MODULI OF 3-DIMENSIONAL DIFFEOMORPHISMS WITH SADDLE-FOCI

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Abstract. We consider a space $\mathcal{U}$ of 3-dimensional diffeomorphisms f with hyperbolic fixed points p the stable and unstable manifolds of which have quadratic tangencies and satisfying some open conditions and such that $Df(p)$ has non-real expanding eigenvalues and a real contracting eigenvalue. The aim of this paper is to study moduli of diffeomorphisms in $\mathcal{U}$. We show that, for a generic element f of $\mathcal{U}$, all the eigenvalues of $Df(p)$ are moduli and the restriction of a conjugacy homeomorphism to a local unstable manifold is a uniquely determined linear conformal map.

The topological classification of structurally unstable diffeomorphisms or vector fields on a manifold M is an important subject in the study of dynamical systems. Palis [Pa] suggested that moduli play important roles in such a classification. For a subspace $\mathcal{N}$ of the diffeomorphism space $\mathrm{Diff}^r(M)$ with $r \geq 1$, we say that a value $m(f)$ determined by $f \in \mathcal{N}$ is a *modulus* in $\mathcal{N}$ if $m(g) = m(f)$ holds for any $g \in \mathcal{N}$ topologically conjugate to f, that is, there exists a homeomorphism $h : M \to M$ with $g = h \circ f \circ h^{-1}$. A modulus for a certain class of vector fields

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is defined similarly. We say that a set $\mu_{\mathcal{N}}$ of moduli is *complete* if any f, $g \in \mathcal{N}$ with $m(f) = m(g)$ for all $m \in \mu_{\mathcal{N}}$ are topologically conjugate. For given vector fields X, Y on M, a candidate for a conjugacy homeomorphism between X and Y is found in a usual manner. In many cases, such a map is well defined in a most part of M. So it remains to show that the map is extended to a homeomorphism on M by using the condition that X and Y have the same value for any moduli in $\mu_{\mathcal{N}}$. On the other hand, in the diffeomorphism case, it would be difficult to find a complete set of moduli except for very restricted classes $\mathcal{N}$ in $\mathrm{Diff}^r(M)$.

First we consider the case that $\dim M = 2$ and f_j $(j = 0, 1)$ are elements of $\mathrm{Diff}^r(M)$ $(r \geq 2)$ with two saddle fixed points p_j, q_j. Suppose moreover that $W^u(p_j)$ and $W^s(q_j)$ have a quadratic heteroclinic tangency r_j and there exists a conjugacy homeomorphism h between f_1 and f_2 with $h(p_0) = p_1$, $h(q_0) = q_1$ and $h(r_0) = r_1$. Then, Palis [Pa] proved that $\dfrac{\log|\lambda_0|}{\log|\mu_0|} = \dfrac{\log|\lambda_1|}{\log|\mu_1|}$ holds under ordinary conditions, where λ_j is the contracting eigenvalue of $Df(p_j)$ and μ_i is the expanding eigenvalue of $Df(q_j)$. In [Po], Posthumus proved that the homoclinic version of Palis' results. In fact, he proved that, if f_j $(j = 0, 1)$ has a saddle fixed point p_j with a homoclinic quadratic tangency, then $\dfrac{\log|\lambda_0|}{\log|\mu_0|} = \dfrac{\log|\lambda_1|}{\log|\mu_1|}$ holds, where λ_j, μ_j are the contracting and expanding eigenvalues of $Df(p_i)$. Moreover, he showed that, by using some results of de Melo [dM], $\lambda_0 = \lambda_1$ and $\mu_0 = \mu_1$ hold if $\dfrac{\log|\lambda_0|}{\log|\mu_0|}$ is irrational. We refer to [dMP, dMvS, PT, MP1, GPvS, Ha] and references therein for more results on moduli of 2-dimensional diffeomorphisms. Moduli for 2-dimensional flows with saddle-connections are studied by Palis [Pa] and Takens [Ta] and so on. In those papers, they present finite sets of moduli which are complete in a neighborhood of the saddle connection in M.

In this paper, we consider 3-dimensional diffeomorphisms f with a hyperbolic fixed point p such that $W^u(p)$ and $W^s(p)$ have a quadratic tangency and $Df(p)$ has non-real expanding eigenvalues $re^{\pm\sqrt{-1}\theta}$ with $r > 1$ and a contracting eigenvalue $0 < \lambda < 1$. Moduli for diffeomorphisms of dimension more than two have been already studied by [NPT, Du2, MP2] and so on.

First we will prove the following theorem.

Theorem A. *Let M be a 3-manifold and f_j $(j = 0, 1)$ elements of $\mathrm{Diff}^r(M)$ for some $r \geq 3$ which have hyperbolic fixed points p_j and homoclinic quadratic tangencies q_j positively associated with p_j and satisfy the following conditions.*

- *For $j = 0, 1$, there exists a neighborhood $U(p_j)$ of p_j in M such that $f_j|_{U(p_j)}$ is linear and $Df_j(p_j)$ has non-real eigenvalues $r_j e^{\pm\sqrt{-1}\theta_j}$ and a real eigenvalue λ_j with $r_j > 1$, $\theta_j \neq 0 \mod \pi$ and $0 < \lambda_j < 1$.*
- *f_0 is topologically conjugate to f_1 by a homeomorphism $h : M \to M$ with $h(p_0) = p_1$ and $h(q_0) = q_1$.*

Then the following (1) and (2) hold.

(1) $\dfrac{\log \lambda_0}{\log r_0} = \dfrac{\log \lambda_1}{\log r_1}$.

(2) *Either $\theta_0 = \theta_1$ or $\theta_0 = -\theta_1 \mod 2\pi$.*

Here we say that a homoclinic quadratic tangency q_0 is *positively associated* with p_0 if both $f_0^n(q_0)$ and $f_0^{-n}(\alpha)$ lie in the same component of $U(p_0) \setminus W^u_{\mathrm{loc}}(p_0)$ for a sufficiently large $n \in \mathbb{N}$ and any small curve α in $W^s(p_0)$ containing q_0. Theorem A holds also in the case when $\theta_0 = 0 \mod \pi$ or $-1 < \lambda_j < 0$ except for some rare case, see Remark 1.1 for details.

Remark 0.1. Assertion (1) of Theorem A is implied in the case (D) of Theorem 1.1 in [NPT, Chapter III]. Assertion (2) is also proved by Dufraine [Du2] under weaker assumptions. The author used non-spiral curves in $W^u_{\mathrm{loc}}(p)$ emanating from p. On the other hand, we employ unstable bent disks defined in Section 1 which are originally introduced by Nishizawa [Ni]. By using such disks, we construct a convergent sequence of mutually parallel straight segments in $W^u_{\mathrm{loc}}(p)$ which are mapped to straight segments in $W^u_{\mathrm{loc}}(h(p))$ by h, see Figure 3.1. An *advantage* of our proof is that these sequences are applicable to prove our main theorem, Theorem B below.

Results corresponding to Theorem A for 3-dimensional flows with Shilnikov cycles are obtained by Togawa [To], Carvalho-Rodrigues [CR] and for those with connections of saddle-foci by Bonatti-Dufraine [BD], Dufraine [Du1], Rodrigues [Ro] and so on. See the Section 2 in [Ro] for details. Moreover Carvalho and Rodrigues [CR] present results on moduli of 3-dimensional flows with Bykov cycles.

Theorem B. *Under the assumptions in Theorem A, suppose moreover that $\theta_0/2\pi$ is irrational. Then the following conditions hold.*

(1) $\lambda_0 = \lambda_1$ *and* $r_0 = r_1$.
(2) *The restriction* $h|_{W^u_{\mathrm{loc}}(p_0)} : W^u_{\mathrm{loc}}(p_0) \to W^u_{\mathrm{loc}}(p_1)$ *is a uniquely determined linear conformal map.*

In contrast to Posthumus' results for 2-dimensional diffeomorphisms, the eigenvalues λ_0 and r_0 are proved to be moduli without the assumption that $\dfrac{\log \lambda_0}{\log r_0}$ is irrational.

The restriction $h|_{W^u_{\mathrm{loc}}(p_0)}$ is said to be a *linear conformal map* if $h|_{W^u_{\mathrm{loc}}(p_0)}$ is represented as $h|_{W^u_{\mathrm{loc}}(p_0)}(z) = \rho e^{\sqrt{-1}\omega} z$ ($z \in W^u_{\mathrm{loc}}(p_0)$) for some $\rho \in \mathbb{R} \setminus \{0\}$ and $\omega \in \mathbb{R}$ under the natural identification of $W^u_{\mathrm{loc}}(p_0)$, $W^u_{\mathrm{loc}}(p_1)$ with neighborhoods of the origin in $\mathbb{C}$ via their linearizing coordinates.

For any $r_j > 1$ and $\theta_j \in \mathbb{R}$ ($j = 0, 1$), let $\varphi_j : \mathbb{C} \to \mathbb{C}$ be the map defined by $\varphi_j(z) = r_j e^{\sqrt{-1}\theta_j} z$. Then there are many choices of conjugacy homeomorphisms on $\mathbb{C}$ for φ_0 and φ_1. For example, we take two-sided Jordan curves Γ_j in $\mathbb{C}$ with $\varphi_j(\Gamma_j) \cap \Gamma_j = \emptyset$ and bounding disks in $\mathbb{C}$ containing the origin arbitrarily. Then there exists a conjugacy homeomorphism $h : \mathbb{C} \to \mathbb{C}$ for φ_0 and φ_1 with $h(\Gamma_0) = \Gamma_1$. On the other hand, Theorem B (2) implies that we have severe constraints in the choice of conjugacy homeomorphisms for 3-dimensional diffeomorphisms as above. Intuitively, it says that only a homeomorphism h with $h|_{W^u_{\mathrm{loc}}}(p)$ linear and conformal can be a candidate for a conjugacy between f_0 and f_1. As an application of the linearity and conformality of $h|_{W^u_{\mathrm{loc}}}(p)$, we will present a new modulus for f_0 other than θ_0, λ_0, r_0, see Corollary C in Section 4.

