HISTORIC BEHAVIOR IN NON-HYPERBOLIC HOMOCLINIC CLASSES

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Dedicated to Hiroshi Kokubu for his 60^{th} birthday

Abstract. We show that C^1-generically for diffeomorphisms of manifolds of dimension $d \geq 3$, a homoclinic class containing saddles of different indices has a residual subset where the orbit of any point has historic behavior.

1. Introduction

The purpose of this paper is to explore historic behavior in non-hyperbolic invariant sets of higher-dimensional diffeomorphisms. Let us begin by explaining what historic behavior is. For a given continuous map f on a manifold M, $x \in M$ and a continuous function $\varphi : M \to \mathbb{R}$, we consider the sequence of partial averages

$$\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x))$$

along the forward orbit $O^+(x, f) = \{x, f(x), f^2(x), \ldots\}$. If the limit of the above averages exists as $n \to \infty$, it is called the *Birkhoff average* associated with (x, φ). Otherwise, such a phenomenon, which was observed in Bowen's example [Gau92, Tak94], is called historic by Ruelle [Rue01]. To be more precise, we say that $O^+(x, f)$ has *historic behavior* if there is a $\varphi \in C^0(M, \mathbb{R})$ such that the Birkhoff average associated with (x, φ) does not exist, see [Tak08].

The motivation behind the study of historic behavior is the following. The Birkhoff ergodic theorem implies that if μ is a probability measure on M which is *f-invariant*, that is, $\mu(B) = \mu(f^{-1}(B))$ for every measurable set $B \subset M$, the Birkhoff average exists for μ-almost every point $x \in M$. Hence the set of initial points whose forward orbits have historic behavior is of μ-measure zero. However since the Lebesgue measure is in general not f-invariant, the set is not always of Lebesgue measure zero. Furthermore, even if the set has Lebesgue measure zero, it is possible for the set to be topologically large.

In the case of uniformly hyperbolic systems, Takens [Tak08] proves that the existence of a Markov partition induces historic behavior in a residual set. However, his method cannot be applied to a non-hyperbolic invariant set, where the Markov partition is not guaranteed. One of causes of the lack of hyperbolicity is existence of a homoclinic tangency, i.e., a non-transverse intersection between stable and unstable manifolds. It is shown in [KS17] that there exist some persistent classes of

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surface diffeomorphisms with homoclinic tangencies and contracting non-trivial wandering domains in which the orbit of every point has historic behavior, see below for further information. This result can be extended to certain 3-dimensional flows with heteroclinic cycles of periodic solutions [LR17]. Meanwhile, for the geometric Lorenz attractor it is known that there is a residual subset in the trapping region with historic behavior [KLS16].

In this paper we consider invariant sets called *homoclinic classes*, which form the basic pieces of a dynamical system. Properly speaking, for a diffeomorphism with a saddle periodic point P, the *homoclinic class* $H(P)$ is defined as the closure of transverse intersections of the unstable and stable manifolds of the orbit of P. Note that every maximal invariant transitive hyperbolic set is a homoclinic class but the converse is not always true. In fact, a homoclinic class may be non-hyperbolic if it contains a homoclinic tangency or as another possibility has periodic points with different indices (the dimension of the stable manifold). This second situation is related to the notion of a heterodimensional cycle (see in Section 4 or [BDV05, §6]), an important topic in non-hyperbolic dynamical systems, and will be the one studied in this article.

From now on, let M be a compact connected manifold without boundary of dimension $d \geq 3$. Let us introduce the following terminology. A homoclinic class $H(P)$ of a diffeomorphism f has *residual historic behavior* if there is a residual subset R of $H(P)$ and there exists a $\varphi \in C^0(M, \mathbb{R})$ such that the Birkhoff average associated with (x, φ) does not exist for every $x \in R$. The main result of the paper is the following:

Theorem A. *For any C^1-generic diffeomorphism of M, a homoclinic class containing saddles of different indices has residual historic behavior.*

We remind that a property holds C^1-*generically* if there is a residual subset of the space of all C^1-diffeomorphisms satisfying this property. Also observe that the above definition of residual historic behavior is somewhat stronger than the initial one because the continuous function φ is independent of the point x.

Let us continue with two questions related to Theorem A. Entropy and dimension of sets with historic behavior was studied in shifts with the specification property in [BS00] (see also [BLV14, BLV18] where the historic set is called an irregular set). In particular, the set of initial points in a uniformly hyperbolic set whose orbits have historic behavior carries full topological entropy and full Hausdorff dimension. So it is natural to ask whether similar properties hold for homoclinic classes in our setting.

