)), \varphi_n^{\bar{i}}(\flat\mathbb{G}(A^u)))$ is a sub-strip of $(\mathbb{G}^u(0), \flat\mathbb{G}^u(0))$. Then

$$\ell^u(\varphi_n) = \pi^m \circ \varphi_n^{\bar{i}}(A^u)$$

holds. There exists a sub-strip $(S(A^u), \sharp S(A^u))$ of $(S(0), \sharp S(0))$ such that $f_n^{-N_1}(S(A^u)) = \varphi_n^{\bar{i}}(\mathbb{G}(A^u)) \cap f_n^{-N_1}(S(0))$. Then we have a sub-strip $(E(A^u), \sharp E(A^u))$ of $(\mathbb{G}(A^u), \sharp\mathbb{G}(A^u))$ with

$$(\varphi_n^{\bar{i}}(E(A^u)), \varphi_n^{\bar{i}}(\flat E(A^u))) = (f_n^{-N_1}(S(A^u)), f_n^{-N_1}(\sharp S(A^u))).$$

The strip $E(A^u)$ has the foliation $\mathcal{F}_{A^u}$ induced from $\mathcal{F}_S$ via $\varphi_n^{-\bar{i}} \circ f_n^{-N_1}$. One can retake the local unstable foliation $\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m)$ on E compatible with $W^s_{\mathrm{loc}}(\Gamma_m)$ and extending $\bigcup_{A^u \in \mathcal{A}} \mathcal{F}_{A^u}$. The reason why we choose the unstable foliation with $\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m) \supset \varphi_n^{-\bar{i}} \circ f_n^{-N_1}(\mathcal{F}_S|_{S(A^u)})$ will be explained in the proof of Lemma 8.2.

Now we take the unstable foliation $\mathcal{F}^u_{\mathrm{loc}}(\Lambda)$ compatible with $W^u_{\mathrm{loc}}(\Lambda)$ carefully as notified in Remark 4.2-(2). The strip $S(0)$ has the foliation $\mathcal{F}^u_{S(0)}$ induced from $\mathcal{F}^u_{\mathrm{loc}}(\Gamma_m)|_{f^{-N_1}(S(0))}$ via f^{N_1}. By (5.1), $S(0)$ is an almost vertical strip disjoint from $W^u_{\mathrm{loc}}(\Lambda)$. The foliation $\mathcal{F}^u_{\mathrm{loc}}(\Lambda)$ can be defined so that $\mathcal{F}^u_{\mathrm{loc}}(\Lambda)|_{S(0)} = \mathcal{F}^u_{S(0)}$.

From the conditions (S-v), (S-vi) of Section 3, there are integers $N_0 > 0$, $N_2 > N_*$ and a C^1-arc L in $U(q)$ meeting $f_n^{-N_0}(\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m))$ exactly and such that $L' = L \cap f_n^{N_2}(\mathcal{F}^u_{\mathrm{loc}}(\Lambda))$ is a sub-arc of L each element of which is a quadratic tangency of leaves of $f_n^{-N_0}(\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m))$ and $f_n^{N_2}(\mathcal{F}^u_{\mathrm{loc}}(\Lambda))$. See Figure 5.3.

Let

$$\pi^u : E \longrightarrow L \tag{5.3}$$

be the projection along the leaves of $f_n^{-N_0}(\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m))$. For the sub-strip $S(A^u)$ of $S(0)$ given as above, consider the projection $\pi_{S,A^u} : S(A^u) \longrightarrow L$ defined by $\pi_{S,A^u} = \pi^u \circ \varphi_n^{-\bar{i}} \circ f_n^{-N_1}|_{S(A^u)}$ and the composition

$$\pi_{A^u} := \pi_{S,A^u} \circ \psi_{A^u} : B^u(0) \longrightarrow L, \tag{5.4}$$

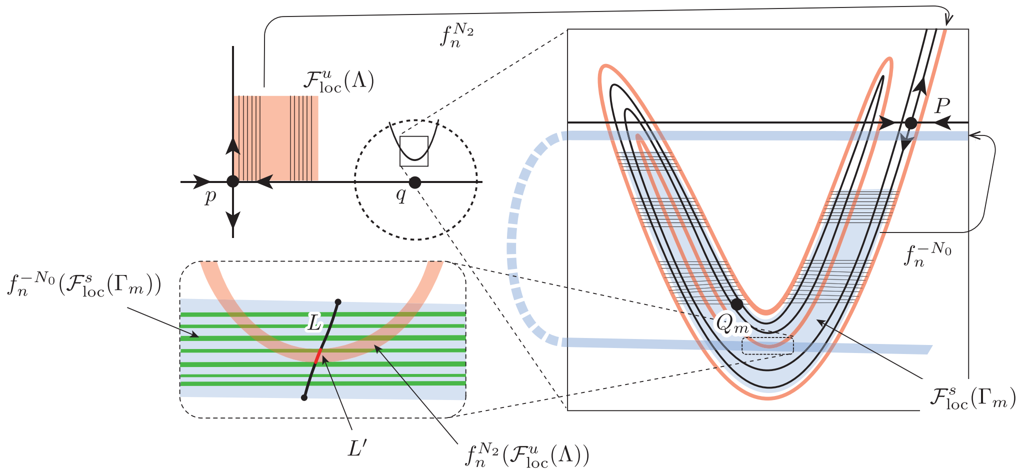

FIGURE 5.3. Heteroclinic tangencies on L' of stable and unstable foliations.

where $\psi_{A^u} : B^u(0) \longrightarrow S(A^u)$ is a diffeomorphism from $B^u(0) = \{0\} \times [0,2]$ onto a component of $\flat S(A^u)$ along the leaves of $\mathcal{F}_S$. See Figure 5.2 and Figure 7.2 with $\widehat{A}^u_k$, $f^{-\widehat{n}_k}$, f^{-N_1} replaced by A^u, $\varphi_n^{-\bar{i}}$, $f_n^{-N_1}$ respectively. Let $K^s_{\Lambda,L}$, $K^u_{m,L}$ be the Cantor sets on L defined as

$$K^s_{\Lambda,L} = \pi^s(K^s_\Lambda) \quad \text{and} \quad K^u_{m,L} = \pi^u(K^u_m), \tag{5.5}$$

where

$$\pi^s : S \to L \tag{5.6}$$

is the projection along the leaves of $f^{N_2}(\mathcal{F}^u_{\mathrm{loc}}(\Lambda))$. For any $B^s \in \mathcal{B}^s$, let $B^s_L := \pi^s(B^s)$ is a bridge of $K^s_{\Lambda,L}$. Similarly, for any $A^u \in \mathcal{A}^u$, let $A^u_L := \pi^s(A^u)$ is a bridge of $K^s_{m,L}$.

5.2. **Encounter of s-bridges and u-bridges, I.** Here we will study the heteroclinical connection between s-bridges B^s of $K^s_{\Lambda,L}$ and u-bridges A^u of $K^u_{m,L}$ in L.

Let $\mathcal{U}(\varphi_{-2,0})$ be a sufficiently small C^r-neighborhood of $\varphi_{-2,0}$. For any sufficiently large n, the return map $\varphi := \varphi_n$ of (4.7) is contained in $\mathcal{U}(\varphi_{-2,0})$. Theorem 3.2 together with Remark 4.8 assures that, if we take the integer m sufficiently large, then the Cantor set K^u_m in $\ell^u(\varphi)$ with respect to

$$\varphi = \pi_m \circ f^{N_*+n_*}_{\mu_{n_*}} \tag{5.7}$$

satisfies

$$\tau(K^u_m) > \max\left\{r^2_{s+},\ \tau(K^s_\Lambda)^{-1},\ 3^8\right\}. \tag{5.8}$$

Here $n_* = n_*(m)$ is a positive integer with $\lim_{m\to\infty} n_*(m) = \infty$ such that $f_{\mu_{n_*}}$ is arbitrarily C^r-close to f. Write $\tau := \tau(K^u_m)$ for short and let

$$\xi_0 := \left(\frac{1}{r_{s+}} - \frac{1}{2\tau^{1/2}}\right) r_{s-}. \tag{5.9}$$

By (5.8) and Lemma 4.3, we have $0 < \xi_0 < 1$.

By Theorem 3.2, one can choose $\mu_{n_*} = \Theta_{n_*}(\bar{\mu}) \neq 0$ so that the Cantor sets $K^s_{\Lambda,L}$ and $K^u_{m,L}$ on $f_{\mu_{n_*}}$ are linked in L. We denote the $f_{\mu_{n_*}}$ again by f. Note that the leaves of $f^{-N_0}(\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m))$ are almost horizontal in $U(q)$. Now we fix a $C^{1+\alpha}$-coordinate on a small neighborhood $U(L)$ of L in $U(q)$ such that each horizontal line is a leaf of $f^{-N_0}(\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m))$ and each vertical line is a leaf of $f^{-N_0}(\mathcal{F}^u_{\mathrm{loc}}(\Gamma_m))$, see Figure 5.4. From the definition of the coordinate and those of $\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m)$ and $\mathcal{F}^u_{\mathrm{loc}}(\Lambda)$

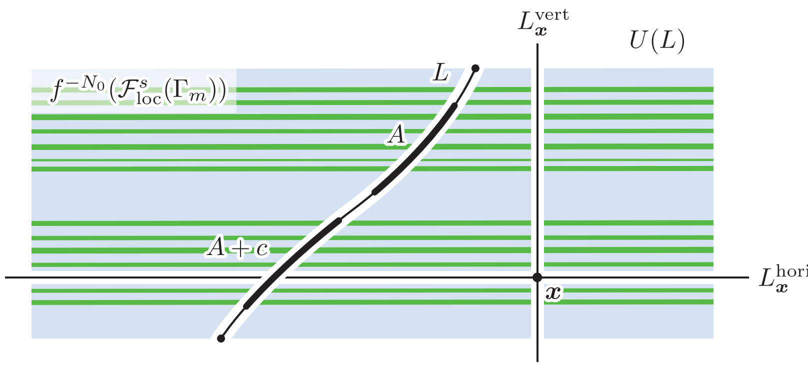

Figure 5.4. A $C^{1+\alpha}$-coordinate on $U(L)$.

in Subsection 5.1, for any $\boldsymbol{x} \in U(L)$ with $f^{(N_0+N_1)}(\boldsymbol{x}) \in S(0)$, the $f^{(N_0+N_1)}$-images of the horizontal and vertical lines passing through $\boldsymbol{x}$ are leaves of $\mathcal{F}_S$ and $\mathcal{F}^u_{\mathrm{loc}}(\Lambda)$ respectively. This fact is used in the proof of Lemma 8.2, see also Remark 7.7.

A parametrization of L. Though C^1-arc L has the parametrization naturally induced from the $C^{1+\alpha}$-coordinate on $U(L)$, we need another parameterization on L suitable to our argument. For the sub-arc $L' = L \cap f^{N_2}(\mathcal{F}^u_{\mathrm{loc}}(\Lambda))$, $\widetilde{L} = f^{-N_2}(L')$ is a C^1-arc in S connecting the components of $\flat S$. See Figures 5.3 and 7.3. Since L meets $f^{N_2}(\{0\} \times [0,2])$ transversely, $\widetilde{L}$ also meets $\{0\} \times [0,2]$ transversely. Thus there exists a small $\delta > 0$ such that $\widetilde{L}_\delta = \widetilde{L} \cap ([0,\delta] \times [0,2])$ is a sub-arc of $\widetilde{L}$ meeting vertical lines in $[0,\delta] \times [0,2]$ transversely. Let $\varpi^u : S \to [0,2] \times \{0\} \subset W^s_{\mathrm{loc}}(p)$ be the vertical projection with respect to the orthogonal coordinate on S. We fix C^1-parametrizations on L and $\widetilde{L}$ extending those on $\widetilde{L}_\delta$, $L_\delta := f^{N_2}(\widetilde{L}_\delta)$ such that $\varpi^u|_{\widetilde{L}_\delta} : \widetilde{L}_\delta \to [0,2] \times \{0\}$ and $f^{N_2}|_{\widetilde{L}_\delta} : \widetilde{L}_\delta \to L'$ are parameter-preserving embeddings. In particular, the parameter value of $\boldsymbol{x} \in L_\delta$ is t if $\varpi^u(f^{-N_2}(\boldsymbol{x})) = (t, 0)$. For any interval A in L represented by $[\alpha_0, \alpha_1]$ with respect to the parametrization and any constant c, the interval $[\alpha_0 + c, \alpha_1 + c]$ in L is denoted by $A + c$ if it is well defined. See Figure 5.4. Then the length $|A| = \alpha_1 - \alpha_0$ of A is equal to the length $|A + c|$ of $A + c$.

For given bridges B^s_L of $K^s_{\Lambda,L}$ and A^u_L of $K^u_{m,L}$, we say that the pair (B^s_L, A^u_L) is *linked* if

- $\mathrm{Int}(B^s_L \cap A^u_L) \neq \emptyset$, and
- B^s_L is not contained in a gap of $A^u_L \cap K^u_{m,L}$ and A^u_L is not contained in a gap of $B^s_L \cap K^s_{\Lambda,L}$.

Moreover the pair is called *ξ-linked* for a constant $0 < \xi \leq 1$ if

$$|B^s_L \cap A^u_L| \geq \xi \min\{|B^s_L|, |A^u_L|\}.$$

The pair (B^s_L, A^u_L) is *γ-proportional* for a constant γ with $0<\gamma<1$ if

$$|A^u_L| \geq |B^s_L| \geq \gamma |A^u_L|,$$

and the pair is *u-dominating* if $|A^u_L| \geq |B^s_L|$.

Remark 5.1. Let B^s, A^u be the bridges of K^s_Λ and K^u_m respectively. Since the projections π^s, π^u of (5.6) and (5.3) are C^1-maps, $\pi^s|_{B^s}: B^s \to B^s_L$ and $\pi^u|_{A^u}: A^u \to A^u_L$ are almost affine if $|B^s|$ and $|A^u|$ are sufficiently small. Thus one can suppose the following conditions without loss of generality.

(i) For bridges $B^s_{L,k}$, $B^s_{L,k+1}$ of bridges of $K^s_{m,L}$ with sufficiently large generations k, $k+1$ and $B^s_{L,k} \supset B^s_{L,k+1}$, the conclusion of Lemma 4.3 holds, that is,

$$r_{s-} \leq |B^s_{L,k}||B^s_{L,k+1}|^{-1} \leq r_{s+} \tag{5.10}$$

if necessary modifying r_{s-} and r_{s+} slightly.

(ii) For bridges $A^u_{L,k}$, $A^u_{L,k+1}$ of bridges of $K^u_{\Lambda,L}$ with sufficiently large generations k, $k+1$ and $A^u_{L,k} \supset A^u_{L,k+1}$ and the gaps $G^u_{L,k+1}$, the interval I^u_k corresponding to those in Lemma 4.5, the conclusions (1)–(3) of Lemma 4.5 hold.

(iii) From the definition of the thickness, the restricted Cantors sets satisfy $\tau(B^s \cap K^s_\Lambda) \geq \tau(K^s_\Lambda)$ and $\tau(A^u \cap K^u_m) \geq \tau(K^u_m)$. Thus, for any $0<\varepsilon<1$ and bridges B^s, A^u with sufficiently large generation, $\tau(B^s_L \cap K^s_{\Lambda,L}) \geq (1-\varepsilon)\tau(B^s \cap K^s_\Lambda)$ and $\tau(A^u_L \cap K^u_{m,L}) \geq (1-\varepsilon)\tau(A^u \cap K^u_m)$ hold. From (5.8), one can suppose that

$$\tau(B^s_L \cap K^s_{\Lambda,L})\tau(A^u_L \cap K^u_{m,L}) > 1. \tag{5.11}$$

See Subsections 4.1 and 4.2 of [24] for similar arguments.

The first perturbation of f. Now we consider the perturbation corresponding to the Δ-sliding in [6, p. 1672]. Since, in [6], the sliding is done along the straight segment, the perturbed diffeomorphism is still of C^r class. However, in our case, $\widetilde{L}$ is guaranteed only to be of class C^1. Hence a perturbed map along $\widetilde{L}$ would not be a C^r-diffeomorphism. So we need another C^r-perturbation which gives an effect similar to the Δ-sliding.

Consider the gap strip $\mathbb{G}^u_\Lambda(0) := \pi^{-1}_{\mathcal{F}^s_{\mathrm{loc}}(\Lambda)}(G^u(0))$, for short $\mathbb{G}^u(0)$, in S associated to the gap $G^u(0) := \mathrm{Gap}(B^u(1;1), B^u(1;2))$ and the projection $\pi_{\mathcal{F}^s_{\mathrm{loc}}(\Lambda)}: S \to \{0\}\times[0,2]$ along the leaves of $\mathcal{F}^s_{\mathrm{loc}}(\Lambda)$. Recall that L' is the sub-arc of L given in Subsection 5.1 meeting $f^{N_2}(\mathcal{F}^u_{\mathrm{loc}}(\Lambda))$ exactly (see Figure 5.3) and $\widetilde{L} = f^{-N_2}(L')$ is a C^1-arc in the strip $\mathbb{G}^u(0)$ disjoint from $\sharp\mathbb{G}^u(0)$ and meeting $\mathcal{F}^u_{\mathrm{loc}}(\Lambda)$ exactly. See the left-hand panel of Figure 5.5 and Figure 7.3. For a sufficiently small $\delta_0>0$, let $\mathcal{B}_{\delta_0/2}$ and $\mathcal{B}_{\delta_0}$ be the disks in M centered at the left edge of $\widetilde{L}$ of radius $\delta_0/2$ and δ_0 respectively such that $\mathcal{B}_{\delta_0} \cap S \subset \mathbb{G}^u(0)$ and $f^{-1}(\mathcal{B}_{\delta_0}) \cap (\mathcal{B}_{\delta_0} \cup X_{n_*}) = \emptyset$, where X_{n_*} is the union of rectangles used in Section 3 to define the basic set Γ_m satisfying the conditions (S-vi) and (S-vii). See Figure 3.2. From the disjointness, any perturbation of f supported on $f^{-1}(\mathcal{B}_{\delta_0})$ dose not affect the invariant set Γ_m and hence the local stable foliation $\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m)$ on E.

For any δ with $|\delta|$ sufficiently smaller than δ_0, consider the perturbation of f supported on $f^{-1}(\mathcal{B}_{\delta_0})$ such that the restriction of the perturbed map f_δ on $f^{-1}(\mathcal{B}_{\delta_0/2})$ is the horizontal δ-shift. Strictly, consider a C^r-diffeomorphism $h_\delta: M \to M$ which is the identify on $M \setminus \mathcal{B}_{\delta_0}$ and the $(\delta,0)$-shift $\boldsymbol{x} \mapsto \boldsymbol{x} + (\delta, 0)$ on $\mathcal{B}_{\delta_0/2}$, and define

$f_\delta = h_\delta \circ f$. Then, for any $\boldsymbol{x} \in f^{-1}(\mathcal{B}_{\delta_0/2})$,

$$f_\delta(\boldsymbol{x}) = h_\delta \circ f(\boldsymbol{x}) = f(\boldsymbol{x}) + (\delta, 0).$$

One can construct the maps f_δ with fixed δ_0 so as to C^r-converge to f as $\delta \to 0$. In particular, any f_δ can be supposed to satisfy the conditions (5.8), (5.10) and (5.11).