1. Front curves and folding curves

For $j = 0, 1$, let f_j be a diffeomorphism and q_j a quadratic tangency associated with a hyperbolic fixed point p_j satisfying the conditions of Theorem A. We will define in this section front curves in $W^u(p_j)$ and folding curves in $W^u_{\mathrm{loc}}(p_j)$ and show in the next section that these curves converge to straight segments which are preserved by any conjugacy homeomorphism between f_0 and f_1.

We set $f_0 = f$, $p_0 = p$, $q_0 = q$, $r_0 = r$, $\theta_0 = \theta$ and $\lambda_0 = \lambda$ for short. Similarly, let $f_1 = f'$, $p_1 = p'$, $q_1 = q'$, $r_1 = r'$, $\theta_1 = \theta'$ and $\lambda_1 = \lambda'$. Suppose that $(z,t) = (x,y,t)$ with $z = x + \sqrt{-1}y$ is a coordinate around p with respect to which f is linear. For a small $a > 0$, let $D_a(p)$ be the disk $\{z \in \mathbb{C}\,; |z| \le a\}$. We may assume that q is contained in the interior of $D_a(p) \times \{0\} \subset W^u_{\mathrm{loc}}(p)$ and $\widehat{q} = f^N(q)$ is in the interior of the upper half $W^{s+}_{\mathrm{loc}}(p) = \{0\} \times [0,a]$ of $W^s_{\mathrm{loc}}(p)$ for some $N \in \mathbb{N}$. See Figure 1.1. Let $U_a(p)$ be the circular column in the coordinate neighborhood defined by

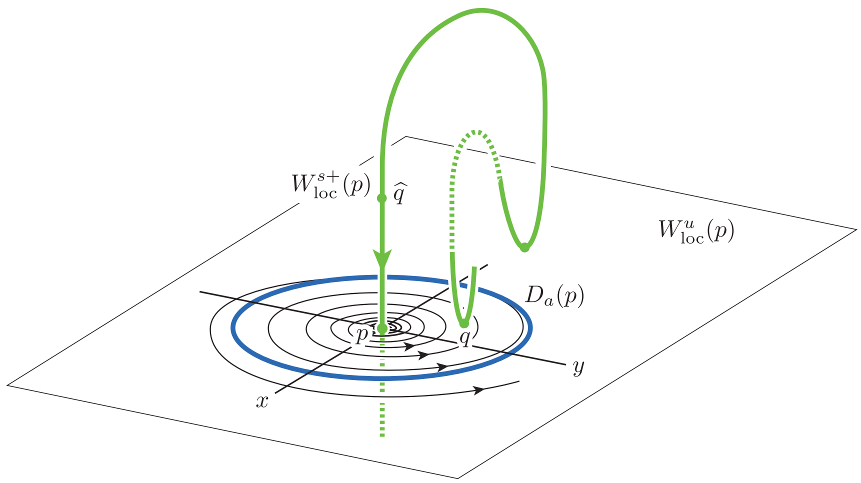

FIGURE 1.1. A saddle-focus p and a homoclinic quadratic tangency q in $D_a(p)$.

$U_a(p) = D_a(p) \times [0,a]$ and $V_{\widehat{q}}$ a small neighborhood of $\widehat{q}$ in $U_a(p)$. Suppose that $U_a(p)$ has the Euclidean metric induced from the linearizing coordinate on $U_a(p)$. By choosing the coordinate suitably and replacing θ by $-\theta$ if necessary, we may assume that the restriction $f|_{D_a(p)}$ is represented as $re^{\sqrt{-1}\theta}z$ for $z \in \mathbb{C}$ with $|z| < a$. Similarly, one can suppose that $f'|_{D_{a'}(p')}$ is represented as $r'e^{\sqrt{-1}\theta'}z$ for some $a' > 0$. The orthogonal projection $\mathrm{pr} : U_a(p) \to D_a(p)$ is defined by $\mathrm{pr}(x,y,t) = (x,y)$.

In this section, we construct an unstable bent disk $\widetilde{H}_0$ in $W^u(p) \cap U_a(p)$, the front curve $\widetilde{\gamma}_0$ in $\widetilde{H}_0$ and the folding curves γ_0 in $U_a(p)$. We also define the sequence of unstable bent disks $\widetilde{H}_m$ in $W^u(p) \cap U_a(p)$ converging to $\widetilde{H}_0$, which will be used in the next section to construct the sequence of front curves converging to $\widetilde{\gamma}_0$.

1.1. **Construction of unstable bent disks, front curves and folding curves.** We set $\widehat{q} = (0, t_0)$. Let $\widetilde{H}$ be the component of $W^u(p) \cap V_{\widehat{q}}$ containing $\widehat{q}$. One can retake the linearizing coordinate on $\mathbb{C}$ if necessary so that the line in $V_{\widehat{q}}$ passing through $\widehat{q}$ and parallel to the x-axis in $U_a(p)$ meets $\widetilde{H}$ transversely. Then $\widetilde{H}$ is represented as the graph of a C^r-function $x = \varphi(y,t)$ with

$$\varphi(0,t_0) = 0, \quad \frac{\partial \varphi}{\partial t}(0,t_0) = 0 \quad \text{and} \quad \frac{\partial^2 \varphi}{\partial t^2}(0,t_0) \neq 0. \tag{1.1}$$

By the implicit function theorem, there exists a C^{r-1}-function $t = \eta(y)$ defined in a small neighborhood V of 0 in the y-axis and satisfying $\eta(0) = t_0$ and $\partial\varphi(y, \eta(y))/\partial t = 0$. Then the curve $\widetilde{\gamma}$ in $V_{\widehat{q}}$ parametrized by $\big(\varphi(y, \eta(y)), y, \eta(y)\big)$ divides $\widetilde{H}$ into two components and $\gamma = \mathrm{pr}(\widetilde{\gamma})$ is a C^{r-1}-curve embedded in $D_a(p)$. Let $\widetilde{H}^+$ (resp. $\widetilde{H}^-$) be the closure of the upper (resp. lower) component of $\widetilde{H} \setminus \widetilde{\gamma}$. For a sufficiently large $n_0 \in \mathbb{N}$, the component $\widetilde{H}_0$ of $f^{n_0}(\widetilde{H}) \cap U_a(p)$ containing $q_0 = f^{n_0}(\widehat{q})$ is an *unstable bent disk* in $U_a(p)$ such that $\partial\widetilde{H}_0$ is a simple closed C^r-curve in $\partial_{\mathrm{side}} U_a(p)$, where

$$\partial_{\mathrm{side}} U_a(p) = \{(x, t) \in \mathbb{C} \times \mathbb{R}\,; |z| = a, 0 \leq t < a\} \subset \partial U_a(p).$$

See Figure 1.2. We set $\widetilde{\gamma}_0 = f^{n_0}(\widetilde{\gamma}) \cap \widetilde{H}_0$, $\widetilde{H}_0^+ = f^{n_0}(\widetilde{H}^+) \cap \widetilde{H}_0$, $\widetilde{H}_0^- = f^{n_0}(\widetilde{H}^-) \cap \widetilde{H}_0$, $H_0 = \mathrm{pr}(\widetilde{H}_0^+) = \mathrm{pr}(\widetilde{H}_0^-)$ and $\gamma_0 = \mathrm{pr}(\widetilde{\gamma}_0)$. Then $\widetilde{\gamma}_0$ is called the *front curve* of $\widetilde{H}_0$ and γ_0 is the *folding curve* of H_0.

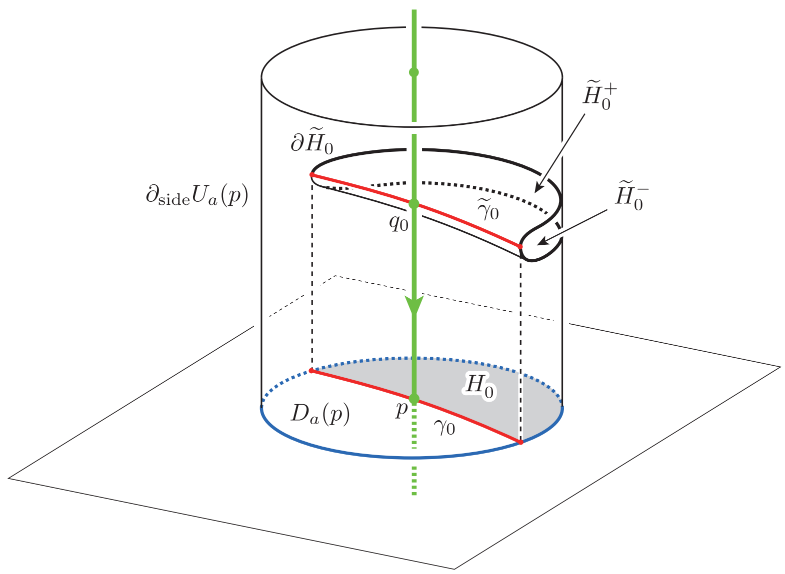

FIGURE 1.2. The front curve $\widetilde{\gamma}_0$ divides $\widetilde{H}_0$ into the two sheets $\widetilde{H}_0^+$ and $\widetilde{H}_0^-$. The folding curve γ_0 of H_0 is the orthogonal image of $\widetilde{\gamma}_0$.

We note that Nishizawa [Ni] has studied unstable bent disks similar to $\widetilde{H}_0$ as above in a different situation. In fact, he considered a 3-dimensional diffeomorphism g which has a saddle fixed point s such that all the eigenvalues of $Dg(s)$ are real and has a homoclinic quadratic tangency associated with s. Here we consider the component $\widetilde{H}_{0;u}^-$ of $f^u(\widetilde{H}_0^-) \cap U_a(p)$ containing $f^u(q_0)$ for $u \in \mathbb{N}$. Since the homoclinic tangency q is positively associated with p, one can show that there exists $\widetilde{H}_{0;u}^-$ which meets $W^s(p)$ transversely at a point $\widehat{z}$ near q by using an argument similar to that in [Ni, Lemma 4.4]. See Figure 1.3. To show the claim, the assumption of $\theta_0 \neq 0 \mod \pi$ in Theorem A is crucial. In fact, the condition implies that the following property:

(P) There exists an arbitrarily large u such that the interior of $H_{0;u} = \mathrm{pr}(\widetilde{H}_{0;u}^-)$ in $D_a(p)$ contains q.