Question 1. *Does the historic set of a generic homoclinic class have full topological entropy or full Hausdorff dimension?*

The other question is a version of Takens' last problem [Tak08]: *are there persistent classes of smooth dynamical systems for which the set of initial states which give rise to orbits with historic behavior has positive Lebesgue measure?* An affirmative answer to this problem is already given in dimension 2 as follows. We say that an open set D is a *contracting non-trivial wandering domain* for a diffeomorphism f if

a) $f^i(D) \cap f^j(D) = \emptyset$ if $i \neq j$,
b) the union of $\omega(x)$ for any $x \in D$ is not equal to a single periodic orbit,
c) the diameter of $f^i(D)$ converges to zero if $i \to \infty$.

As mentioned previously, it is proven in [KS17] that every two-dimensional diffeomorphism in any Newhouse open set (i.e., an open set of C^2 diffeomorphisms which have persistent tangencies associated with some basic sets, see [PT93, §6.1]) is contained in the C^r-closure ($2 \leq r < \infty$) of diffeomorphisms having contracting non-trivial wandering domains, where the orbit of any point has historic behavior. According to a conjecture of Palis, see in [BDV05, §5.5], the cause for lack of hyperbolicity besides homoclinic tangencies is the existence of heterodimensional cycles. Thus the question is as follows.

Question 2. *Does there exist a persistent class near every diffeomorphism having a heterodimensional cycle in which any diffeomorphism has a non-trivial wandering domain D such that the orbit of every point in D has historic behavior?*

Note that [KNS17] gives a condition ensuring that 3-dimensional diffeomorphisms with heterodimensional cycles are C^1-approximated by diffeomorphisms having contracting non-trivial wandering domains along some attracting invariant circles. However, these domains contain no points whose orbits have historic behavior due to the Denjoy-like construction used in [KNS17].

We close the introduction by explaining the structure of the paper and how to obtain Theorem A proven in Section 4. It will be a consequence of our Theorem 3.1 in Section 3 and a key result of [BD12] on the generation of special type of hyperbolic sets, called *blenders*. These sets appear inside homoclinic classes with index variation (containing periodic points of different indices) for generic diffeomorphisms, as is explained in Section 4. Thus in Section 2 we recall the definitions of blenders according to the various levels (Definitions 2.1, 2.3), all of which can be realized by using the covering property (Definition 2.4). Then in Section 3 we give a key result (Theorem 3.1) of this paper, which contains the essential ingredients for obtaining residual historic behavior associated with dynamical blenders.

2. Blenders

Blenders (with codimension one) were initially defined by Bonnatti and Díaz in [BD96] and were used to construct robustly transitive non-hyperbolic diffeomorphisms (see also [BDV05, BD12]). Blenders with larger codimension were studied in [NP12, BKR14, BR17]. We will use the following definitions coming from [BBD16].

Definition 2.1 (*cu*-blender)**.** Let f be a diffeomorphism of a manifold M. A compact set $\Gamma \subset M$ is a *cu-blender of codimension $c > 0$* if

(1) Γ is a transitive maximal f-invariant hyperbolic set in a relatively compact open set U,
$$\Gamma = \bigcap_{n\in\mathbb{Z}} f^n(\overline{U}) \quad \text{and} \quad T_\Gamma M = E^s \oplus E^c \oplus E^{uu},$$
where $E^u = E^c \oplus E^{uu}$ is the unstable bundle, $uu = \dim E^{uu} \geq 1$ and $c = \dim E^c \geq 1$,
(2) there exists an open set $\mathcal{D}$ of C^1-embeddings of uu-dimensional discs, and
(3) there exists a C^1-neighborhood $\mathscr{U}$ of f such that, for all $D \in \mathcal{D}$ and $g \in \mathscr{U}$,
$$W^s_{\mathrm{loc}}(\Gamma_g) \cap D \neq \emptyset,$$
where Γ_g is the continuation of Γ for g.

See in [BBD16, Sec. 3.1] for the topology of the set of such embeddings in (2). The set $\mathcal{D}$ is called the *superposition region* of the blender. A *cs-blender of codimension c* of f is a *cu*-blender of codimension c for f^{-1}. The term of "codimension" above comes from the bifurcation theory as it is explained in [BR19].

2.1. **Dynamical blender.** In [BBD16], the authors introduce the notion of a strictly invariant family of discs as a criterion to obtain a blender, which we explain in what follows.

Let $\mathscr{D}(M)$ be the set of *uu*-dimensional (closed) discs C^1-embedded into M and endowed with the C^1-topology.