Note that this perturbation moves the arc $\widetilde{L}$ of tangencies. One can choose local unstable and stable foliations $\mathcal{F}^u_{\rm loc}(\Lambda;\delta)$, $\mathcal{F}^s_{\rm loc}(\Gamma_m;\delta)$ of f_δ compatible with $W^u_{\rm loc}(\Lambda)$ and $W^s_{\rm loc}(\Gamma_m)$ so that

$$\mathcal{F}^u_{\rm loc}(\Lambda;\delta)|_{\mathcal{B}_{\delta_0}} = h_\delta(\mathcal{F}^u_{\rm loc}(\Lambda)|_{\mathcal{B}_{\delta_0}}), \quad f_\delta^{-N_0}(\mathcal{F}^s_{\rm loc}(\Gamma_m;\delta))|_{U(L)} = f^{-N_0}(\mathcal{F}^s_{\rm loc}(\Gamma_m))|_{U(L)}.$$

The latter equality implies

$$f_\delta^{-(N_0+N_2)}(\mathcal{F}^s_{\rm loc}(\Gamma_m;\delta))|_{\mathcal{B}_{\delta_0}} = f^{-(N_0+N_2)}(\mathcal{F}^s_{\rm loc}(\Gamma_m))|_{\mathcal{B}_{\delta_0}}.$$

However $f_\delta^{-(N_0+N_2)}(\mathcal{F}^s_{\rm loc}(\Gamma_m;\delta))|_{f_\delta^{-1}(\mathcal{B}_{\delta_0})}$ is not equal to $f^{-(N_0+N_2)}(\mathcal{F}^s_{\rm loc}(\Gamma_m))|_{f^{-1}(\mathcal{B}_{\delta_0})}$. Let $\widetilde{L}(\delta)$ be the arc consisting of tangencies between $\mathcal{F}^u_{\rm loc}(\Lambda;\delta)$ and $f_\delta^{-(N_0+N_2)}(\mathcal{F}^s_{\rm loc}(\Gamma_m;\delta))$. See Figure 5.5. From the Implicit Function Theorem, the arcs $\widetilde{L}(\delta)$ C^1-depend

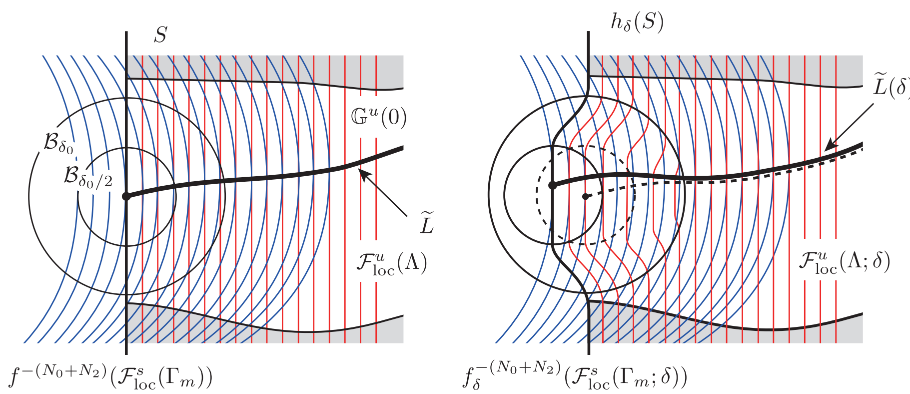

FIGURE 5.5. Shifting of $\mathcal{F}_{\rm loc}(\Lambda)$. The case of $\delta < 0$.

on δ and $\widetilde{L}(0) = \widetilde{L}$. Let $B^s(\delta)$ be the bridge and $K^s_\Lambda(\delta)$ the Cantor set in $\widetilde{L}(\delta)$ which are projected respectively onto the bridge B^s and the Cantor K^s_Λ in $W^s_{\rm loc}(p)$ along the leaves of $\mathcal{F}^u_{\rm loc}(\Lambda;\delta)$. Let $L(\delta)$ be an arc in $U(L)$ containing $f_\delta^{N_2}(\widetilde{L}(\delta))$ and crossing $f_\delta^{-N_0}(\mathcal{F}^s_{\rm loc}(\Gamma_m;\delta))$ exactly, and let $\pi^u_\delta : E \to L(\delta)$ be the projection along the leaves of $f_\delta^{-N_0}(\mathcal{F}^s_{\rm loc}(\Gamma_m;\delta))$ such that π^u_0 is equal to π^u of (5.3). We also consider the bridge $A^u(\delta)$ and the Cantor set $K^u_m(\delta)$ in $\widetilde{L}(\delta)$ with $f_\delta^{N_2}(A^u(\delta)) = \pi^u_\delta(A^u) =: A^u_L(\delta)$ and $f_\delta^{N_2}(K^u_m(\delta)) = \pi^u_\delta(K^u_m) =: K^u_{m,L}(\delta)$. Then the situation is illustrated as follows.

$$\begin{array}{ccccc}
\widetilde{L}(\delta) & \xrightarrow{f_\delta^{N_2}|_{\widetilde{L}(\delta)}} & L(\delta) & \xleftarrow{\pi^u_\delta} & E \\
\cup & & \cup & & \cup \\
A^u(\delta), K^u_m(\delta) & \longrightarrow & A^u_L(\delta), K^u_{m,L}(\delta) & \longleftarrow & A^u, K^u_m
\end{array}$$

Let $\widetilde{\pi}^s_\delta : \widetilde{L} \to \widetilde{L}(\delta)$ be the composition of the δ-shift map $\boldsymbol{x} \mapsto \boldsymbol{x} + (\delta, 0)$ followed by the projection along the leaves of $\mathcal{F}^u_{\mathrm{loc}}(\Lambda;\delta)$, and let $\widetilde{\pi}^u_\delta : \widetilde{L} \to \widetilde{L}(\delta)$ be the projection along the leaves of $f_\delta^{-(N_0+N_2)}(\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m;\delta))$. From our construction of f_δ, $B^s(\delta) = \widetilde{\pi}^s_\delta(B^s(0))$ for any s-bridge $B^s(0)$ in $\widetilde{L} \cap \mathcal{B}_{\delta_0/2}$. Similarly, $\widetilde{\pi}^u_\delta(A^u(0)) = A^u(\delta)$ for any u-bridge $A^u(0)$ in $\widetilde{L} \cap \mathcal{B}_{\delta_0/2}$. Strictly, $\widetilde{\pi}^u_\delta(\widetilde{L})$ is not contained in $\widetilde{L}(\delta)$ when $\delta > 0$. But it is not a crucial problem since, in our arguments below, $\delta = \Delta_k, \Delta$ can be taken sufficiently small so that the $\widetilde{\pi}^u_\delta$-images of any bridges in $\widetilde{L}$ used later are contained in $\widetilde{L}(\delta)$.

Recall that $\varpi^u : S \to [0,2]$ with the identification $[0,2] = [0,2] \times \{0\}$ is the vertical projection used to parametrize $\widetilde{L}$ and L. For $\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}, \boldsymbol{y}'$ in $\widetilde{L}$, let $\boldsymbol{x}_\delta, \boldsymbol{x}'_\delta \in \widetilde{L}(\delta)$ be the $\widetilde{\pi}^s_\delta$-images of $\boldsymbol{x}$, $\boldsymbol{x}'$, while let $\boldsymbol{y}_\delta, \boldsymbol{y}'_\delta \in \widetilde{L}(\delta)$ be the $\widetilde{\pi}^u_\delta$-images of $\boldsymbol{y}, \boldsymbol{y}'$. Since $\widetilde{L}(\delta)$ C^1-converges to $\widetilde{L}$ as $\delta \to 0$, it follows from the property (F-iii) of compatible local foliations in Subsection 4.4 that there exists a constant $C > 0$ independent of δ and satisfying

(5.12)
$$\begin{aligned}
&(1 - C|\delta|)|\varpi^u(\boldsymbol{x}) - \varpi^u(\boldsymbol{x}')| \le |\varpi^u(\boldsymbol{x}_\delta) - \varpi^u(\boldsymbol{x}'_\delta)| \le (1 + C|\delta|)|\varpi^u(\boldsymbol{x}) - \varpi^u(\boldsymbol{x}')|,\\
&(1 - C|\delta|)|\varpi^u(\boldsymbol{y}) - \varpi^u(\boldsymbol{y}')| \le |\varpi^u(\boldsymbol{y}_\delta) - \varpi^u(\boldsymbol{y}'_\delta)| \le (1 + C|\delta|)|\varpi^u(\boldsymbol{y}) - \varpi^u(\boldsymbol{y}')|,\\
&(1 - C|\delta|)|\varpi^u(\boldsymbol{x}) + \delta - \varpi^u(\boldsymbol{y})| - O(\delta^2)\\
&\qquad\qquad \le |\varpi^u(\boldsymbol{x}_\delta) - \varpi^u(\boldsymbol{y}_\delta)| \le (1 + C|\delta|)|\varpi^u(\boldsymbol{x}) + \delta - \varpi^u(\boldsymbol{y})| + O(\delta^2).
\end{aligned}$$

Here we explain the reason why the third inequalities include the terms of $O(\delta^2)$. Let l_0, l_1 be the leaves of $\mathcal{F}^u_{\mathrm{loc}}(\Lambda;\delta)$ and $f_\delta^{-(N_0+N_2)}(\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m;\delta))$, respectively, passing through $\boldsymbol{x}_\delta$. Let $\boldsymbol{x}'_\delta$ be the intersection point of $\widetilde{L}$ and l_1. See Figure 5.6. From

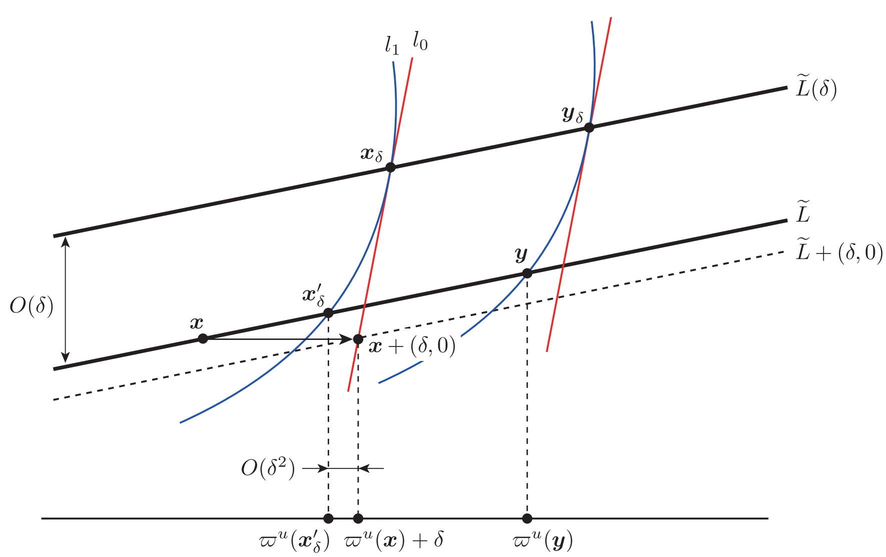

FIGURE 5.6. Explanation of the third inequalities of (5.12).

the second inequalities of (5.12), we have

$$(1 - C|\delta|)|\varpi^u(\boldsymbol{x}'_\delta) - \varpi^u(\boldsymbol{y})| \le |\varpi^u(\boldsymbol{x}_\delta) - \varpi^u(\boldsymbol{y}_\delta)| \le (1 + C|\delta|)|\varpi^u(\boldsymbol{x}'_\delta) - \varpi^u(\boldsymbol{y})|.$$

Since l_0 and l_1 have a quadratic tangency at $\boldsymbol{x}_\delta$, we have

$$|\varpi^u(\boldsymbol{x}+(0,\delta)) - \varpi^u(\boldsymbol{x}'_\delta)| = |\varpi^u(\boldsymbol{x}) + \delta - \varpi^u(\boldsymbol{x}'_\delta)| = O(\delta^2).$$

This shows the third inequalities of (5.12).

Lemma 5.2 (Linking Lemma)**.** *Suppose that the dynamically defined Cantor sets $K^s_{\Lambda,L}$ and $K^u_{m,L}$ satisfy (5.8). Let $B^s(0)$ be an s-bridge of $K^s_\Lambda(0)$ and $A^u(0)$ a u-bridge of $K^u_m(0)$ contained in $\mathcal{B}_{\delta_0/2}\cap\widetilde{L}$ such that the pair $(B^s(0), A^u(0))$ is linked. Moreover, suppose that the generations of $B^s(0)$ and $A^u(0)$ are so large that the conditions of Lemma 4.5 and Remark 5.1 hold. Then, for any ε with $0<\varepsilon\le|B^s(0)\cap A^u(0)|$ and sufficiently smaller than the radius δ_0 of $\mathcal{B}_{\delta_0}$, there exist bridges B^s_1, $\widetilde{B}^s_1$, A^u_1, $\widetilde{A}^u_1$ and gaps G^s, G^u in $\widetilde{L}$ with*

$$B^s_1, \widetilde{B}^s_1 \subset B^s(0),\ G^s = \mathrm{Gap}(B^s_1, \widetilde{B}^s_1),\ A^u_1, \widetilde{A}^u_1 \subset A^u(0),\ G^u = \mathrm{Gap}(A^u_1, \widetilde{A}^u_1)$$

and satisfying the following properties (1)–(2) *for any ν with $|\nu|<\varepsilon$ and* (3)–(4) *for some δ with $|\delta|<\varepsilon$:*

(1) $r_{s+}^{-1}\tau^{-5/4}\varepsilon \le |B^s_1(\nu)| < r_{s-}^{-1}\tau^{-3/4}\varepsilon,\quad r_{s+}^{-1}\tau^{-5/4}\varepsilon \le |\widetilde{B}^s_1(\nu)| < r_{s-}^{-1}\tau^{-3/4}\varepsilon$;
(2) *both $(B^s_1(\nu), A^u_1(\nu))$ and $(\widetilde{B}^s_1(\nu), \widetilde{A}^u_1(\nu))$ are τ^{-1}-proportional;*
(3) *both $(B^s_1(\delta), A^u_1(\delta))$ and $(\widetilde{B}^s_1(\delta), \widetilde{A}^u_1(\delta))$ are ξ_0-linked pairs, where ξ_0 is the constants given in (5.9);*
(4) *$G^u(\delta)$ and $G^s(\delta)$ have a common middle point.*

Note that ξ_0 of (5.9) is a universal constant independent of the choices of B^s, A^u or ε. Here we set B^s_1, $\widetilde{B}^s_1$, A^u_1, $\widetilde{A}^u_1$ for simplicity instead of $B^s_1(0)$, $\widetilde{B}^s_1(0)$, $A^u_1(0)$, $\widetilde{A}^u_1(0)$ respectively.

The following lemma is essential in the proof of Lemma 5.2.

Lemma 5.3. *Let $(B^s(0), A^u(0))$ be the linked pair given in Lemma 5.2. For any ε with $0<\varepsilon\le|B^s(0)\cap A^u(0)|$ and sufficiently smaller than δ_0, there exist an interval J_1 with $J_1\subset(-\varepsilon,\varepsilon)$, sub-bridges $\widehat{B}^s_1\subset B^s(0)$ and $\widehat{A}^u_1\subset A^u(0)$ satisfying the following conditions.*

(1) $\tau^{-5/4}\varepsilon \le |\widehat{B}^s_1(\nu)| < \tau^{-3/4}\varepsilon$ *for any ν with $|\nu|\le\varepsilon$;*
(2) $\tau^{-1/2}|\widehat{A}^u_1(\nu)| \le |\widehat{B}^s_1(\nu)| < \tau^{-1/4}|\widehat{A}^u_1(\nu)|$ *for any ν with $|\nu|\le\varepsilon$;*
(3) $\widehat{B}^s_1(\nu)\cap\widehat{A}^u_1(\nu)\neq\emptyset$ *if and only if $\nu\in J_1$.*

Proof. (1) First we consider the case of $\nu=0$. Since the pair $(B^s(0), A^u(0))$ is linked and $\tau(B^s(0)\cap K^s_\Lambda(0))\tau(A^u(0)\cap K^u_m(0))>1$ by (5.11), it follows from Gap Lemma that $(B^s(0)\cap K^s_\Lambda(0))\cap(A^u(0)\cap K^u_m(0))$ contains a point, say a_0. Take an s-bridge $B^s\langle i\rangle$ with $B^s\langle i\rangle\ni a_0$ and $|B^s\langle i\rangle|<\tau^{-3/4}\varepsilon$, where i represents the generation of $B^s\langle i\rangle$. If $|B^s\langle i\rangle|\ge\tau^{-5/3}\varepsilon$, then we set $\widehat{B}^s_1=B^s\langle i\rangle$. Otherwise, consider the s-bridge $B^s\langle i-1\rangle$ with $B^s\langle i-1\rangle\ni a_0$. Since $r_{s+}^2<\tau$ by (5.8), we have from Lemma 4.3 that

$$|B^s\langle i-1\rangle| < r_{s+}|B^s\langle i\rangle| < r_{s+}\tau^{-5/4}\varepsilon < \tau^{-3/4}\varepsilon \quad\text{and}\quad |B^s\langle i-1\rangle| \ge 2|B^s\langle i\rangle|.$$

If $|B^s\langle i-1\rangle|\ge\tau^{-5/4}\varepsilon$, then we set $\widehat{B}^s_1=B^s\langle i-1\rangle$. Otherwise, we repeat the same process until we get the s-bridge containing a_0 and satisfying the inequality of (1). We adopt the bridge as $\widehat{B}^s_1$. This shows (1) for the case of $\nu=0$. From (5.12), we know that it is not hard to generalize this result to the case of $|\nu|\le\varepsilon$.

(2) First we consider the case of $\nu = 0$. We mean by $A^u\langle i\rangle$ that the generation of the u-bridge is i. Suppose that $A^u = A^u\langle j\rangle$. First we show that there exists a sub-bridge A_0^u of A^u with

$$\frac{\varepsilon}{3\tau^{1/4}} \le |A_0^u| \le \frac{\varepsilon}{3} \tag{5.13}$$

and contained in a closed sub-arc of A^u of width $\varepsilon/3$ and containing a_0. Here we do not necessarily require that A_0^u contains a_0. Let $A^u\langle j+1\rangle$ be the sub-bridge of A^u containing a_0. If $A^u\langle j+1\rangle \ge \varepsilon/3$, then we repeat the argument using $A^u\langle j+1\rangle$ instead of A^u. So it suffices to consider the case of $A^u\langle j+1\rangle < \varepsilon/3$. Suppose that I is a sub-arc of A^u with $|I| = \varepsilon/3$ and containing a_0 as a boundary point. If $A^u\langle j+1\rangle \ge \varepsilon/3\tau^{1/4}$, then one can set $A_0^u = A^u\langle j+1\rangle$. Otherwise, consider the maximum sub-arc I' of A^u with a_1 as a boundary point and containing I, where a_1 is the boundary point of I other than a_0, see Figure 5.7. Let $A_1^u\langle j+1\rangle$ be the sub-

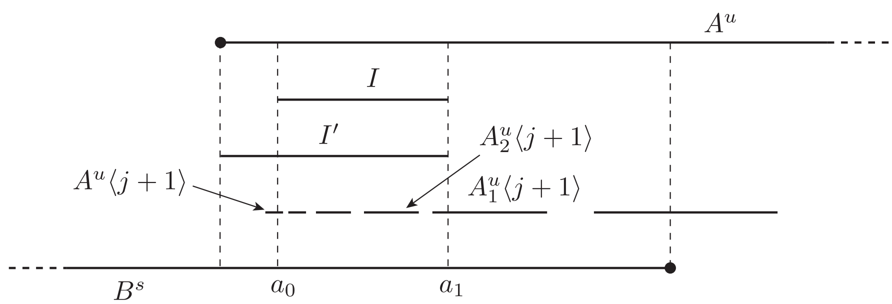

FIGURE 5.7. Detecting an unstable bridge of next generation.

bridge of A^u closest to a_0 among all u-bridges not contained in I'. By Lemma 4.5-(2) (together with Remark 5.1-(ii) for strictly), $|A_1^u\langle j+1\rangle| \ge |I'|/3 \ge |I|/3 = \varepsilon/9$. By Lemma 4.5-(3) and (5.8), the u-bridge $A_2^u\langle j+1\rangle$ closest to a_1 among all u-bridges contained in I' satisfies $|A_2^u\langle j+1\rangle| \ge |A_1^u\langle j+1\rangle|/3 > \varepsilon/27 > \varepsilon/3\tau^{1/4}$. Thus $A_0^u := A_2^u\langle j+1\rangle$ satisfies (5.13).

Consider the sequence of u-bridges

$$A_0^u = A^u\langle k\rangle \supset A^u\langle k+1\rangle \supset \cdots \supset A^u\langle k+i\rangle \supset A^u\langle k+i+1\rangle \supset \cdots$$

such that, for any integer $i \ge 0$, the bottom point of $A^u\langle k+i+1\rangle$ is equal to the leading point of $A^u\langle k+i\rangle$. By (5.13),

$$|A_0^u| \ge \frac{\varepsilon}{3\tau^{1/4}} = \tau^{1/4}\frac{\varepsilon}{3\tau^{1/2}} > \tau^{1/4}\frac{\varepsilon}{\tau^{3/4}} \ge \tau^{1/4}|\widehat{B}_1^s|.$$

Thus there exists $i \ge 0$ such that $|A^u\langle k+i+1\rangle| \le \tau^{1/4}|\widehat{B}_1^s| < |A^u\langle k+i\rangle|$. By Lemma 4.5-(1),

$$|\widehat{B}_1^s| \ge \tau^{-1/4}|A^u\langle k+i+1\rangle| \ge \tau^{-1/4}\frac{|A^u\langle k+i\rangle|}{5} > \tau^{-1/2}|A^u\langle k+i\rangle|.$$

Here we used the inequality $\tau > 3^8 > 5^4$ derived from (5.8). Thus $\widehat{A}_1^u := A^u\langle k+i\rangle$ satisfies the inequality of (2) for $\nu = 0$. Again by using (5.12), one can generalize this result to the case of $|\nu| \le \varepsilon$.