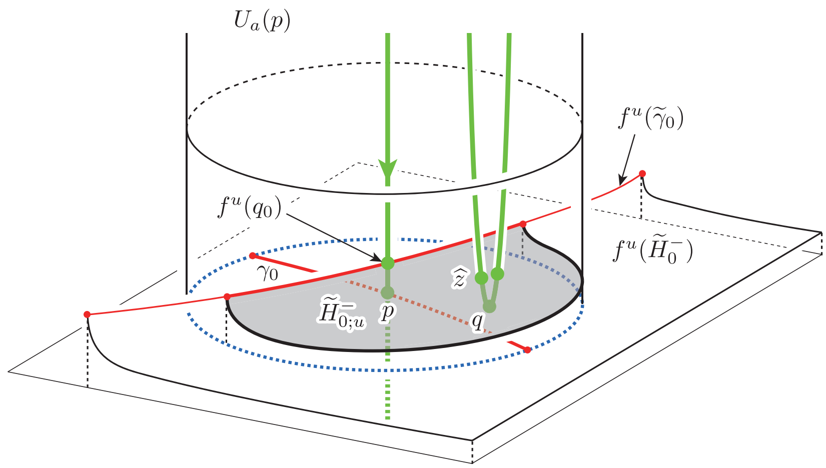

FIGURE 1.3. The half disk $\widetilde{H}_{0;u}^-$ meets $W^s(p)$ transversely at two points near q, one of which is $\widehat{z}$.

Remark 1.1. (1) We here suppose $\theta = 0 \mod \pi$. Even in this case, if f has the property (P), then the component of $W^s(p)$ containing q and $W^u(p)$ have a homoclinic transverse intersection point. Then Theorems A and B will be proved quite similarly. Since $\theta = 0 \mod \pi$, all $f^u(\gamma_0)$ are tangent to a unique straight segment γ_∞ in $D_a(p)$ at p. Thus the property (P) is satisfied if γ_∞ does not pass through q.

(2) Even in the case of $-1 < \lambda < 0$, one can show that f has the property (P) similarly by using f^2 instead of f if $2\theta \neq 0 \mod \pi$. Moreover, since either q or $f(q)$ is a homoclinic tangency positively associated with p, Theorems A and B hold without the assumption that q is positively associated with p.

1.2. **Construction of convergent sequence of unstable bent disks.** Take $v \in \mathbb{N}$ such that $\widehat{z}_0 = f^v(\widehat{z})$ is a point $(0, \widehat{t})$ contained in $U_a(p)$, where $\widehat{z}$ is the transverse intersection point of $\widetilde{H}_{0;u}^-$ and $W^s(p)$ given in the previous subsection. Let D be a small disk in $W^u(p) \cap U_a(p)$ whose interior contains $\widehat{z}_0$. The *absolute slope* $\sigma(\boldsymbol{v})$ of a vector $\boldsymbol{v} = (v_1, v_2, v_3)$ in $U_a(p)$ with $(v_1, v_2) \neq (0,0)$ is given as

$$\sigma(\boldsymbol{v}) = \frac{|v_3|}{\sqrt{v_1^2 + v_2^2}}.$$

The *maximum absolute slope* $\sigma(D)$ of D is defined by

$$\sigma(D) = \max\{\sigma(\boldsymbol{v})\,;\, \text{unit vectors } \boldsymbol{v} \text{ in } U_a(p) \text{ tangent to } D\}.$$

Fix $m_0 \in \mathbb{N}$ such that, for any $m \in \mathbb{N} \cup \{0\}$, the component D_m of $f^{m_0+m}(D) \cap U(p)$ containing $f^{m_0+m}(\widehat{z}_0)$ is a properly embedded disk in $U_a(p)$ with $\partial D_m \subset \partial_{\mathrm{side}} U_a(p)$. Note that D_m intersects $W^s_{\mathrm{loc}}(p)$ transversely at $(0, \lambda^m t_0)$, where $t_0 = \lambda^{m_0}\widehat{t}$. See Figure 1.4. The maximum absolute slope of D_m satisfies

$$\sigma(D_m) \leq \sigma_0 \lambda^m r^{-m}, \tag{1.2}$$

where $\sigma_0 = \sigma(D)\lambda^{m_0} r^{-m_0}$. Consider a short straight segment ρ in $U_a(p)$ meeting $\widetilde{H}_0$ orthogonally at q_0. Then $\widetilde{\rho} = f^{-(N+n_0)}(\rho)$ is a C^r-curve meeting $D_a(p)$

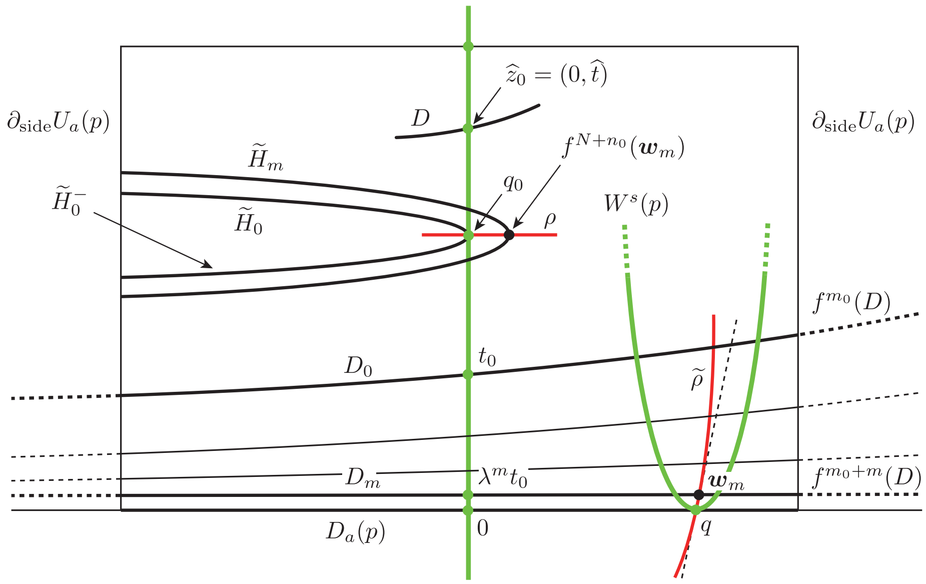

FIGURE 1.4. Trip from $\widetilde{H}_0^-$ to $\widetilde{H}_m$: $f^{u+v}(\widetilde{H}_0^-) \supset D$, $f^{m_0}(D) \supset D_0$, $f^m(D_0) \supset D_m$ and $f^{N+n_0}(D_m) \supset \widetilde{H}_m$, where N, n_0 are the positive integers with $f^N(q) = \widetilde{q}$ and $f^{n_0}(\widetilde{q}) = q_0$. The dotted line passing through q represents a straight segment tangent to $\widetilde{\rho}$ at q.

transversely at q, where N, n_0 are the positive integers given as above. One can choose $m_0 \in \mathbb{N}$ so that, for any $m \in \mathbb{N} \cup \{0\}$, $\widetilde{\rho}$ meets D_m transversely at a single point $\boldsymbol{w}_m = (z_m, s_m)$. Then (1.2) implies that $|t_0\lambda^m - s_m| \le \widetilde{a}\sigma_0\lambda^m r^{-m}$, where $\widetilde{a} = \sup_{m\ge 0}\{|z_m|\} < \infty$. It follows that $s_m = t_0\lambda^m + O(\lambda^m r^{-m})$. Since $\widetilde{\rho}$ has a tangency of order at least two with a straight segment at q,

$$\mathrm{dist}(\boldsymbol{w}_m, q) = \widetilde{t}_0\lambda^m + O(\lambda^m r^{-m}) + O(\lambda^{2m}) = \widetilde{t}_0\lambda^m + o(\lambda^m) \tag{1.3}$$

for some constant $\widetilde{t}_0 > 0$. By the inclination lemma, D_m uniformly C^r-converges to $D_a(p)$. A short curve in $W^s(p)$ containing q as an interior point meets D_m transversely in two points for all sufficiently large m. Let $\widetilde{H}_m$ be the component of $f^{N+n_0}(D_m) \cap U_a(p)$ containing $f^{N+n_0}(\boldsymbol{w}_m)$. Then $\widetilde{H}_m$ C^r-converges to $\widetilde{H}_0$ as $m \to \infty$. By (1.1), there exist C^r-functions $\varphi_m(y,t)$ C^r-converging to φ and representing $\widetilde{H}_m$ as the graph of $x = \varphi_m(y,t)$. Then the front curve $\widetilde{\gamma}_m$ in $\widetilde{H}_m$ is defined as the front curve $\widetilde{\gamma}_0$ in $\widetilde{H}_0$. Since $\partial\varphi_m(y,t)/\partial t$ C^{r-1}-converges to $\partial\varphi(y,t)/\partial t$, $\widetilde{\gamma}_m$ also C^{r-1}-converges to $\widetilde{\gamma}_0$. Note that $\widetilde{\gamma}_m$ divides $\widetilde{H}_m$ into the upper surface $\widetilde{H}_m^+$ and the lower surface $\widetilde{H}_m^-$ with $\widetilde{\gamma}_m = \widetilde{H}_m^+ \cap \widetilde{H}_m^-$ and $H_m = \mathrm{pr}(\widetilde{H}_m) = \mathrm{pr}(\widetilde{H}_m^+) = \mathrm{pr}(\widetilde{H}_m^-)$. The image $\gamma_m = \mathrm{pr}(\widetilde{\gamma}_m)$ is called the folding curve of H_m.

2. Limit straight segments

A curve γ in $D_a(p)$ is called a *straight segment* if γ is a segment with respect to the Euclidean metric on $D_a(p)$. In this section, we will construct a proper straight segment $\gamma_0^\natural$ in $D_a(p)$ with $p \notin \gamma_0^\natural$ which is mapped to a straight segment in $U_{a'}(p')$ by h.