Definition 2.2 (f-invariant family, see Def. 3.7 in [BBD16]). A family $\mathcal{D}$ of discs in $\mathscr{D}(M)$ is said to be *strictly f-invariant* if there exists a neighborhood $\mathscr{N}$ of $\mathcal{D}$ in $\mathscr{D}(M)$ such that for every disc $D_0 \in \mathscr{N}$ there is a disc $D_1 \in \mathcal{D}$ with $D_1 \subset f(D_0)$.

Suppose that Γ is a transitive hyperbolic set having a partially hyperbolic splitting $T_\Gamma M = E^s \oplus E^c \oplus E^{uu}$ with $E^u = E^c \oplus E^{uu}$ and $uu = \dim E^{uu}$. Moreover, assume that there exists a strictly f-invariant family of *uu*-dimensional C^1-discs tangent to an invariant expanding cone-field $\mathcal{C}^{uu}$ around E^{uu} (see the precise definition in [BBD16, Sec. 3.2]). Then Γ is actually a *cu*-blender of codimension $c = \dim E^c > 0$, see [BBD16, Lem. 3.14]. Motivated by this result, they introduced the following class of blenders:

Definition 2.3 (Dynamical *cu*-blender). Let f be a diffeomorphism of a manifold M. A compact set $\Gamma \subset M$ is a *dynamical cu-blender of codimension $c > 0$* if

(1) Γ is a transitive maximal f-invariant hyperbolic set in a relatively compact open set U,
$$\Gamma = \bigcap_{n\in\mathbb{Z}} f^n(\overline{U}) \quad \text{and} \quad T_\Gamma M = E^s \oplus E^c \oplus E^{uu}$$
where $E^u = E^c \oplus E^{uu}$ is the unstable bundle, $uu = \dim E^{uu} \geq 1$ and $c = \dim E^c \geq 1$,
(2) there is a strictly Df-invariant cone-field $\mathcal{C}^{uu}$ around E^{uu} which can be extended to $\overline{U}$,
(3) there is a strictly f-invariant family $\mathcal{D}$ of *uu*-dimensional discs in $\mathscr{D}(M)$ such that every disc D in a neighborhood of $\mathcal{D}$ is contained in U and tangent to $\mathcal{C}^{uu}$, i.e.,
$$D \subset U \quad \text{and} \quad T_x D \subset \mathcal{C}^{uu}(x) \quad \text{for all } x \in D.$$

A *dynamical cs-blender of codimension c* for f is a dynamical *cu*-blender of codimension c for f^{-1}.

2.2. **Covering property.** Although Definition 2.3 is very useful, the difficulty is in showing the existence of a strictly invariant family of discs. The following covering criterion helps us to conclude when a hyperbolic set is a dynamical blender. This criterion appeared in [BD96, BDV05] in the case of codimension $c = 1$ and it was extended for large codimension in [NP12, BKR14, BR17, ACW17].

Let Γ be a horseshoe, i.e., a locally maximal invariant hyperbolic set of a diffeomorphism $f : M \to M$ conjugated with a full shift. Assume that f restricted to Γ has a partially hyperbolic splitting $T_\Gamma M = E^s \oplus E^{cu} \oplus E^{uu}$ such that $E^u = E^{cu} \oplus E^{uu}$

is the unstable bundle, $c = \dim E^{cu} \geq 1$, $uu = \dim E^{uu} \geq 1$, and there are positive constants $\hat{\gamma}, \hat{\nu} \in \mathbb{R}$ such that

$$\|Df|_{E^s}\| < 1 < \|Df|_{E^{cu}}\| \leq \hat{\gamma}^{-1} < \hat{\nu}^{-1} < m(Df|_{E^{uu}}).$$

Here $\|F\|$ stands for the operator norm and $m(F) := \|F^{-1}\|^{-1}$ for a given invertible linear map F. Moreover, we also assume that Γ is contained in a chart of M which is in local coordinates $\mathbb{R}^d = \mathbb{R}^{s+uu} \times \mathbb{R}^c$.

For what follows, let us define the sets of vertical and horizontal rectangles. A *vertical* rectangle V on $[-1,1]^{s+uu}$ is a set of the form

$$V = \bigcup_{y\in[-1,1]^{uu}} I_y \times \{y\},$$

where for each y, I_y is a product of $\{uu\}$-tuple closed intervals which depend C^1-continuously on y. A *horizontal* rectangle H is defined in an analogous manner, the union now being taken over $\{x\} \times I_x,\ x \in [-1,1]^s$.

The covering property consists of the conditions we describe next. See Figure 1.