(3) Let ν be any number with $|\nu| \le \varepsilon$. Since $\widehat{A}_1^u \subset I$, $\widehat{A}_1^u(\nu)$ is contained in $I(\nu)$. By (1), $|\widehat{B}_1^s(\nu)| < \varepsilon/3$. Let $\boldsymbol{x}_B$ be the left edge of $\widehat{B}_1^s$ and $\boldsymbol{x}_I$ the right

edge of I. Since $|\widehat{B}_1^s| < \varepsilon/3$ and I is an arc of length $\varepsilon/3$ with $I \cap \widehat{B}_1^s \neq \emptyset$, $\varpi^u(\boldsymbol{x}_B) - \varpi^u(\boldsymbol{x}_I) > -2\varepsilon/3$. Let $\boldsymbol{x}_B(\nu)$, $\boldsymbol{x}_I(\nu)$ be the points of $\widetilde{L}(\nu)$ corresponding to $\boldsymbol{x}_B$, $\boldsymbol{x}_I$ respectively. By (5.12),

$$\varpi^u(\boldsymbol{x}_B(\nu)) - \varpi^u(\boldsymbol{x}_I(\nu)) > \nu - \frac{2\varepsilon}{3}(1 + C\nu) - O(\nu^2)$$

for any $0 \leq \nu \leq \varepsilon$. This implies that, if the right hand side of this inequality is positive or equivalently

$$\nu > \frac{2\varepsilon}{3 - 2\varepsilon C + 3O(\nu)}$$

by regarding $O(\nu^2) = \nu O(\nu)$, then $\widehat{B}_1^s(\nu)$ lies in the right component of $\widetilde{L}(\nu) \setminus I(\nu)$. One can choose $\varepsilon > 0$ so small that the right hand side of the preceding inequality is smaller than ε. Similarly, if $\nu < -2\varepsilon/(3 - 2\varepsilon C + 3O(\nu))$, then $\widehat{B}_1^s(\nu)$ lies in the left component of $\widetilde{L}(\nu) \setminus I(\nu)$. Thus the interval J_1 satisfying the condition (3) is contained in $(-\varepsilon, \varepsilon)$. This completes the proof. □

Proof of (1) and (2) of Lemma 5.2. To show (1), we will present a procedure how to define our desired sub-bridges and gaps. Suppose that the generations of $\widehat{A}_1^u(\nu)$ and $\widehat{B}_1^s(\nu)$ given in Lemma 5.3 are k and l, respectively. Let A_1^u, $\widetilde{A}_1^u$ be sub-bridges of $\widehat{A}_1^u$ of generation $k+1$ with the connecting gap G^u and such that one of A_1^u and $\widetilde{A}_1^u$ contains the leading point of $\widehat{A}_1^u$, that is, G^u is the leading gap of $\widehat{A}_1^u$. Let B_1^s and $\widetilde{B}_1^s$ be sub-bridges of $\widehat{B}_1^s$ of generation $l+1$ with the connecting gap G^s. We may assume that B_1^s and A_1^u lie in the left sides of G_1^s and G_1^u respectively if necessary exchanging notations. By Lemmas 4.3 and 5.3-(1), for any ν with $|\nu| < \varepsilon$,

$$r_{s+}^{-1}\tau^{-5/4}\varepsilon \leq |B_1^s(\nu)| < r_{s-}^{-1}\tau^{-3/4}\varepsilon.$$

The inequality concerning $|\widetilde{B}_1^s(\nu)|$ is proved in the same manner. This shows (1).

(2) By Lemmas 4.3, 4.5 and 5.3-(2), for any ν with $|\nu| < \varepsilon$,

$$|A_1^u(\nu)| \geq \frac{|\widehat{A}_1^u(\nu)|}{5} \geq \frac{\tau^{1/4}|\widehat{B}_1^s(\nu)|}{5} \geq \frac{\tau^{1/4} r_{s-}|B_1^s(\nu)|}{5} \geq |B_1^s(\nu)|,$$
$$|B_1^s(\nu)| \geq r_{s+}^{-1}|\widehat{B}_1^s(\nu)| \geq r_{s+}^{-1}\tau^{-1/2}|\widehat{A}_1^u(\nu)| \geq r_{s+}^{-1}\tau^{-1/2}|A_1^u(\nu)| \geq \tau^{-1}|A_1^u(\nu)|.$$

This shows that $(B_1^s(\nu), A_1^u(\nu))$ is τ^{-1}-proportional. The τ^{-1}-proportionality of $(\widetilde{B}_1^s(\nu), \widetilde{A}_1^u(\nu))$ is shown quite similarly. This proves (2). □

We need the following inequality in the proof of (3):

$$\tau^{-1/2} \geq \frac{|G^u(\nu)|}{|\widehat{B}_1^s(\nu)|} \qquad (|\nu| < \varepsilon). \tag{5.14}$$

In fact, by Lemma 5.3-(2), $|\widehat{B}_1^s(\nu)| \geq \tau^{-1/2}|\widehat{A}_1^u(\nu)| \geq \tau^{-1/2}|A_1^u(\nu)|$. From the definition of thickness, $\tau \leq |A_1^u(\nu)|/|G^u(\nu)|$. It follows that

$$\tau^{-1/2}|A_1^u(\nu)| > \tau^{-1/2}\tau|G^u(\nu)| = \tau^{1/2}|G^u(\nu)|.$$

Hence (5.14) holds.

Proof of (3) and (4) of Lemma 5.2. Since $G^s(\delta) \subset \widehat{B}_1^s(\delta)$ and $G^u(\delta) \subset \widehat{A}_1^u(\delta)$, there is a $\delta \in J_1$ such that the middle point of $G^s(\delta)$ is equal to that of $G^u(\delta)$. Again by Lemmas 4.5-(1) and 5.3-(2), $|A_1^u(\delta)| \geq |\widehat{A}_1^u(\delta)|/5 \geq \tau^{1/4}|\widehat{B}_1^s(\delta)|/5 > |\widehat{B}_1^s(\delta)|$. Similarly $|\widetilde{A}_1^u(\delta)| \geq |\widehat{B}_1^s(\delta)|$. Thus we have $\mathrm{Int}\widehat{A}_1^u(\delta) \supset \widehat{B}_1(\delta)$. By (2), $|A_1^u(\delta)| \geq |B_1^s(\delta)|$.

By Lemma 4.3 and (5.14), $|B_1^s(\delta)| \geq r_{s+}^{-1}|\widehat{B}_1^s(\delta)| \geq r_{s+}^{-1}\tau^{1/2}|G^u(\delta)| \geq |G^u(\delta)|$. This implies that $B_1^s(\delta)$ is not contained in $G^u(\delta)$.

To show that the pair $(B_1^s(\delta), A_1^u(\delta))$ is ξ_0-linked, we need to consider the two cases of (a) $G^s(\delta) \subsetneq G^u(\delta)$ and (b) $G^s(\delta) \supset G^u(\delta)$, see Figure 5.8.

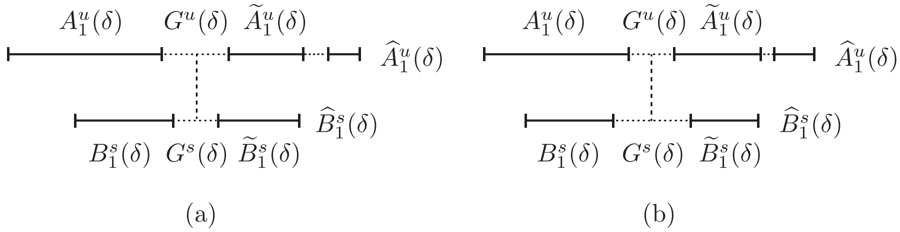

FIGURE 5.8. Stable and unstable gaps with the same middle point and bridges near the gaps.

First we consider the case (a). One of the boundary points of $B_1^s(\delta)$ is contained in $A_1^u(\delta)$ and the other is contained in $G^u(\delta)$. It follows that $B_1^s(\delta) \cap A_1^u(\delta) \neq \emptyset$, $B_1^s(\delta)$ is not contained in a gap of $A_1^u(\delta) \cap K_m^u(\delta)$ and $A_1^u(\delta)$ is not contained in a gap of $B_1^s(\delta) \cap K_\Lambda^s(\delta)$. This implies that $(B_1^s(\delta), A_1^u(\delta))$ is a linked pair. By Lemmas 5.3-(2) and 4.3,

$$|B_1^s(\delta) \cap A_1^u(\delta)| = |B_1^s(\delta)| + \frac{|G^s(\delta)|}{2} - \frac{|G^u(\delta)|}{2} > |B_1^s(\delta)| - \frac{|G^u(\delta)|}{2} \geq \frac{|\widehat{B}_1^s(\delta)|}{r_{s+}} - \frac{|G^u(\delta)|}{2}.$$

Since $\min\{|B_1^s(\delta)|, |A_1^u(\delta)|\} = |B_1^s(\delta)|$, we have by (5.9)

$$\frac{|B_1^s(\delta) \cap A_1^u(\delta)|}{\min\{|B_1^s(\delta)|, |A_1^u(\delta)|\}} = \frac{|B_1^s(\delta) \cap A_1^u(\delta)|}{|B_1^s(\delta)|} \geq \left(\frac{|\widehat{B}_1^s(\delta)|}{r_{s+}} - \frac{|G^u(\delta)|}{2}\right)\frac{r_{s-}}{|\widehat{B}_1^s(\delta)|}$$
$$\geq \left(\frac{1}{r_{s+}} - \frac{1}{2\tau^{1/2}}\right) r_{s-} = \xi_0.$$

In the case (b), it is immediately seen that $B_1^s(\delta) \cap A_1^u(\delta) \neq \emptyset$ and $A_1^u(\delta)$ is not contained in a gap of $B_1^s(\delta) \cap K_\Lambda^s(\delta)$. Moreover, we will show that $B_1^s(\delta)$ is not contained in a gap of $A_1^u(\delta) \cap K_m^u(\delta)$ by contradiction. Suppose that there would exist a gap $G_1^u(\delta)$ of $A_1^u(\delta) \cap K_m^u(\delta)$ with $G_1^u(\delta) \supset B_1^s(\delta)$. This implies that there is a u-bridge $A_*^u(\delta)$ which is adjacent to $G_1^u(\delta)$ and contained in $G^s(\delta)$. Thus,

$$\frac{|A_*^u(\delta)|}{|G^s(\delta)|}\frac{|B_1^s(\delta)|}{|G_1^u(\delta)|} = \frac{|A_*^u(\delta)|}{|G^s(\delta)|}\frac{|B_1^s(\delta)|}{|G_1^u(\delta)|} < 1.$$

On the other hand, we have from (5.11)

$$\frac{|A_*^u(\delta)|}{|G_1^u(\delta)|}\frac{|B_1^s(\delta)|}{|G^s(\delta)|} > \tau(B^s(\delta) \cap K_\Lambda^s(\delta))\tau(A^u(\delta) \cap K_m^u(\delta)) > 1.$$

This is a contradiction. Hence, we conclude that $B_1^s(\delta)$ is not contained in a gap of $A_1^u(\delta) \cap K_m^u(\delta)$. Since $B_1^s(\delta) \subset A_1^u(\delta)$, one has

$$\frac{|B_1^s(\delta) \cap A_1^u(\delta)|}{\min\{|B_1^s(\delta)|, |A_1^u(\delta)|\}} = \frac{|B_1^s(\delta)|}{|B_1^s(\delta)|} = 1 > \xi_0.$$

This completes the proof of (3). □

6. Linear growth property of linked pairs

As in the preceding section, any s-bridge $B^s(\delta)$ here means a bridge with respect to $K^s_\Lambda(\delta)$ and any u-bridge $A^u(\delta)$ means a bridge with respect to $K^u_m(\delta)$ for any δ with $|\delta| < \varepsilon$. The main result of this section is as follows:

Lemma 6.1 (Linear Growth Lemma). *Let ξ_0 be the constant of (5.9) and let $(B^s(0), A^u(0))$ be a linked pair in $\widetilde{L} \cap \mathcal{B}_{\delta_0/2}$. For any $0 < \varepsilon < |B^s(0) \cap A^u(0)|$, there exist a constant Δ with $|\Delta| < \varepsilon\tau^{-3/4}/2$, collections of sub-bridges $\{B^s_k\}_{k\geq 1}$ of $B^s(0)$ and $\{A^u_k\}_{k\geq 1}$ of $A^u(0)$, positive integers N_s and N_u independent of ε which satisfy the following* (1)–(3) *for every $k \geq 1$.*

(1) *$(B^s_k(\Delta), A^u_k(\Delta))$ is a u-dominating $\xi_0/2$-linked pair.*

(2) *For the union*

$$I_k = A^u_k(\Delta) \cup B^s_k(\Delta), \tag{6.1}$$

which is an arc in $\widetilde{L}(\Delta)$, there exists a positive constant α_0 independent of k such that, for any integer $l > k$, the $\alpha_0|A^u_k(\Delta)|$-neighborhood of I_k in $\widetilde{L}(\Delta)$ is disjoint from the $\alpha_0|A^u_l(\Delta)|$-neighborhood of I_l in $\widetilde{L}(\Delta)$.

(3) *If n_k and i_k are generations of B^s_k and A^u_k, respectively, then*

$$n_k < n_{k+1} \leq n_k + N_s, \quad i_k < i_{k+1} \leq i_k + N_u.$$

Here it is crucial that Δ is an arbitrarily small constant independent of k. Lemma 6.1 follows immediately from the next technical lemma.

Lemma 6.2. *Under the assumptions same as in Lemma 6.1, there exist sequences*

- *$\{n_k\}_{k\geq 1}$, $\{i_k\}_{k\geq 1}$ of positive integers;*
- *$\{\delta_k\}_{k\geq 1}$ of real numbers with*

$$|\delta_k| \leq 2^{-1}\xi_0\tau^{-3/4}\varepsilon r_{s-}^{-k};$$

- *$\{B^s_k\}_{k\geq 1}$, $\{\widetilde{B}^s_k\}_{k\geq 1}$ of s-bridges of generation n_k with $B^s_k, \widetilde{B}^s_k \subset \widetilde{B}^s_{k-1}$, $\widetilde{B}^s_0 = B^s(0)$ which have the connecting gaps $G^s_k = \mathrm{Gap}(B^s_k, \widetilde{B}^s_k)$;*
- *$\{A^u_k\}_{k\geq 1}$, $\{\widetilde{A}^u_k\}_{k\geq 1}$ of u-bridges of generation i_k with $A^u_k, \widetilde{A}^u_k \subset \widetilde{A}^u_{k-1}$, $\widetilde{A}^u_0 = A^u(0)$ which have the connecting gaps $G^u_k = \mathrm{Gap}(A^u_k, \widetilde{A}^u_k)$;*

satisfying the following (1)–(3) *for each $k \geq 1$.*

(1) *For any $t = 1, \ldots, k$ and the positive number ξ_{k-t} defined as (6.2), both $(B^s_t(\Delta_k), A^u_t(\Delta_k))$ and $(\widetilde{B}^s_t(\Delta_k), \widetilde{A}^u_t(\Delta_k))$ are u-dominating ξ_{k-t}-linked pairs, where $\Delta_k = \delta_1 + \cdots + \delta_k$.*

(2) *$G^u_k(\Delta_k)$ and $G^s_k(\Delta_k)$ have a common middle point.*

(3) *There exist integers $1 \leq N_s$, $N_u < \infty$ independent of k such that*

$$n_k < n_{k+1} \leq n_k + N_s, \quad i_k < i_{k+1} \leq i_k + N_u.$$

Moreover, $\Delta := \sum_{k=1}^{\infty} \delta_k$ is an absolutely convergent series with $\Delta_* := \sum_{k=1}^{\infty} |\delta_k| < \varepsilon\tau^{-3/4}/2$.

Let $\{n_k\}_{k=1}^{\infty}$ be the strictly increasing sequence of generations given in Lemma 6.2 and $n_0 = 0$. For any integer $k \geq 1$, let

$$\xi_k := \xi_0\Big(1 - \frac{1}{2}\sum_{i=1}^{k} r_{s-}^{-\widetilde{n}_i}\Big), \tag{6.2}$$

where $\{\widetilde{n}_i\}_{i=1}^{\infty}$ is the sequence defined by

$$\widetilde{n}_i = \inf\{n_{i+l} - n_l\,;\, l = 0, 1, 2, \dots\}.$$

Since $n_{l+1} \geq n_l + 1$, we have $\widetilde{n}_i \geq i$ for any $i \geq 1$. The inequality $r_{s-} > 2$ of Lemma 4.3 implies

$$\xi_k \geq \xi_0\Big(1 - \frac{1}{2}\sum_{i=1}^{k} r_{s-}^{-i}\Big) \geq \xi_0\Big(1 - \frac{1}{2}\frac{1}{r_{s-} - 1}\Big) \geq \xi_0\Big(1 - \frac{1}{2}\Big) = \frac{\xi_0}{2} > 0. \tag{6.3}$$

Proof of Lemma 6.2. Applying Lemma 5.2 to the linked pair $(B^s(0), A^u(0))$, we obtain a constant δ_1 with $|\delta_1| < \varepsilon$, sub-bridges $B_1^s, \widetilde{B}_1^s$ of $B^s(0)$ and $A_1^u, \widetilde{A}_1^u$ of $A^u(0)$ such that $(B_1^s(\delta_1), A_1^u(\delta_1))$ and $(\widetilde{B}_1^s(\delta_1), \widetilde{A}_1^u(\delta_1))$ are u-dominating ξ_0-linked pairs and the connecting gaps $G_1^s(\delta_1)$ and $G_1^u(\delta_1)$ have a common middle point. Let a be the constant defined as

$$a = \max\left\{2,\ \frac{4\xi_0(1 + C\varepsilon)}{r_{s+}(1 - r_{s-}^{-1})(1 - 2r_{s-}^{-1})}\right\}, \tag{6.4}$$

which will be used later to prove Lemma 6.1-(2). Again applying Lemma 5.2 to the linked pair $(\widetilde{B}_1^s(\delta_1), \widetilde{A}_1^u(\delta_1))$ for $\varepsilon_1 := \xi_0|\widetilde{B}_1^s|/2ar_{s+}$ instead of ε, we obtain a constant δ_2 with $|\delta_2| \leq \varepsilon_1$, sub-bridges $B_2^s(\delta_1), \widetilde{B}_2^s(\delta_1)$ of $\widetilde{B}_1^s(\delta_1)$ and $A_2^u(\delta_1), \widetilde{A}_2^u(\delta_1)$ of $\widetilde{A}_1^u(\delta_1)$ such that $(B_2^s(\Delta_2), A_2^u(\Delta_2))$ and $(\widetilde{B}_2^s(\Delta_2), \widetilde{A}_2^u(\Delta_2))$ are u-dominating ξ_0-linked pairs and the connecting gaps $G_2^s(\Delta_2)$ and $G_2^u(\Delta_2)$ have a common middle point. Similarly, for any $k \geq 2$, there exist a constant δ_k with $|\delta_k| \leq \varepsilon_{k-1} := \xi_0|\widetilde{B}_{k-1}^s|/2ar_{s+}$, sub-bridges $B_k^s(\Delta_{k-1}), \widetilde{B}_k^s(\Delta_{k-1})$ of $\widetilde{B}_{k-1}^s(\Delta_{k-1})$ and $A_k^u(\Delta_{k-1}), \widetilde{A}_k^u(\Delta_{k-1})$ of $\widetilde{A}_{k-1}^u(\Delta_{k-1})$ such that $(B_k^s(\Delta_k), A_k^u(\Delta_k))$ and $(\widetilde{B}_k^s(\Delta_k), \widetilde{A}_k^u(\Delta_k))$ are u-dominating ξ_0-linked pairs and the connecting gaps $G_k^s(\Delta_k)$ and $G_k^u(\Delta_k)$ have a common middle point. This shows (2).

Now we will show that, for any $k \geq 1$, $(B_t^s(\Delta_k), A_t^u(\Delta_k))$ is a u-dominating ξ_{k-t}-linked pair for each $t = 1, \dots, k$. Suppose that the assertion holds until the k-th step and consider the $(k+1)$-st step. When $t = k+1$, the proof is already done. So we may suppose that $t \leq k$. Since $(B_t^s(\Delta_k), A_t^u(\Delta_k))$ is a u-dominating ξ_{k-t}-linked pair,

$$|B_t^s(\Delta_k) \cap A_t^u(\Delta_k)| \geq \xi_{k-t}|B_t^s(\Delta_k)|.$$

By this inequality together with (5.12),

$$\begin{aligned}
|B_t^s(\Delta_{k+1}) \cap A_t^u(\Delta_{k+1})| &\geq (1 - C|\delta_{k+1}|\,)\big(\,|B_t^s(\Delta_k) \cap A_t^u(\Delta_k)| - |\delta_{k+1}|\,\big) - O(\delta_{k+1}^2)\\
&\geq (1 - C|\delta_{k+1}|\,)\big(\,\xi_{k-t}|B_t^s(\Delta_k)| - |\delta_{k+1}|\,\big) - O(\delta_{k+1}^2)\\
&\geq (1 - C|\delta_{k+1}|\,)\left(\xi_{k-t}\frac{|B_t^s(\Delta_{k+1})|}{1 + C|\delta_{k+1}|} - |\delta_{k+1}|\right) - O(\delta_{k+1}^2)\\
&= \xi_{k-t}|B_t^s(\Delta_{k+1})| - \left(1 + \frac{2C\xi_{k-t}|B_t^s(\Delta_{k+1})|}{1 + C|\delta_{k+1}|} + O(\delta_{k+1})\right)|\delta_{k+1}|.
\end{aligned}$$

Since $|B_t^s(\Delta_{k+1})| < \varepsilon$ by Lemma 5.2 and $\xi_{k-t} < 1$ by (6.2), one can choose $\varepsilon > 0$ so that the contribution of the last parenthesis is smaller than two. Then

$$\begin{aligned}
|B_t^s(\Delta_{k+1}) \cap A_t^u(\Delta_{k+1})| &\geq \xi_{k-t}|B_t^s(\Delta_{k+1})| - 2|\delta_{k+1}|\\
&= (\xi_{k-t} - 2|\delta_{k+1}|\,|B_t^s(\Delta_{k+1})|^{-1})|B_t^s(\Delta_{k+1})|.
\end{aligned}$$

Let n_k be the generation of $\widetilde{B}^s_k$. By Lemma 4.3, $|\widetilde{B}^s_k(\Delta_{k+1})| \le r_{s-}^{-(n_k-n_{t-1})}|\widetilde{B}^s_{t-1}(\Delta_{k+1})|$ and $|\widetilde{B}^s_{t-1}(\Delta_{k+1})| \le r_{s+}|B^s_t(\Delta_{k+1})|$. Since $a \ge 2$ by (6.4),

$$\begin{aligned}
2|\delta_{k+1}|\,|B^s_t(\Delta_{k+1})|^{-1} &\le 2\frac{\xi_0|\widetilde{B}^s_k(\Delta_{k+1})|}{2ar_{s+}}|B^s_t(\Delta_{k+1})|^{-1}\\
&\le \frac{\xi_0 r_{s-}^{-(n_k-n_{t-1})}|\widetilde{B}^s_{t-1}(\Delta_{k+1})|}{ar_{s+}}r_{s+}|\widetilde{B}^s_{t-1}(\Delta_{k+1})|^{-1}\\
&\le \frac{\xi_0 r_{s-}^{-(n_k-n_{t-1})}}{2}.
\end{aligned}$$

Then

$$\begin{aligned}
\xi_{k-t} - 2|\delta_{k+1}|\,|B^s_t(\Delta_{k+1})|^{-1} &\ge \xi_0\Big(1-\frac{1}{2}\sum_{i=1}^{k-t} r_{s-}^{-\widetilde{n}_i}\Big) - \xi_0\frac{r_{s-}^{-(n_k-n_{t-1})}}{2}\\
&= \xi_0\Big(1-\frac{1}{2}\Big(\sum_{i=1}^{k-t} r_{s-}^{-\widetilde{n}_i} + r_{s-}^{-(n_k-n_{t-1})}\Big)\Big)\\
&\ge \xi_0\Big(1-\frac{1}{2}\Big(\sum_{i=1}^{k-t} r_{s-}^{-\widetilde{n}_i} + r_{s-}^{-\widetilde{n}_{k+1-t}}\Big)\Big) = \xi_{k+1-t}.
\end{aligned}$$

Since $\xi_{k+1-t} \ge \xi_0/2$ by (6.3), it follows that $(B^s_t(\Delta_{k+1}), A^u_t(\Delta_{k+1}))$ is a $\xi_0/2$-linked pair. This shows (1).