2.1. **Sequences of folding curves converging to straight segments.** Let α be an oriented C^{r-1}-curve in $D_a(p)$ of bounded length. Since $r-1\geq 2$, there exists the maximum absolute curvature $\kappa(\alpha)$ of α. If α passes near the center 0 of $D_a(p)$ and satisfies $\kappa(\alpha)<1/a$, then α has a unique point $z(\alpha)$ with $\mathrm{dist}(0,z(\alpha))=\mathrm{dist}(0,\alpha)$. In fact, if α had two points z_i $(i=1,2)$ with $\mathrm{dist}(0,z_i)=\mathrm{dist}(0,\alpha)$, then for a point z_3 in α with the maximum $\mathrm{dist}(0,z_3)$ between z_1 and z_2, the curvature of α at z_3 is not less than $1/\mathrm{dist}(0,z_3)\geq 1/a$, a contradiction. We denote by $\vartheta(\alpha) \mod 2\pi$ the angle between $\widehat{\alpha}$ and the positive direction of the x-axis at 0, where $\widehat{\alpha}$ is the oriented curve in $D_a(p)$ obtained from α by the parallel translation taking $z(\alpha)$ to 0.

By (1.3), there exists a constant $\widetilde{d}_0>0$ such that

$$\mathrm{dist}(\widetilde{\gamma}_m,\text{the }t\text{-axis})=\widetilde{d}_0(\widetilde{t}_0\lambda^m+o(\lambda^m))+o(\lambda^m)=\widetilde{d}_0\widetilde{t}_0\lambda^m+o(\lambda^m). \tag{2.1}$$

Since γ_m C^{r-1}-converges to γ_0, $\kappa(\gamma_m)$ also converges to $\kappa(\gamma_0)$ as $m\to\infty$. This shows that

$$\sup_m\{\kappa(\gamma_m)\}=\kappa_0<\infty. \tag{2.2}$$

It follows that, for all sufficiently large m, there exists a unique point c_m of γ_m with

$$\mathrm{dist}(c_m,0)=\mathrm{dist}(\gamma_m,0)=\mathrm{dist}(\widetilde{c}_m,\text{the }t\text{-axis})=\mathrm{dist}(\widetilde{\gamma}_m,\text{the }t\text{-axis}),$$

where $\widetilde{c}_m$ is the point of $\widetilde{\gamma}_m$ with $\mathrm{pr}(\widetilde{c}_m)=c_m$.

Fix w with $0<w<a/2$ arbitrarily. For any $n\in\mathbb{N}$, let $m(n)$ be the minimum positive integer such that $f^n(c_m)$ is contained in $D_w(p)$ for any $m\geq m(n)$. Then $\lim_{n\to\infty}m(n)=\infty$ holds. For any $m\geq m(n)$, the component $\widetilde{H}_{m,n}$ of $f^n(\widetilde{H}_m)\cap U_a(p)$ containing $\widetilde{c}_{m,n}=f^n(\widetilde{c}_m)$ is a proper disk in $U_a(p)$ with $\partial\widetilde{H}_{m,n}\subset\partial_{\mathrm{side}}U_a(p)$. Then $\widetilde{\gamma}_{m,n}=f^n(\widetilde{\gamma}_m)\cap\widetilde{H}_{m,n}$ is the front curve of $\widetilde{H}_{m,n}$ and $\gamma_{m,n}=\mathrm{pr}(\widetilde{\gamma}_{m,n})$ is the folding curve of $H_{m,n}=\mathrm{pr}(\widetilde{H}_{m,n})$. Then $c_{m,n}=\mathrm{pr}(\widetilde{c}_{m,n})$ is a unique point of $\gamma_{m,n}$ closest to 0. Here we orient $\widetilde{\gamma}_m=\widetilde{\gamma}_{m,0}$ so that $\widetilde{\gamma}_{m,0}$ C^{r-1}-converges as oriented curves to $\widetilde{\gamma}_0$ as $m\to\infty$. Suppose that $\gamma_{m,n}$ has the orientation induced from that on $\widetilde{\gamma}_{m,0}$ via $\mathrm{pr}\circ f^n$. In particular, it follows that

$$\lim_{m\to\infty}\vartheta(\gamma_{m,0})=\vartheta(\gamma_0). \tag{2.3}$$

We set $d_{m,n}=\mathrm{dist}(c_{m,n},0)$. By (2.1),

$$d_{m,n}=r^n(\widetilde{d}_0\widetilde{t}_0\lambda^m+o(\lambda^m)). \tag{2.4}$$

There exist subsequences $\{m_j\}$, $\{n_j\}$ of $\mathbb{N}$ and $w\lambda/2\leq w_0\leq w$ such that

$$\lim_{j\to\infty}\widetilde{d}_0\widetilde{t}_0\lambda^{m_j}r^{n_j}=w_0. \tag{2.5}$$

If necessary taking subsequences of $\{m_j\}$ and $\{n_j\}$ simultaneously, we may also assume that $\vartheta(\gamma_{m_j,n_j})$ has a limit $\theta^\natural$. Since $f(z)=re^{\sqrt{-1}\theta}z$ on $D_a(p)$, by (2.2) we have

$$\kappa(\gamma_{m_j,n_j})\leq r^{-n_j}\kappa(\gamma_{m_j,0})\leq r^{-n_j}\kappa_0\to 0\quad\text{as}\quad j\to\infty.$$

Thus the following lemma is obtained immediately.

Lemma 2.1. *The sequence γ_{m_j,n_j} uniformly converges as oriented curves to an oriented straight segment $\gamma_0^\natural$ in $D_a(p)$ with $\vartheta(\gamma_0^\natural)=\theta^\natural$ and $\mathrm{dist}(\gamma_0^\natural,0)=w_0$.*

We say that $\gamma_0^\natural$ is the *limit straight segment* of γ_{m_j,n_j}.

2.2. **Limit straight segments preserved by the conjugacy.** Let $U_{a'}(p')$, $U_{b'}(p')$ be the circular columns defined as $U_a(p)$ for some $0 < a' < b'$ which are contained in a coordinate neighborhood around p' with respect to which f' is linear. One can retake $a > 0$ and choose such a', b' so that $U_{a'}(p') \subset h(U_a(p)) \subset U_{b'}(p')$.

Let $\widetilde{H}'_{m,n}$ be the component of $h(\widetilde{H}_{m,n}) \cap U_{a'}(p')$ defined as $\widetilde{H}_{m,n}$ and $\mathrm{pr}(\widetilde{H}'_{m,n}) = H'_{m,n}$. One can define the front and folding curves $\widetilde{\gamma}'_{m,n}$, $\gamma'_{m,n}$ in $\widetilde{H}'_{m,n}$ and $H'_{m,n}$ as $\widetilde{\gamma}_{m,n}$, $\gamma_{m,n}$ in $\widetilde{H}_{m,n}$ and $H_{m,n}$ respectively. See Figure 2.1.

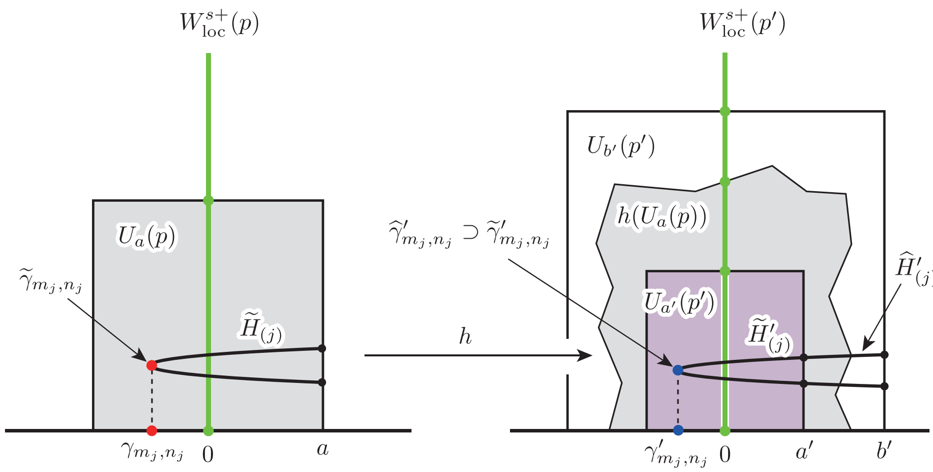

FIGURE 2.1. The image $h(\widetilde{H}_{(j)})$ is contained in $\widehat{H}'_{(j)}$, but $h(\widetilde{H}^{\pm}_{(j)})$ is not necessarily contained in $\widehat{H}'^{\pm}_{(j)}$.

Since h is only supposed to be a homeomorphism, $h(\widetilde{\gamma}_{m,n}) \cap U_{a'}(p')$ would not be equal to $\widetilde{\gamma}'_{m,n}$. We will show that this equality holds in the limit. For the sequences $\{m_j\}$, $\{n_j\}$ given in Section 2, we set $\widetilde{H}_{m_j,n_j} = \widetilde{H}_{(j)}$, $H_{m_j,n_j} = H_{(j)}$, $\widetilde{H}'_{m_j,n_j} = \widetilde{H}'_{(j)}$ and $H'_{m_j,n_j} = H'_{(j)}$ for simplicity. Similarly, suppose that $\widehat{H}'_{(j)}$ is the component of $W^u(p') \cap U_{b'}(p')$ containing $\widetilde{H}'_{(j)}$ and $\widehat{\gamma}'_{m_j,n_1}$ is the front curve of $\widehat{H}'_{(j)}$. The distance between $\boldsymbol{x}$, $\boldsymbol{y}$ in $U_a(p)$ is denoted by $d(\boldsymbol{x}, \boldsymbol{y})$ and that between $\boldsymbol{x}'$, $\boldsymbol{y}'$ in $U_{a'}(p')$ by $d'(\boldsymbol{x}', \boldsymbol{y}')$.

The *path metric* on $\widetilde{H}_{(j)}$ is denoted by $d_{\widetilde{H}_{(j)}}$. That is, for any $\boldsymbol{x}$, $\boldsymbol{y} \in \widetilde{H}_{(j)}$, $d_{\widetilde{H}_{(j)}}(\boldsymbol{x}, \boldsymbol{y})$ is the length of a shortest piecewise smooth curve in $\widetilde{H}_{(j)}$ connecting $\boldsymbol{x}$ with $\boldsymbol{y}$. The path metrics $d_{\widetilde{H}'_{(j)}}$ on $\widetilde{H}'_{(j)}$ and $d_{\widehat{H}'_{(j)}}$ on $\widehat{H}'_{(j)}$ are defined similarly.