Definition 2.4 (Covering property)**.** There are open sets $B, B_1, \dots, B_k \subset (-1,1)^c$, horizontal and vertical rectangles $H_1, \dots, H_k$ and $V_1, \dots, V_k$ respectively in $[-1,1]^{s+uu}$, satisfying the following:

(1) Γ is the maximal invariant set in $V \times [-1,1]^c$ where $V = V_1 \cup \cdots \cup V_k$.
(2) $H_i \times \overline{B_i} \subset f^{-1}(V_i \times B)$ and $f(H_i \times B_i)$ is a vertical rectangle in $V_i \times B$. The vertical rectangle here is defined similarly as above having the expression

$$\bigcup_{y\in[-1,1]^{uu}} I_y \times \{y\} \times J_y.$$

(3) $\overline{B} \subset B_1 \cup \cdots \cup B_k$ with Lebesgue number $L > 0$.
(4) The local strong unstable manifolds are C^1-embedded graphs in $[-1,1]^d$ of the form

$$D(y) = (h_s(y), y, h_c(y)),\ y \in [-1,1]^{uu},$$

and having the Lipschitz constant $\hat{C}$ of h_c satisfying $\hat{C} < L$.

Notice that condition (4) relates the variation of the cone-field C^{uu} to the Lebesgue number of the cover in condition (3). Moreover, (4) always holds when Γ is a standard affine horseshoe (see [ACW17, Def. 7.4]) or f is a one-step skew-product (see [BR17, Def. 5.1]), since in these examples $\hat{C} = 0$. In both cases the dynamics of f is associated with an iterated function system (IFS for short) generated by contracting maps $\phi_1, \dots, \phi_k$ of $[-1,1]^c$, and then the fourth condition above can be summarized to asking that

$$\overline{B} \subset \phi_1(B) \cup \cdots \cup \phi_k(B). \tag{2.1}$$

Remark 2.5. It follows from [BR17, proof of Thm. A.2] that a horseshoe satisfying the covering property is a dynamical *cu*-blender. Namely, the set $\mathcal{D}$ of strictly f-invariant discs is given by the set of almost-vertical C^1-embedded discs in $V \times B$. These *almost-vertical discs* are defined to be close to the so-called constant vertical discs, which are *uu*-dimensional discs projecting into a single point on $\mathbb{R}^c$. For more details and the precise definitions see [BR17, BR19].

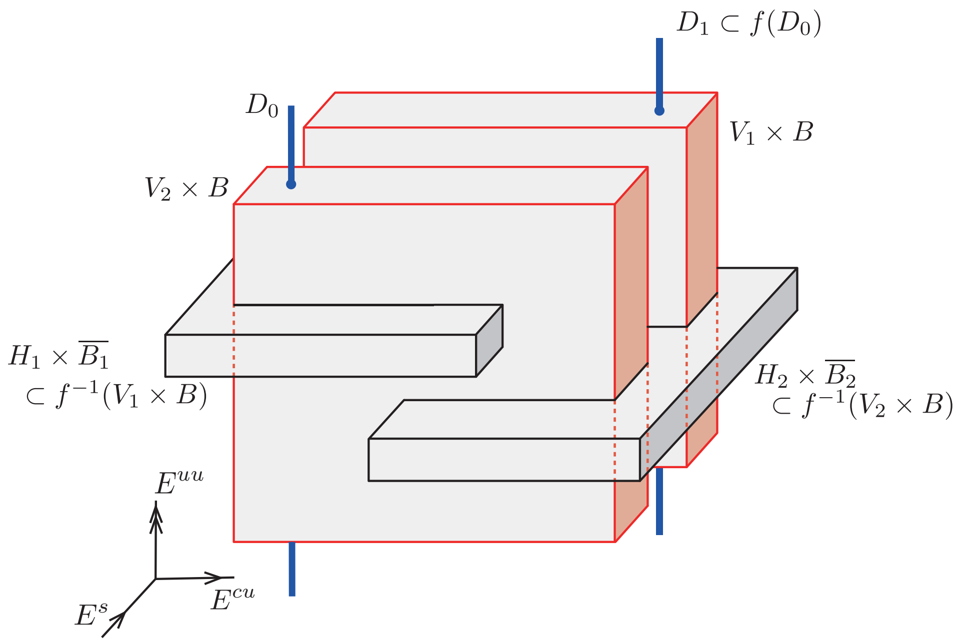

FIGURE 1. Covering property as $k = 2$

2.2.1. *Prototypical blender-horseshoe.* Any known example of a horseshoe which is a blender satisfies the covering property. This is the case of the important model used in many articles called the *prototypical blender-horseshoe* (see [BD12, Sec. 5.1] and also [BD96, BDV05, BD08, BDK12]).