By Lemma 5.2-(1), the length of $\widetilde{B}^s_{k+1}$ is evaluated as follows:

$$|\widetilde{B}^s_{k+1}| \ge r_{s+}^{-1}\tau^{-5/4}\varepsilon_k \ge r_{s+}^{-1}\tau^{-5/4}\frac{\xi_0|\widetilde{B}^s_k|}{2r_{s+}} = 2^{-1}r_{s+}^{-2}\tau^{-5/4}\xi_0|\widetilde{B}^s_k|. \tag{6.5}$$

Since the generation of $\widetilde{B}^s_k$ is n_k, by Lemma 4.3,

$$r_{s-}^{-(n_{k+1}-n_k)} \ge |\widetilde{B}^s_{k+1}||\widetilde{B}^s_k|^{-1}.$$

This implies that

$$n_{k+1}-n_k \le \frac{\log(2^{-1}r_{s+}^{-2}\tau^{-5/4}\xi_0)}{\log(r_{s-}^{-1})}.$$

Thus, the maximum integer N_s not greater than the right hand side of this inequality satisfies $n_{k+1} \le n_k + N_s$ for any $k \ge 1$. It follows from (6.5) and the proportionality condition in Lemma 5.2-(2) that

$$\begin{aligned}
|\widetilde{A}^u_{k+1}| \ge |\widetilde{B}^s_{k+1}| &\ge 2^{-1}r_{s+}^{-2}\tau^{-5/4}\xi_0|\widetilde{B}^s_k| \ge 2^{-1}r_{s+}^{-2}\tau^{-5/4}\xi_0\tau^{-1}|\widetilde{A}^u_k|\\
&= 2^{-1}r_{s+}^{-2}\tau^{-9/4}\xi_0|\widetilde{A}^u_k|.
\end{aligned}$$

We suppose that the generation of $\widetilde{A}^u_k$ is i_k. Since $|\widetilde{A}^u_{k+1}||\widetilde{A}^u_k|^{-1} \le \big(\frac{2}{3}\big)^{i_{k+1}-i_k}$ by Lemma 4.5-(1), one has a positive integer N_u independent of k and satisfying

$$i_{k+1}-i_k \le N_u \le \frac{\log(2^{-1}r_{s+}^{-2}\tau^{-9/4}\xi_0)}{\log\big(\frac{2}{3}\big)}$$

for any $k \ge 1$.

Since $a \ge 1$, $n_{k-1} - n_1 \ge (k-1) - 1 = k-2$ and $0 < \xi_0 < 1$, it follows from Lemmas 4.3 and 5.2-(1) that

$$|\delta_k| \le \frac{\xi_0 |\widetilde{B}^s_{k-1}(\Delta_{k-1})|}{2ar_{s+}} \le \frac{\xi_0}{2r_{s+}} r_{s-}^{-n_{k-1}+n_1}(r_{s-}^{-1}\tau^{-3/4}\varepsilon) \le \frac{\xi_0}{2r_{s+}} r_{s-}^{-k+2}(r_{s-}^{-1}\tau^{-3/4}\varepsilon)$$
$$= \frac{\xi_0}{2r_{s+}r_{s-}^{-1}} r_{s-}^{-k}\tau^{-3/4}\varepsilon < \frac{\varepsilon}{2} r_{s-}^{-k}\tau^{-3/4}.$$

This shows that

$$\sum_{k=1}^{\infty} |\delta_k| < \frac{\varepsilon}{2}\tau^{-3/4}\sum_{k=1}^{\infty} r_{s-}^{-k} = \frac{\varepsilon}{2}\tau^{-3/4}\frac{1}{r_{s-}-1} < \frac{\varepsilon}{2}\tau^{-3/4}.$$

In particular, $\Delta = \sum_{k=1}^{\infty}\delta_k$ is an absolutely convergent series with $\Delta_* = \sum_{k=1}^{\infty}|\delta_k| < \varepsilon\tau^{-3/4}/2$. This shows (3) and completes the proof. □

The proof of (1) and (3) of Lemma 6.1 is obtained immediately from Lemma 6.2. So it remains to prove (2).

Proof of (2) of Lemma 6.1. Since $A^u_k(\Delta) \cap B^s_k(\Delta) \ne \emptyset$, $I_k = A^u_k(\Delta) \cup B^s_k(\Delta)$ is an arc in $\widetilde{L}(\Delta)$. The union $\widehat{B}^s_k(\Delta) = B^s_k(\Delta) \cup G^s_k(\Delta) \cup \widetilde{B}^s_k(\Delta)$ is the smallest s-bridge containing $B^s_k(\Delta)$ and $\widetilde{B}^s_k(\Delta)$. By Lemma 4.3, $|B^s_k(\Delta)|$, $|\widetilde{B}^s_k(\Delta)| \le r_{s-}^{-1}|\widehat{B}^s_k(\Delta)|$. It follows that

$$|G^s_k(\Delta)| = |\widehat{B}^s_k(\Delta)| - (|B^s_k(\Delta)| + |\widetilde{B}^s_k(\Delta)|) \ge |\widehat{B}^s_k(\Delta)|(1-2r_{s-}^{-1}) \ge |\widetilde{B}^s_k(\Delta)|(1-2r_{s-}^{-1}).$$

Similarly we have $|G^s_k(\Delta)| \ge |B^s_k(\Delta)|(1-2r_{s-}^{-1})$. As in the proof of Lemma 6.2, for any integer $l \ge k+1$,

$$|\delta_l| < \frac{\xi_0|\widetilde{B}^s_{l-1}(\Delta_{l-1})|}{2ar_{s+}} \le \frac{(1+C\varepsilon)\xi_0|\widetilde{B}^s_{l-1}(\Delta)|}{2ar_{s+}} \le \frac{(1+C\varepsilon)\xi_0}{2ar_{s+}} r_{s-}^{-n_{l-1}+n_k}|\widetilde{B}^s_k(\Delta)|$$
$$\le \frac{(1+C\varepsilon)\xi_0}{2ar_{s+}} r_{s-}^{-(l-1-k)}|\widetilde{B}^s_k(\Delta)|.$$

Thus, by (6.4), we have

$$|\Delta - \Delta_k| \le \sum_{l=k+1}^{\infty}|\delta_l| \le \frac{(1+C\varepsilon)\xi_0|\widetilde{B}^s_k(\Delta)|}{2ar_{s+}}\frac{1}{1-r_{s-}^{-1}} \le \frac{|G^s_k(\Delta)|}{8}. \tag{6.6}$$

Define the constant α_0 as

$$\alpha_0 = \max\left\{\frac{1-2r_{s-}^{-1}}{8\tau}, \frac{1}{2^{m+3}}\right\}.$$

Then we have

$$\alpha_0|A^u_k(\Delta)| \le \alpha_0\tau|B^s_k(\Delta)| \le \alpha_0\frac{\tau}{1-2r_{s-}^{-1}}|G^s_k(\Delta)| = \frac{|G^s_k(\Delta)|}{8}. \tag{6.7}$$

Since $|A^u_k(\Delta)| \le 2^{m+1}|G^u_k(\Delta)|$ by Lemma 4.5-(3),

$$\alpha_0|A^u_k(\Delta)| \le \alpha_0 2^{m+1}|G^u_k(\Delta)| \le \frac{|G^u_k(\Delta)|}{4}. \tag{6.8}$$

We have chosen Δ_k so that $G^s_s(\Delta_k)$ and $G^u_k(\Delta_k)$ have the common middle point $\boldsymbol{x}_k$, see Figure 6.1. By (5.12) and (6.6), for any $\boldsymbol{x} \in B^s_k(\Delta_k)$ and all sufficiently

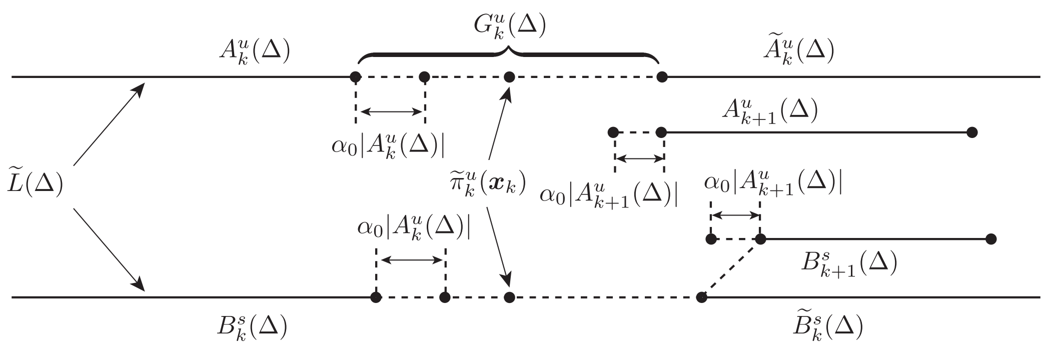

FIGURE 6.1. Detecting a sequence of $\xi_0/2$-proportional linked pairs.

large k,

$$\begin{aligned}|\varpi^u(\widetilde{\pi}_k^s(\boldsymbol{x})) - \varpi^u(\widetilde{\pi}_k^u(\boldsymbol{x}_k))| &\geq (1 - C|\Delta - \Delta_k|)|\varpi^u(\boldsymbol{x}) - \varpi^u(\boldsymbol{x}_k)| - |\Delta - \Delta_k| - O(|\Delta - \Delta_k|^2) \\ &\geq \frac{(1 - C|\Delta - \Delta_k|)|G_k^s(\Delta)|}{2} - |\Delta - \Delta_k| - O(|\Delta - \Delta_k|^2) \\ &\geq \frac{|G_k^s(\Delta)|}{4},\end{aligned}$$

where $\widetilde{\pi}_k^s : \widetilde{L}(\Delta_k) \to \widetilde{L}(\Delta)$ is the composition of the shift map $\boldsymbol{x} \mapsto \boldsymbol{x} + (\Delta - \Delta_k, 0)$ followed by the projection along the leaves of $\mathcal{F}^u_{\mathrm{loc}}(\Lambda; \Delta)$ and $\widetilde{\pi}_k^u : \widetilde{L}(\Delta_k) \to \widetilde{L}(\Delta)$ is the projection along the leaves of $f_\Delta^{-(N_0+N_2)}(\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m; \Delta))$. Thus, by (6.7), $\mathcal{N}(B_k^s(\Delta); \alpha_0|A_k^u|)$ does not contain $\widetilde{\pi}_k^u(\boldsymbol{x}_k)$, where $\mathcal{N}(J, \eta)$ denotes the η-neighborhood of J in $\widetilde{L}(\Delta)$ for $\eta > 0$ and a compact subset J of $\widetilde{L}(\Delta)$. Since $B_{k+1}(\Delta_k)$ is contained in $\widetilde{B}_k(\Delta_k)$ and $|A_{k+1}^u(\Delta)| \leq |A_k^u(\Delta)|$, one can show similarly that $\mathcal{N}(B_{k+1}^s(\Delta); \alpha_0|A_{k+1}^u(\Delta)|)$ does not contain $\widetilde{\pi}_k^u(\boldsymbol{x}_k)$. By (6.8), $\mathcal{N}(A_k^u(\Delta); \alpha_0|A_k^u(\Delta)|)$ also does not contain $\widetilde{\pi}_k^u(\boldsymbol{x}_k)$. Since $A_{k+1}^u(\Delta) \subset \widetilde{A}_k^u(\Delta)$ and $B_{k+1}^s(\Delta) \subset \widetilde{B}_k^s(\Delta)$, we have similarly $\mathcal{N}(A_{k+1}^u(\Delta); \alpha_0|A_{k+1}^u(\Delta)|) \not\ni \widetilde{\pi}_k^u(\boldsymbol{x}_k)$. This shows that

$$\mathcal{N}(I_k; \alpha_0|A_k^u(\Delta)|) \cap \mathcal{N}(I_{k+1}; \alpha_0|A_{k+1}^u(\Delta)|) = \emptyset. \tag{6.9}$$

Since $\widetilde{A}_{k+1}^u(\Delta) \subset \widetilde{A}_k^u(\Delta)$ and $\widetilde{B}_{k+1}^s(\Delta) \subset \widetilde{B}_k^s(\Delta)$, one can also prove that

$$\mathcal{N}(I_k; \alpha_0|A_k^u(\Delta)|) \cap \mathcal{N}(\widetilde{I}_{k+1}; \alpha_0|A_{k+1}^u(\Delta)|) = \emptyset,$$

where $\widetilde{I}_{k+1} = \widetilde{A}_{k+1}^u(\Delta) \cup \widetilde{B}_{k+1}^s(\Delta)$. From this fact together with $I_{k+2} \subset \widetilde{I}_{k+1}$, it follows that

$$\mathcal{N}(I_k; \alpha_0|A_k^u(\Delta)|) \cap \mathcal{N}(I_{k+2}; \alpha_0|A_{k+2}^u(\Delta)|) = \emptyset. \tag{6.10}$$

The assertion (2) is completed by applying an argument similar to that from (6.9) to (6.10) repeatedly. □

7. CRITICAL CHAINS

Since $|\Delta| < \varepsilon\tau^{-3/4}/2$ by Lemma 6.1, we may suppose that the perturbed diffeomorphism f_Δ given in Subsection 5.2 is arbitrarily C^r-close to the original diffeomorphism f. So we reset the notations and write $f_\Delta = f$, $\widetilde{L}(\Delta) = \widetilde{L}$, $L(\Delta) = L$, $\mathcal{F}^u_{\mathrm{loc}}(\Lambda; \Delta) = \mathcal{F}^u_{\mathrm{loc}}(\Lambda)$, $\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m; \Delta) = \mathcal{F}^s_{\mathrm{loc}}(\Gamma_m)$ and so on. Since $\widetilde{L}$

and L are parametrized so that $f^{N_2}|_{\widetilde{L}} : \widetilde{L} \to L$ is parameter-preserving, Linear Growth Lemma (Lemma 6.1) holds for the bridges $B^s_{L,k} = f^{N_2}(B^s_k(\Delta))$ and $A^u_{L,k} = f^{N_2}(A^u_k(\Delta))$ with respect to the Cantors set $K^s_{\Lambda,L} = f^{N_2}(K^s_\Lambda(\Delta))$ and $K^u_{m,L} = f^{N_2}(K^u_m(\Delta))$ in L.

7.1. **Encounter of s-bridges and u-bridges, II.** In Subsection 5.2, we have studied the heteroclinical connection between s-bridges $B^s(\Delta)$ of $K^s_\Lambda(\Delta)$ and u-bridges $A^u(\Delta)$ of $K^u_m(\Delta)$ in $\widetilde{L}$ for $\delta = \Delta$. To construct wandering domains, we also need to study the homoclinical connection between s-bridges B^{s*}_L of $K^s_{\Lambda,L}$ and u-bridges B^u_L of $K^u_{\Lambda,L}$ in L.

We write $\underline{1}^{(n)} := \underbrace{1\dots\dots 1}_{n}$, $\underline{2}^{(n)} := \underbrace{2\dots\dots 2}_{n}$ and prove the following key lemma.

Consider a positive integer z_0 independent of k and satisfying the conditions (8.4) and (8.5) which are given later. Let $\{z_k\}_{k=1}^\infty$ be any sequence of integers such that each entry z_k is either z_0 or $z_0 + 1$.

Lemma 7.1 (Critical Chain Lemma). *Let $(B^s_k(\Delta), A^u_k(\Delta))$ $(k = 0, 1, 2, \dots)$ be the sequence of the u-dominating $\xi_0/2$-linked pairs of bridges for $(K^s_\Lambda(\Delta), K^u_m(\Delta))$ given in Lemma 6.1. Then there exists a constant $T_0 > 0$ such that, for any $T \ge T_0$ and integers $k \ge 1$, there are*

- *a u-bridge $\widehat{A}^u_{L,k}$ of $K^u_{m,L}$ with $\widehat{A}^u_{L,k} \subset A^u_{L,k}$ and a u-bridge $B^u_{L,k}$ of $K^u_{\Lambda,L}$ contained in the leading gap of $\widehat{A}^u_{L,k}$,*
- *s-bridges $\widehat{B}^s_{L,k+1}$, $B^{s*}_{L,k+1}$ of $K^s_{\Lambda,L}$ with $B^{s*}_{L,k+1} \subset \widehat{B}^s_{L,k+1} \subset B^s_{L,k+1}$;*
- *positive constants C_1, C_2 independent of k*

satisfying the following conditions.

(1) *There exists an interval $J^*_{k+1} \subset (-\varepsilon_0 r_{s-}^{-T(k+1)}, \varepsilon_0 r_{s-}^{-T(k+1)})$ such that $(B^{s*}_{L,k+1} + t) \cap B^u_{L,k+1} \neq \emptyset$ if and only if $t \in J^*_{k+1}$, where $\varepsilon_0 = \xi_0 |B^s_0(\Delta)|/2$.*

(2) *Let $\widehat{A}^u_k$ be the bridge of K^u_m with $\pi^u(\widehat{A}^u_k) = \widehat{A}^u_{L,k}$ and $\mathbb{G}(\widehat{A}^u_k)$ the strip of the leading gap $G(\widehat{A}^u_k)$ of defined as in Subsection 5.1, where $\pi^u : E \to L$ is the projection of (5.3). See Figure 5.2. Suppose that $\widehat{i}_k$ is the minimum positive integer satisfying $f^{\widehat{i}_k}(\mathbb{G}(\widehat{A}^u_k)) \subset \mathbb{G}^u(0)$ and having $N_* + n_*$ as a divisor, where N_* and n_* are the integers given in Theorem 3.2 and Subsection 5.2, respectively. In other words, $\bar{i}_k = \widehat{i}_k/(N_* + n_*)$ is the minimum integer with $(\varphi^{\bar{i}_k}(\mathbb{G}(\widehat{A}^u_k)), \varphi^{\bar{i}_k}(\flat\mathbb{G}(\widehat{A}^u_k))) \subset (\mathbb{G}^u(0), \flat\mathbb{G}^u(0))$. Let $\widehat{n}_{k+1}$ be the generation of $\widehat{B}^s_{k+1}$. Then the inequality $\widehat{i}_k + \widehat{n}_{k+1} < C_1 T k + C_2$ holds.*

Moreover, if the itinerary of $\widehat{B}^s_{k+1}$ is $\widehat{\underline{w}}_{k+1}$, then one can suppose that $B^u_{L,k+1} = \pi_{\widehat{A}^u_k}(B^u_{k+1})$ and $B^{s*}_{L,k+1} = \pi^s(B^{s*}_{k+1})$ for the projections $\pi_{\widehat{A}^u_k} : B^u(0) \to L$ of (5.4) and $\pi^s : S \to L$ of (5.6) and for the bridges B^u_{k+1} of $K^u_\Lambda(\Delta)$ and B^{s*}_{k+1} of $K^s_\Lambda(\Delta)$ with the itineraries $\underline{1}^{(z_k k^2)} \underline{2}^{(k^2)} \underline{v}_{k+1} [\widehat{\underline{w}}_{k+1}]^{-1}$ and $\widehat{\underline{w}}_{k+1} [\underline{v}_{k+1}]^{-1} \underline{2}^{(k^2)} \underline{1}^{(z_k k^2)}$ respectively. That is,

$$B^u_k = B^u(z_k k^2 + \langle k \rangle; \underline{1}^{(z_k k^2)} \underline{2}^{(k^2)} \underline{v}_{k+1} [\widehat{\underline{w}}_{k+1}]^{-1}),$$
$$B^{s*}_{k+1} = B^s(z_k k^2 + \langle k \rangle; \widehat{\underline{w}}_{k+1} [\underline{v}_{k+1}]^{-1} \underline{2}^{(k^2)} \underline{1}^{(z_k k^2)}), \tag{7.1}$$

where

$$\langle k \rangle = \widehat{n}_{k+1} + k^2 + k \tag{7.2}$$

and $\underline{v}_{k+1}$ *is an arbitrarily chosen element of* $\{1,2\}^k$*. In other words,* $\underline{v}_{k+1}$ *is a non-specified itinerary of length* k*.*

See Figure 7.1 for the situation of Lemma 7.1, where $\widehat{G}^u_{L,k+1} = \pi^u(G(\widehat{A}^u_{k+1}))$. Figure 7.2 illustrates the transition from B^u_k to $B^u_{L,k}$ via $f^{-N_0} \circ f^{-\widehat{i}_k} \circ f^{-N_1}$ schematically.