Lemma 2.2. (i) *For any $\varepsilon > 0$, there exists a constant $\eta(\varepsilon) > 0$ independent of $j \in \mathbb{N}$ and satisfying the following conditions.*

- $\lim_{\varepsilon \to 0} \eta(\varepsilon) = 0$.
- *Let $\boldsymbol{x}$, $\boldsymbol{y}$ be any points of $\widetilde{H}_{(j)}$ both of which are contained in one of $\widetilde{H}^{+}_{(j)}$ and $\widetilde{H}^{-}_{(j)}$. If $d(\boldsymbol{x}, \boldsymbol{y}) < \eta(\varepsilon)$, then $d_{\widetilde{H}_{(j)}}(\boldsymbol{x}, \boldsymbol{y}) < \varepsilon$.*

(ii) *For any $\varepsilon > 0$, there exists a constant $\delta(\varepsilon) > 0$ independent of $j \in \mathbb{N}$ and satisfying the following conditions.*

- $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$.

- *Let $\boldsymbol{x}$, $\boldsymbol{y}$ be any points of $\widetilde{H}_{(j)}$ both of which are contained in one of $\widetilde{H}^{+}_{(j)}$ and $\widetilde{H}^{-}_{(j)}$. If $d_{\widetilde{H}_{(j)}}(\boldsymbol{x},\boldsymbol{y}) < \delta(\varepsilon)$ and $\boldsymbol{x}' = h(\boldsymbol{x})$ and $\boldsymbol{y}' = h(\boldsymbol{y})$ are contained in $\widetilde{H}'_{(j)}$, then $d_{\widetilde{H}'_{(j)}}(\boldsymbol{x}',\boldsymbol{y}') < \varepsilon$.*

One can take these constants $\eta(\varepsilon)$, $\delta(\varepsilon)$ so that they work also for $d_{\widetilde{H}'_{(j)}}$ and $d_{\widehat{H}'_{(j)}}$.

Proof. (i) The assertion is proved immediately from the fact that $\widetilde{H}^{\pm}_{(j)}$ uniformly converges to a disk $H^{\natural}$ in $D_a(p)$ such that $d(\boldsymbol{x},\boldsymbol{y}) = d_{H^{\natural}}(\boldsymbol{x},\boldsymbol{y})$ for any $\boldsymbol{x},\boldsymbol{y} \in H^{\natural}$.
(ii) Suppose that $\boldsymbol{x},\boldsymbol{y} \in \widetilde{H}^{+}_{(j)}$. First we consider the case that both $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are contained in one of $\widetilde{H}'^{+}_{(j)}$ and $\widetilde{H}'^{-}_{(j)}$, say $\widetilde{H}'^{+}_{(j)}$. If $d_{\widetilde{H}'_{(j)}}(\boldsymbol{x}',\boldsymbol{y}') \geq \varepsilon$, then it follows from the assertion (i) that $d'(\boldsymbol{x}',\boldsymbol{y}') \geq \eta(\varepsilon)$. Since h is uniformly continuous on $U_a(p)$, there exists a constant $\delta_1(\varepsilon) > 0$ with $\lim_{\varepsilon\to 0}\delta_1(\varepsilon) = 0$ and $d(\boldsymbol{x},\boldsymbol{y}) \geq \delta_1(\varepsilon)$. Hence, in particular, $d_{\widetilde{H}_{(j)}}(\boldsymbol{x},\boldsymbol{y}) \geq \delta_1(\varepsilon)$. Thus $d_{\widetilde{H}_{(j)}}(\boldsymbol{x},\boldsymbol{y}) < \delta_1(\varepsilon)$ implies $d_{\widetilde{H}'_{(j)}}(\boldsymbol{x}',\boldsymbol{y}') < \varepsilon$.

Next we suppose that $\boldsymbol{x}' \in \widetilde{H}'^{+}_{(j)}$ and $\boldsymbol{y}' \in \widetilde{H}'^{-}_{(j)}$. Consider a shortest curve α in $\widetilde{H}_{(j)}$ connecting $\boldsymbol{x}$ and $\boldsymbol{y}$. Since $\alpha' = h(\alpha)$ is contained in $\widehat{H}'_{(j)}$, α' intersects $\widehat{\gamma}'_{m_j,n_j}$ non-trivially. Let $\boldsymbol{z}$ be one of the intersection points of α with $h^{-1}(\widehat{\gamma}'_{m_j,n_j})$. See Figure 2.2. Suppose that $d_{\widetilde{H}_{(j)}}(\boldsymbol{x},\boldsymbol{y}) < \delta_1(\varepsilon/2)$. Since $d_{\widetilde{H}_{(j)}}(\boldsymbol{x},\boldsymbol{y}) = d_{\widetilde{H}_{(j)}}(\boldsymbol{x},\boldsymbol{z}) +$

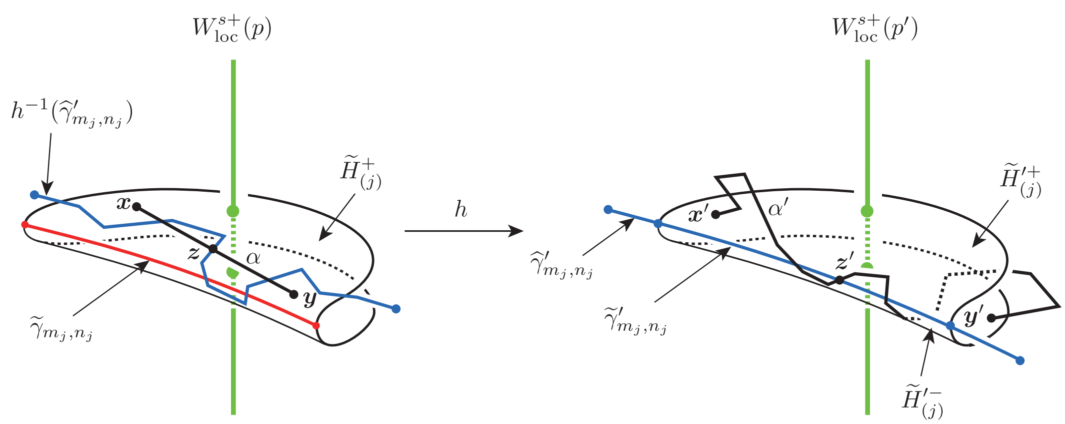

FIGURE 2.2. The case of $\boldsymbol{x},\boldsymbol{y} \in \widetilde{H}^{+}_{(j)}$, $\boldsymbol{x}' \in \widetilde{H}'^{+}_{(j)}$ and $\boldsymbol{y}' \in \widetilde{H}'^{-}_{(j)}$.

$d_{\widetilde{H}_{(j)}}(\boldsymbol{z},\boldsymbol{y})$,

$$d_{\widetilde{H}_{(j)}}(\boldsymbol{x},\boldsymbol{z}) < \delta_1(\varepsilon/2) \quad \text{and} \quad d_{\widetilde{H}_{(j)}}(\boldsymbol{z},\boldsymbol{y}) < \delta_1(\varepsilon/2).$$

Since $\boldsymbol{x}',\boldsymbol{z}' \in \widehat{H}'^{+}_{(j)}$ and $\boldsymbol{z}',\boldsymbol{y}' \in \widehat{H}'^{-}_{(j)}$, by the result in the previous case we have $d_{\widehat{H}'_{(j)}}(\boldsymbol{x}',\boldsymbol{z}') < \varepsilon/2$ and $d_{\widehat{H}'_{(j)}}(\boldsymbol{z}',\boldsymbol{y}') < \varepsilon/2$, and hence

$$d_{\widetilde{H}'_{(j)}}(\boldsymbol{x}',\boldsymbol{y}') = d_{\widehat{H}'_{(j)}}(\boldsymbol{x}',\boldsymbol{y}') < \varepsilon.$$

Thus $\delta(\varepsilon) := \delta_1(\varepsilon/2)$ satisfies the conditions of (ii). □

The following result is a key of this paper.

Lemma 2.3. *For any $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that, for any $j \geq j_0$,*

$$h(\widetilde{\gamma}_{m_j,n_j}) \cap \widetilde{H}'_{(j)} \subset \mathcal{N}_\varepsilon(\widetilde{\gamma}'_{m_j,n_j}, \widetilde{H}'_{(j)}),$$

where $\mathcal{N}_\varepsilon(\widetilde{\gamma}'_{m_j,n_j}, \widetilde{H}'_{(j)})$ is the ε-neighborhood of $\widetilde{\gamma}'_{m_j,n_j}$ in $\widetilde{H}'_{(j)}$.

Figure 2.3 illustrates the situation of Lemma 2.3.

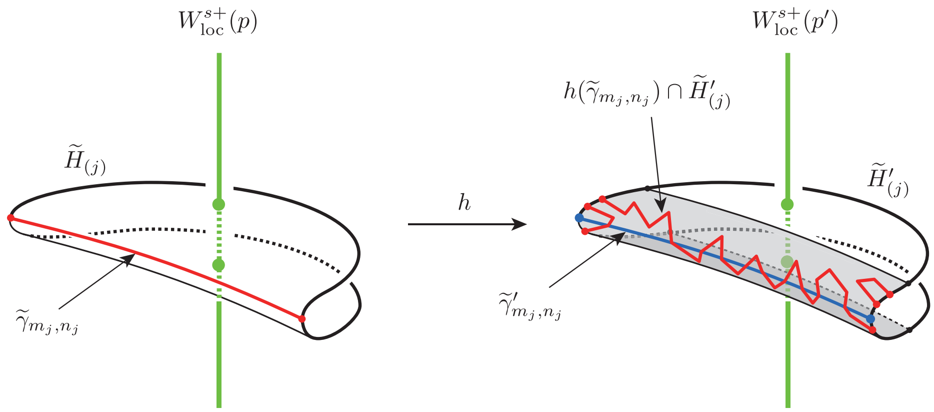

FIGURE 2.3. The shaded region represents $\mathcal{N}_\varepsilon(\widetilde{\gamma}'_{m_j,n_j}, \widetilde{H}'_{(j)})$.