In this model, f is locally defined as a one-step skew-product of the form $f = F \ltimes (g_1, g_2)$ on $\mathbb{R}^d = \mathbb{R}^{s+uu} \times \mathbb{R}$. Here, F is a map having a horseshoe, Λ, in $[-1,1]^{s+uu}$ conjugated with a full shift on two symbols, while

$$g_1(x) = \lambda x, \qquad \text{and} \qquad g_2(x) = \lambda x - \mu$$

for $x \in \mathbb{R}$ with $1 < \lambda < 2$ and $0 < \mu < 1$. Namely,

$$f(\omega, x) = \begin{cases} (F(\omega), g_1(x)) & \text{if } \omega \in H_1 \\ (F(\omega), g_2(x)) & \text{if } \omega \in H_2 \end{cases} \tag{2.2}$$

where H_1 and H_2 are the horizontal rectangles in $[-1,1]^{s+uu}$ containing Λ. The associated contracting IFS is given by the inverse maps of the expanding g_1 and g_2. Then, for any $\varepsilon > 0$ small enough, the open set $B = (\varepsilon, (\mu+1)/\lambda - \varepsilon)$ satisfies the covering property (2.1). See Figure 2. Hence, according to Remark 2.5, the maximal invariant set Γ of f in $H \times [-1,1]$, where $H = H_1 \cup H_2$ is a dynamical *cu*-blender of codimension $c = 1$.

In what follows we will need to consider dynamical blenders having an extra assumption. For this reason we introduce the following definition:

Definition 2.6. A periodic point P in a dynamical *cu*-blender Γ of a diffeomorphism f is said to be *distal* if the orbit of P is far from the strictly f-invariant family of disc and whose unstable manifold contains a disc in this family.

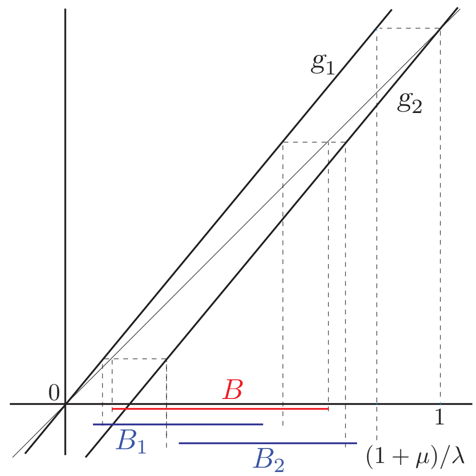

FIGURE 2. Covering property for (g_1, g_2)

The following lemma shows that prototypical blender-horseshoes have a distal periodic point. Moreover, since this property is open, it also holds by any nearby dynamical blender.

Lemma 2.7. *The dynamical cu-blender Γ of the map f in (2.2) has a distal point.*

Proof. Let $\mathcal{D}$ be the strictly f-invariant family of uu-dimensional discs of Γ. According to Remark 2.5, this corresponds with the region $V \times B$ in $[-1,1]^d$ where $V = F(H) \cap [-1,1]^{s+uu}$. Let $p \in \Lambda$ be the fixed point of F in H_1. Then $P = (p, 0)$ is a fixed point of f in Γ. Observe that P does not belong to $V \times B$. Moreover, the intersection of $W^u(P)$ and $V \times B$ contains a disc in the strictly f-invariant family $\mathcal{D}$ of uu-dimensional discs. Thus, P is a distal point of Γ. □

3. HISTORIC BEHAVIOR IN A HOMOCLINIC CLASS WITH A BLENDER

The following result is the main ingredient to get Theorem A. We say a periodic point Q is *homoclinically related* to a periodic point P if $W^u(Q) \pitchfork W^s(P) \neq \emptyset$ and $W^u(P) \pitchfork W^s(Q) \neq \emptyset$.

Theorem 3.1. *Consider a C^1-diffeomorphism f of M having a homoclinic class $H(P)$, which contains a dynamical cu-blender Γ with a distal periodic point in Γ homoclinically related to P. Then $H(P)$ has residual historic behavior.*

Proof. Denote by $\mathcal{D}$ the strictly f-invariant family of discs of the dynamical *cu*-blender Γ. See Figure 3. By abuse of notation, we also denote by $\mathcal{D}$ the region in M where the discs of this family are embedded. Since the homoclinic classes of homoclinically related saddle points coincide, relabeling if necessary, we will denote by P the periodic (distal) point of f in Γ whose distance from $\mathcal{D}$ is larger than some $r > 0$. For simplicity, we assume that P is a fixed point of f.

