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On the Evolution of Compressible and Incompressible Viscous Fluids with a Sharp Interface

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Abstract: In this paper, we consider some two phase problems of compressible and incompressible viscous fluids' flow without surface tension under the assumption that the initial domain is a uniform $W_q^{2-1/q}$ domain in $\mathbb{R}^N$ $(N \geq 2)$. We prove the local in the time unique existence theorem for our problem in the L_p in time and L_q in space framework with $2 < p < \infty$ and $N < q < \infty$ under our assumption. In our proof, we first transform an unknown time-dependent domain into the initial domain by using the Lagrangian transformation. Secondly, we solve the problem by the contraction mapping theorem with the maximal L_p-L_q regularity of the generalized Stokes operator for the compressible and incompressible viscous fluids' flow with the free boundary condition. The key step of our proof is to prove the existence of an $\mathcal{R}$-bounded solution operator to resolve the corresponding linearized problem. The Weis operator-valued Fourier multiplier theorem with $\mathcal{R}$-boundedness implies the generation of a continuous analytic semigroup and the maximal L_p-L_q regularity theorem.

Keywords: Navier–Stokes equations; two phase problem; local in time unique existence theorem; $\mathcal{R}$-bounded operator

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1. Introduction

It is an important mathematical problem to consider the unsteady motion of a bubble in an incompressible viscous fluid or that of a drop in a compressible viscous one. The problem is, in general, formulated mathematically by the Navier–Stokes equations in a time-dependent domain separated by an interface, where one part of the domain is occupied by a compressible viscous fluid and another part by an incompressible viscous fluid. More precisely, we consider two fluids that fill a region $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$. Let $\Gamma \subset \Omega$ be a given surface that bounds the region Ω_+ occupied by a compressible barotropic viscous fluid and the region Ω_- occupied by an incompressible viscous one. We assume that the boundary of $\Omega_\pm$ consists of two parts, Γ and $\Gamma_\pm$, where $\partial\Omega_\pm = \Gamma \cup \Gamma_\pm$, $\Gamma_\pm \cap \Gamma = \emptyset$, $\Gamma_+ \cap \Gamma_- = \emptyset$, and $\Omega = \Omega_+ \cup \Gamma \cup \Omega_-$. Let Γ_t, Γ_{t-}, Ω_{t+}, and Ω_{t-} with time variable $t > 0$ be the time evolution of Γ, Γ_-, Ω_+, and Ω_-, respectively. We assume that the two fluids are immiscible, so that $\Omega_{t+} \cap \Omega_{t-} = \emptyset$ for any $t \geq 0$. Moreover, we assume that no phase transitions occur, and we do not consider the surface tension at the interface Γ_t and the boundary Γ_{t-}. Thus, in this paper, we consider that the motion of the fluids is governed by the following system of equations:

$$\begin{cases} \partial_t\rho_+ + \mathrm{div}\,(\rho_+\mathbf{v}_+) = 0 & \text{in } \Omega_{t+}, \\ \rho_+(\partial_t\mathbf{v}_+ + \mathbf{v}_+\cdot\nabla\mathbf{v}_+) - \mathrm{Div}\,\mathbf{S}_+(\mathbf{v}_+) + \nabla p(\rho_+) = 0 & \text{in } \Omega_{t+}, \\ \mathrm{div}\,\mathbf{v}_- = 0 & \text{in } \Omega_{t-}, \\ \rho_{0-}(\partial_t\mathbf{v}_- + \mathbf{v}_-\cdot\nabla\mathbf{v}_-) - \mathrm{Div}\,\mathbf{S}_-(\mathbf{v}_-) + \nabla\pi_- = 0, & \text{in } \Omega_{t-} \end{cases} \tag{1}$$

subject to the interface condition:

$$\begin{cases} (\mathbf{S}_+(\mathbf{v}_+) - p(\rho_+)\mathbf{I})\mathbf{n}_t|_{\Gamma_t+0} - (\mathbf{S}_-(\mathbf{v}_-) - \pi_-\mathbf{I})\mathbf{n}_t|_{\Gamma_t-0} = -p(\rho_{0+})\mathbf{n}_t|_{\Gamma_t-0}, \\ \mathbf{v}_+|_{\Gamma_t+0} - \mathbf{v}_-|_{\Gamma_t-0} = 0 \end{cases} \tag{2}$$

on Γ_t, boundary conditions:

$$\mathbf{v}_+|_{\Gamma_+} = 0, \quad (\mathbf{S}_-(\mathbf{v}_-) - \pi_-\mathbf{I})\mathbf{n}_{t-}|_{\Gamma_{t-}} = 0, \tag{3}$$

kinematic conditions:

$$V_n = \mathbf{u}_- \cdot \mathbf{n}_t \quad \text{on } \Gamma_t, \quad V_{n-} = \mathbf{u}_- \cdot \mathbf{u}_{t-} \quad \text{on } \Gamma_{t-} \tag{4}$$

for any $t > 0$, and initial conditions:

$$(\rho_+, \mathbf{v}_+)|_{t=0} = (\rho_{0+} + \theta_{0+}, \mathbf{v}_{0+}) \ \text{ in } \Omega_+, \quad \mathbf{v}_-|_{t=0} = \mathbf{v}_{0-} \ \text{ in } \Omega_-. \tag{5}$$

Here, $\mathbf{v}_\pm = (v_{1\pm}, \dots, v_{N\pm})$ are the unknown velocity fields of the fluids, $\rho_{0\pm}$ positive numbers describing the mass densities of $\Omega_\pm$, ρ_+ the unknown mass density of Ω_{t+}, π_- the unknown pressure, θ_{0+} and $\mathbf{v}_{0\pm}$ the prescribed initial data, $p(s)$ the prescribed pressure, which is a C^∞ function defined on an open interval $(\rho_{0+}/2, 2\rho_{0+})$ satisfying the condition: $p'(s) \geq 0$ on $(\rho_{0+}/2, 2\rho_{0+})$, $\mathbf{n}_t$ the unit outward normal to Γ_t, pointing from Ω_{t-} to Ω_{t+}, $\mathbf{n}_{t-}$ the unit outward normal to Γ_{t-}, V_n the evolution speed of Γ_t along $\mathbf{n}_t$, and V_{n-} the evolution speed of Γ_{t-} along $\mathbf{n}_{t-}$.

Moreover, for any point $x_0 \in \Gamma_t$, $f|_{\Gamma_t\pm 0}(x_0, t)$ is defined by:

$$f|_{\Gamma_t\pm 0}(x_0, t) = \lim_{\substack{x \to x_0 \\ x \in \Omega_{t\pm}}} f(x, t),$$

and the stress tensors $\mathbf{S}_\pm$ are defined by:

$$\mathbf{S}_+(\mathbf{v}_+) = \mu_+\mathbf{D}(\mathbf{v}_+) + (\nu_+ - \mu_+)\mathrm{div}\, \mathbf{v}_+\mathbf{I}, \quad \mathbf{S}_-(\mathbf{v}_-) = \mu_-\mathbf{D}(\mathbf{v}_-)$$

with viscosity coefficients $\mu_\pm$ and ν_+, which are positive constants in this paper, where $\mathbf{D}(\mathbf{v})$ denotes the deformation tensor whose (j,k) components are $D_{jk}(\mathbf{v}) = \partial_j v_k + \partial_k v_j$ with $\partial_j = \partial/\partial x_j$ and $\mathbf{I}$ is the $N \times N$ identity matrix. Finally, for an $N \times N$ matrix function $\mathbf{K} = (K_{ij})$, $\mathrm{Div}\, \mathbf{K}$ is an N-vector whose ith components are $\sum_{j=1}^N \partial_j K_{ij}$, and also, for any vector of functions $\mathbf{v} = (v_1, \dots, v_N)$, we set $\mathrm{div}\, \mathbf{v} = \sum_{j=1}^N \partial_j v_j$ and $\mathbf{v} \cdot \nabla \mathbf{v} = (\sum_{j=1}^N v_j \partial_j v_1, \dots, \sum_{j=1}^N v_j \partial_j v_N)$. For any functions $f_\pm$ defined on $\Omega_\pm$, f denotes a function defined by $f = f_\pm$ in $\Omega_\pm$.