Proof. For $\sigma = \pm$, we will show that $h^{-1}(\widetilde{H}'^{\sigma}_{(j)} \setminus \mathcal{N}_\varepsilon(\gamma'_{m_j,n_j}, \widetilde{H}'_{(j)})) \subset \widetilde{H}^{\sigma}_{(j)}$ for all sufficiently large j. Since $h^{-1}|_{U_{a'}(p')}$ is uniformly continuous, there exists $\nu(\varepsilon) > 0$ such that, for any $\boldsymbol{x}', \boldsymbol{y}' \in U_{a'}(p')$ with $d'(\boldsymbol{x}', \boldsymbol{y}') < \nu(\varepsilon)$, the inequality $d(\boldsymbol{x}, \boldsymbol{y}) < \eta(\delta(\varepsilon))$ holds, where $\boldsymbol{x} = h^{-1}(\boldsymbol{x}')$, $\boldsymbol{y} = h^{-1}(\boldsymbol{y}')$. Since both $\widetilde{H}'^{+}_{(j)}$ and $\widetilde{H}'^{-}_{(j)}$ uniformly converge to the same half disk $H'^{\natural}$ in $D_{a'}(p')$, there exists $j_0 \in \mathbb{N}$ such that, for any $j \geq j_0$ and any $\boldsymbol{x}' \in \widetilde{H}'^{\sigma}_{(j)} \setminus \mathcal{N}_\varepsilon(\widetilde{\gamma}'_{(j)}, \widetilde{H}'_{(j)})$, $d'(\boldsymbol{x}', \boldsymbol{y}')$ is less than $\nu(\varepsilon)$, where $\boldsymbol{y}'$ is the element of $\widetilde{H}'^{-\sigma}_{(j)}$ with $\mathrm{pr}(\boldsymbol{x}') = \mathrm{pr}(\boldsymbol{y}')$. Then we have $d(\boldsymbol{x}, \boldsymbol{y}) < \eta(\delta(\varepsilon))$. If both $\boldsymbol{x}$ and $\boldsymbol{y}$ were contained in one of $\widetilde{H}^{\sigma}_{(j)}$ and $\widetilde{H}^{-\sigma}_{(j)}$, then by Lemma 2.2 (i) $d_{\widetilde{H}_{(j)}}(\boldsymbol{x}, \boldsymbol{y}) < \delta(\varepsilon)$. Then, by Lemma 2.2 (ii), $d_{\widetilde{H}'_{(j)}}(\boldsymbol{x}', \boldsymbol{y}')$ would be less than ε. This contradicts that $\boldsymbol{x}' \in \widetilde{H}'^{\sigma}_{(j)} \setminus \mathcal{N}_\varepsilon(\widetilde{\gamma}'_{m_j,n_j}, \widetilde{H}'_{(j)})$ and $\boldsymbol{y}' \in \widetilde{H}'^{-\sigma}_{(j)}$. See Figure 2.4. Thus, if $\boldsymbol{y}$ is contained in $\widetilde{H}^{\sigma}_{(j)}$, then $\boldsymbol{x}$ is not in $\widetilde{H}^{\sigma}_{(j)}$. In particular, $\boldsymbol{x}$ is not contained in $\widetilde{\gamma}_{m_j,n_j} = \widetilde{H}^{+}_{(j)} \cap \widetilde{H}^{-}_{(j)}$, and so $\widetilde{\gamma}_{m_j,n_j} \cap h^{-1}(\widetilde{H}'^{\sigma}_{(j)} \setminus \mathcal{N}_\varepsilon(\widetilde{\gamma}'_{m,n}, \widetilde{H}'_{(j)})) = \emptyset$. Since $h^{-1}(\widetilde{H}'^{\sigma}_{(j)} \setminus \mathcal{N}_\varepsilon(\widetilde{\gamma}'_{m,n}, \widetilde{H}'_{(j)}))$ is connected, it follows that $h^{-1}(\widetilde{H}'^{\sigma}_{(j)} \setminus \mathcal{N}_\varepsilon(\widetilde{\gamma}'_{m,n}, \widetilde{H}'_{(j)})) \subset \widetilde{H}^{\sigma}_{(j)}$ for $\sigma = \pm$, and hence $h^{-1}(\mathcal{N}_\varepsilon(\widetilde{\gamma}'_{m_j,n_j}, \widetilde{H}'_{(j)})) \supset \widetilde{\gamma}_{m_j,n_j} \cap h^{-1}(\widetilde{H}'_{(j)})$. This completes the proof. □

From the proof of Lemma 2.3, we know that there exists a simple curve in $h(\widetilde{\gamma}_{m_j,n_j}) \cap \widetilde{H}'_{(j)}$ connecting the two components of $\partial \widetilde{H}'_{(j)} \cap \partial \mathcal{N}_\varepsilon(\widetilde{\gamma}'_{m_j,n_j}, \widetilde{H}'_{(j)})$. The following corollary says that the images of certain straight segments in $D_a(p)$ by the homeomorphism h are naturally straight segments in $D_{a'}(p')$.

Corollary 2.4. *For the limit straight segment $\gamma_0^{\natural}$ of γ_{m_j,n_j}, $h(\gamma_0^{\natural}) \cap D_{a'}(p')$ is the limit straight segment of γ'_{m_j,n_j}, i.e. $h(\gamma_0^{\natural}) \cap D_{a'}(p') = \gamma_0'^{\natural}$.*

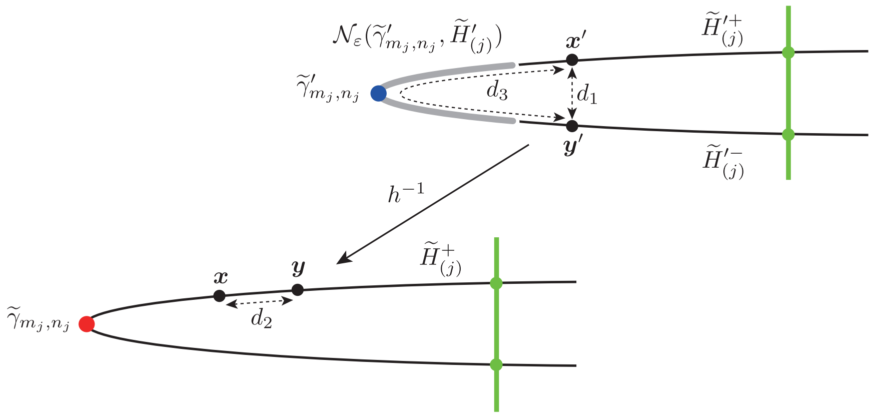

FIGURE 2.4. The situation which does not actually occur. $d_1 := \mathrm{dist}(\boldsymbol{x}', \boldsymbol{y}') < \nu(\varepsilon)$, $d_2 := \mathrm{dist}_{\widetilde{H}_{(j)}}(\boldsymbol{x}, \boldsymbol{y}) < \delta(\varepsilon)$ and $d_3 := \mathrm{dist}_{\widetilde{H}'_{(j)}}(\boldsymbol{x}', \boldsymbol{y}') < \varepsilon$.

Proof. Since $\gamma_0^\natural$ is the limit straight segment of $\widetilde{\gamma}_{m_j,n_j}$ and h is uniformity continuous, $h(\gamma_0^\natural) \cap D_{a'}(p')$ is the limit of $h(\widetilde{\gamma}_{m_j,n_j}) \cap \widetilde{H}'_{(j)}$. It follows from Lemma 2.3 that $h(\gamma_0^\natural) \cap D_{a'}(p')$ is also the limit of $\mathrm{pr}(\widetilde{\gamma}'_{m_j,n_j}) = \gamma'_{m_j,n_j}$, that is, $h(\gamma_0^\natural) \cap D_{a'}(p')$ is equal to the limit straight segment of γ'_{m_j,n_j}. □

For any straight segment l in $D_a(p)$ such that $h(l)$ is also a straight segment in $D_{b'}(p')$, we denote $h(l) \cap D_{a'}(p')$ simply by $h(l)$. In particular, Corollary 2.4 implies that $h(\gamma_0^\natural) = \gamma_0'^\natural$.

3. PROOF OF THEOREM A

Suppose that $\mathrm{St}_a(p)$ is the set of oriented proper straight segments in $D_a(p)$ passing through 0, that is, each element of $\mathrm{St}_a(p)$ is an oriented diameter of the disk $D_a(p)$. For any $l \in \mathrm{St}_a(p)$ and $n \in \mathbb{N}$, the component of $f^n(l) \cap U_a(p)$ containing 0 is also an element of $\mathrm{St}_a(p)$. We denote the element simply by $f^n(l)$.