Consider a continuous function $\varphi : M \to [0,1]$ such that $\varphi(x) = 1$ for all x in the open ball $B_r(P)$ of radius r centered at P and $\varphi(x) = 0$ if x belongs to $\mathcal{D}$. Fix $\varepsilon > 0$, $N \in \mathbb{N}$ and $x \in H(P)$. We define $\Delta(x, \varepsilon, N)$ as the set of points $y \in B_\varepsilon(x)$ which satisfies the following condition: there are $n_1, n_2 \geq N$ such that

$$\frac{1}{n_1}\sum_{i=0}^{n_1-1}\varphi(f^i(y)) - \frac{1}{n_2}\sum_{i=0}^{n_2-1}\varphi(f^i(y)) > \frac{1}{2}. \tag{3.1}$$

The first trivial observation is that $\Delta(x,\varepsilon,N)$ is open. Moreover, it holds that

Claim 1. $\Delta(x,\varepsilon,N)\cap H(P)\neq\emptyset$.

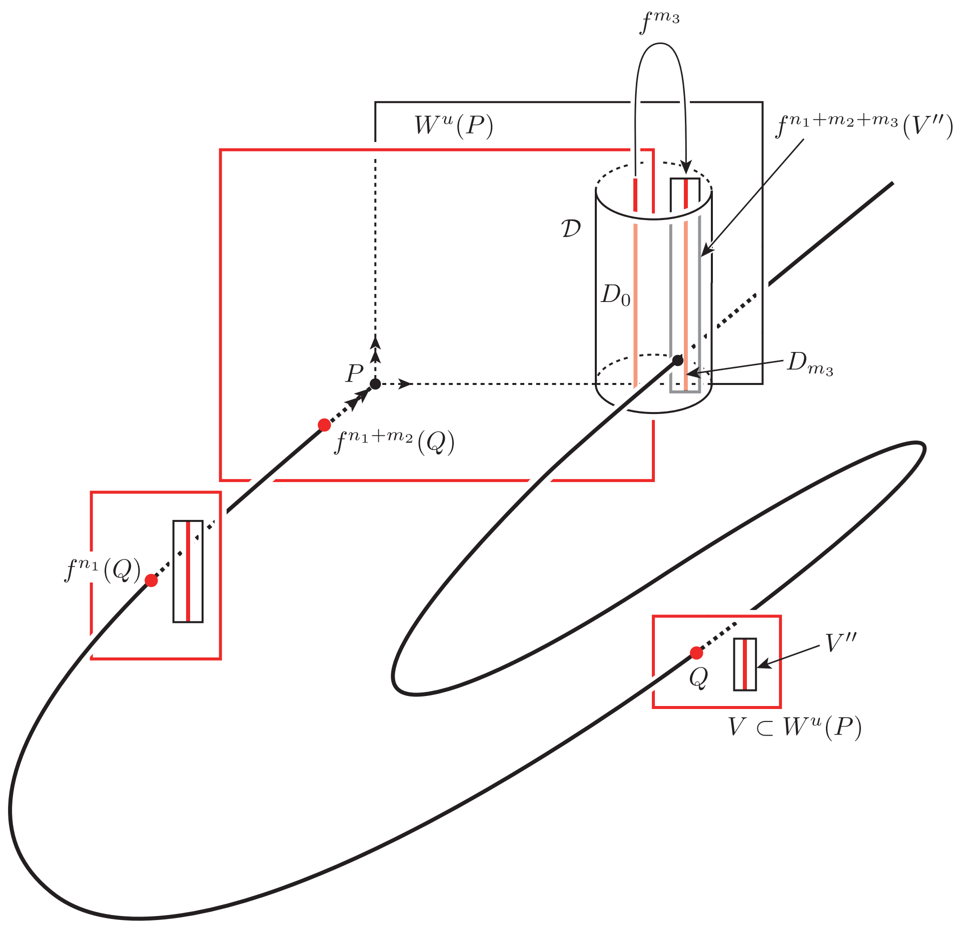

FIGURE 3. Sketch of the proof of Theorem 3.1.

Let us postpone the proof of this claim to first conclude the theorem. Take R_N as the union of $\Delta(x,\varepsilon,N)$ for $\varepsilon>0$ and $x\in H(P)$. Clearly $R_N\cap H(P)$ is an open and dense set in $H(P)$. Hence the set $R\cap H(P)$, where $R=\cap R_N$, is a residual set in $H(P)$. Moreover, if $y\in R$ then for every $N\in\mathbb{N}$, the forward orbit of y has N-conditional historic behavior, i.e., (3.1) holds. This implies that the forward orbit of y has historic behavior, concluding the theorem.