Aside from the dynamical system (1) subject to (2), (3), and (5), a kinematic condition (4) for Γ_t and Γ_{t-} gives:

$$\Gamma_t = \{x = \mathbf{x}(\xi, t) \mid \xi \in \Gamma\}, \ \ \Gamma_{t-} = \{x = \mathbf{x}(\xi, t) \mid \xi \in \Gamma_-\}, \tag{6}$$

where $\mathbf{x}(\xi, t)$ is the solution of the Cauchy problem:

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t) = \begin{cases} \mathbf{v}_+ & \text{in } \Omega_{t+}, \\ \mathbf{v}_- & \text{in } \Omega_{t-}, \end{cases} \quad \mathbf{x}|_{t=0} = \xi \in \Omega. \tag{7}$$

This expresses the fact that the interface Γ_t and the free surface Γ_{t-} consist for all $t > 0$ of the same fluid particles, which do not leave them and are not incident on them from inside Ω_t. It is clear that $\Omega_{t\pm}$ is given by:

$$\Omega_{t\pm} = \{x = \mathbf{x}(\xi, t) \mid \xi \in \Omega_\pm\}. \tag{8}$$

Problem (1) with (2)–(5) can therefore be written as an initial boundary value problem with interface Γ in the given domain Ω if we go over the Euler coordinates $x \in \Omega_{t\pm}$ to Lagrange coordinates $\xi \in \Omega_\pm$ with $\mathbf{x}$ by (7). If velocity vector fields $\mathbf{u}_\pm(\xi,t)$ defined on $\Omega_\pm$ are known as functions of the Lagrange coordinates $\xi \in \Omega_\pm$, then this connection can be written in the form:

$$x = \xi + \int_0^t \mathbf{u}_\pm(\xi,s)\,ds \equiv X_{\mathbf{u}_\pm}(\xi,t) \tag{9}$$

and $\mathbf{u}_\pm(\xi,t) = \mathbf{v}(X_{\mathbf{u}_\pm}(\xi,t),t)$. Let $A_\pm$ be the Jacobi matrix of the transformation (9) with element $a_{ij}^\pm = \delta_{ij} + \int_0^t(\partial_{\xi_j}u_{\pm,i})(\xi,s)\,ds$ with δ_{ij} being the Kronecker delta symbols. There exists a small number $\sigma > 0$ such that $A_\pm$ is invertible, that is $\det A_\pm \neq 0$, whenever:

$$\max_{i,j=1,\ldots,N}\sup_{\xi\in\Omega_+}\left|\int_0^t(\partial_{\xi_j}u_{+,i})(\xi,s)ds\right| < \sigma \quad (t>0), \tag{10}$$

while $\det A_- = 1$ in Ω_-, because of the incompressibility. Whenever (10) is valid, we have:

$$\nabla_x = A_\pm^{-1}\nabla_\xi = \left(\mathbf{I} + \mathbf{V}_0\left(\int_0^t \nabla\mathbf{u}_\pm(\xi,s)\,ds\right)\right)\nabla_\xi$$

with $\nabla_x = {}^T(\partial_{x_1},\ldots,\partial_{x_N})$ (${}^TM$ denotes the transposed M) and $\nabla\xi = {}^T(\partial_{\xi_1},\ldots,\partial_{\xi_N})$, where $\mathbf{V}_0 = \mathbf{V}_0(\mathbf{w})$ is the $N\times N$ matrix of C^∞ functions with respect to $\mathbf{w} = (w_1,\ldots,w_N)$ defined on $|\mathbf{w}| < \sigma$ and $\mathbf{V}_0(0) = 0$. Let $\mathbf{n}$ and $\mathbf{n}_-$ be unit outward normals to Γ and Γ_-, respectively, and then, by (8), we have:

$$\mathbf{n}_t = \frac{A_-^{-1}\mathbf{n}}{|A_-^{-1}\mathbf{n}|}, \quad \mathbf{n}_{t-} = \frac{A_-^{-1}\mathbf{n}_-}{|A_-^{-1}\mathbf{n}_-|}.$$

Setting $\rho_+(X_{\mathbf{u}_+}(\xi,t),t) = \rho_{0+} + \theta_{0+}(\xi) + \theta_+(\xi,t)$ and $p_- = \pi_-(X_{\mathbf{u}_-}(\xi,t),t)$ and using the facts that $\rho_+(X_{\mathbf{u}_+}(\xi,t),t) = \mathcal{J}_+(\xi,t)^{-1}(\rho_{0+} + \theta_{0+}(\xi))$ with $\mathcal{J}_\pm = \det A_\pm$ and $\mathrm{div}_x\mathbf{v}_\pm = \mathcal{J}_\pm^{-1}\mathrm{div}_\xi(\mathcal{J}_\pm {}^TA_\pm^{-1}\hat{\mathbf{v}}_\pm)$ with $\hat{\mathbf{v}}_\pm(\xi,t) = \mathbf{v}_\pm(X_{\mathbf{u}_\pm}(\xi,t),t)$, we can write Equations (1)–(5) with Lagrange coordinates in the form:

$$\begin{cases} \partial_t\theta_+ + (\rho_{0+}+\theta_{0+})\mathrm{div}\,\mathbf{u}_+ = F_+ & \text{in } \Omega_+, \\ (\rho_{0+}+\theta_{0+})\partial_t\mathbf{u}_+ - \mathrm{Div}\,S_+(\mathbf{u}_+) + \nabla(p'(\rho_{0+}+\theta_{0+})\theta_+) = \mathbf{g}_+ + \mathbf{G}_+ & \text{in } \Omega_+, \\ \rho_{0-}\partial_t\mathbf{u}_- - \mathrm{Div}\,\mathbf{S}_-(\mathbf{u}_-) + \nabla p_- = \mathbf{G}_- & \text{in } \Omega_-, \\ \mathrm{div}\,\mathbf{u}_- = F_- & \text{in } \Omega_-, \\ (\mathbf{S}_+(\mathbf{u}_+) - p'(\rho_{0+}+\theta_{0+})\theta_+\mathbf{I})\mathbf{n}|_{\Gamma+0} - (\mathbf{S}_-(\mathbf{u}_-) - p_-\mathbf{I})\mathbf{n}|_{\Gamma-0} = \mathbf{h} + \mathbf{H}, \\ \mathbf{u}_+|_{\Gamma+0} - \mathbf{u}_-|_{\Gamma-0} = 0, \quad \mathbf{u}_+|_{\Gamma_+} = 0, \quad (\mathbf{S}_-(\mathbf{u}_-) - p_-\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = \mathbf{H}_- \end{cases} \tag{11}$$

for $t > 0$ subject to the initial condition:

$$(\theta_+, \mathbf{v}_+)|_{t=0} = (0, \mathbf{v}_{0+}) \ \text{ in } \Omega_+, \quad \mathbf{u}_-|_{t=0} = \mathbf{v}_{0-} \ \text{ in } \Omega_-. \tag{12}$$

Here, $\mathbf{g}_+ = -p'(\rho_{0+}+\theta_{0+})\nabla\theta_{0+}$, $\mathbf{h} = -(p(\rho_{0+}+\theta_{0+}) - p(\rho_{0+}))\mathbf{n}$, and $F_\pm$, $\mathbf{G}_\pm$, $\mathbf{H}$ and $\mathbf{H}_-$ are nonlinear functions with respect to θ_+, $\mathbf{u}_\pm$, $\mathbf{w}_\pm = \int_0^t \nabla\mathbf{u}_\pm(\xi,s)ds$ of the forms:

$$\begin{aligned} F_+ &= -\{\theta_+\mathrm{div}\,\mathbf{u}_+ + (\rho_{0+}+\theta_{0+}+\theta_+)\mathrm{tr}\,(\mathbf{V}_{0+}\nabla\mathbf{u}_+)\}, \\ \mathbf{G}_+ &= -\theta_+\partial_t\mathbf{u}_+ + \mathrm{Div}\,(\mu_+\mathbf{V}_{D+}\nabla\mathbf{u}_+ + (\nu_+-\mu_+)\mathrm{tr}\,(\mathbf{V}_{0+}\nabla\mathbf{u}_+)\mathbf{I}) \\ &\quad + \mathbf{V}_0\nabla\{\mu_+(\mathbf{D}(\mathbf{u}_+) + \mathbf{V}_{D+}\nabla\mathbf{u}_+) + (\nu_+-\mu_+)(\mathrm{div}\,\mathbf{u}_+ + \mathrm{tr}\,(\mathbf{V}_{0+}\nabla\mathbf{u}_+))\mathbf{I}\} \\ &\quad - \nabla\Big(\int_0^1 p''(\rho_{0+}+\theta_{0+}+\tau\theta_+)(1-\tau)\,d\tau\,\theta_+^2\Big) - \mathbf{V}_{0+}p'(\rho_{0+}+\theta_{0+}+\theta_+)\nabla(\theta_{0+}+\theta_+), \\ \mathbf{G}_- &= -\rho_{0-}\mathbf{V}_{-1}\partial_t\mathbf{u}_- + \mu_-\{\mathrm{Div}\,(\mathbf{V}_{D-}\nabla\mathbf{u}_-) + \mathbf{V}_{-1}\mathrm{Div}\,(\mathbf{D}(\mathbf{u}_-) + V_{D-}\nabla\mathbf{u}_-)\}, \\ F_- &= (1-J_-)\mathrm{div}\,\mathbf{u} - \mathrm{tr}(\mathbf{V}_{0-}\nabla\mathbf{u}_-) = \mathrm{div}\,((1-J_-)\mathbf{u}_- - {}^TV_{0-}J_-\mathbf{u}_-), \\ \mathbf{H} &= -\mu_+[\mathbf{V}_{D+}\nabla\mathbf{u}_+ + \mathbf{V}_{-1}(\mathbf{D}(\mathbf{u}_+) + \mathbf{V}_{D+}\nabla\mathbf{u}_+) + (\mathbf{I}+\mathbf{V}_{-1})(\mathbf{D}(\mathbf{u}_+) + \mathbf{V}_{D+}\nabla\mathbf{u}_+)\mathbf{V}_{0+}]\mathbf{n} \end{aligned} \tag{13}$$