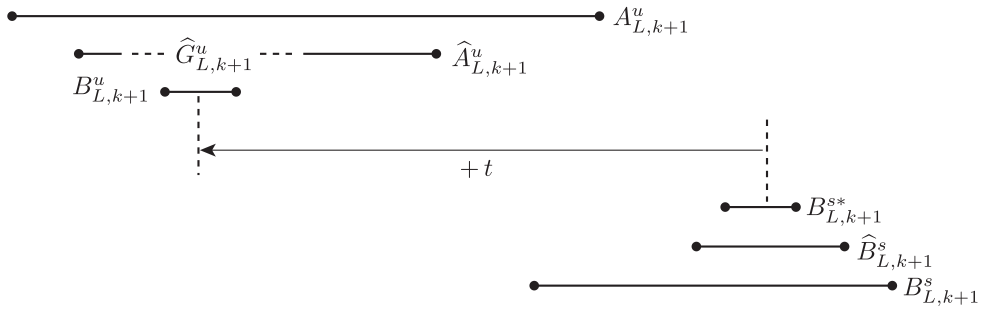

FIGURE 7.1. Sliding of $B^{s*}_{L,k+1}$ by $+t$.

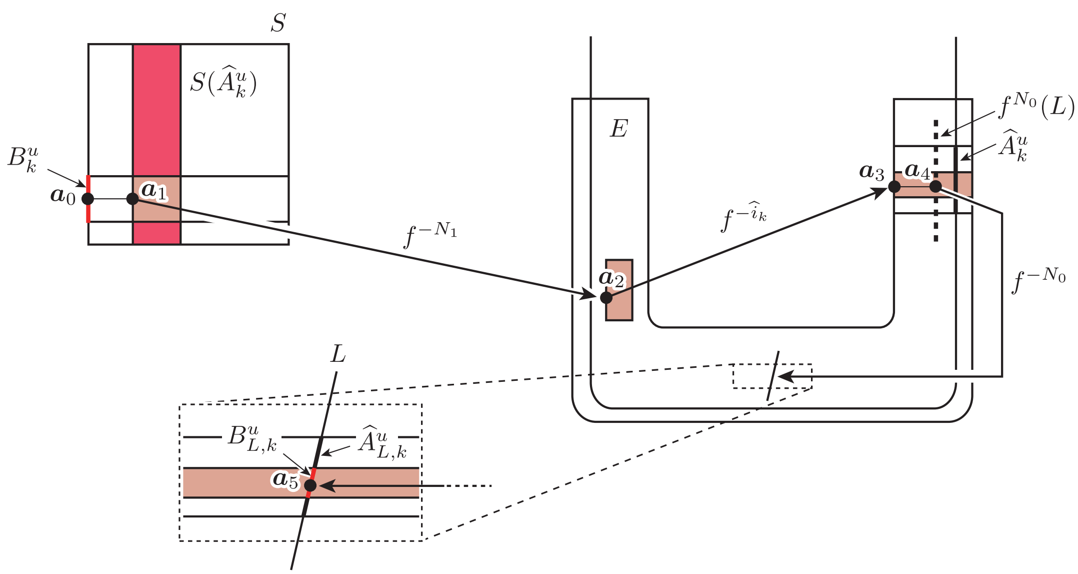

FIGURE 7.2. Backward transition from B^u_k to $B^u_{L,k}$. $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ lie on the same horizontal line and $\boldsymbol{a}_3$ and $\boldsymbol{a}_4$ also do.

Proof of Lemma 7.1. Let T_0 be the constant defines as

$$T_0 = \frac{N_s \log(r_{s+})}{\log(r_{s-})}.$$

Since $\varepsilon_0 = \xi_0 |B^s_0(\Delta)|/2$, by Lemmas 4.3 and 6.1-(2),

$$|B^s_{L,k} \cap A^u_{L,k}| \geq \frac{\xi_0 |B^s_k(\Delta)|}{2} \geq \frac{\xi_0 |B^s_0(\Delta)|}{2} r_{s+}^{-N_s k} = \varepsilon_0 r_{s-}^{-T_0 k}$$

for any integer $k \geq 1$. Here we fix an integer $T \geq T_0$. By Lemma 5.3, there exist an interval J_k with $J_k \subset (-\varepsilon_0 r_{s-}^{-Tk}, \varepsilon_0 r_{s-}^{-Tk})$, sub-bridges $\widehat{B}^s_{L,k} \subset B^s_{L,k}$ and $\widehat{A}^u_{L,k} \subset A^u_{L,k}$ satisfying the following conditions;

(a) $\tau^{-5/4}\varepsilon_0 r_{s-}^{-Tk} \leq |\widehat{B}^s_{L,k}| < \tau^{-3/4}\varepsilon_0 r_{s-}^{-Tk}$;

(b) $\tau^{-1/2}|\widehat{A}^u_{L,k}| \leq |\widehat{B}^s_{L,k}| < \tau^{-1/4}|\widehat{A}^u_{L,k}|$;

(c) $(\widehat{B}^s_{L,k} + t) \cap \widehat{A}^u_{L,k} \neq \emptyset$ if and only if $t \in J_k$.

Suppose that $\widehat{A}^u_{L,k}$ is of generation $i_k > 0$. By (5.2), we have the integer $\bar{i}_k$ with $(\varphi^{\bar{i}_k}(\mathbb{G}(A^u_k)), \varphi^{\bar{i}_k}(\flat\mathbb{G}(A^u_k))) \subset (\mathbb{G}^u(0), \flat\mathbb{G}^u(0))$ and $i_k \leq \bar{i}_k \leq (m-1)i_k$. We set $\widehat{i}_k = \bar{i}_k(N_* + i_*)$. Let B^{s*}_{k+1} and B^u_{k+1} be the bridges defined as (7.1). Since $B^{s*}_{L,k+1} = \pi^s(B^{s*}_{k+1}) \subset \widehat{B}^s_{L,k+1}$ and $B^u_{L,k+1} = \pi_{\widehat{A}^u_k}(B^u_{k+1}) \subset \widehat{A}^u_{L,k+1}$, by the condition (c), there is a sub-interval J^*_{k+1} of J_{k+1} such that $(B^{s*}_{L,k+1} + t) \cap B^u_{L,k+1} \neq \emptyset$ if and only if $t \in J^*_{k+1}$, see Figures 7.1 and 7.2. This completes the proof of (1).

Now we will show (2). By Lemma 4.3, one has

$$r_{s+}^{-\widehat{n}_k}|B^s_0(\Delta)| \leq |\widehat{B}^s_k(\Delta)| \leq r_{s-}^{-\widehat{n}_k}|B^s_0(\Delta)|,$$

and from (a)

$$\tau^{-5/4}\varepsilon_0 r_{s-}^{-Tk} \leq r_{s-}^{-\widehat{n}_k}|B^s_0(\Delta)|.$$

Hence, for every $l \geq 0$,

$$\widehat{n}_{k+l} \leq T(k+l) + \frac{\log(\tau^{5/4}\varepsilon_0^{-1}|B^s_0(\Delta)|)}{\log r_{s-}}. \tag{7.3}$$

It follows from (b) together with Lemma 4.5 that

$$\tau^{-5/4}\varepsilon_0 r_{s-}^{-Tk} \leq |\widehat{B}^s_{L,k}| < \tau^{-1/4}\Big(\frac{3}{2}\Big)^{-i_k}|A^u_{L,0}|.$$

By using the inequalities as above, we have

$$\begin{aligned}\log\Big(\frac{3}{2}\Big)i_k &\leq \log(r_{s-})Tk + \log(\tau\varepsilon_0^{-1}|A^u_{L,0}|) + \widehat{n}_k\log(r_{s+}/r_{s-})\\ &\leq \log(r_{s-})Tk + \log(\tau\varepsilon_0^{-1}|A^u_{L,0}|) + \left(Tk + \frac{\log(\tau^{5/4}\varepsilon_0^{-1}|B^s_0(\Delta)|)}{\log(r_{s-})}\right)\log(r_{s+}/r_{s-})\\ &= \log(r_{s+})Tk + C_3,\end{aligned}$$

where $C_3 = \log(\tau\varepsilon_0^{-1}|A^u_{L,0}|) + \dfrac{\log(\tau^{5/4}\varepsilon_0^{-1}|B^s_0(\Delta)|)\log(r_{s+}/r_{s-})}{\log(r_{s-})}$. Hence

$$\widehat{i}_k \leq (m-1)i_k(N_* + n_*) \leq \frac{\log(r_{s+}^{(N_*+n_*)(m-1)})}{\log\big(\frac{3}{2}\big)}Tk + \frac{C_3(N_*+n_*)(m-1)}{\log\big(\frac{3}{2}\big)}. \tag{7.4}$$

By (7.3) and (7.4), one can get positive constants C_1 and C_2 satisfying (2). □

7.2. **The second perturbation of f.** In Subsection 5.2, we perturbed f by performing the 'Δ-sliding' of f along the arc $\widetilde{L}$. The second perturbation presented here is 'switchback slidings' in neighborhoods of the brides $B^s_k(\Delta)$ $(k = 1, 2, \dots)$ which are done individually by using bump functions with mutually disjoint supports in S.

Consider bridges B^s in $[0,2] \times \{0\} \subset W^s_{\rm loc}(p)$ and B^u in $\{0\} \times [0,2] \subset W^u_{\rm loc}(p)$ associated with the Cantor sets K^s_Λ and K^u_Λ of (4.2), respectively. Set $\mathbb{B}^s = (\pi_{\mathcal{F}^u_{\rm loc}(\Lambda)})^{-1}(B^s)$ and $\mathbb{B}^u = (\pi_{\mathcal{F}^s_{\rm loc}(\Lambda)})^{-1}(B^u)$, where $\pi_{\mathcal{F}^u_{\rm loc}(\Lambda)}$ and $\pi_{\mathcal{F}^s_{\rm loc}(\Lambda)}$ are the projections given in Subsection 4.2. Note that $\mathbb{B}^s$ is a sub-strip of $(S,\sharp S)$ and $\mathbb{B}^u$ is a sub-strip of $(S,\flat S)$. We call that $\mathbb{B}^{u(s)}$ is the *bridge strip* for $B^{u(s)}$. From our definition of B_k^u and B_{k+1}^{s*} in (7.1), $f^{z_k k^2+\langle k\rangle}(\mathbb{B}_k^u)=\mathbb{B}_{k+1}^{s*}$, see Figure 7.3. The *gap strip* $\mathbb{G}^{u(s)}$ for a gap $G^{u(s)}$ of $K_\Lambda^{u(s)}$ is defined similarly.

Recall that the bridges $B^s(\Delta)$, $A^u(\Delta)$ in Lemma 6.1 are taken in $\widetilde{L}\cap\mathcal{B}_{\delta_0/2}$ so that our perturbation of the diffeomorphisms f does not affect the invariant set Γ_m and local stable foliation $\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m)$ on E.

Now we need to choose them more carefully. Let D_{n_*} be the rectangle used in Section 3 to define the basic set Γ_m, which satisfies the conditions (S-vi) and (S-vii). The dynamics of $\varphi=\varphi_{n_*}$ on D_{n_*} is determined by that of f on $X_{n_*}=D_{n_*}\cup f(D_{n_*})\cup\cdots\cup f^{N_*+n_*}(D_{n_*})$, where N_* and n_* are the integers given in Theorem 3.2 and Subsection 5.2, respectively. So our aim is accomplished by perturbing f in the complement of a neighborhood of X_{n_*}. By (S-vi) and Theorem 3.2, one can choose the parameter $\mu_*=\Theta_{n_*}(\bar\mu_*)$ and bridges $B^s(\Delta)$ of $K_\Lambda(\Delta)$ and $A^u(\Delta)$ of $K_m(\Delta)$ so that $f=f_{\mu_*}$ satisfies the conditions of Lemma 5.2 and the following extra conditions for a small constant $\alpha_1>0$. Recall that $\varpi^u:S\to[0,2]\times\{0\}\subset W^s_{\mathrm{loc}}(p)$ is the vertical projection with respect to the orthogonal coordinate on S, which is given in Subsection 5.2.

- (B-i) For the sub-arc $I=B^s(\Delta)\cup A^u(\Delta)$ of $\widetilde{L}$, the $\alpha_1|I|$-neighborhood $\widehat{\mathbb{I}}$ of the strip $\mathbb{I}=(\varpi^u)^{-1}\circ\varpi^u(I)$ in S is disjoint from X_{n_*}.
- (B-ii) For the sub-bridges $B^s_k(\Delta)$ of $B^s(\Delta)$ and $A^u_k(\Delta)$ of $A^u(\Delta)$ given in Lemma 6.1 and the sub-arc $I_k=B^s_k(\Delta)\cup A^u_k(\Delta)$ of (6.1), the strips $\widehat{\mathbb{I}}_k=(\varpi^u)^{-1}(\widehat{I}_k)$ $(k=1,2,\dots)$ in S are mutually disjoint, where $\widehat{I}_k$ is the $\alpha_1|\varpi^u(I_k)|$-neighborhood of $\varpi^u(I_k)$ in $[0,2]\times\{0\}$.

In fact, the assertion (B-i) is guaranteed by choosing the parameter $\bar\mu_*$ and the initial linked pair $B^s(0)$ and $A^u(0)$ of Lemma 5.2 suitably so that the sub-arc I is contained in an arbitrarily small neighborhood of the left component of $\flat S$ in S, which is represented by the shaded region in Figure 3.2. We refer to Subsection 6.5 in [33] (and also Subsection 5.3 in [24]) for such an argument. The assertion (B-ii) holds if we take α_1 sufficiently small comparing with α_0 of Lemma 6.1-(2).

We will define an auxiliary stable foliation on S for Lemma 7.2 below. Consider a C^1-foliation $\mathcal{G}^s(0)$ on $\mathbb{G}^u(0)$ such that $\widetilde{L}$ and any components of $\sharp\mathbb{G}^u(0)$ are leaves of $\mathcal{G}^s(0)$ and each leaf meets $\mathcal{F}^u_{\mathrm{loc}}(\Lambda)$ exactly. Then $\mathcal{G}^s(0)$ is uniquely extended to a local stable C^1-foliation $\mathcal{G}^s_{\mathrm{loc}}(\Lambda)$ on S compatible with $W^s_{\mathrm{loc}}(\Lambda)$.

Let $\widehat{A}^u_{L,k}$ and $B^u_{L,k}$ be the bridges of $K^u_{m,L}$ and $K^u_{\Lambda,L}$ given in Lemma 7.1, respectively. Note that $\widehat{A}^u_{L,k}$ contains $B^u_{L,k}$, see Figure 7.1. Consider the curves $\widetilde{L}_k$ in S and L_k in $U(q)$ defined by

$$\widetilde{L}_k=f^{-(z_k k^2+\langle k\rangle)}(\mathbb{B}^{s*}_{k+1}\cap\widetilde{L}),\quad L_k=f^{-N_0}\circ f^{-\widehat{i}_k}\circ f^{-N_1}(\widetilde{L}_k\cap S(\widehat{A}^u_k)),$$

see Figure 7.3, where $S(\widehat{A}^u_k)$ is the strip in S associated with $\widehat{A}^u_k$, see also Figure 5.2. Note that $\widetilde{L}_k$ is a leaf of the foliation $\mathcal{G}^s_{\mathrm{loc}}(\Lambda)$.

Suppose that l is a compact C^1-curve in S such that any line tangent to l is not vertical. For any $\boldsymbol{a}\in l$, let $\mathrm{slope}_{\boldsymbol{a}}(l)\ge 0$ be the (absolute) slope of l at $\boldsymbol{a}$ and define the maximum slope $\mathrm{slope}(l)$ of l by $\max\{\mathrm{slope}_{\boldsymbol{a}}(l)\,;\,\boldsymbol{a}\in l\}$. The slope is defined similarly for compact C^1-curves in $U(L)$ which do not have vertical tangent lines, where $U(L)$ is supposed to have the $C^{1+\alpha}$-coordinate introduced in Subsection 5.2.

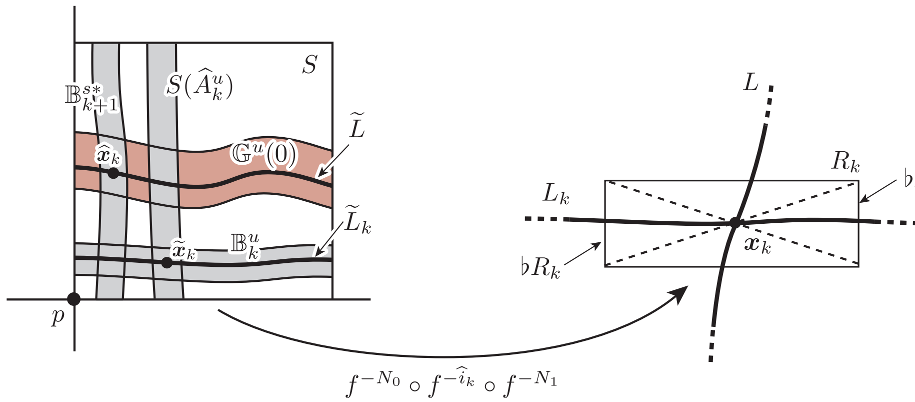

FIGURE 7.3. Backward transition from L to L_k via $\widetilde{L}$ and $\widetilde{L}_k$.

Lemma 7.2. *There exists a constant $\alpha > 1$ satisfying the following inequalities for any integer $k > 0$ and any leaf l of $\mathcal{G}^s_{\mathrm{loc}}(\Lambda)$ contained in $\mathbb{B}^u_k$.*

$$\mathrm{slope}(l) < \alpha(\sigma^{-1}\lambda)^{z_k k^2}, \quad \mathrm{slope}(L_k) < \alpha^k(\sigma^{-1}\lambda)^{z_k k^2}.$$

In particular, $\mathrm{slope}(\widetilde{L}_k) < \alpha(\sigma^{-1}\lambda)^{z_k k^2}$.

Proof. Since the left component $\{0\} \times [0,2]$ of $\flat S$ is a leaf of $\mathcal{F}^u_{\mathrm{loc}}(\Lambda)$, any leaf of $\mathcal{G}^s_{\mathrm{loc}}(\Lambda)$ meets $\{0\} \times [0,2]$ transversely. If necessary replacing δ by a smaller positive number, we may assume that, for any leaf l of $\mathcal{G}^s_{\mathrm{loc}}(\Lambda)$, the restriction $l_{[0,\delta]} = l \cap ([0,\delta] \times [0,2])$ has no vertical tangent line. From the compactness of $\mathcal{G}^s_{\mathrm{loc}}(\Lambda)$, we have a constant $\alpha_1 > 0$ independent of $l \in \mathcal{G}^s_{\mathrm{loc}}(\Lambda)$ such that $\mathrm{slope}(l_{[0,\delta]}) < \alpha_1$. We denote the left edge of l by $e(l)$. Let n_0 be the smallest integer with $n_0 \geq \log\delta/\log\lambda$. For any $n \geq n_0$, the component l_n of $f^{-n}(l_{[0,\delta]}) \cap S$ containing $f^{-n}(e(l))$ is a leaf of $\mathcal{G}^s_{\mathrm{loc}}(\Lambda)$. Since f^n is the linear map $(\sigma^n x, \lambda^n y)$ on a small neighborhood of l_n in S, $\mathrm{slope}(l_n) < \alpha_1(\sigma^{-1}\lambda)^n$.

Now we suppose that l is any leaf of $\mathcal{G}^s_{\mathrm{loc}}(\Lambda)$ contained in $\mathbb{B}^u_k$. It follows from the form (7.1) of B^u_k that, for any integer $k > 0$ with $z_k k^2 \geq z_0 k^2 > n_0$, $f^{z_k k^2}$ is the linear map $(\sigma^{z_k k^2} x, \lambda^{z_k k^2} y)$ on a small neighborhood of l with $f^{z_k k^2}(l) \subset l'_{[0,\delta]}$ for some leaf l' of $\mathcal{G}^s_{\mathrm{loc}}(\Lambda)$. This shows that $\mathrm{slope}(l) < \alpha_1(\sigma^{-1}\lambda)^{z_k k^2}$. By replacing α_1 by a larger constant $\alpha > 1$ if necessary, one can suppose that $\mathrm{slope}(l) < \alpha(\sigma^{-1}\lambda)^{z_k k^2}$ for all $k > 0$. Thus the first inequality holds.