Proof of Claim 1. Since $H(P)$ is a homoclinic class, there exists $Q\in B_\varepsilon(x)$ belonging to $W^s(P)\pitchfork W^u(P)$. Since $Q\in W^s(P)$, there is $m_0\in\mathbb{N}$ such that $f^{m_0}(Q)\in B_r(P)$.

Moreover, assuming r sufficiently small, we have that $f^{m_0+n}(Q) \in B_r(P)$ for all $n \geq 0$. Take $m_1 \in \mathbb{N}$ such that $n_1 = m_0 + m_1 \geq N$ and $m_0/n_1 < 1/8$. Then

$$\frac{1}{n_1}\sum_{i=0}^{n_1-1}\varphi(f^i(Q)) \geq \frac{1}{n_1}\sum_{i=m_0}^{n_1-1}\varphi(f^i(Q)) \geq \frac{n_1 - m_0}{n_1} > \frac{7}{8}.$$

By continuity, we have that

$$\frac{1}{n_1}\sum_{i=0}^{n_1-1}\varphi(f^i(y)) > \frac{7}{8} \quad \text{for all } y \in B_\varepsilon(x) \text{ close enough to } Q. \tag{3.2}$$

Now consider a small disc V in $W^u(P) \cap B_\varepsilon(x)$ transverse to $W^s(P)$ at Q such that $f^{n_1}(V) \subset B_r(P)$. By the Inclination Lemma, the forward iteration of V approaches $W^u(P)$. Since P is distal, the unstable manifold of P contains a disc in the open family $\mathcal{D}$. Then we find $m_2 \in \mathbb{N}$ such that $f^{n_1+m_2}(V)$ also contains a disc D_0 in $\mathcal{D}$. See Figure 3. Moreover, as $\mathcal{D}$ is strictly f-invariant, there is a sequence of discs $\{D_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}$ such that $D_n \subset f(D_{n-1})$ for all $n \geq 1$. Take $m_3 \in \mathbb{N}$ such that $(n_1 + m_2)/n_2 < 1/8$, where $n_2 = n_1 + m_2 + m_3$. Let $V' = f^{-m_3-m_2-n_1}(D_{m_3}) \subset V$. Then,

$$\frac{1}{n_2}\sum_{i=0}^{n_2-1}\varphi(f^i(y)) \leq \frac{1}{n_2}\sum_{i=0}^{n_1+m_2}\varphi(f^i(y)) \leq \frac{n_1+m_2}{n_2} < \frac{1}{8} \quad \text{for all } y \in V'.$$

By continuity, we have that

$$\frac{1}{n_2}\sum_{i=0}^{n_2-1}\varphi(f^i(y)) < \frac{1}{8} \quad \text{for all } y \in B_\varepsilon(x) \text{ close enough to } V'. \tag{3.3}$$

Therefore, Equations (3.2) and (3.3) imply (3.1) concluding that $\Delta(x,\varepsilon,N)$ is not empty.

To conclude the proof we actually need to show that $\Delta(x,\varepsilon,N) \cap H(P) \neq \emptyset$. To do this, we will use that Γ is a *cu*-blender. Take a small neighborhood V'' of V' in $V \subset W^u(P) \cap B_\varepsilon(x)$ such that $f^{n_2}(V'')$ is a strip foliated by discs in $\mathcal{D}$. Since Γ is a *cu*-blender and $P \in \Gamma$ is a fixed point, we get that $W^s(P)$ transversally intersects $f^{n_2}(V'')$ at a point Y. In particular, Y belongs to $H(P)$ since $f^{n_2}(V'') \subset W^u(P)$. Thus, $f^{-n_2}(Y) \in H(P) \cap V'' \subset H(P) \cap \Delta(x,\varepsilon,N)$. This ends the proof of Claim 1. □

Now the proof of Theorem 3.1 is complete. □

4. Blenders in generic non-hyperbolic homoclinic classes

We first give a result on historic behavior but with respect to a *fixed* homoclinic class. This will follow from the existence of blenders and Theorem 3.1. Thus, as a part of the proof we review the known arguments for obtaining blenders in the C^1-generic context. Denote by $\mathrm{ind}(P)$ the stable index of a saddle periodic point P for f, and by P_g the continuation of P if g is close to f.