$$
\begin{aligned}
&-(\nu_+ - \mu_+)[\mathrm{tr}\,(\mathbf{V}_{0+}\nabla\mathbf{u}_+)]\mathbf{n} + [\int_0^1 (1-\tau)p''(\rho_{0+}+\theta_{0+}+\tau\theta_+)\,d\tau\,\theta_+^2]\mathbf{n}\\
&+\mu_-[\mathbf{V}_{D-}\nabla\mathbf{u}_- + \mathbf{V}_{-1}\mathbf{D}(\mathbf{u}_-) + \mathbf{V}_{D-}\nabla\mathbf{u}_- + (\mathbf{I}+\mathbf{V}_{-1})(\mathbf{D}(\mathbf{u}_-)+\mathbf{V}_{D-}\nabla\mathbf{u}_-)\mathbf{V}_{0-}]\mathbf{n}\\
\mathbf{H}_- &= -\mu_-[\mathbf{V}_{D-}\nabla\mathbf{u}_- + \mathbf{V}_{-1}(\mathbf{D}(\mathbf{u}_-)+\mathbf{V}_{D-}\nabla\mathbf{u}_-) + (\mathbf{I}+\mathbf{V}_{-1})(\mathbf{D}(\mathbf{u}_-)+\mathbf{V}_{D-}\nabla\mathbf{u}_-)\mathbf{V}_{0-}]\mathbf{n}_-
\end{aligned}
$$

with $\mathbf{V}_{0\pm} = \mathbf{V}_0(\mathbf{w}_\pm)$, $\mathbf{V}_{D\pm} = \mathbf{V}_D(\mathbf{w}_\pm)$, $\mathbf{V}_{-1} = \mathbf{V}_{-1}(\mathbf{w}_\pm)$, and $J_- = \det(\nabla X_{\mathbf{u}_-})$. In the formula (13), $\mathbf{V}_{-1}$ is defined by $\mathbf{V}_{-1} = (\mathbf{I}+\mathbf{V}_0)^{-1} - \mathbf{I}$, $\mathrm{tr}\,B$ means the trace of $N \times N$ matrix B, and $\mathbf{V}_D(\mathbf{w})$ is a matrix of the C^∞ function with respect to $\mathbf{w}$ defined on $|\mathbf{w}| < \sigma$, which satisfies $\mathbf{V}_D(0) = 0$ and relations: $\mathbf{D}(\mathbf{v}) = \mathbf{D}(\hat{\mathbf{v}}) + \mathbf{V}_D(\int_0^t \nabla\hat{\mathbf{v}}\,ds)\nabla\hat{\mathbf{v}}$ with $\hat{\mathbf{v}} = \mathbf{v}(X_{\mathbf{u}}(\xi,t),t)$.

Since the pioneering work [1] on the well-posedness of Navier–Stokes equations around a free surface, there have been many studies on the free boundary problem. Here, we introduce the known results concerning compressible and incompressible viscous two-phase fluids.

Denisova [2,3] proved the local well-posedness theorem and the global well-posedness theorem for Equations (1)–(3) and (5) in the L_2 framework. The purpose of this paper is to prove the local well-posedness for Equations (1)–(3) and (5) in the L_p in time and L_q in space framework with $2 < p < \infty$ and $N < q < \infty$ under the physically reasonable assumption on the viscosity coefficients, that is $\mu_\pm > 0$ and $\nu_+ > 0$. The regularity of solutions in our result is optimal in the sense of the maximal regularity, while the L_2 framework used by Denisova [2,3] loses regularity from the point of view of Sobolev's imbedding theorem.

Moreover, we consider the problem with full generality about the domain. Namely, we consider the problem in a uniform $W_q^{2-1/q}$ domain, the conditions of which are satisfied by bounded domains, exterior domains, half-spaces, perturbed half-spaces, and layer domains (cf. Shibata [4]).

Symbols 1. To state our theorem on the local in time unique existence of solutions to Equations (1)–(3) and (5), we introduce some functional spaces and the definition of the uniform $W_r^{2-1/r}$ domain. For the differentiations of scalar functions f and N-vector functions $\mathbf{g}$, we use the following symbols:

$$
\begin{aligned}
\nabla f &= (\partial_1 f,\ldots,\partial_N f), & \nabla^2 f &= (\partial_i\partial_j f \mid i,j = 1,\ldots,N),\\
\nabla\mathbf{g} &= (\partial_i g_j \mid i,j=1,\ldots,N), & \nabla^2\mathbf{g} &= (\partial_i\partial_j g_k \mid i,j,k = 1,\ldots,N),
\end{aligned}
$$

where $\partial_i = \partial/\partial x_i$. For any domain D and $1 \le q \le \infty$, $L_q(D)$, $W_q^m(D)$, and $B_{q,p}^s(D)$ denote the standard Lebesgue space, Sobolev space, and Besov space, while $\|\cdot\|_{L_q(D)}$, $\|\cdot\|_{W_q^m(D)}$, and $\|\cdot\|_{B_{q,p}^s(D)}$ denote their norms. We set $W_q^0(D) = L_q(D)$ and $W_q^s(D) = B_{q,q}^s(D)$. In addition, $(a,b)_D$ denotes the inner product on D defined by $(a,b)_D = \int_D a(x)b(x)\,dx$. Let X be any Banach space with norm $\|\cdot\|_X$. We set $X^d = \{f = (f,\ldots,f_d) \mid f_i \in X$ $(i=1,\ldots,d)\}$, while its norm is denoted by $\|\cdot\|_X$ instead of $\|\cdot\|_{X^d}$ for short. Let $\hat W_q^1(D)$ and $\hat W_{q,0}^1(D)$ be homogeneous spaces defined by $\hat W_q^1(D) = \{v \in L_{q,\mathrm{loc}}(D) \mid \nabla v \in L_q(D)^N\}$ and $\hat W_{q,0}^1(D) = \{v \in \hat W_q^1(D) \mid v|_{\partial D} = 0\}$, respectively, where ∂D is the boundary of D. Moreover, we set $W_{q,0}^1(D) = \{v \in W_q^1(D) \mid v|_{\partial D} = 0\}$. For $1 \le p \le \infty$, $L_p((a,b),X)$ and $W_p^m((a,b),X)$ denote the usual Lebesgue space and Sobolev space of X-valued functions defined on an interval (a,b), while $\|\cdot\|_{L_p((a,b),X)}$ and $\|\cdot\|_{W_p^m((a,b),X)}$ denote their norms, respectively. For any N-vector $\mathbf{w} = (w_1,\ldots,w_N)$ and $\mathbf{z} = (z_1,\ldots,z_N)$, we define $<\mathbf{w},\mathbf{z}>$, $\mathcal{T}_{\mathbf{z}}[\mathbf{w}]$, and $\mathcal{N}_{\mathbf{z}}[\mathbf{w}]$ by:

$$
<\mathbf{w},\mathbf{z}> = \sum_{j=1}^N w_j z_j, \qquad \mathcal{T}_{\mathbf{z}}[\mathbf{w}] = \mathbf{w} - <\mathbf{w},\mathbf{z}>\mathbf{z}, \qquad \mathcal{N}_{\mathbf{z}}[\mathbf{w}] = <\mathbf{w},\mathbf{z}>\mathbf{z}, \tag{14}
$$

respectively. Here, $\mathcal{T}_{\mathbf{z}}[\mathbf{w}]$ denotes the tangential part of $\mathbf{w}$ with respect to $\mathbf{z}$. For $1 < q < \infty$, q' denotes the dual exponent defined by $q' = q/(q-1)$. We use the letter C to denote generic constants, and $C_{a,b,\cdot}$ denotes that the constant $C_{a,b,\cdot}$ essentially depends on the quantities a, b, $\cdots$. Constants C, $C_{a,b,\cdots}$ may change from line to line.