Since $f^n|_{D_a(p)}$ preserves angles on $D_a(p)$, by (2.3), for any $k, n \in \mathbb{N}$,

$$\vartheta(\gamma_{m,n}) - \vartheta(\gamma_{m+k,n}) = \vartheta(\gamma_{m,0}) - \vartheta(\gamma_{m+k,0}) \to \vartheta(\gamma_0) - \vartheta(\gamma_0) = 0$$

as $m \to \infty$. Moreover it follows from (2.4) that $\lim_{j\to\infty} d_{m_j+k,n_j} = w_0\lambda^k$. By these facts together with Lemma 2.1, one can show that γ_{m_j+k,n_j} uniformly converges as $m \to \infty$ to a straight segment $\gamma_k^\natural$ in $U_a(p)$ with

$$\vartheta(\gamma_k^\natural) = \theta^\natural \quad \text{and} \quad d(0, \gamma_k^\natural) = w_0\lambda^k. \tag{3.1}$$

Thus we have obtained the parallel family $\{\gamma_k^\natural\}$ of oriented straight segments in $D_a(p)$. See Figure 3.1. By Corollary 2.4, $\{\gamma_k'^\natural\}$ with $\gamma_k'^\natural = h(\gamma_k^\natural)$ is also a parallel family of oriented straight segments in $D_{a'}(p')$. Since $\gamma_k'^\natural$ is the limit of γ'_{m_j+k,n_j} as $j \to \infty$, we have the equations

$$\vartheta(\gamma_k'^\natural) = \theta'^\natural \quad \text{and} \quad d(0, \gamma_k'^\natural) = w_0'\lambda'^k. \tag{3.2}$$

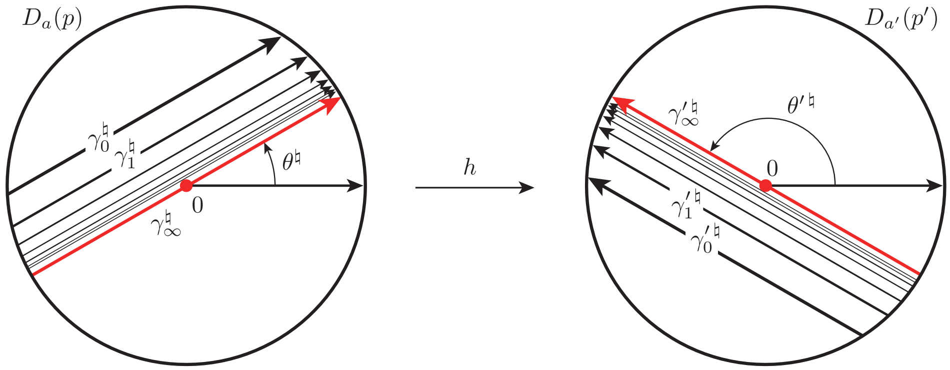

Figure 3.1. The images of the parallel straight segments $\gamma_k^\natural$ in $D_a(p)$ by h.

corresponding to (3.1) for some $\theta'^\natural$ and $w_0' > 0$. Let $\gamma_\infty^\natural \in \mathrm{St}_a(p)$ (resp. $\gamma_\infty'^\natural \in \mathrm{St}_{a'}(p')$) be the limit of $\gamma_k^\natural$ (resp. $\gamma_k'^\natural$).

Proof of Theorem A. By Lemma 2.1 and (2.4), $w_0 = \lim_{j\to\infty} \widetilde{d}_0\widetilde{t}_0\lambda^{m_j}r^{n_j}$. This implies that

$$\lim_{j\to\infty}\left(\frac{m_j}{n_j}\log\lambda + \log r\right) = \lim_{j\to\infty}\frac{1}{n_j}\log\frac{w_0}{\widetilde{d}_0\widetilde{t}_0} = 0$$

and hence $\lim_{j\to\infty}\dfrac{m_j}{n_j} = -\dfrac{\log r}{\log\lambda}$. Applying the same argument to $\gamma'^\natural_{m_j,n_j}$, we also have $\lim_{j\to\infty}\dfrac{m_j}{n_j} = -\dfrac{\log r'}{\log\lambda'}$. This shows the part (1) of Theorem A.

Now we will prove the part (2). For any $n \in \mathbb{N}\cup\{0\}$, we set $f^n(\gamma_\infty^\natural) = \gamma_{\infty,n}^\natural$ and $f'^n(\gamma_\infty'^\natural) = \gamma_{\infty,n}'^\natural$. By Corollary 2.4,

$$h(\gamma_{\infty,n}^\natural) = h(f^n(\gamma_\infty^\natural)) = f'^n(h(\gamma_\infty^\natural)) = f'^n(\gamma_\infty'^\natural) = \gamma_{\infty,n}'^\natural. \tag{3.3}$$

We identify $\mathrm{St}_a(p)$ with the unit circle $S^1 = \{z \in \mathbb{C}\,;\,|z| = 1\}$ by corresponding $l \in \mathrm{St}_a(p)$ to $e^{\sqrt{-1}\vartheta(l)}$. Then the action of f on $\mathrm{St}_a(p)$ is equal to the θ-rotation R_θ on S^1 defined by $R_\theta(z) = e^{\sqrt{-1}\theta}z$.

If $\theta/2\pi = v/u$ for coprime positive integers u, v with $0 \le v < u$. Since $h(\gamma_\infty^\natural) = \gamma_\infty'^\natural$, we have $f'^k(\gamma_\infty'^\natural) \ne \gamma_\infty'^\natural$ for $k = 1,\dots,u-1$ and $f'^u(\gamma_\infty'^\natural) = \gamma_\infty'^\natural$. This implies that $\theta'/2\pi = v'/u$ for some $v' \in \mathbb{N}$ with $0 \le v' < u$. Since $h|_{D_a(p)} : D_a(p) \to D_{a'}(p')$ is a homeomorphism with the correspondence $h(R_\theta^k(\gamma_\infty^\natural)) = R_{\theta'}^k(\gamma_\infty'^\natural)$ $(k = 0,1,\dots,u-1)$, there exists an orientation-preserving homeomorphism $\eta_0 : S^1 \to S^1$ with $\eta_0(e^{\sqrt{-1}(\theta^\natural + k\theta)}) = e^{\sqrt{-1}(\theta'^\natural + k\theta')}$ for $k = 0,1,\dots,u-1$. We set $\Gamma = \{e^{\sqrt{-1}(\theta^\natural+k\theta)};\,k = 0,1,\dots,u-1\}$ and $\Gamma' = \{e^{\sqrt{-1}(\theta'^\natural+k\theta')};\,k = 0,1,\dots,u-1\}$. Then $\left[e^{\sqrt{-1}\theta^\natural}, e^{\sqrt{-1}(\theta^\natural+\theta)}\right) \cap \Gamma$ consists of v points, where $[a,b)$ denotes the positively oriented half-open interval in S^1 for $a, b \in S^1$ with $a \ne b$. Since moreover $\eta_0\big(\left[e^{\sqrt{-1}\theta^\natural}, e^{\sqrt{-1}(\theta^\natural+\theta)}\right) \cap \Gamma\big) = \left[e^{\sqrt{-1}\theta'^\natural}, e^{\sqrt{-1}(\theta'^\natural+\theta')}\right) \cap \Gamma'$ consists of v' points, it follows that $v = v'$, and hence $\theta = \theta'$.

Next we suppose that $\theta/2\pi$ is irrational. Then, for any $l \in \mathrm{St}_a(p)$, there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that the sequence $\gamma_{\infty,n_k}^\natural$ uniformly converges to l as

$k \to \infty$. By (3.3), $\gamma'^{\natural}_{\infty,n_k}$ uniformly converges to $l' = h(l)$. Since $\gamma'^{\natural}_{\infty,n_k} \in \mathrm{St}_{a'}(p')$, l' is also an element of $\mathrm{St}_{a'}(p')$. Thus we have a homeomorphism $\eta : S^1 \to S^1$ with respect to which R_θ and $R_{\theta'}$ are conjugate. Since the rotation number is invariant under topological conjugations, $\theta/2\pi = \theta'/2\pi \mod 1$ holds. This completes the proof of the part (2). □

4. Proof of Theorem B

In this section, we will prove Theorem B. Suppose that f, f' are elements of $\mathrm{Diff}^r(M)$ satisfying the conditions of Theorems A and $\theta/2\pi$ is irrational.

Since $\theta = \theta' \mod 2\pi$, for any $k, j \in \mathbb{N}$,

$$\vartheta(\gamma^{\natural}_{\infty,k}) - \vartheta(\gamma^{\natural}_{\infty,j}) = \vartheta(\gamma'^{\natural}_{\infty,k}) - \vartheta(\gamma'^{\natural}_{\infty,j}) = (k-j)\theta \mod 2\pi. \tag{4.1}$$

Let l_j $(j = 1, 2)$ be any elements of $\mathrm{St}_a(p)$. As in the proof of Theorem A, there exist subsequences $\{n_k\}$, $\{n_j\}$ of $\mathbb{N}$ such that the sequencers $\{\gamma^{\natural}_{\infty,n_k}\}$, $\{\gamma^{\natural}_{\infty,n_j}\}$ uniformly converge to l_1 and l_2 respectively. Then, $\{\gamma'^{\natural}_{\infty,n_k}\}$, $\{\gamma'^{\natural}_{\infty,n_j}\}$ also uniformly converge to the elements $l'_1 = h(l_1)$ and $l'_2 = h(l_2)$ of $\mathrm{St}_{a'}(p')$ respectively. Then, by (4.1),

$$\vartheta(l_2) - \vartheta(l_1) = \vartheta(l'_2) - \vartheta(l'_1) \mod 2\pi. \tag{4.2}$$

For the proof of Theorem B, we need another family of straight segments in $D_a(p)$. Fix an integer a_0 with

$$a_0 > \max\left\{\frac{\log(2r)}{\log(\lambda^{-1})}, \frac{\log(2r')}{\log(\lambda'^{-1})}\right\}.$$

For any $k \geq 0$, we consider the straight segment $\xi^{\natural}_k = f^k(\gamma^{\natural}_{a_0 k}) \cap D_a(p)$. By (3.1),

$$\vartheta(\xi^{\natural}_k) - \vartheta(\xi^{\natural}_0) = k\theta \mod 2\pi \quad \text{and} \quad d(0, \xi^{\natural}_k) = w_0 \lambda^{a_0 k} r^k < 2^{-k} w_0. \tag{4.3}$$

Similarly, by (3.2), $\xi'^{\natural}_k = h(\xi^{\natural}_k)$ is a straight segment in $D_{a'}(p')$ with

$$\vartheta(\xi'^{\natural}_k) - \vartheta(\xi'^{\natural}_0) = k\theta \mod 2\pi \quad \text{and} \quad d(0, \xi'^{\natural}_k) = w'_0 \lambda'^{a_0 k} r'^k < 2^{-k} w'_0. \tag{4.4}$$

Proof of Theorem B. Let α be the element of $\mathrm{St}_a(p)$ with $\vartheta(\xi^{\natural}_0) - \vartheta(\alpha) = \pi/2$ and $\alpha' = h(\alpha) \in \mathrm{St}_{a'}(p')$. We will show that $\theta_{\alpha'} := \vartheta(\xi'^{\natural}_0) - \vartheta(\alpha')$ is also equal to $\pi/2 \mod 2\pi$. See Figure 4.1. In fact, since $\theta/2\pi$ is irrational, by (4.3) there exists a