Theorem 4.1. *Let f be a C^1-diffeomorphism of M and fix a hyperbolic periodic point P for f. Assume that the homoclinic class $H(P)$ contains a hyperbolic saddle Q with* $\mathrm{ind}(Q) = \mathrm{ind}(P) + 1$. *Then there is an open set $\mathcal{U}$ of C^1-diffeomorphisms with $f \in \overline{\mathcal{U}}$ such that $H(P_g)$ has residual historic behavior for all $g \in \mathcal{U}$.*

Proof. Since the homoclinic class of P must be non-trivial, according to [BDK12, Thm. 1] we can approximate f by a diffeomorphism having a C^1-robust heterodimensional cycle associated with hyperbolic sets containing the continuations of P and Q (see also [BCDG13, Cor. 2.4]).

Recall that a diffeomorphism has a *heterodimensional cycle* if there exists a pair of transitive hyperbolic sets Λ and Γ with different indices such that their invariant manifolds meet cyclically. Since $\mathrm{ind}(P) < \mathrm{ind}(Q)$, by construction of the cycle as done in [BD08, Sect.4], there exists a prototypical *cu*-blender-horseshoe Γ homoclinically related with P. The fact that the heterodimensional cycle persists under perturbations comes from the robustness of the blender. In particular, from Lemma 2.7, there is a C^1-open subset $\mathcal{V}$ arbitrarily close to f such that any $g \in \mathcal{V}$ has a dynamical *cu*-blender Γ_g and a distal periodic point in Γ_g homoclinically related with P_g. Thus, we are in the assumptions of Theorem 3.1 and consequently $H(P_g)$ has residual historic behavior. To conclude, it is enough to take $\mathcal{U}$ as the union of all of these open sets $\mathcal{V}$. □

Remark 4.2. Observe that for a C^1-generic diffeomorphism the following facts are known. For every pair of hyperbolic periodic points the homoclinic classes either coincide or are disjoint. And if a homoclinic class $H(P)$ contains saddles of different indices, then it also contains periodic points of all intermediate indices [ABC$^+$07, Thm. 1 and Lem. 2.1]. In particular, for a C^1-generic f and a given homoclinic class $H(P)$, we may assume there exists a point Q with $\mathrm{ind}(Q) = \mathrm{ind}(P) + 1$. Thus, the previous Theorem 4.1 can be applied in this context.

In the above theorem the homoclinic class is fixed, but observe that the set of diffeomorphisms obtained is open. However, we would like to show C^1-generic historic behavior for *any* homoclinic class having index variation (containing saddles of different indices). To this end, we present the following result due to Bonatti and Díaz [BD12] on generation of blenders inside any homoclinic class with certain index variation.

Theorem 4.3 ([BD12, Thm. 6.4])**.** *There is a residual subset $\mathcal{R}$ of C^1-diffeomorphisms f of M such that for every homoclinic class $H(P)$ containing a hyperbolic saddle Q with* $\mathrm{ind}(Q) > \mathrm{ind}(P)$, *there is a transitive hyperbolic set Λ containing P and a cu-blender Γ.*

This theorem follows from standard genericity arguments by first proving the statement for a fixed homoclinic class $H(P)$. On the other hand, this is done by means of similar reasoning as in the proof of Theorem 4.1 and Remark 4.2. We also would like to emphasize the following:

Remark 4.4. The *cu*-blenders constructed in Theorem 4.3 come from the perturbations of prototypical *cu*-blender horseshoes in the sense of Section 2.2.1, and in particular are dynamical blenders.

Now we prove our main Theorem.

Proof of Theorem A. Consider the residual set $\mathcal{R}$ given in Theorem 4.3. We can assume that for any $f \in \mathcal{R}$ and every pair of hyperbolic periodic points P and Q of f either $H(P) = H(Q)$ or $H(P) \cap H(Q) = \emptyset$ (see [ABC$^+$07, Lem. 2.1]). Now, fix $f \in \mathcal{R}$ and let P be a saddle periodic point of f whose homoclinic class contains a saddle Q of different stable index.

If $\mathrm{ind}(Q) > \mathrm{ind}(P)$ then by Theorem 4.3 we get a *cu*-blender Γ in $H(P)$, whose saddles are homoclinically related with P. By Remark 4.4, Γ is a dynamical *cu*-blender and from Lemma 2.7 has a distal periodic point related with P. Thus, we are in the assumptions of Theorem 3.1 and consequently $H(P)$ has residual historic behavior.

Next we suppose that $\mathrm{ind}(Q) < \mathrm{ind}(P)$. Since $H(P) = H(Q)$, $H(Q)$ contains points of different indices and similarly as above we can apply Theorem 4.3 for this homoclinic class. Hence, we get that $H(Q)$ has residual historic behavior. As $H(Q) = H(P)$ we also get the same conclusion for $H(P)$ and conclude the proof. □

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