In this paper, let $J_q(\Omega_-)$ be a solenoidal space defined by setting:

$$J_q(D) = \{\mathbf{u}_- \in L_q(D)^N \mid (\mathbf{u}_-, \nabla\varphi)_D = 0 \quad \text{for any } \varphi \in \hat{W}^1_{q',0}(D)\}. \tag{15}$$

We write $\operatorname{div}\mathbf{u} = f = \operatorname{div}\mathbf{f}$ in D for $f \in W^1_q(D)$, $\mathbf{f} \in L_q(D)^N$, and $\mathbf{u} \in W^1_q(D)$, if:

$$(f, \varphi)_D = -(\mathbf{f}, \nabla\varphi)_D \quad \text{for any } \varphi \in W^1_{q',0}(D), \quad \operatorname{div}\mathbf{u} = f \ \text{ in } D, \ \text{ and } \quad \mathbf{u} - \mathbf{f} \in J_q(D). \tag{16}$$

We now introduce a few definitions.

Definition 1. *Let $1 < r < \infty$, and let D be a domain in $\mathbb{R}^N$ with boundary ∂D. We say that D is a uniform $W^{2-1/r}_r$ domain, if there exist positive constants α, β, and K such that for any $x_0 = (x_{01}, \ldots, x_{0N}) \in \partial D$, there exist a coordinate number j and a $W^{2-1/r}_r$ function $h(\check{x})$ $(\check{x} = (x_1, \ldots, \check{x}_j, \ldots, x_N))$ defined on $B'_\alpha(\check{x}'_0)$ with $\check{x}'_0 = (x_{01}, \ldots \check{x}_{0j}, \ldots, x_{0N})$ and $\|h\|_{W^{2-1/r}_r(B'_\alpha(\check{x}'_0))} \leq K$ such that:*

$$\begin{aligned} D \cap B_\beta(x_0) &= \{x \in \mathbb{R}^N \mid x_j > h(x') \ \ (x' \in B'_\alpha(\check{x}'_0))\} \cap B_\beta(x_0), \\ \partial D \cap B_\beta(x_0) &= \{x \in \mathbb{R}^N \mid x_j = h(x') \ \ (x' \in B'_\alpha(\check{x}'_0))\} \cap B_\beta(x_0). \end{aligned} \tag{17}$$

Here, $(x_1, \ldots, \check{x}_j, \ldots, x_N) = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N)$, $B'_\alpha(\check{x}'_0) = \{\check{x}' \in \mathbb{R}^{N-1} \mid |\check{x}' - \check{x}'_0| < \alpha\}$, and $B_\beta(x_0) = \{x \in \mathbb{R}^N \mid |x - x_0| < \beta\}$.

Second, we introduce the assumption of the solvability of the weak Dirichlet problem, which is needed to treat the divergence condition for the incompressible part.

Definition 2. *Let $1 < q < \infty$. We say that the weak Dirichlet problem is uniquely solvable on $\hat{W}^1_{q,0}(\Omega_-)$ with exponents q, if for any $\mathbf{f} \in L_q(\Omega_-)^N$, there exists a unique solution $\theta \in \hat{W}^1_{q,0}(\Omega_-)$ of the variational problem:*

$$(\nabla\theta, \nabla\varphi)_{\Omega_-} = (\mathbf{f}, \nabla\varphi)_{\Omega_-} \quad \textit{for any } \varphi \in \hat{W}^1_{q',0}(\Omega_-). \tag{18}$$

Remark 1. (1) *Since $\partial\Omega_- = \Gamma \cup \Gamma_-$ with $\Gamma \cap \Gamma_- = \emptyset$, $\hat{W}^1_{q,0}(\Omega)_- = \{v \in \hat{W}^1_q(\Omega_-) \mid v|_\Gamma = v|_{\Gamma_-} = 0\}$.* (2) *When $q = 2$, the weak Dirichlet problem is uniquely solvable on Ω_- without any restriction, but for $q \in (1, \infty) \setminus \{2\}$, we do not know the unique solvability in general. For example, we know the unique solvability of the weak Dirichlet problem in bounded domains, exterior domains, half-space, layer, and tube domains. (cf. Galdi [5], as well as Shibata [4,6]).*

Remark 2. *Let $\mathcal{K}$ be a linear operator defined by $\mathcal{K}(\mathbf{f}) = \theta$. Then, combining the unique solvability with Banach's closed range theorem implies the estimate:*

$$\|\nabla\mathcal{K}(\mathbf{f})\|_{L_q(\Omega_-)} \leq C\|\mathbf{f}\|_{L_q(\Omega_-)}. \tag{19}$$

Moreover, for any $\mathbf{f} \in L_q(\Omega_-)^N$ and $g \in W^1_q(\Omega_-)$, $v = g + \mathcal{K}(\mathbf{f} - \nabla g) \in W^1_q(\Omega_-) + \hat{W}^1_{q,0}(\Omega_-)$ satisfies the variational equation: $(\nabla v, \nabla\varphi)_{\Omega_-} = (\mathbf{f}, \nabla\varphi)_{\Omega_-}$ for any $\varphi \in \hat{W}^1_{q',0}(\Omega_-)$, subject to $v = g$ on Γ and Γ_-. Here, we set $W^1_q(\Omega_-) + \hat{W}^1_{q,0}(\Omega_-) = \{p_1 + p_2 \mid p_1 \in W^1_q(\Omega_-),\ p_2 \in \hat{W}^1_{q,0}(\Omega_-)\}$, which is the space for the pressure term p_- in the incompressible part.

The following theorem is our main result about local in time unique existence of solutions to Equations (11) with (12).

Theorem 1. *Let $2 < p < \infty$, $N < q < \infty$, $2/p + N/q < 1$, and $R > 0$. Let $\rho_{0\pm}$ be positive constants describing the reference mass density on $\Omega_\pm$, and let $p(s)$ be a C^∞ function defined on $(\rho_{0+}/2, 2\rho_{0+})$ such that $0 \leq p'(s) \leq \rho_{1+}$ with some positive constant ρ_{1+} for any $\rho \in (\rho_{0+}/2, 2\rho_{0+})$. Let $\Omega_\pm$ be uniform $W^{2-1/q}_q$ domains in $\mathbb{R}^N$ $(N \geq 2)$. Assume that the weak Dirichlet problem is uniquely solvable on $\hat{W}^1_{q,0}(\Omega_-)$ with exponents q and q'. Let $\theta_{0+} \in W^1_q(\Omega_+)$ and $\mathbf{v}_{0\pm} \in B^{2(1-1/p)}_{q,p}(\Omega_\pm)^N$ be initial data with:*

$$\|\theta_{0+}\|_{W^1_q(\Omega)} + \|\mathbf{v}_{0+}\|_{B^{2(1-1/p)}_{q,p}(\Omega_+)} + \|\mathbf{v}_{0-}\|_{B^{2(1-1/p)}_{q,p}(\Omega_-)} \leq R,$$

which satisfy the compatibility condition:

$$\begin{aligned} &\mathcal{T}_{\mathbf{n}}[\mathbf{S}_+(\mathbf{v}_{0+})\mathbf{n}]|_{\Gamma+0} - \mathcal{T}_{\mathbf{n}}[\mathbf{S}_-(\mathbf{v}_{0-})\mathbf{n}]|_{\Gamma-0} = 0, \ \ \mathbf{v}_{0+}|_{\Gamma+0} - \mathbf{v}_{0-}|_{\Gamma-0} = 0, \\ &\mathbf{v}_{0+}|_{\Gamma_+} = 0, \ \ \mathcal{T}_{\mathbf{n}_-}[\mathbf{S}_-(\mathbf{v}_{0-})]|_{\Gamma_-} = 0, \ \ \operatorname{div}\mathbf{v}_{0-} \in J_q(\Omega_-), \end{aligned} \tag{20}$$

and the range condition:

$$\frac{3}{4}\rho_{0+} < \rho_{0+} + \theta_{0+}(x) < \frac{7}{4}\rho_{0+} \quad (x \in \Omega_+). \tag{21}$$

Then, there exists a $T > 0$ depending on R such that the system of Equations (11) with (12) admits a unique solution $(\theta_+, \mathbf{u}_\pm)$ with:

$$\theta_+ \in W^1_p((0,T), W^1_q(\Omega_+)), \quad \mathbf{u}_\pm \in W^1_p((0,T), L_q(\Omega_\pm)^N) \cap L_p((0,T), W^2_q(\Omega_\pm)^N)$$

satisfying (10) and the estimate:

$$\|\theta_+\|_{W^1_p((0,T),W^1_q(\Omega_+))} + \sum_{\ell=+,-} \Big(\|\mathbf{u}_\ell\|_{L_p((0,T),W^2_q(\Omega_\ell))} + \|\partial_t \mathbf{u}_\ell\|_{L_p((0,T),L_q(\Omega_\ell))} \Big) \leq C_R$$

with some constant C_R depending on R, $\rho_{0\pm}$, p, and q.