Since M is compact, there exists a constant $\gamma > 1$ satisfying

$$\gamma^{-1} < \|Df^{-1}_{\boldsymbol{x}}(\boldsymbol{v})\| < \gamma \tag{7.5}$$

for any point $\boldsymbol{x}$ of M and any unit tangent vector $\boldsymbol{v} \in T_{\boldsymbol{x}}(M)$. By Lemma 7.1-(2), there exists a constant $H > 0$ with $N_0 + \widehat{i}_k + N_1 < Hk$ for any $k > 0$. Note that $f^{-N_0} \circ f^{-\widehat{i}_k} \circ f^{-N_1}$ maps $\widetilde{L}_k \cap S(\widehat{A}_k)$ onto L_k and the horizontal segment passing through any point of $\widetilde{L}_k \cap S(\widehat{A}_k)$ to the horizontal segment passing through a point of L_k. Since moreover $\mathrm{slope}(\widetilde{L}_k) < \alpha(\sigma^{-1}\lambda)^{z_k k^2}$,

$$\mathrm{slope}(L_k) < \alpha\gamma^{2Hk}(\sigma^{-1}\lambda)^{z_k k^2} < (\alpha\gamma^{2H})^k(\sigma^{-1}\lambda)^{z_k k^2}.$$

Thus we have the second inequality by denoting $\alpha\gamma^{2H}$ newly by α. □

In particular, Lemma 7.2 implies that, for all sufficiently large k, L_k is almost horizontal and hence L_k meets L transversely at a single point $\boldsymbol{x}_k = (x_k, y_k)$. Note that $\widetilde{\boldsymbol{x}}_k = f^{N_1} \circ f^{\widehat{i}_k} \circ f^{N_0}(\boldsymbol{x}_k)$ is a point of $\widetilde{L}_k$. There exists a neighborhood $\mathcal{N}_0$ of $\widetilde{L}$ in $\mathbb{G}^u(0) \setminus \sharp\mathbb{G}^u(0)$ consisting of leaves of $\mathcal{G}^s(0)$. In particular, $\mathcal{N}_0$ is a sub-strip of $(\mathbb{G}^u(0), \flat\mathbb{G}^u(0))$ and the components $l_{1,0}$, $l_{2,0}$ of $\sharp\mathcal{N}_0$ meet $\mathcal{F}^u_{\mathrm{loc}}(\Lambda)$ exactly. We set

$$\mathrm{dist}(\sharp\mathcal{N}_0, \widetilde{L}) = \zeta > 0. \tag{7.6}$$

Let $\mathcal{N}_k$ be the component of $f^{-(z_k k^2 + \langle k \rangle)}(\mathcal{N}_0) \cap S$ containing $\widetilde{L}_k$. Since $\mathcal{N}_0 \subset \mathbb{G}^u(0)$, $\mathcal{N}_k$ is contained in some gap strip in $\mathbb{B}^u_k$.

For sequences $\{u_k\}$, $\{v_k\}$ of positive numbers, $u_k \asymp v_k$ means that there exist constants $0 < c_1 < c_2$ independent of k such that $c_1 \le \dfrac{u_k}{v_k} \le c_2$ holds for all k.

Lemma 7.3. *There exists a constant $0 < \nu < 1$ such that, for all sufficiently large k, $[0,2] \times \left[\widetilde{y}_k - \nu^{k^2}\sigma^{-z_k k^2}, \widetilde{y}_k + \nu^{k^2}\sigma^{-z_k k^2}\right]$ is contained in $\mathcal{N}_k$, where $\widetilde{y}_k$ is the y-entry of $\widetilde{\boldsymbol{x}}_k = (\widetilde{x}_k, \widetilde{y}_k)$, see Figure 7.4.*

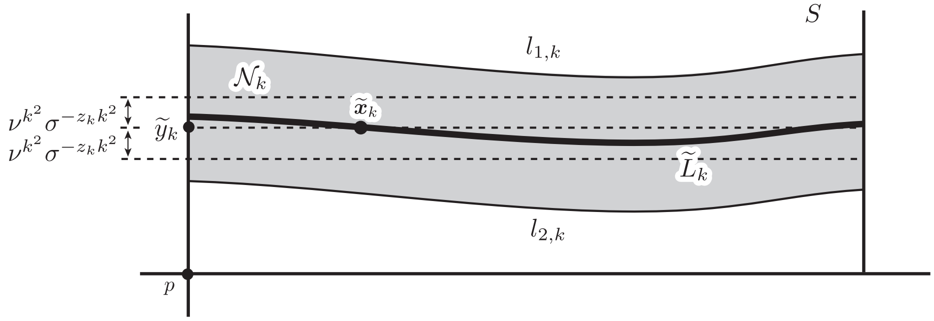

FIGURE 7.4. Situation in $\mathcal{N}_k$.

Proof. By (7.5) and (7.6), we have

$$\mathrm{dist}(\sharp\mathcal{N}_k, \widetilde{L}_k) \ge \gamma^{-\langle k \rangle}\sigma^{-z_k k^2}\zeta.$$

By Lemma 7.1-(2), $\widehat{n}_{k+1} \asymp k$ and hence $\langle k \rangle = \widehat{n}_{k+1} + k^2 + k \asymp k^2$. Thus one can take a constant $0 < \nu < 1$ satisfying

$$\mathrm{dist}(\sharp\mathcal{N}_k, \widetilde{L}_k) \ge 2\nu^{k^2}\sigma^{-z_k k^2}$$

for any $k > 0$. Since the components $l_{1,k}$, $l_{2,k}$ of $\sharp\mathcal{N}_k$ are leaves of $\mathcal{G}^s_{\mathrm{loc}}(\Lambda)$, $\mathrm{slope}(l_{i,k}) < \alpha(\sigma^{-1}\lambda)^{z_k k^2}$ for $i = 1, 2$ by Lemma 7.2. Here we suppose that the integer z_0 satisfies

$$\lambda^{z_0} < \nu. \tag{7.7}$$

We will see later that the condition (7.7) is implied by the condition (8.5). Since

$$\frac{\mathrm{slope}(l_{i,k})}{2\nu^{k^2}\sigma^{-z_k k^2}} < \frac{\alpha(\nu^{-1})^{k^2}\lambda^{z_0 k^2}}{2} \to 0 \quad (k \to \infty)$$

for $i = 1, 2$, it follows that

$$[0,2] \times \left[\widetilde{y}_k - \nu^{k^2}\sigma^{-z_k k^2}, \widetilde{y}_k + \nu^{k^2}\sigma^{-z_k k^2}\right] \subset \mathcal{N}_k$$

for all sufficiently large k. □

Consider the map h_k defined by

$$h_k := f^{z_k k^2+\langle k\rangle} \circ f^{N_1} \circ f^{\widehat{i}_k} \circ f^{N_0} \tag{7.8}$$

and the sequence $\{\widehat{\boldsymbol{x}}_k\}$ with $\widehat{\boldsymbol{x}}_k = f^{z_k k^2+\langle k\rangle}(\widetilde{\boldsymbol{x}}_k) = h_k(\boldsymbol{x}_k)$. Let $\boldsymbol{u}_k = (u_k, v_k)$ be the vector with

$$\widehat{\boldsymbol{x}}_k + \boldsymbol{u}_k = f^{-N_2}(\boldsymbol{x}_{k+1}). \tag{7.9}$$

Figure 7.5 illustrates the transition of base points schematically.

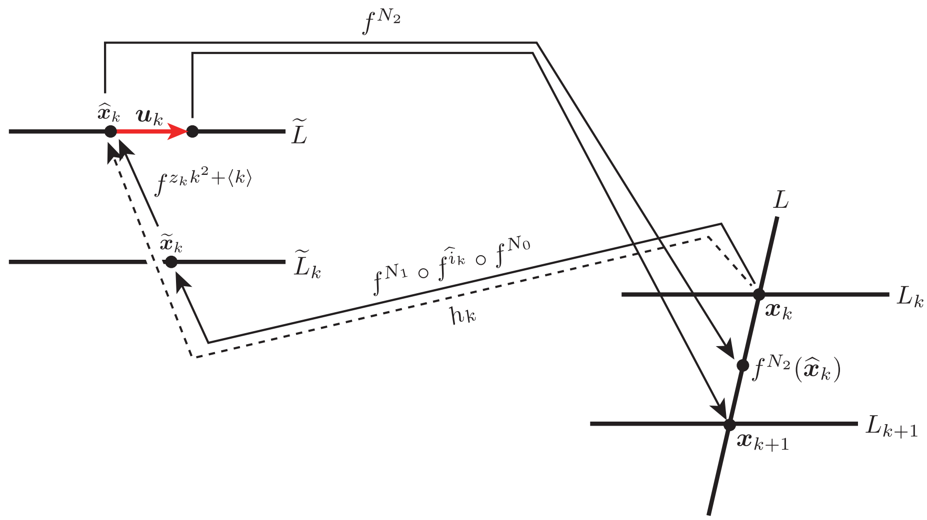

FIGURE 7.5. A transition from $\boldsymbol{x}_k$ to $f^{N_2}(\widehat{\boldsymbol{x}}_k)$ and a shifted transition from $\boldsymbol{x}_k$ to $\boldsymbol{x}_{k+1}$.

Lemma 7.4. *There exists a constant $\beta > 0$ independent of k such that $\|\boldsymbol{u}_k\| \leq \beta r_{s-}^{-Tk}$, where T is a number not smaller than the constant T_0 given in Lemma 7.1.*

Proof. Since $f^{N_2}(\widehat{\boldsymbol{x}}_k) \in B^{s*}_{L,k+1}$ and $\boldsymbol{x}_{k+1} \in B^u_{k+1}$, it follows from Lemma 7.1-(1) that there exists a vector $\boldsymbol{t}_{k+1} = (s_{k+1}, t_{k+1})$ with $|t_{k+1}| < 2^{-1}\xi_0 |B^s_0(\Delta)| r_{s-}^{-T(k+1)}$ and $f^{N_2}(\widehat{\boldsymbol{x}}_k) + \boldsymbol{t}_{k+1} = \boldsymbol{x}_{k+1}$. Since N_2 is independent of k, $\|\boldsymbol{u}_k\| \asymp \|\boldsymbol{t}_{k+1}\| \asymp |t_{k+1}|$. Thus we have a constant $\beta > 0$ satisfying $\|\boldsymbol{u}_k\| = \|\boldsymbol{x}_{k+1} - f^{-N_2}(\widehat{\boldsymbol{x}}_k)\| \leq \beta r_{s-}^{-Tk}$ for any $k \geq 1$. □

This proof suggests that $\boldsymbol{t}_{k+1}$ and hence $\boldsymbol{u}_k$ depend on T. So we will write

$$\boldsymbol{t}_k = \boldsymbol{t}_k(T) \quad \text{or} \quad \boldsymbol{u}_k = \boldsymbol{u}_k(T)$$

when we emphasize the dependence.

Recall that we supposed that r is an integer with $3 \leq r < \infty$. The projection $\pi_{\mathcal{F}^s_{\mathrm{loc}}(\Lambda)}$ is a $C^{1+\alpha}$-function but not necessary of C^r-class. So we need a suitable substitute for the map. Let $\varpi^s : \mathbb{G}^u(0) \to \{0\} \times [0,2] \subset W^u_{\mathrm{loc}}(p)$ be a C^r-map arbitrarily C^1-close to $\pi_{\mathcal{F}^s_{\mathrm{loc}}(\Lambda)}|_{\mathbb{G}^u(0)}$. Since $\mathcal{N}_0$ is contained in $\mathbb{G}^u(0) \setminus \sharp\mathbb{G}^u(0)$, there exist a $d > 0$ and sub-intervals H, $\widehat{H}$ of $G^u(0)$ such that $G^u(0) = [\min H - 2d|H|, \max H + 2d|H|]$, $\widehat{H} = [\min H - d|H|, \max H + d|H|]$ and $\pi^{-1}_{\mathcal{F}^s_{\mathrm{loc}}(\Lambda)}(H)$ contains

$\mathcal{N}_0$. Then one can choose the C^r-map ϖ^s so that $\mathbb{H} = (\varpi^s)^{-1}(H)$ and $\widehat{\mathbb{H}} = (\varpi^s)^{-1}(\widehat{H})$ are strips in S with

$$\mathcal{N}_0 \subsetneq \mathbb{H} \subsetneq \widehat{\mathbb{H}} \subsetneq \mathbb{G}^u(0).$$

See Figure 7.6.

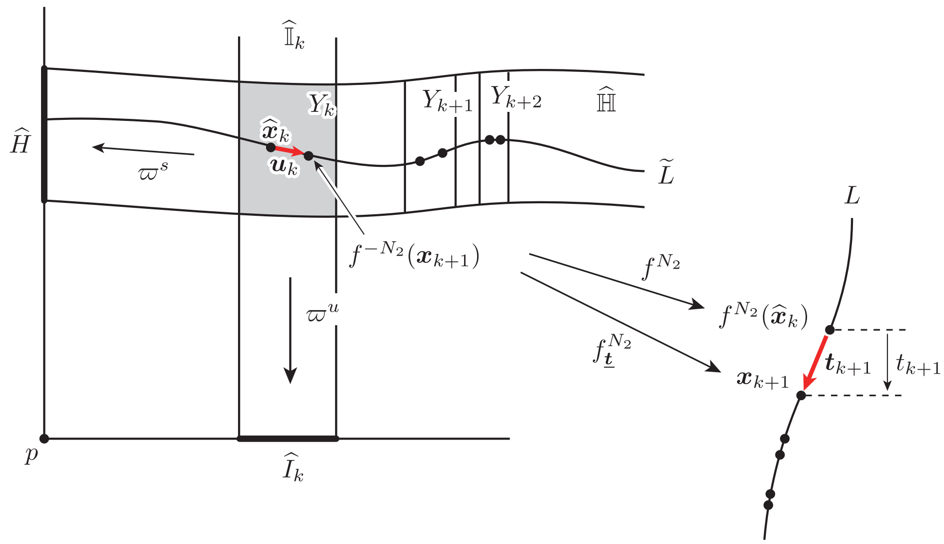

FIGURE 7.6. A shifting from $\widehat{\boldsymbol{x}}_k$ to $f^{-N_2}(\boldsymbol{x}_{k+1})$ and the resultant shifting from $f^{N_2}(\widehat{\boldsymbol{x}}_k)$ to $\boldsymbol{x}_{k+1}$.

Now we define the bump functions θ_k and $\widetilde{\theta}$ supported on $\widehat{I}_k$ and $\widehat{H}$ respectively, where $\widehat{I}_k$ is the interval in $[0,2]\times\{0\}$ given in (B-ii). First, consider a non-decreasing C^∞ function on $\mathbb{R}$ with

$$s(x) = \begin{cases} 0 & \text{if } x \leq -1; \\ 1 & \text{if } x \geq 0. \end{cases}$$

For $\rho > 0$ and the interval $[a,b]$, let $S_{\rho,[a,b]}$ be the non-negative C^∞ function on $\mathbb{R}$ defined as

$$S_{\rho,[a,b]}(x) := s\left(\frac{x-a}{\rho(b-a)}\right) + s\left(\frac{b-x}{\rho(b-a)}\right) - 1.$$

The support of $S_{\rho,[a,b]}$ is $[a-\rho(b-a), b+\rho(b-a)]$ and the height on $[a,b]$ is identical to 1. Since $S_{\rho,[a,b]}$ is symmetric with respect to $x = \frac{a+b}{2}$, one has

$$\|S_{\rho,[a,b]}\|_r \leq \frac{1}{\big(\rho(b-a)\big)^r}\|s\|_r, \tag{7.10}$$

where $\|\cdot\|_r$ is the norm given by the derivative until order r. Then our desired bump functions are defined by

$$\theta_k := S_{\alpha_1/2,\varpi^u(I_k)}, \quad \widetilde{\theta} := S_{d,H}, \tag{7.11}$$

where I_k is in $\widetilde{L}$ defined as (6.1) and α_1 is the constant given in (B-ii).

Lemma 7.5. *Let T be any real number with*

$$T > N_u r \log(5 \cdot 2^{m-3})/\log(r_{s-}),$$

where N_u is the integer given in Lemma 6.2-(3). Then there exists a constant $C_T > 0$ satisfying

$$\lim_{T\to+\infty} C_T = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{\|\boldsymbol{u}_k(T)\|}{|A^u_{L,k+1}|^r} < C_T.$$

Proof. By Lemmas 4.5-(1) and 6.2-(3), we have

$$|A^u_{L,k}| \geq (5\cdot 2^{m-3})^{-i_k}|A^u_{L,0}| \quad \text{and} \quad i_k \leq N_u k + i_0.$$

Then

$$\frac{1}{|A^u_{L,k}|} < \frac{1}{(5\cdot 2^{m-3})^{-i_k}|A^u_{L,0}|} < \frac{(5\cdot 2^{m-3})^{i_0}}{(5\cdot 2^{m-3})^{-N_u k}|A^u_{L,0}|}.$$

By Lemma 7.4, one has

$$\sum_{k=1}^{\infty} \frac{\|\boldsymbol{u}_k(T)\|}{|A^u_{L,k+1}|^r} < \frac{(5\cdot 2^{m-3})^{i_0+N_u r}\beta}{|A^u_{L,0}|^r} \sum_{k=1}^{\infty} \left(\frac{(5\cdot 2^{m-3})^{N_u r}}{r_{s-}^T}\right)^k.$$

Since $T > N_u r \log(5\cdot 2^{m-3})/\log(r_{s-})$, the right-hand side of the inequality is equal to

$$C_T := \frac{(5\cdot 2^{m-3})^{i_0+N_u r}\beta}{|A^u_{L,0}|^r (r_{s-}^T - (5\cdot 2^{m-3})^{N_u r})}.$$

Since $r_{s-} \geq 2$, $\lim_{T\to\infty} C_T = 0$. □

The square $S = [0,2]\times[0,2]$ is naturally supposed to be embedded in $\mathbb{R}^2$. We may assume that the ambient surface M has a Riemannian metric whose restriction on S coincides with the standard Euclidean metric on $\mathbb{R}^2$. The curvature of a leaf of $\mathcal{F}^u_{\text{loc}}(\Lambda)$ (resp. $f^{-(N_0+N_2)}(\mathcal{F}^s_{\text{loc}}(\Gamma_m))$) at $\boldsymbol{x} \in \widetilde{L}$ is denoted by $\kappa_\Lambda(\boldsymbol{x})$ (resp. $\kappa_{\Gamma_m}(\boldsymbol{x})$). By (F-iii) in Subsection 4.4, both $\kappa_\Lambda(\boldsymbol{x})$ and $\kappa_{\Gamma_m}(\boldsymbol{x})$ vary C^1 along $\widetilde{L}$. Since $\mathcal{F}^u_{\text{loc}}(\Lambda)$ and $f^{-(N_0+N_2)}(\mathcal{F}^s_{\text{loc}}(\Gamma_m))$ have quadratic tangencies along $\widetilde{L}$, there exists a constant $K > 0$ with $|\kappa_\Lambda(\boldsymbol{x}) - \kappa_{\Gamma_m}(\boldsymbol{x})| \geq K$ for any $\boldsymbol{x} \in \widetilde{L}$. Moreover, by Lemma 7.4,

$$|\kappa_\Lambda(\widehat{\boldsymbol{x}}_k) - \kappa_{\Gamma_m}(\widehat{\boldsymbol{x}}_k + \boldsymbol{u}_k)| \geq K/2 \tag{7.12}$$

for all sufficiently large k. Let l_k, $\widehat{l}_k$ be the leaves of $\mathcal{F}^u_{\text{loc}}(\Lambda)$ passing though $\widehat{\boldsymbol{x}}_k$ and $\widehat{\boldsymbol{x}}_k + \boldsymbol{u}_k$ respectively, and let $-\pi \leq \omega_k \leq \pi$ be the angle of $l_k + \boldsymbol{u}_k$ and $\widehat{l}_k$ at $\widehat{\boldsymbol{x}}_k + \boldsymbol{u}_k$. See Figure 7.7. Since the directions of leaves of $\mathcal{F}^u_{\text{loc}}(\Lambda)$ also vary C^1 along $\widetilde{L}$ by (F-iii) in Subsection 4.4, there exists a constant $C > 0$ independent of k with

$$|\omega_k| \leq C\|\boldsymbol{u}_k\|. \tag{7.13}$$

Consider the orientation-preserving isometry $\xi_k : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\xi_k(\boldsymbol{x}) = A_k(\boldsymbol{x} - \widehat{\boldsymbol{x}}_k) + \widehat{\boldsymbol{x}}_k + \boldsymbol{u}_k,$$

where A_k is the orthogonal matrix of rotation ω_k. Since

$$A_k - E = \begin{pmatrix} \cos\omega_k - 1 & -\sin\omega_k \\ \sin\omega_k & \cos\omega_k - 1 \end{pmatrix},$$

the inequality (7.13) implies that the C^r-norm of $A_k - E$ as a linear map is

$$\|A_k - E\|_r = \|A_k - E\| = O(\|\boldsymbol{u}_k\|), \tag{7.14}$$

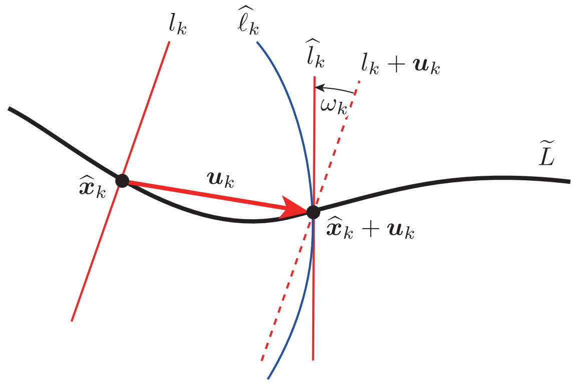

FIGURE 7.7. A small parallel translation and a small rotation.

where E is the unit matrix of order two.