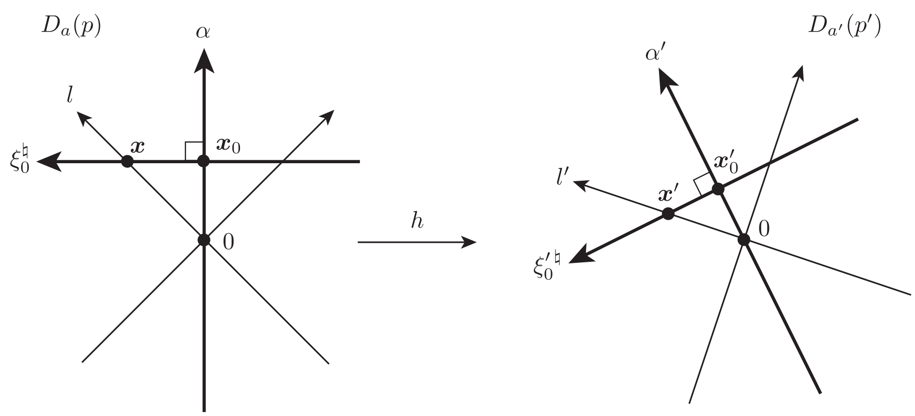

FIGURE 4.1. Correspondence of straight segments via h.

subsequence $\xi^{\natural}_{k_j}$ uniformly converges to α. Since $h|_{D_a(p)}$ is uniformly continuous, $\xi'^{\natural}_{k_j}$ also uniformly converges to α'. On the other hand, since $\vartheta(\xi^{\natural}_{k_j})-\vartheta(\alpha)=k_j\theta+\pi/2 \mod 2\pi$ and $\vartheta(\xi'^{\natural}_{k_j})-\vartheta(\alpha')=k_j\theta+\theta_{\alpha'} \mod 2\pi$,

$$\theta_{\alpha'}-\frac{\pi}{2}=\big(\vartheta(\xi'^{\natural}_{k_j})-\vartheta(\alpha')\big)-\big(\vartheta(\xi^{\natural}_{k_j})-\vartheta(\alpha)\big)\to 0 \mod 2\pi$$

as $j\to\infty$. Thus we have $\theta_{\alpha'}=\pi/2 \mod 2\pi$.

We denote by $z(\boldsymbol{x})\in\mathbb{C}$ the entry of $\boldsymbol{x}\in D_a(p)$ with respect to the linearizing coordinate on $D_a(p)$. Similarly, the entry of $\boldsymbol{x}'\in D_{a'}(p')$ is denoted by $z'(\boldsymbol{x}')$. Let $\boldsymbol{x}_0$ be the intersection point of α and $\xi^{\natural}_0$, and let $\boldsymbol{x}'_0=h(\boldsymbol{x}_0)$. One can set $z(\boldsymbol{x}_0)=\rho_0 e^{\sqrt{-1}\omega_0}$ and $z'(\boldsymbol{x}'_0)=\rho'_0 e^{\sqrt{-1}\omega'_0}$ for some $\rho_0>0$, $\rho'_0>0$ and $\omega_0,\omega'_0\in\mathbb{R}$. We define the new linearizing coordinate on $D_{a'}(p')$ by using the linear conformal map such that, for any $\boldsymbol{x}'\in D_{a'}(p')$, $z'^{\,\mathrm{new}}(\boldsymbol{x}')=\rho_0\rho_0'^{-1}e^{\sqrt{-1}(\omega_0-\omega'_0)}z'(\boldsymbol{x}')$. Then $z(\boldsymbol{x}_0)=z'^{\,\mathrm{new}}(\boldsymbol{x}'_0)$ holds.

For any $\boldsymbol{x}\in\xi^{\natural}_0$, there exists $l\in\mathrm{St}_a(p)$ with $\{\boldsymbol{x}\}=\xi^{\natural}_0\cap l$. Then $\boldsymbol{x}'=h(\boldsymbol{x})$ is the intersection of $\xi'^{\natural}_0$ and $l'=h(l)$. By (4.2), $\vartheta(l)-\vartheta(\alpha)=\vartheta(l')-\vartheta(\alpha') \mod 2\pi$ and hence $z(\boldsymbol{x})=z'^{\,\mathrm{new}}(\boldsymbol{x}')$. We say the property that h is *identical* on $\xi^{\natural}_0$. Since $\theta/2\pi$ is irrational, there exists $k_*\in\mathbb{N}$ satisfying

$$\frac{\pi}{3}\le\vartheta(\xi^{\natural}_{k_*})-\vartheta(\xi^{\natural}_0)\le\frac{\pi}{2} \mod 2\pi.$$

Then $\xi^{\natural}_{k_*}$ meets $\xi^{\natural}_0$ at a single point $\boldsymbol{x}_{k_*}$ in $D_a(p)$. For $\alpha_{k_*}=f^{k_*}(\alpha)$ and $\alpha'_{k_*}=h(\alpha_{k_*})$, we have $\vartheta(\xi^{\natural}_{k_*})-\vartheta(\alpha_{k_*})=\vartheta(\xi'^{\natural}_{k_*})-\vartheta(\alpha'_{k_*})=\pi/2$. Since h is identical at $\boldsymbol{x}_{k_*}$, h is proved to be identical on $\xi^{\natural}_{k_*}$ by an argument as above. Then one can show inductively that, for any $n\in\mathbb{N}$, h is identical on $\xi^{\natural}_{nk_*}$. See Figure 4.2. By (4.3),

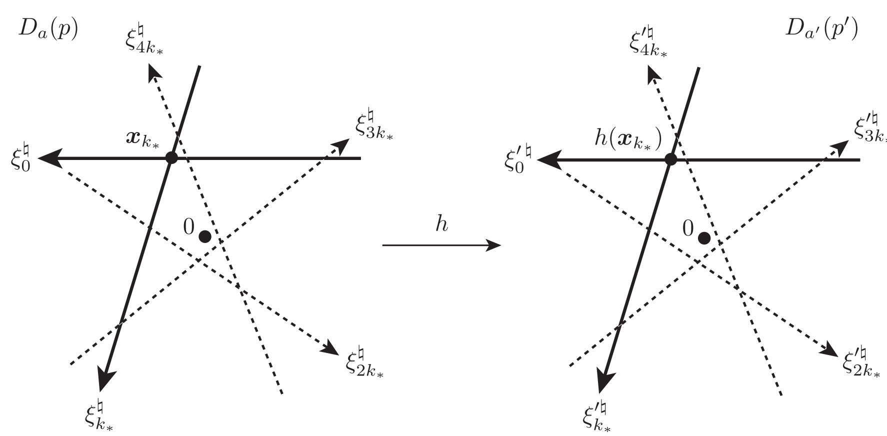

Figure 4.2. Correspondence via h with respect to the new coordinate on $D_{a'}(p')$.

$\lim_{n\to\infty}d(0,\xi^{\natural}_{nk_*})=0$. Since moreover $k_*\theta/2\pi$ is irrational, $\overline{\bigcup_{n=1}^{\infty}\xi^{\natural}_{nk_*}}$ is equal to $D_a(p)$. This shows that h is identical on $D_a(p)$. In particular, this implies that $h|_{D_a(p)}$ is a linear conformal map with respect to the original coordinates. We write $z(q)=\rho_1 e^{\sqrt{-1}\omega_1}$ and $z'(q')=\rho'_1 e^{\sqrt{-1}\omega'_1}$. It follows from the assumption of $h(q)=q'$

in our theorems that $h(z) = \rho_1'\rho_1^{-1}e^{\sqrt{-1}(\omega_1'-\omega_1)}z$ for any $z \in \mathbb{C}$ with $|z| \le a$. In particular, this implies that $h|_{W^u_{\mathrm{loc}}(p)}$ is a linear conformal map. Let $\widetilde{h}$ be any other conjugacy homeomorphism between f and f' satisfying the conditions in Theorems A and B. In particular, $\widetilde{h}(p) = p'$ and $\widetilde{h}(q) = q'$ hold. Since $z(q) = \rho_1 e^{\sqrt{-1}\omega_1}$ and $z'(q') = \rho_1' e^{\sqrt{-1}\omega_1'}$, one can show as above that $\widetilde{h}(z) = \rho_1'\rho_1^{-1}e^{\sqrt{-1}(\omega_1'-\omega_1)}z$ for any $z \in \mathbb{C}$ with $|z| \le a$ and hence $\widetilde{h}|_{D_a(p)} = h|_{D_a(p)}$. This shows the assertion (2) of Theorem B and $r = r'$. Then, by the assertion (1) of Theorem A, we also have $\lambda = \lambda'$. This completes the proof. □

Let $\widehat{z}$ be the homoclinic transverse point of $W^u(p)$ and $W^s(p)$ given in Subsection 1.1. Fix a sufficiently large $n \in \mathbb{N}$ with $s = f^{-n}(\widehat{z}) \in D_p(a)$. Then $s' = h(s)$ is contained in $D_{b'}(p')$. The following corollary shows that $z(s)/z(q)$ is a modulus for f. Recall that $z(\boldsymbol{x}) \in \mathbb{C}$ is the entry of $\boldsymbol{x}$ with respect to the complex linearizing coordinate on $D_a(a)$. The complex number $z'(\boldsymbol{x}')$ is defined similarly for $\boldsymbol{x}' \in D_{a'}(p')$.

Corollary C. *Let f, f' be elements of $\mathrm{Diff}^r(M)$ satisfying the conditions of Theorems A and B, and let h be a conjugacy homeomorphism between f and f' with $h(p) = p'$ and $h(q) = q'$. If $h|_{W^u_{\mathrm{loc}}(p)}$ is orientation-preserving, then $z(s)/z(q) = z'(s')/z'(q')$. Otherwise, $z(s)/z(q) = \overline{z'(s')/z'(q')}$.*

Proof. Here we only consider the case that h is orientation-preserving. Since $h|_{D_a(p)}$ is a linear conformal map, the triangle with vertices $0, z(q), z(s)$ is similar to that with vertices $0, z'(q'), z'(s')$ with respect to Euclidean geometry. This shows $z(s)/z(q) = z'(s')/z'(q')$. □

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