Using the argument due to Ströhmer [7], we can show the injectivity of the map $x = X_{\mathbf{u}_\pm}(\xi, t)$, so that we have the following local in time unique existence theorem for (1)–(5).

Theorem 2. *Let $N < q < \infty$, $2 < p < \infty$, $2/p + N/q < 1$, and $R > 0$. Assume that $\Omega_\pm$ are uniform $W^{2-1/q}_q$ domains. Assume that the weak Dirichlet problem is uniquely solvable on $\hat{W}^1_{q,0}(\Omega_-)$ with exponents q and q'. Let $\theta_{0+} \in W^1_q(\Omega_+)$ and $\mathbf{v}_{0\pm} \in B^{2(1-1/p)}_{q,p}(\Omega_\pm)^N$ be initial data that satisfy the compatibility condition (20), range condition (21), and:*

$$\|\theta_{0+}\|_{W^1_q(\Omega_+)} + \|\mathbf{v}_{0+}\|_{B^{2(1-1/p)}_{q,p}(\Omega_+)} + \|\mathbf{v}_{0-}\|_{B^{2(1-1/p)}_{q,p}(\Omega_-)} \leq R.$$

Then, there exists a $T > 0$ depending on R such that Equation (1) subject to the interface condition (2), boundary condition (3), kinematic condition (4), and initial condition (5) admits a unique solution $(\rho_+, \mathbf{v}_\pm)$ with:

$$\begin{aligned} \rho - \rho_{0+} &\in W^1_p((0,T), L_q(\Omega_{t+})) \cap L_p((0,T), W^1_q(\Omega_{t+})), \\ \mathbf{v}_\pm &\in W^1_p((0,T), L_q(\Omega_{t\pm})^N) \cap L_p((0,T), W^2_q(\Omega_{t\pm})^N). \end{aligned}$$

Remark 3. *Here, $f \in W^m_p((0,T), W^n_q(\Omega_{t\pm}))$ denotes that for almost all $t \in (0,T)$, $\partial^k_t f(\cdot, t) \in W^n_q(\Omega_{t\pm})$ and:*

$$\|f\|_{W^m_p((0,T),W^n_q(\Omega_{t\pm}))} := \sum_{k=0}^m \Big(\int_0^T \|\partial^k_t f(\cdot,t)\|^p_{W^n_q(\Omega_{t\pm})}\,dt \Big)^{1/p} < \infty.$$

Theorem 1 is proven by using a standard fixed point argument based on the maximal L_p-L_q regularity for solutions to the linear problem:

$$\begin{cases} \partial_t\theta_+ + \gamma_{2+}\mathrm{div}\,\mathbf{u}_+ = f_+ & \text{in } \Omega_+, \\ \gamma_{0+}\partial_t\mathbf{u}_+ - \mathrm{Div}\,\mathbf{S}_+(\mathbf{u}_+) + \nabla(\gamma_{1+}\theta_+) = \mathbf{g}_+ & \text{in } \Omega_+, \\ \rho_{0-}\partial_t\mathbf{u}_- - \mathrm{Div}\,\mathbf{S}_-(\mathbf{u}_-) + \nabla p_- = \mathbf{g}_- & \text{in } \Omega_-, \\ \mathrm{div}\,\mathbf{u}_- = f_- = \mathrm{div}\,\mathbf{f}_- & \text{in } \Omega_-, \\ (\mathbf{S}_+(\mathbf{u}_+) - \gamma_{1+}\theta_+\mathbf{I})\mathbf{n}|_{\Gamma_t+0} - (\mathbf{S}_-(\mathbf{u}_-) - p_-\mathbf{I})\mathbf{n}|_{\Gamma_t-0} = \mathbf{h} & \text{for } t>0, \\ \mathbf{u}_+|_{\Gamma_t+0} - \mathbf{u}_-|_{\Gamma_t-0} = 0 & \text{for } t>0, \\ \mathbf{u}_+|_{\Gamma_+} = 0, \quad (\mathbf{S}_-(\mathbf{u}_-) - p_-\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = \mathbf{h}_- & \text{for } t>0, \\ (\theta_+, \mathbf{u}_+)|_{t=0} = (\theta_{0+}, \mathbf{u}_{0+}) \text{ in } \Omega_+, \quad \mathbf{u}_-|_{t=0} = \mathbf{u}_{0-} & \text{in } \Omega_-. \end{cases} \tag{22}$$

Here, $\gamma_i = \gamma_i(x)$ $(i = 0,1,2)$ are uniformly continuous functions defined on $\overline{\Omega_+}$ such that:

$$\frac{1}{2}\rho_{0+} \le \gamma_{0+}(x) \le 2\rho_{0+}, \;\; 0 \le \gamma_{k+}(x) \le \rho_{2+} \;\; (x \in \Omega), \quad \|\nabla\gamma_{\ell+}\|_{L_r(\Omega_+)} \le \rho_{2+} \tag{23}$$

for $k = 1,2$ and $\ell = 0,1,2$ with some positive constant ρ_{2+} and $N < r < \infty$. We may consider the case where $\gamma_{1+} = 0$, which corresponds to the Lamé system.

Symbols 2. To state our main result for linear Equation (22), we introduce more symbols and functional spaces used throughout this paper. Set:

$$W^m_{p,\mathrm{loc}}((a,b),X) = \{f(t) \mid f(t) \in W^m_p((c,d),X) \text{ for any } c, d \text{ with } a<c<d<b\}$$

and $W^0_{p,\mathrm{loc}}(\mathbb{R},X) = L_{p,\mathrm{loc}}(\mathbb{R},X)$. Moreover, we set:

$$W^m_{p,\gamma}(\mathbb{R},X) = \{f(t) \in L_{p,\mathrm{loc}}(\mathbb{R},X) \mid e^{-\gamma t}\partial_t^j f(t) \in L_p(\mathbb{R},X) \;\; (j = 0,1,\ldots,m)\}$$

with $\partial_t^0 f(t) = f(t)$ and $W^0_{p,\gamma}(\mathbb{R},X) = L_{p,\gamma}(\mathbb{R},X)$ and $W^0_{p,\gamma,0}(\mathbb{R},X) = L_{p,\gamma,0}(\mathbb{R},X)$.

Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace transform and the Laplace inverse transform defined by:

$$\mathcal{L}[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t)\,dt, \qquad \mathcal{L}^{-1}[g](t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda t} g(\tau)\,d\tau$$

with $\lambda = \gamma + i\tau \in \mathbb{C}$, respectively. Given $s \in \mathbb{R}$ and X-valued function $f(t)$, we set:

$$\Lambda^s_\gamma f(t) = \mathcal{L}^{-1}_\lambda[\lambda^s\mathcal{L}[f](\lambda)](t).$$

We introduce a Bessel potential space of X-valued functions of order $s > 0$ as follows:

$$H^s_{p,\gamma}(\mathbb{R},X) = \{f \in L_p(\mathbb{R},X) \mid e^{-\gamma' t}\Lambda^s_{\gamma'}[f](t) \in L_p(\mathbb{R},X) \text{ for any } \gamma' \ge \gamma\}.$$

We have the following theorem.

Theorem 3. *Let $1 < p,q < \infty$, $N < r < \infty$, $2/p + N/q \ne 1$, and $2/p + N/q \ne 2$. Assume that $r \ge \max(q,q')$, that $\Omega_\pm$ are uniformly $W^{2-1/q}_q$ domains, and that the weak Dirichlet problem is uniquely solvable on Ω_- with exponents q and q'. Then, there exists a positive number γ_0 such that the following three assertions are valid.*