It follows from the definition of ξ_k that $\xi_k(l_k)$ is a curve tangent to $\widehat{l}_k$ at $\widehat{\boldsymbol{x}}_k+\boldsymbol{u}_k$ and hence to the leaf $\widehat{\ell}_k$ of $f^{-(N_0+N_2)}(\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m))$ passing through $\widehat{\boldsymbol{x}}_k+\boldsymbol{u}_k$. Note that $\xi_k(l_k)$ is in general not identical to $\widehat{l}_k$. Since any orientation-preserving isometry on $\mathbb{R}^2$ preserves curvature, by (7.12), $\widehat{\boldsymbol{x}}_k+\boldsymbol{u}_k$ is a quadratic tangency of $\xi_k(l_k)$ and $\widehat{\ell}_k$. Since $\xi_k(\boldsymbol{x})-\boldsymbol{x}=(A_k-E)(\boldsymbol{x}-\widehat{\boldsymbol{x}}_k)+\boldsymbol{u}_k$ and S is bounded, (7.14) implies

$$\|(\xi_k-\mathrm{Id}_{\mathbb{R}^2})|_S\|_r=O(\|\boldsymbol{u}_k\|). \tag{7.15}$$

The sequence $\underline{\boldsymbol{t}}=(\boldsymbol{t}_2,\boldsymbol{t}_3,\dots,\boldsymbol{t}_k,\dots)$ of vectors with $f^{N_2}(\widehat{\boldsymbol{x}}_k)+\boldsymbol{t}_{k+1}=\boldsymbol{x}_{k+1}$ is called the *perturbation sequence*. Note that $\underline{\boldsymbol{t}}$ depends on the constant T given in Lemma 7.1 and each entry $\boldsymbol{t}_k=\boldsymbol{t}_k(T)$ converges to the zero vector as $T\to\infty$. Using the bump functions $\{\theta_k\}_{k\geq 1}$, $\widetilde{\theta}$ and the isometries $\xi_k(x,y)$, we define the sequence of the C^r-perturbation maps $\psi_{\underline{\boldsymbol{t}},a}: M\to M$ $(a=1,2,\dots)$ supported on the disjoint union $\bigcup_{k=1}^a Y_k\subset S$ as

$$\psi_{\underline{\boldsymbol{t}},a}(\boldsymbol{x}):=\boldsymbol{x}+\sum_{k=1}^{a}\vartheta_k(\boldsymbol{x})(\xi_k(\boldsymbol{x})-\boldsymbol{x}), \tag{7.16}$$

where

$$\vartheta_k(\boldsymbol{x})=\theta_k(\varpi^u(\boldsymbol{x}))\widetilde{\theta}(\varpi^s(\boldsymbol{x}))\quad\text{and}\quad Y_k=\widehat{\mathbb{I}}_k\cap\ \widehat{\mathbb{H}}.$$

Each Y_k is a rectangle with curvilinear top and bottom as illustrated in Figure 7.6.

Lemma 7.6. *For any $\underline{\boldsymbol{t}}=\underline{\boldsymbol{t}}(T)$, the sequence $\{\psi_{\underline{\boldsymbol{t}},a}\}_{a=1}^{\infty}$ C^r-converges to the C^r-map $\psi_{\underline{\boldsymbol{t}}}$ with*

$$\psi_{\underline{\boldsymbol{t}}}(\boldsymbol{x}):=\boldsymbol{x}+\sum_{k=1}^{\infty}\vartheta_k(\boldsymbol{x})(\xi_k(\boldsymbol{x})-\boldsymbol{x}) \tag{7.17}$$

for $(x,y)\in S$. Moreover, $\psi_{\underline{\boldsymbol{t}}}=\psi_{\underline{\boldsymbol{t}}(T)}$ are C^r-diffeomorphisms on M for all sufficiently large T which C^r-converge to the identity as $T\to\infty$.

Proof. Recall that $I_{k+1}=A^u_{k+1}(\Delta)\cup B^s_{k+1}(\Delta)$ is the arc of (6.1) in $\widetilde{L}$. Since N_2 is independent of k, it follows from $|A^u_{k+1}(\Delta)|\geq|B^s_{k+1}(\Delta)|$ that

$$|I_{k+1}|=|A^u_{k+1}(\Delta)\cup B^s_{k+1}(\Delta)|\asymp|A^u_{k+1}(\Delta)|\asymp|A^u_{L,k+1}|.$$

By this fact together with (7.10) and (7.15), for any integers a, b with $1 \le a < b$,

$$\|\psi_{\underline{t},b} - \psi_{\underline{t},a}\|_r \le C_0 \|S_{\rho,[-1,1]}\|_r \sum_{k=a+1}^{b} \frac{\|\boldsymbol{u}_k\|}{|A^u_{L,k+1}|^r}, \tag{7.18}$$

where C_0 is a constant independent of $\underline{t}$. By Lemma 7.5, $\{\psi_{\underline{t},a}\}_{a=1}^{\infty}$ is a Cauchy sequence in the space $(\mathrm{Map}^r(M), \|\cdot\|_r)$ of C^r-maps on M, which is a complete metric space. Thus $\psi_{\underline{t},a}$ C^r-converges to the C^r-map $\psi_{\underline{t}} = \psi_{\underline{t}(T)}$ defined by (7.17) as $a \to \infty$. Again by Lemma 7.5,

$$\|\psi_{\underline{t}(T)} - \mathrm{id}_M\|_r \le C_0 \|S_{\rho,[-1,1]}\|_r \sum_{k=1}^{\infty} \frac{\|\boldsymbol{u}_k\|}{|A^u_{L,k+1}|^r} \le C_0 C_T.$$

Since $\lim_{T\to\infty} C_T = 0$, the map $\psi_{\underline{t}(T)}$ C^r-converges to the identity as $T \to \infty$. Since the identity is a diffeomorphism, $\psi_{\underline{t}(T)}$ is also a diffeomorphism for all sufficiently large T. This completes the proof. □

Remark 7.7. We note that $\widetilde{L}$ is no longer a tangency curve of $\psi_{\underline{t}}(\mathcal{F}^u_{\mathrm{loc}}(\Lambda))$ and $f^{-(N_0+N_2)}(\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m))$. However, from our construction (7.17) of $\psi_{\underline{t}}$, the leaves of them passing through $\widehat{\boldsymbol{x}}_k + \boldsymbol{u}_k$ still have a quadratic tangency. In fact, (7.11) implies $\vartheta_k(\boldsymbol{x}) = 1$ and $\vartheta_{k'}(\boldsymbol{x}) = 0$ ($k' \ne k$) for any $\boldsymbol{x} \in S$ sufficiently close to $\widehat{\boldsymbol{x}}_k$, and hence

$$\psi_{\underline{t}}(\boldsymbol{x}) = \boldsymbol{x} + (\xi_k(\boldsymbol{x}) - \boldsymbol{x}) = \xi_k(\boldsymbol{x}).$$

It follows that $\psi_{\underline{t}}(l_k)$ and $\widehat{\ell}_k$ have a quadratic tangency at $\widehat{\boldsymbol{x}}_k + \boldsymbol{u}_k$.

Let J_k be a short segment in $U(L)$ which is vertical with respect to the $C^{1+\alpha}$-coordinate on $U(L)$ given in Subsection 5.2 and passes through $\boldsymbol{x}_k$, for example see Figure 8.1. From the definition of the foliation $\mathcal{F}^u_{\mathrm{loc}}(\Lambda)$, we know that $h_k(J_k)$ is a segment intersecting $\widetilde{L}$ transversely at $\widehat{\boldsymbol{x}}_k$ and contained in a leaf l_k of $\mathcal{F}^u_{\mathrm{loc}}(\Lambda)$, where h_k is the diffeomorphism defined as (7.8). Thus $\psi_{\underline{t}}(h_k(J_k))$ and some leaf of $f^{-(N_0+N_2)}(\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m))$ has a quadratic tangency at $\widehat{\boldsymbol{x}}_k + \boldsymbol{u}_k$.

From the definition (7.11) of θ_k, $\widetilde{\theta}$ and the form (7.17) of $\psi_{\underline{t}}$, we know that the support $\mathrm{supp}(\psi_{\underline{t}})$ of $\psi_{\underline{t}}$ is contained in the disjoint union $\bigcup_{k=1}^{\infty} Y_k$. We now define the C^r-map $f_{\underline{t}}$ by

$$f_{\underline{t}} := f \circ \psi_{\underline{t}}, \tag{7.19}$$

for the perturbation sequence $\underline{t} = (\boldsymbol{t}_2, \boldsymbol{t}_3, \ldots, \boldsymbol{t}_k, \ldots)$. Since $\mathrm{supp}(\psi_{\underline{t}})$ is contained in a small region in S sufficiently close to the left component of $\flat S$, $\mathrm{supp}(\psi_{\underline{t}})$ is disjoint from $\bigcup_{i=1}^{N_2} f^i(\mathrm{supp}(\psi_{\underline{t}}))$, see Figure 3.2. It follows that $f_{\underline{t}}$ is equal to f on $\bigcup_{i=1}^{N_2} f^i(\mathrm{supp}(\psi_{\underline{t}}))$. Hence, by (7.9) and (7.17),

$$f_{\underline{t}}^{N_2}(\widehat{\boldsymbol{x}}_k) = f_{\underline{t}}^{N_2-1} \circ (f \circ \psi_{\underline{t}})(\widehat{\boldsymbol{x}}_k) = f^{N_2-1} \circ f(\widehat{\boldsymbol{x}}_k + \boldsymbol{u}_k) = \boldsymbol{x}_{k+1}, \tag{7.20}$$

see Figure 7.6. By Lemma 7.6, one can suppose that $f_{\underline{t}}$ is a C^r-diffeomorphism arbitrarily C^r-close to f if we take T sufficiently large.

Remark 7.8. (1) Since $\bigcup_{k=1}^{\infty} Y_k \subset \widehat{\mathbb{I}}$, by (B-i) the support $\mathrm{supp}(\psi_{\underline{t}})$ is disjoint from X_{n_*}. Thus both the invariant set Γ_m and local stable foliation $\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m)$ for the perturbed diffeomorphism $f_{\underline{t}}$ are the same as the originals.

(2) It is possible to rearrange our construction so that both f and $\psi_{\underline{t},a}$ are of C^∞-class. However, even in the case, Lemma 7.6 only asserts that $\psi_{\underline{t}}$ is of C^r-class with $2 \le r < \infty$ but not necessarily of C^∞-class. In fact, though $\psi_{\underline{t}}$ is a C^∞-map on

the neighborhood Y_k of any $\widehat{\boldsymbol{x}}_k$, the authors do not know whether it holds also at the limit point of $\{\widehat{\boldsymbol{x}}_k\}$. So it may be impossible to suppose that the composition $f_{\underline{t}} := f \circ \psi_{\underline{t}}$ of (7.19) is a C^∞-diffeomorphism. The same problem would occur in the original example in [6] though they assert that it is of class C^∞.

8. Detection of wandering domains with historic behavior

8.1. **Some constants for wandering domains.** From now on, we write $f_{\underline{t}} = f$ for short. Recall that σ is the unstable eigenvalue of the derivative Df_p of f at p given in (S-iii). Define the constant ω by

$$\omega = \max\bigl\{\nu^{-1},\ \sup\bigl\{\|Df_{\boldsymbol{x}}\|\,;\,\boldsymbol{x}\in S\bigr\},\ \sup\bigl\{\|(Df_{\boldsymbol{x}})^{-1}\|\,;\,\boldsymbol{x}\in S\bigr\}\bigr\}, \tag{8.1}$$

where ν is the constant given in Lemma 7.3. In this section, the constant

$$b_k := \varepsilon(5\cdot 2^{m-3})^{-p_k}\omega^{-q_k}\sigma^{-r_k} \tag{8.2}$$

plays an important role, where '$5\cdot 2^{m-3}$' is the number presented in Lemma 4.5-(1) and

$$p_k = \sum_{i=0}^{\infty}\frac{\widehat{i}_{k+i}}{2^i},\quad q_k = \sum_{i=0}^{\infty}\frac{\langle k+i\rangle}{2^i},\quad r_k = \sum_{i=0}^{\infty}\frac{z_{k+i}(k+i)^2}{2^i}, \tag{8.3}$$

where $\widehat{i}_{k+i}$ is the constant given in Lemma 7.1-(2). The factor ε in (8.2) is a positive constant independent of k which will be fixed later. Remember that each z_k is either z_0 or z_0+1 for the integer z_0 given in Subsection 7.1 and $\langle k+i\rangle = \widehat{n}_{k+1+i} + (k+i)^2 + k + i$ by (7.2).

Lemma 8.1. (1) *For any $\eta > 0$, suppose that z_0 satisfies (8.4). Then there exists an integer $k_* > 0$ such that, for any $k \ge k_*$,*

$$b_k^{\frac{1}{2}} \le \varepsilon^{-\frac{1}{2}}(5\cdot 2^{m-3})^{-\frac{\widehat{i}_k}{2}+\frac{3}{4}p_{k+1}}\omega^{-\frac{\langle k\rangle}{2}+\frac{3}{4}q_{k+1}}\sigma^{(1+\eta)z_k k^2}b_{k+1},$$

$$\frac{\langle k\rangle}{2} + \frac{3}{4}q_{k+1} < 4k^2.$$

(2) *For any integer $k > 0$,*

$$b_{k+1} = \varepsilon^{-1}(5\cdot 2^{m-3})^{2\widehat{i}_k}\omega^{2\langle k\rangle}\sigma^{2z_k k^2}b_k^2.$$

Proof. (1) By (8.2),

$$\frac{b_k^{\frac{1}{2}}}{b_{k+1}} = \varepsilon^{-\frac{1}{2}}(5\cdot 2^{m-3})^{p_{k+1}-\frac{p_k}{2}}\omega^{q_{k+1}-\frac{q_k}{2}}\sigma^{r_{k+1}-\frac{r_k}{2}}.$$

It is immediate from (8.3) that $p_{k+1} - \frac{p_k}{2} = -\frac{\widehat{i}_k}{2} + \frac{3}{4}p_{k+1}$ and $q_{k+1} - \frac{q_k}{2} = -\frac{\langle k\rangle}{2} + \frac{3}{4}q_{k+1}$.

Since $f(x) = \frac{4}{2-(1+x)^2} - \frac{1}{1+x} - 2$ is a monotone increasing function on $0 \le x < \sqrt{2}-1$ with $f(0) = 1$ and $\lim_{x\nearrow\sqrt{2}-1} f(x) = \infty$, for any $\eta > 0$, there exists a unique $0 < \eta_1 < \sqrt{2}-1$ with

$$\frac{4}{2-(1+\eta_1)^2} - \frac{1}{1+\eta_1} - 2 = 1 + \eta.$$

Since $\lim_{k\to\infty}\frac{(k+1)^2}{k^2} = 1$, there exists an integer $k_* > 0$ such that $(k+1)^2 \le (1+\eta_1)k^2$ for any $k \ge k_*$. We choose the integer z_0 so as to satisfy

$$z_0 + 1 \le (1+\eta_1)z_0. \tag{8.4}$$

Then $z_{k+1}(k+1)^2 \le (z_0+1)(k+1)^2 \le (1+\eta_1)^2 z_k k^2$ holds. Repeating a similar argument, one can have $z_{k+1+i}(k+1+i)^2 \le z_k(1+\eta_1)^{2(i+1)}k^2$ for any $k \ge k_*$ and $i \ge 0$. Since $0 < \left(\frac{1+\eta_1}{2}\right)^2 < 1$,

$$r_{k+1} \le \left(\sum_{i=0}^{\infty} \frac{(1+\eta_1)^{2(i+1)}}{2^i}\right) z_k k^2 = \left(\frac{4}{2-(1+\eta_1)^2} - 2\right) z_k k^2.$$

On the other hand, since $r_k \ge \sum_{i=0}^{\infty} \frac{1}{2^i}\frac{z_k}{1+\eta_1}k^2 = \frac{2}{1+\eta_1} z_k k^2$,

$$r_{k+1} - \frac{r_k}{2} \le \left(\frac{4}{2-(1+\eta_1)^2} - 2 - \frac{1}{1+\eta_1}\right) z_k k^2 = (1+\eta) z_k k^2.$$

This shows the first inequality of (1).

Since $\widehat{n}_{k+1} \asymp k$ by Lemma 7.1-(2), one can choose k_* so that $\langle k\rangle = \widehat{n}_{k+1} + k^2 + k \le 2k^2$ for any $k \ge k_*$. Since

$$q_{k+1} - \sum_{i=0}^{\infty} \frac{k^2}{2^i} = \sum_{i=0}^{\infty} \frac{\langle k+1+i\rangle}{2^i} - \sum_{i=1}^{\infty} \frac{k^2}{2^i} \asymp k,$$

we may assume that $q_{k+1} < 2\sum_{i=0}^{\infty} \frac{k^2}{2^i} = 4k^2$ for any $k \ge k_*$. It follows that

$$\frac{\langle k\rangle}{2} + \frac{3}{4} q_{k+1} < k^2 + \frac{3}{4}\cdot 4k^2 = 4k^2.$$

This shows the second inequality of (1).

(2) The equality of (2) is derived immediately from (8.2) together with the equalities

$$2p_k - p_{k+1} = 2\widehat{i}_k, \quad 2q_k - q_{k+1} = 2\langle k\rangle, \quad 2r_k - r_{k+1} = 2z_k k^2.$$

This completes the proof. □

Since z_0 is required to satisfy (8.4), we need to choose z_0 sufficiently large according as $\eta > 0$ is taken small. By (S-iii) in Section 3, $\lambda\sigma < 1$. We take and fix a sufficiently small $\eta > 0$ with $\lambda\sigma^{1+\eta} < 1$. So one can choose z_0 so as to satisfy

$$\omega^4 (\lambda\sigma^{1+\eta})^{z_0} < 1. \tag{8.5}$$

Since $\nu^{-1} \le \omega$, it is not hard to show that the condition (8.5) implies (7.7). We will see later that the superscript '4' of ω corresponds to the coefficient '4' of k^2 in the second inequality of Lemma 8.1-(1).

8.2. **Critical chain of rectangles.** For every $k \ge 1$, let $\boldsymbol{x}_k = (x_k, y_k)$ be the intersection point of L_k and L given in Subsection 7.1. We consider the rectangle

$$R_k = \left[x_k - b_k^{\frac{1}{2}}, x_k + b_k^{\frac{1}{2}}\right] \times [y_k - b_k, y_k + b_k]$$

centered at $\boldsymbol{x}_k$ with respect to the $C^{1+\alpha}$-coordinate on $U(L)$ defined in Subsection 5.2, see Figures 5.4 and 7.3. The absolute slope of the diagonals of R_k is $b_k^{\frac{1}{2}}$. By (8.2), there exists a constant $0 < \gamma < 1$ such that $b_k^{\frac{1}{2}} > \gamma^k \omega^{-k^2} \sigma^{-z_k k^2}$. By Lemma 7.2 and (8.5),

$$\frac{\mathrm{slope}(L_k)}{b_k^{\frac{1}{2}}} < \frac{\alpha^k (\sigma^{-1}\lambda)^{z_k k^2}}{\gamma^k \omega^{-k^2} \sigma^{-z_k k^2}} = (\alpha\gamma^{-1})^k (\omega\lambda^{z_k})^{k^2} \to +0 \quad (k \to \infty).$$

Thus one can suppose that $L_k \cap R_k$ is an arc in R_k passing through $\boldsymbol{x}_k$ and well approximating the horizontal line $y = y_k$ if $k \ge k_*$. In particular, the arc connects the components of the edge $\flat R_k$. The strip $\mathbb{B}_k^u$ is divided by $\widetilde{L}_k$ into the two strips

$\mathbb{B}_k^{u\pm}$. Similarly, R_k is divided by L_k into the two strips $R_k^{\pm}$. See Figure 7.3 again. By Lemma 4.5-(1), one can have a constant $C > 0$ satisfying

$$\mathrm{dist}\big(f^{N_1} \circ f^{\widehat{i}_k} \circ f^{N_0}(l_k^{\pm}), \widetilde{L}_k\big) \le C(5 \cdot 2^{m-3})^{\widehat{i}_k} b_k,$$

where $l_k^{\pm} = \sharp R_k \cap R_k^{\pm}$. By (8.1) and (8.2),

$$(5 \cdot 2^{m-3})^{\widehat{i}_k} b_k \le (5 \cdot 2^{m-3})^{\widehat{i}_k} \varepsilon (5 \cdot 2^{m-3})^{-\widehat{i}_k} \omega^{-k^2} \sigma^{-z_k k^2} \le \varepsilon \nu^{k^2} \sigma^{-z_k k^2}.$$

It follows from Lemma 7.3 that $f^{N_1} \circ f^{\widehat{i}_k} \circ f^{N_0}(R_k^{\pm}) \subset \mathbb{B}_k^{u\pm} \cap \mathcal{N}_k$ holds for any $k \ge k_*$ if we take $\varepsilon > 0$ sufficiently small. In particular, this implies that $h_k(R_k) \subset \mathbb{B}_{k+1}^{s*} \cap \widehat{\mathbb{H}}$, where h_k is the diffeomorphism defined as (7.8). See Figure 7.3 for $\mathbb{B}_{k+1}^{s*}$ and Figure 7.6 for $\widehat{\mathbb{H}}$.