EXISTENCE *For any initial data $\theta_{0+} \in W^1_q(\Omega_+)$ and $\mathbf{u}_{0\pm} \in B^{2(1-1/p)}_{q,p}(\Omega_\pm)$, and any right members f_+, $f_- = \mathrm{div}\,\mathbf{f}_-$, $\mathbf{g}_\pm$, $\mathbf{h}$, and $\mathbf{h}_-$ with:*

$$
\begin{aligned}
&f_+ \in L_{p,\gamma_0}(\mathbb{R}, W^1_q(\Omega_+)), \quad \mathbf{g}_\pm \in L_{p,\gamma_0}(\mathbb{R}, L_q(\Omega_\pm)^N), \quad \mathbf{h} \in L_{p,\gamma_0}(\mathbb{R}, W^1_q(\Omega)^N) \cap H^{1/2}_{p,\gamma_0}(\mathbb{R}, L_q(\Omega)^N), \\
&f_- \in L_{p,\gamma_0}(\mathbb{R}, W^1_q(\Omega_-)) \cap H^{1/2}_{p,\gamma_0,0}(\mathbb{R}, L_q(\Omega_-)), \quad \mathbf{f}_- \in W^1_p(\mathbb{R}, L_q(\Omega_-)), \\
&\mathbf{h}_- \in L_{p,\gamma_0}(\mathbb{R}, W^1_q(\Omega_-)^N) \cap H^{1/2}_{p,\gamma_0}(\mathbb{R}, L_q(\Omega_-)^N),
\end{aligned} \tag{24}
$$

satisfying the compatibility conditions:

$$
\begin{aligned}
&\mathcal{T}_\mathbf{n}[\mathbf{S}_+(\mathbf{u}_{0+})\mathbf{n}]|_{\Gamma+0} - \mathcal{T}_\mathbf{n}[\mathbf{S}_-(\mathbf{u}_{0-})\mathbf{n}]|_{\Gamma-0} = \mathcal{T}_\mathbf{n}[\mathbf{h}|_\Gamma]|_{t=0}, \quad \mathbf{u}_{0+}|_{\Gamma+0} - \mathbf{u}_{0-}|_{\Gamma-0} = 0, \\
&\mathbf{u}_{0+}|_{\Gamma_+} = 0, \quad \mathcal{T}_{\mathbf{n}_-}[\mathbf{S}_-(\mathbf{u}_{0-})\mathbf{n}_-]|_{\Gamma_-} = \mathcal{T}_{\mathbf{n}_-}[\mathbf{h}_-]|_{t=0}, \\
&\operatorname{div} \mathbf{u}_{0-} = f_-|_{t=0} \ \text{in } \Omega_-, \quad (\mathbf{u}_{0-}, \nabla\varphi)_{\Omega_-} = (\mathbf{f}_-, \nabla\varphi)_{\Omega_-} \quad \text{for any } \varphi \in \hat{W}^1_{q',0}(\Omega_-).
\end{aligned} \tag{25}
$$

Equation (22) admits solutions θ_+ and $\mathbf{u}_\pm$ with:

$$
\theta_+ \in W^1_{p,\gamma_0}(\mathbb{R}, W^1_q(\Omega_+)), \quad \mathbf{u}_\pm \in L_{p,\gamma_0}(\mathbb{R}, W^2_q(\Omega_\pm)^N) \cap W^1_{p,\gamma_0}(\mathbb{R}, L_q(\Omega_\pm)^N) \tag{26}
$$

possessing the estimate:

$$
\begin{aligned}
&\|e^{-\gamma t}(\partial_t\theta_+, \gamma\theta_+)\|_{L_p(\mathbb{R}_+, W^1_q(\Omega_+))} + \sum_{\ell=+,-} \|e^{-\gamma t}(\partial_t \mathbf{u}_\ell, \gamma\mathbf{u}_\ell, \Lambda^{1/2}_\gamma \nabla\mathbf{u}_\ell, \nabla^2\mathbf{u}_\ell)\|_{L_p(\mathbb{R}_+, L_q(\Omega_\ell))} \\
&\leq C\Big(\|\theta_{0+}\|_{W^1_q(\Omega_+)} + \sum_{\ell=+,-} \|\mathbf{u}_{0\ell}\|_{B^{2(1-1/p)}_{q,p}(\Omega_\ell)} \\
&\quad + \|e^{-\gamma t} f_+\|_{L_p(\mathbb{R}_+, W^1_q(\Omega_+))} + \|e^{-\gamma t}(\Lambda^{1/2}_\gamma f_-, \nabla f_-)\|_{L_p(\mathbb{R}, L_q(\Omega_-))} \\
&\quad + \|e^{-\gamma t}\partial_t \mathbf{f}_-\|_{L_p(\mathbb{R}, L_q(\Omega_-))} + \|e^{-\gamma t}\mathbf{g}_+\|_{L_p(\mathbb{R}, L_q(\Omega_+))} + \|e^{-\gamma t}\mathbf{g}_-\|_{L_p(\mathbb{R}, L_q(\Omega_-))} \\
&\quad + \|e^{-\gamma t}(\Lambda^{1/2}_\gamma \mathbf{h}, \nabla\mathbf{h})\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{-\gamma t}(\Lambda^{1/2}_\gamma \mathbf{h}_-, \nabla\mathbf{h}_-)\|_{L_p(\mathbb{R}, L_q(\Omega_-))}\Big)
\end{aligned} \tag{27}
$$

for any $\gamma \geq \gamma_0$, where C is a constant independent of γ.

UNIQUENESS *Let θ_+ and $\mathbf{u}_\pm$ satisfy (26) and Equation (22) with $\theta_{0+} = 0$, $\mathbf{u}_{0\pm} = 0$, $f_\pm = 0$, $\mathbf{g}_\pm = 0$, $\mathbf{f}_- = 0$, and $\mathbf{h} = 0$, then $\theta_+ = 0$ and $\mathbf{u}_\pm = 0$.*

To prove Theorem 3, Problem (22) is divided into two parts: One is the case where the right side in (22) is considered for all $t \in \mathbb{R}$, while the initial conditions are not taken into account. The other case is non-homogeneous initial conditions and a zero right side in (22). In the first case, solutions are represented by the Laplace inverse transform of solution formulas represented by using $\mathcal{R}$-bounded solution operators for the generalized resolvent problem corresponding to (22). Combining the $\mathcal{R}$-boundedness and Weis's operator-valued Fourier multiplier theorem yields the maximal L_p-L_q estimate of solutions to Equation (22) with zero initial conditions. Moreover, the $\mathcal{R}$-bounded solution operators yield the generation of the continuous analytic semigroup associated with Equation (22), which, combined with some real interpolation technique, yields the L_p-L_q maximal regularity for the initial problem for Equation (22). Combining these two results gives Theorem 3. To prove the generation of the continuous analytic semigroup, we have to eliminate the pressure term p_- in Equation (22), and so, using the assumption of the unique existence of the weak Dirichlet problem, we define the reduced generalized resolvent problem (RGRP) (cf. (41) in Section 2 below) according to Grubb and Solonnikov [8], which is the equivalent system to the generalized resolvent problem (GRP) corresponding to (22).

The paper is organized as follows. In Section 2, first we introduce (GRP) and state main results for (GRP). Secondly, we drive (RGRP) and discuss some equivalence between (GRP) and (RGRP). Thirdly, we state the main results for (RGRP), which implies the results for (GRP) according to the equivalence between (GRP) and (RGRP). In Section 3, we discuss the model problems in $\mathbb{R}^N$. In Section 4, we discuss the bent half space problems for (RGRP). In Section 5, we prove the main result for (RGRP) and also Theorem 3. In Section 6, we prove Theorem 1 by the Banach fixed point argument based on Theorem 3.

2. $\mathcal{R}$-Bounded Solution Operators

To prove the generation of the continuous analytic semigroup and the maximal L_p-L_q regularity for the linear problem (22), we show the existence of $\mathcal{R}$-bounded solution operators to the following generalized resolvent problem (GRP) corresponding to: (22):

$$\begin{cases} \lambda\theta_+ + \gamma_{2+}\mathrm{div}\,\mathbf{u}_+ = f_+ & \text{in } \Omega_+, \\ \lambda\mathbf{u}_+ - \gamma_{0+}^{-1}(\mathrm{Div}\,\mathbf{S}_+(\mathbf{u}_+) - \nabla(\gamma_{1+}\theta_+)) = \mathbf{g}_+ & \text{in } \Omega_+, \\ \lambda\mathbf{u}_- - \rho_{0-}^{-1}(\mathrm{Div}\,\mathbf{S}_-(\mathbf{u}_-) - \nabla p_-) = \mathbf{g}_- & \text{in } \Omega_-, \\ \mathrm{div}\,\mathbf{u}_- = f_- = \mathrm{div}\,\mathbf{f}_- & \text{in } \Omega_- \\ (\mathbf{S}_+(\mathbf{u}_+) - \gamma_{1+}\theta_+\mathbf{I})\mathbf{n}|_{\Gamma+0} - (\mathbf{S}_-(\mathbf{u}_-) - p_-\mathbf{I})\mathbf{n}|_{\Gamma-0} = \mathbf{h}|_\Gamma, & \\ (\mathbf{S}_-(\mathbf{u}_-) - p_-\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = \mathbf{h}_-|_{\Gamma_-}, & \\ \mathbf{u}_+|_{\Gamma+0} - \mathbf{u}_-|_{\Gamma-0} = 0, \qquad \mathbf{u}_+|_{\Gamma_+} = 0. & \end{cases} \tag{28}$$

When $\lambda \neq 0$, setting $\theta_+ = \lambda^{-1}(f_+ - \gamma_{2+}\mathrm{div}\,\mathbf{u}_+)$, we transfer the second equation and the fifth equation in (28) to:

$$\lambda\mathbf{u}_+ - \gamma_0^{-1}(\mathrm{Div}\,\mathbf{S}_+(\mathbf{u}_+) + \lambda^{-1}\nabla(\gamma_{1+}\gamma_{2+}\mathrm{div}\,\mathbf{u}_+)) = \mathbf{g}_+ - \lambda^{-1}\gamma_{0+}^{-1}\nabla(\gamma_{1+}f_+) \quad \text{in } \Omega_+,$$
$$(\mathbf{S}_+(\mathbf{u}_+) + \lambda^{-1}\gamma_{1+}\gamma_{2+}\mathrm{div}\,\mathbf{u}_+\mathbf{I})\mathbf{n} - (\mathbf{S}_-(\mathbf{u}_-) - p_-\mathbf{I})\mathbf{n}|_{\Gamma-0} = \mathbf{h} + \lambda^{-1}\gamma_{1+}f_+\mathbf{n}|_{\Gamma+0},$$

respectively. Thus, $\mathbf{g}_+ - \lambda^{-1}\gamma_{0+}^{-1}\nabla(\gamma_{1+}f_+)$ and $\mathbf{h} + \lambda^{-1}\gamma_{1+}f_+\mathbf{n}|_{\Gamma+0}$, being renamed $\mathbf{g}_+$ and $\mathbf{h}$, respectively, and setting $\gamma_{1+}\gamma_{2+} = \gamma_{3+}$, from now on, we consider the following problem:

$$\begin{cases} \lambda\mathbf{u}_+ - \gamma_{0+}^{-1}(\mathrm{Div}\,\mathbf{S}_+(\mathbf{u}_+) + \delta\nabla(\gamma_{3+}\mathrm{div}\,\mathbf{u}_+)) = \mathbf{g}_+ & \text{in } \Omega_+, \\ \lambda\mathbf{u}_- - \rho_{0-}^{-1}(\mathrm{Div}\,\mathbf{S}_-(\mathbf{u}_-) - \nabla p_-) = \mathbf{g}_- & \text{in } \Omega_-, \\ \mathrm{div}\,\mathbf{u}_- = f_- = \mathrm{div}\,\mathbf{f}_- & \text{in } \Omega_-, \\ (\mathbf{S}_+(\mathbf{u}_+) + \delta\gamma_{3+}\mathrm{div}\,\mathbf{u}_+\mathbf{I})\mathbf{n}|_{\Gamma+0} - (\mathbf{S}_-(\mathbf{u}_-) - p_-\mathbf{I})\mathbf{n}|_{\Gamma-0} = \mathbf{h}|_\Gamma, & \\ (\mathbf{S}_-(\mathbf{u}_-) - p_-\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = \mathbf{h}_-|_{\Gamma_-}, & \\ \mathbf{u}_+|_{\Gamma+0} = \mathbf{u}_-|_{\Gamma-0}, \qquad \mathbf{u}_+|_{\Gamma_+} = 0. & \end{cases} \tag{29}$$

Here, δ and λ satisfy one of the following three conditions:

(C1) $\delta = \lambda^{-1}$, $\quad \lambda \in \Lambda_{\epsilon,\lambda_0} = K_\epsilon \cap \Sigma_{\epsilon,\lambda_0}$,

(C2) $\delta = \delta_0 \in \Sigma_\epsilon$ with $\mathrm{Re}\,\delta_0 < 0$, $\quad \lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq \lambda_0$ and $\mathrm{Re}\,\lambda \geq |\mathrm{Im}\,\lambda||\frac{\mathrm{Re}\,\delta_0}{\mathrm{Im}\,\delta_0}|$,

(C3) $\delta = \delta_0$ with $\mathrm{Re}\,\delta_0 \geq 0$, $\quad \lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq \lambda_0$,

where we set $\Sigma_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \leq \pi - \epsilon\}$ with $0 < \epsilon < \pi/2$, $\Sigma_{\epsilon,\lambda_0} = \{\lambda \in \Sigma_\epsilon \mid |\lambda| \geq \lambda_0\}$, and:

$$K_\epsilon = \{\lambda \in \mathbb{C} \mid (\mathrm{Re}\,\lambda + \rho_3\nu^{-1} + \epsilon)^2 + (\mathrm{Im}\,\lambda)^2 \geq (\rho_3\nu^{-1} + \epsilon)^2\} \tag{30}$$

with $\rho_3 = \sup_{x\in\overline{\Omega}} \gamma_{1+}(x)\gamma_{2+}(x)(\leq \rho_2^2)$. We may include the case where $\gamma_1 = 0$, which corresponds to the Lamé system. The former case (C1) is used to prove the existence of $\mathcal{R}$-bounded solution operators to (28), and the latter cases (C2) and (C3) enable the application of a homotopic argument for proving the exponential stability of the analytic semigroup in bounded domains. For the sake of simplicity, we introduce the set $\Gamma_{\epsilon,\lambda_0}$ defined by:

$$\Gamma_{\epsilon,\lambda_0} = \begin{cases} \Lambda_{\epsilon,\lambda_0} & \text{when } \delta = \lambda^{-1}, \\ \left\{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \geq \lambda_0,\ \mathrm{Re}\,\lambda \geq \left|\frac{\mathrm{Re}\,\delta_0}{\mathrm{Im}\,\delta_0}\right||\mathrm{Im}\,\lambda|\right\} & \text{when } \delta = \delta_0 \in \Sigma_\epsilon \text{ with } \mathrm{Re}\,\delta_0 < 0, \\ \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \geq \lambda_0\} & \text{when } \delta = \delta_0 \text{ with } \mathrm{Re}\,\delta_0 \geq 0. \end{cases} \tag{31}$$

Note that $|\delta| \leq \max(|\delta_0|, \lambda_0^{-1})$.

Before stating our main results for the linear problem, we introduce a few symbols and the definition of the $\mathcal{R}$-bounded operator family and the operator-valued Fourier multiplier theorem due to Weis [9].

Symbols 3. For any two Banach spaces X and Y, $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X to Y, and we write $\mathcal{L}(X) = \mathcal{L}(X, X)$ for short. Hol (U, X) denotes the set of all X-valued holomorphic functions defined on a complex domain U. Let $\mathcal{D}(\mathbb{R}, X)$ and $\mathcal{S}(\mathbb{R}, X)$ be the set of all X-valued C^∞-functions having compact support and the Schwartz space of rapidly decreasing X-valued functions, respectively, while $\mathcal{S}'(\mathbb{R}, X) = \mathcal{L}(\mathcal{S}(\mathbb{R}, \mathbb{C}), X)$. Given $M \in L_{1,\mathrm{loc}}(\mathbb{R} \setminus \{0\}, X)$, we define the operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X) \to \mathcal{S}'(\mathbb{R}, Y)$ by:

$$T_M \varphi = \mathcal{F}^{-1}[M\mathcal{F}[\varphi]] \qquad (\mathcal{F}[\varphi] \in \mathcal{D}(\mathbb{R}, X)). \tag{32}$$

Here, $\mathcal{F}_x$ and $\mathcal{F}_x^{-1}$ denote the Fourier transform and its inversion defined by:

$$\mathcal{F}_x[u](\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} u(x)dx, \quad \mathcal{F}_\xi^{-1}[v](x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix\cdot\xi} v(\xi)d\xi,$$

respectively.

Definition 3. *Let X and Y be Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X, Y)$, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for any $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{x_j\}_{j=1}^n \subset X$, and sequences $\{r_j(u)\}_{j=1}^n$ of independent, symmetric, $\{-1, 1\}$-valued random variables on $[0, 1]$, there holds the inequality:*

$$\int_0^1 \| \sum_{j=1}^n r_j(u) T_j x_j \|_Y^p \, du \le C \int_0^1 \| \sum_{j=1}^n r_j(u) x_j \|_X^p \, du.$$

The smallest such C is called the $\mathcal{R}$-bound of $\mathcal{T}$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

The following theorem was obtained by Weis [9].

Theorem 4. *Let X and Y be two UMD spaces and $1 < p < \infty$. Let M be a function in $C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X, Y))$ such that:*

$$\mathcal{R}_{\mathcal{L}(X,Y)}\Big(\Big\{ (\tau \frac{d}{d\tau})^\ell M(\tau) \mid \tau \in \mathbb{R} \setminus \{0\} \Big\}\Big) \le \kappa < \infty \quad (\ell = 0, 1)$$

with some constant κ. Then, the operator T_M defined in (32) may uniquely be extended to a bounded linear operator from $L_p(\mathbb{R}, X)$ to $L_p(\mathbb{R}, Y)$. Moreover, denoting this extension by T_M, we have:

$$\|T_M\|_{\mathcal{L}(L_p(\mathbb{R},X), L_p(\mathbb{R},Y))} \le C\kappa$$

for some positive constant C depending on p, X, and Y.

Remark 4. *For the definition of the UMD space, we refer to the monograph by Amann [10]. For $1 < q < \infty$ and $m \in \mathbb{N}$, Lebesgue spaces $L_q(\Omega)$ and Sobolev spaces $W_q^m(\Omega)$ are UMD spaces.*

For the calculation of the $\mathcal{R}$-norm, we use the following lemmas.