For any integer $k > 0$, consider the composition

$$g_k := f^{N_2} \circ f^{z_k k^2 + \langle k \rangle} \circ f^{N_1} \circ f^{\widehat{i}_k} \circ f^{N_0} = f^{N_2} \circ h_k. \tag{8.6}$$

By (7.20), $f^{N_2}(\widehat{\boldsymbol{x}}_k) = \boldsymbol{x}_{k+1}$ and hence $g_k(\boldsymbol{x}_k) = \boldsymbol{x}_{k+1}$. Moreover f is chosen so that $f^{N_2}(\mathcal{F}^u_{\mathrm{loc}}(\Lambda))$ and $f^{-N_0}(\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m))$ have a quadratic tangency at $\boldsymbol{x}_{k+1}$.

For sequences $\{u_k(\varepsilon)\}$, $\{v_k(\varepsilon)\}$ of positive numbers, $u_k(\varepsilon) \prec v_k(\varepsilon)$ means that there exists a positive constant a independent of k, ε and satisfying $u_k(\varepsilon) < a v_k(\varepsilon)$ for all k and $\varepsilon > 0$.

The main result of this section is as follows:

Lemma 8.2 (Rectangle Lemma)**.** *There exist an integer $k_0 \ge k_*$ and $\varepsilon > 0$ such that, for any $k \ge k_0$, $g_k(R_k) \subset \mathrm{Int} R_{k+1}$.*

Proof. For any $\boldsymbol{x} \in U(L)$, let $L_{\boldsymbol{x}}^{\mathrm{hori}}$ and $L_{\boldsymbol{x}}^{\mathrm{vert}}$ be the horizontal and vertical lines in $U(L)$ passing through $\boldsymbol{x}$ respectively. See Figure 5.4. Let

$$\pi_{\boldsymbol{x}}^{\mathrm{hori}} : U(L) \to L_{\boldsymbol{x}}^{\mathrm{vert}}, \quad \pi_{\boldsymbol{x}}^{\mathrm{vert}} : U(L) \to L_{\boldsymbol{x}}^{\mathrm{hori}}$$

be the projections along the horizontal and vertical lines on the $C^{1+\alpha}$-coordinate.

First we show that the g_k-image of the center vertical segment $J_k = \{x_k\} \times [y_k - b^k, y_k + b_k]$ of R_k is contained in

$$\tfrac{1}{2} R_{k+1} = \big[x_{k+1} - \tfrac{1}{2} b_{k+1}^{\frac{1}{2}}, x_{k+1} + \tfrac{1}{2} b_{k+1}^{\frac{1}{2}}\big] \times \big[y_{k+1} - \tfrac{1}{2} b_{k+1}, y + \tfrac{1}{2} b_{k+1}\big].$$

See Figure 8.1. We set $\widetilde{J}_k = h_k(J_k)$. By Remark 7.7, $\widetilde{J}_k$ is in a leaf of $\mathcal{F}^u_{\mathrm{loc}}(\Lambda)$. Note that, by (8.2), b_k and hence $|J_k|$ depend on ε. Since $f^{z_k k^2}$ coincides with the linear map $(x, y) \mapsto (\lambda^{z_k k^2} x, \sigma^{z_k k^2} y)$ on $f^{N_1} \circ f^{\widehat{i}_k} \circ f^{N_0}(R_k)$ and N_0, N_1 are independent of k,

$$|\widetilde{J}_k| \prec (5 \cdot 2^{m-3})^{\widehat{i}_k} \omega^{\langle k \rangle} \sigma^{z_k k^2} b_k.$$

Since $g_k(J_k) = f^{N_2}(\widetilde{J}_k)$, by Lemma 8.1-(2), we have

$$|\pi^{\mathrm{vert}}_{\boldsymbol{x}_{k+1}}(g_k(J_k))| \prec (5 \cdot 2^{m-3})^{\widehat{i}_k} \omega^{\langle k \rangle} \sigma^{z_k k^2} b_k = \varepsilon^{\frac{1}{2}} b_{k+1}^{\frac{1}{2}}.$$

By Remark 7.7, $g_k(J_k)$ and $L^{\mathrm{hori}}_{\boldsymbol{x}_{k+1}}$ have a quadratic tangency at $\boldsymbol{x}_{k+1}$. Since the curvatures of leaves of $f^{N_2}(\mathcal{F}^u_{\mathrm{loc}}(\Lambda))$ and $f^{-N_0}(\mathcal{F}^s_{\mathrm{loc}}(\Gamma_m))$ vary C^1 along L by (F-iii) in Subsection 4.4, there exists a constant $A > 0$ independent of k with $|\pi^{\mathrm{hori}}_{\boldsymbol{x}_{k+1}}(g_k(J_k))| \le A |\pi^{\mathrm{vert}}_{\boldsymbol{x}_{k+1}}(g_k(J_k))|^2$. This implies that

$$|\pi^{\mathrm{hori}}_{\boldsymbol{x}_{k+1}}(g_k(J_k))| \prec \varepsilon b_{k+1}.$$

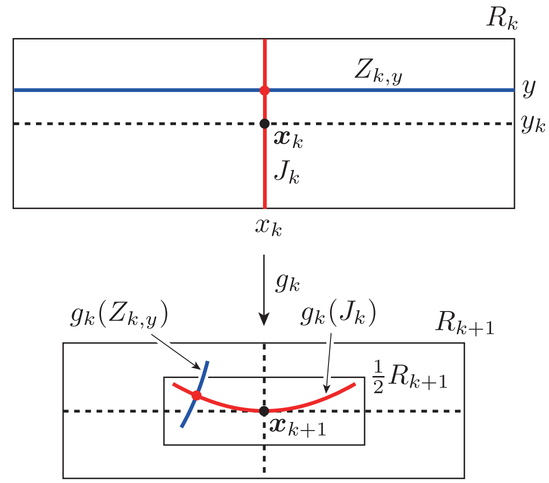

FIGURE 8.1. A transition from R_k into R_{k+1}.

One can choose $\varepsilon > 0$ so that

$$|\pi^{\mathrm{vert}}_{\boldsymbol{x}_{k+1}}(g_k(J_k))| \le \frac{1}{2} b_{k+1}^{\frac{1}{2}} \quad \text{and} \quad |\pi^{\mathrm{hori}}_{\boldsymbol{x}_{k+1}}(g_k(J_k))| \le \frac{1}{2} b_{k+1}.$$

It follows that $g_k(J_k) \subset \frac{1}{2}R_{k+1}$.

Consider next the horizontal segment $Z_{k,y} = \left[x_k - b_k^{\frac{1}{2}}, x_k + b_k^{\frac{1}{2}}\right] \times \{y\}$ in R_k with $y_k - b_k \le y \le y_k + b_k$. Note that the intersection $Z_{k,y} \cap J_k$ consists of the single point (x_k, y). By our constructions of the unstable foliation $\mathcal{F}^u_{\mathrm{loc}}(\Gamma_m)$ in Subsection 5.1 and the horizontal coordinate on $U(L)$ in Subsection 5.2, the image $f^{N_1} \circ f^{\widehat{i}_k} \circ f^{N_0}(Z_{k,y})$ is a strictly horizontal segment in S with

$$|f^{N_1} \circ f^{\widehat{i}_k} \circ f^{N_0}(Z_{k,y})| \prec (5 \cdot 2^{m-3})^{\widehat{i}_k} b_k^{\frac{1}{2}}.$$

Since $f^{z_k k^2}$ is the linear map $(\sigma^{z_k k^2} x, \lambda^{z_k k^2} y)$ in a small neighborhood of $f^{N_1} \circ f^{\widehat{i}_k} \circ f^{N_0}(Z_{k,y})$ in S, the curve $\widetilde{Z}_{k,y} = h_k(Z_{k,y})$ satisfies

$$|\widetilde{Z}_{k,y}| \prec (5 \cdot 2^{m-3})^{\widehat{i}_k} \omega^{\langle k \rangle} \lambda^{z_k k^2} b_k^{\frac{1}{2}}.$$

By Lemma 8.1-(1),

$$\begin{aligned} |g_k(Z_{k,y})| &\prec (5 \cdot 2^{m-3})^{\widehat{i}_k} \omega^{\langle k \rangle} \lambda^{z_k k^2} b_k^{\frac{1}{2}} \\ &\le \varepsilon^{-\frac{1}{2}} (5 \cdot 2^{m-3})^{\frac{\widehat{i}_k}{2} + \frac{3}{4} p_{k+1}} \omega^{\frac{\langle k \rangle}{2} + \frac{3}{4} q_{k+1}} \left(\lambda \sigma^{1+\eta}\right)^{z_k k^2} b_{k+1} \\ &< \varepsilon^{-\frac{1}{2}} (5 \cdot 2^{m-3})^{\frac{\widehat{i}_k}{2} + \frac{3}{4} p_{k+1}} \omega^{4k^2} \left(\lambda \sigma^{1+\eta}\right)^{z_k k^2} b_{k+1}. \end{aligned}$$

By Lemma 7.1-(2) and (8.3), $\frac{\widehat{i}_k}{2} + \frac{3}{4} p_{k+1} \asymp k$. Since $\omega^4 (\lambda \sigma^{1+\eta})^{z_k} < 1$ by (8.5), there exists an integer $k_0 \ge k_*$ such that, for any $k \ge k_0$, $|g_k(Z_{k,y})| < \frac{1}{2} b_{k+1}$. Since $g_k(Z_{k,y}) \cap g_k(J_k) \ne \emptyset$ and $g_k(J_k) \subset \frac{1}{2}R_{k+1}$, $g_k(Z_{k,y})$ is contained in $\mathrm{Int}R_{k+1}$ for all $y \in [y_k - b_k, y_k + b_k]$. It follows that $g_k(R_k) \subset \mathrm{Int}R_{k+1}$. This completes the proof. □

8.3. **Proof of Theorem A.** Now we will present the proof of Theorem A under the notation same as in the preceding subsection.

Proposition 8.3. *Let f be a C^r diffeomorphism on M contained in a Newhouse open set $\mathcal{N}$ of* $\mathrm{Diff}^r(M)$. *Then there exist C^r-diffeomorphisms f_n on M which admit wandering domains and C^r-converge to f.*

Proof. Let $f_{\underline{\boldsymbol{t}}}$ be the diffeomorphism given in (7.19). By Lemmas 7.6 and 8.2, there exists a perturbation sequence $\underline{\boldsymbol{t}_n} = (\boldsymbol{t}_{n,2}, \boldsymbol{t}_{n,3}, \dots)$ such that $f_n := f_{\underline{\boldsymbol{t}_n}}$ C^r-converge to f. Then the interior D of the rectangle R_{k_0} given in Lemma 8.2 is a wandering domain of f_n. If not, there would exist an $a \in \mathbb{N}$ such that $f_n^a(D) \cap D \neq \emptyset$. Take an integer k with $k \geq k_0$ and $z_k k^2 > a$. By (7.1), there exists an integer $q \geq 1$ such that

$$f_n^{q+z_k k^2}(D) \subset \mathbb{B}^u(\langle k \rangle; \underline{2}^{(k^2)}\, \underline{v}_{\,k+1}\, [\underline{\widehat{w}}_{\,k+1}]^{-1}) \subset \mathbb{B}^u(1;2),$$

see Figure 4.1. Moreover we have

$$f^{q+z_k k^2-a}(D) \subset \mathbb{B}^u(\langle k \rangle + a; \underline{1}^{(a)} \underline{2}^{(k^2)}\, \underline{v}_{\,k+1}\, [\underline{\widehat{w}}_{\,k+1}]^{-1}) \subset \mathbb{B}^u(1;1).$$

Hence the intersection $f_n^{q+z_k k^2-a}(f_n^a(D) \cap D) = f_n^{q+z_k k^2}(D) \cap f_n^{q+z_k k^2-a}(D)$ is empty. This contradicts that $f_n^a(D) \cap D \neq \emptyset$. Thus D is a wandering domain of f_n. It is immediate from our construction of D that $\lim_{j\to\infty} \mathrm{diam}\, f_n^j(D) = 0$. Moreover, for a fixed $x_0 \in D$, one can suppose that the ω-limit set $\omega(x_0)$ contains Λ by taking the words $\underline{v}_{\,k+1} \in \{1,2\}^k$ of (7.1) suitably. Since $\lim_{j\to\infty} \mathrm{diam}\, f_n^j(D) = 0$, we also have $\omega(x) \supset \Lambda$ for any $x \in D$. This shows that D is a wandering domain of f_n. □

Note that the first perturbation of f in Subsection 5.2 does not depend on the sequence $\boldsymbol{z} = \{z_k\}_{k=1}^\infty$ such that each entry z_k is either z_0 or $z_0 + 1$ but the second perturbation in Subsection 7.2 does. The wandering domain $D = \mathrm{Int}R_{k_0}$ also depends on $\boldsymbol{z}$. We express the dependence by $f_{n,\boldsymbol{z}}$ and $D_{\boldsymbol{z}} = \mathrm{Int}R_{k_0,\boldsymbol{z}}$. On the other hand, since the support of the second perturbation is fully contained in $\mathbb{G}^u(0)$, $\mathbb{B}^s \cap \mathbb{B}^u$ is independent of the sequence $\boldsymbol{z}$ for any bridges B^s of K^s_Λ and B^u of K^u_Λ.

Remark 8.4. For any integer $j \geq k_0$, consider the integer m_j defined by

$$\begin{aligned} m_j &= N_2 + z_j j^2 + \langle j \rangle + N_1 + \widehat{i}_j + N_0 \\ &= N_2 + z_j j^2 + \widehat{n}_{j+1} + j^2 + j + N_1 + \widehat{i}_j + N_0, \end{aligned}$$

and $\widehat{m}_k = \sum_{j=k_0}^k m_j$ if $k \geq k_0$. By (8.6), $g_j = f^{m_j}$. Our wandering domain $D_{\boldsymbol{z}}$ satisfies $f_{n,\boldsymbol{z}}^{\widehat{m}_k}(D_{\boldsymbol{z}}) \subset \mathrm{Int}R_{k,\boldsymbol{z}}$ for any integer $k \geq k_0$ and $f_{n,\boldsymbol{z}}^j(D_{\boldsymbol{z}})$ stays in S for j with $\widehat{u}_k \leq j < \widehat{m}_k - N_2$, where $\widehat{u}_k = \widehat{m}_{k-1} + N_0 + \widehat{i}_k + N_1$. See Figure 7.3. Fix a sufficiently large positive integer l and suppose that $k > \max\{k_0, l\}$. From the form of B^u_k of (7.1) and the fact that $f_{n,\boldsymbol{z}}^{\widehat{u}_k}(D_{\boldsymbol{z}}) \subset \mathbb{B}^u_k$, $f_{n,\boldsymbol{z}}^j(D_{\boldsymbol{z}})$ is contained in $\mathbb{B}^u(l;\, \underline{1}^{(l)}) \cap \mathbb{B}^s(l;\, \underline{1}^{(l)})$ for any j with $\widehat{u}_k + l \leq j < \widehat{u}_k + z_k k^2 - l$ and in $\mathbb{B}^u(l;\, \underline{2}^{(l)}) \cap \mathbb{B}^s(l;\, \underline{2}^{(l)})$ for any j with $\widehat{u}_k + z_k k^2 + l \leq j < \widehat{u}_k + z_k k^2 + k^2 - l$. Since by Lemma 7.1-(2) $N_2 + \widehat{n}_{j+1} + j + \widehat{i}_j + N_0 \asymp j$, it follows that

$$\lim_{k\to\infty} \frac{(z_k k^2 - 2l) + (k^2 - 2l)}{\widehat{m}_k - \widehat{m}_{k-1}} = \lim_{k\to\infty} \frac{z_k k^2 + k^2 - 4l}{m_k} = 1.$$

Thus, for any integer $l > 0$ and almost all $j > 0$, we have

$$f_{n,\boldsymbol{z}}^{j}(D_{\boldsymbol{z}}) \subset (\mathbb{B}^u(l;\, \underline{1}^{(l)}) \cap \mathbb{B}^s(l;\, \underline{1}^{(l)})) \cup (\mathbb{B}^u(l;\, \underline{2}^{(l)}) \cap \mathbb{B}^s(l;\, \underline{2}^{(l)})). \tag{8.7}$$

Here '*almost all j*' means that, if $d_l(a)$ is the number of integers j $(0 < j \le a)$ satisfying (8.7) for $a > 0$, then $\lim_{a\to\infty} d_l(a)/a = 1$ holds.

Now we are ready to prove our main theorem.

Proof of Theorem A. Let $f_{n,\boldsymbol{z}}$ be the diffeomorphism and $D_{\boldsymbol{z}}$ the wandering domain of $f_{n,\boldsymbol{z}}$ as above. We will show that the sequence $\boldsymbol{z} = \{z_k\}_{k=1}^{\infty}$ can be chosen so that, for any $x \in D_{\boldsymbol{z}}$, the forward orbit $\{x, f_{n,\boldsymbol{z}}(x), f_{n,\boldsymbol{z}}^2(x), \dots\}$ has historic behavior.

Note that the horseshoe Λ has two fixed points p, $\widehat{p}$ with

$$\{p\} = \bigcap_{l=1}^{\infty} \mathbb{B}^u(l;\, \underline{1}^{(l)}) \cap \mathbb{B}^s(l;\, \underline{1}^{(l)}) \quad \text{and} \quad \{\widehat{p}\} = \bigcap_{l=1}^{\infty} \mathbb{B}^u(l;\, \underline{2}^{(l)}) \cap \mathbb{B}^s(l;\, \underline{2}^{(l)}).$$

Consider the space $\mathcal{P}(M)$ of probability measures on M with the weak topology. For any $x \in D_{\boldsymbol{z}} = \mathrm{Int}R_{k_0,\boldsymbol{z}}$ and any non-negative integer m, the element $\mu_x(m)$ of $\mathcal{P}(M)$ is defined as

$$\mu_x(m) = \frac{1}{m+1} \sum_{i=0}^{m} \delta_{f_{n,\boldsymbol{z}}^i(x)}.$$

We are concerned with the subsequence $\{\mu_x(\widehat{m}_k)\}_{k=k_0}^{\infty}$ of $\{\mu_x(m)\}$. Let ν_0 and ν_1 be the elements of $\mathcal{P}(M)$ defined as

$$\nu_0 = \frac{1}{z_0+1}\big(z_0\delta_p + \delta_{\widehat{p}}\big) \quad \text{and} \quad \nu_1 = \frac{1}{z_0+2}\big((z_0+1)\delta_p + \delta_{\widehat{p}}\big),$$

and U_0, U_1 arbitrarily small neighborhoods of ν_0 and ν_1 in $\mathcal{P}(M)$ with $U_0 \cap U_1 = \emptyset$ respectively.

Let k_1 be an integer sufficiently larger than k_0. Then one can suppose that the integer $l < k_1$ in (8.7) is also large enough. If we take the entries of $\boldsymbol{z}$ so that $z_k = z_0$ for $k = 1, \dots, k_1$, then $\mu_x(\widehat{m}_{k_1})$ is contained in U_0.

For any integer $m' > m$, let $\mu_x(m', m)$ be the measure on M defined as $\mu_x(m', m) = \frac{1}{m'+1} \sum_{i=m+1}^{m'} \delta_{f_n^i(x)}$. If $k > k_1$, then

$$\mu_x(\widehat{m}_k) = \frac{\widehat{m}_{k_1}+1}{\widehat{m}_k+1}\mu_x(\widehat{m}_{k_1}) + \mu_x(\widehat{m}_k, \widehat{m}_{k_1}).$$

Thus the contribution of the first term goes to zero as $k \to \infty$. It follows that, if k_2 is sufficiently larger than k_1, then for any sequence $\boldsymbol{z} = \{z_j\}_{j=1}^{\infty}$ with $z_j = z_0$ $(j = 1, \dots, k_1)$ and $z_j = z_0 + 1$ $(j = k_1 + 1, \dots, k_2)$, $\mu_x(\widehat{m}_{k_2})$ is contained in U_1.

Repeating a similar argument, we have a monotone increasing sequence $\{k_a\}_{a=1}^{\infty}$ such that $\{\mu_x(\widehat{m}_{k_a}, \boldsymbol{z})\}_{a=1}^{\infty}$ has two subsequences one of which converges to ν_0 and the other to ν_1, where $\boldsymbol{z} = \{z_j\}_{j=1}^{\infty}$ is the sequence with

$$\begin{cases} z_j = z_0 & \text{for} \;\; j = 1, \dots, k_1,\ k_{2a}+1, \dots, k_{2a+1}\ (a = 1, 2, \dots) \\ z_j = z_0 + 1 & \text{for} \;\; j = k_{2a-1}+1, \dots, k_{2a}\ (a = 1, 2, \dots). \end{cases}$$

In particular, this implies that the limit of $\mu_x(m)$ does not exist. It follows that, for any $x \in D_{\boldsymbol{z}}$, the forward orbit of x under $f_{n,\boldsymbol{z}}$ is historic. This completes the proof of our main theorem. □

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