Lemma 1. (1) *Let X and Y be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X, Y)$. Then, $\mathcal{T} + \mathcal{S} = \{T + S \mid T \in \mathcal{T}, \ S \in \mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X, Y)$ and:*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T} + \mathcal{S}) \le \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}) + \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).$$

(2) *Let X, Y, and Z be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X, Y)$ and $\mathcal{L}(Y, Z)$, respectively. Then, $\mathcal{S}\mathcal{T} = \{ST \mid T \in \mathcal{T},\ S \in \mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X, Z)$ and:*

$$\mathcal{R}_{\mathcal{L}(X,Z)}(\mathcal{S}\mathcal{T}) \le \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})\mathcal{R}_{\mathcal{L}(Y,Z)}(\mathcal{S}).$$

Lemma 2. *Let $1 < p,\ q < \infty$, and let D be a domain in $\mathbb{R}^N$.*
(1) *Let $m(\lambda)$ be a bounded function defined on a subset $\Lambda \subset \mathbb{C}$, and let $M_m(\lambda)$ be a multiplication operator with $m(\lambda)$ defined by $M_m(\lambda)f = m(\lambda)f$ for any $f \in L_q(D)$. Then,*

$$\mathcal{R}_{\mathcal{L}(L_q(D))}(\{M_m(\lambda) \mid \lambda \in \Lambda\}) \le C_{N,q,D}\|m\|_{L_\infty(\Lambda)}.$$

(2) *Let $n(\tau)$ be a C^1 function defined on $\mathbb{R} \setminus \{0\}$ that satisfies the conditions: $|n(\tau)| \le \gamma$ and $|\tau n'(\tau)| \le \gamma$ with some constant $\gamma > 0$ for any $\tau \in \mathbb{R} \setminus \{0\}$. Let T_n be an operator-valued Fourier multiplier defined by $T_n f = \mathcal{F}^{-1}[n\mathcal{F}[f]]$ for any f with $\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, X)$. Then, T_n is extended to a bounded linear operator from $L_p(\mathbb{R}, L_q(D))$ into itself. Moreover, denoting this extension also by T_n, we have:*

$$\|T_n\|_{\mathcal{L}(L_p(\mathbb{R},L_q(D)))} \le C_{p,q,D}\gamma.$$

Remark 5. *For the proofs of Lemma 1 and Lemma 2, we refer to [11], p.28, 3.4. Proposition and p.27, 3.2. Remarks* (4) *(cf. also Bourgain [12]), respectively.*

2.1. Existence of $\mathcal{R}$-Bounded Solution Operators for Problems (28) *and* (29)

We state two theorems about the existence of $\mathcal{R}$-bounded solution operators to Problems (28) and (29).

Theorem 5. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$ and $N < r < \infty$. Assume that $r \ge \max(q, q')$, that $\Omega_\pm$ are uniform $W^{2-1/r}_r$ domains, and that the weak Dirichlet problem is uniquely solvable on Ω_- with exponents q and q'. Let $X^0_q(\Omega)$ and $\mathcal{X}^0_q(\Omega)$ be the spaces defined by:*

$$\begin{aligned} X^0_q(\Omega) = \{\mathbf{G}^0 = (\mathbf{g}_+, \mathbf{g}_-, \mathbf{h}, \mathbf{h}_-, f_-, \mathbf{f}_-) \mid \mathbf{g}_\pm \in L_q(\Omega_\pm)^N,\ \mathbf{h} \in W^1_q(\Omega)^N,\ \mathbf{h}_- \in W^1_q(\Omega_-)^N, \\ f_- \in W^1_q(\Omega_-),\ \mathbf{f}_- \in L_q(\Omega_-)^N,\ f_- = \operatorname{div}\mathbf{f}_-\}, \\ \mathcal{X}^0_q(\Omega) = \{\mathbf{F}^0 = (\mathbf{F}_1, \ldots, \mathbf{F}_9) \mid \mathbf{F}_1 \in L_q(\Omega_+)^N,\ \mathbf{F}_2, \mathbf{F}_5, \mathbf{F}_9 \in L_q(\Omega_-)^N, \\ \mathbf{F}_3 \in L_q(\Omega)^N,\ \mathbf{F}_4 \in W^1_q(\Omega)^N,\ \mathbf{F}_6 \in W^1_q(\Omega_-)^N,\ \mathbf{F}_7 \in L_q(\Omega_-),\ \mathbf{F}_8 \in W^1_q(\Omega_-)\}. \end{aligned}$$

Then, there exist a constant $\lambda_0 > 0$ and operator families $\mathcal{A}^0_\pm(\lambda)$ and $\mathcal{B}^0_-(\lambda)$ with:

$$\mathcal{A}^0_\pm(\lambda) \in \mathrm{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}^0_q(\Omega), W^2_q(\Omega_\pm)^N)),\ \ \mathcal{B}^0_-(\lambda) \in \mathrm{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}^0_q(\Omega), W^1_q(\Omega_-) + \hat{W}^1_{q,0}(\Omega_-)))$$

such that $\mathbf{u}_\pm = \mathcal{A}^0_\pm(\lambda)F^0_\lambda\mathbf{G}^0$ and $p_- = \mathcal{B}^0_-(\lambda)F^0_\lambda\mathbf{G}^0$ are unique solutions to Problem (29) *for any $\lambda \in \Gamma_{\epsilon,\lambda_0}$ and $\mathbf{G}^0 = (\mathbf{g}_+, \mathbf{g}_-, \mathbf{h}, \mathbf{h}_-, f_-, \mathbf{f}_-) \in X^0_q(\Omega)$, and:*

$$\begin{aligned} \mathcal{R}_{\mathcal{L}(\mathcal{X}^0_q(\Omega), W^{2-j}_q(\Omega_\pm)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{A}^0_\pm(\lambda)) \mid \lambda = \gamma + i\tau \in \Gamma_{\epsilon,\lambda_0}\}) \le C, \\ \mathcal{R}_{\mathcal{L}(\mathcal{X}^0_q(\Omega), L_q(\Omega_-)^N)}(\{(\tau\partial_\tau)^\ell\nabla\mathcal{B}^0_-(\lambda) \mid \lambda = \gamma + i\tau \in \Gamma_{\epsilon,\lambda_0}\}) \le C \end{aligned}$$

for $j = 0, 1, 2$ and $\ell = 0, 1$. Here, we set $F^0_\lambda\mathbf{G}^0 = (\mathbf{g}_+, \mathbf{g}_-, \lambda^{1/2}\mathbf{h}, \mathbf{h}, \lambda^{1/2}\mathbf{h}_-, \mathbf{h}_-, \lambda^{1/2}f_-, f_-, \lambda\mathbf{f}_-)$.

Remark 6. (i) *The constants depend on ϵ, q, r, ρ_0, ρ_2, $\mu_\pm$, ν_+, and δ_0, but we do not mention this dependence.*
(ii) *The variables $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$, $\mathbf{F}_5$, $\mathbf{F}_6$, $\mathbf{F}_7$, $\mathbf{F}_8$, and $\mathbf{F}_9$ correspond to $\mathbf{g}_+$, $\mathbf{g}_-$, $\lambda^{1/2}\mathbf{h}$, $\mathbf{h}$, $\lambda^{1/2}\mathbf{h}_-$,*

$\mathbf{h}_-$, $\lambda^{1/2}f_-$, f_-, *and* $\lambda\mathbf{f}_-$.
(iii) *The norms* $\|\cdot\|_{X_q^0(\Omega)}$ *and* $\|\cdot\|_{\mathcal{X}_q^0(\Omega)}$ *are defined by:*

$$\begin{aligned}\|\mathbf{G}^0\|_{X_q^0(\Omega)} &= \|\mathbf{g}_+\|_{L_q(\Omega_+)} + \|(\mathbf{g}_-, \lambda^{1/2}\mathbf{h}_-, \lambda^{1/2}f_-, \lambda\mathbf{f}_-)\|_{L_q(\Omega_-)} + \|(\mathbf{h}_-, f_-)\|_{W_q^1(\Omega_-)} \\ &\quad + \|\lambda^{1/2}\mathbf{h}\|_{L_q(\Omega)}, +\|\mathbf{h}\|_{W_q^1(\Omega)}, \\ \|\mathbf{F}^0\|_{\mathcal{X}_q^0(\Omega)} &= \|\mathbf{F}_1\|_{L_q(\Omega_+)} + \|(\mathbf{F}_2, \mathbf{F}_5, \mathbf{F}_7, \mathbf{F}_9)\|_{L_q(\Omega_-)} + \|\mathbf{F}_3\|_{L_q(\Omega)} + \|\mathbf{F}_4\|_{W_q^1(\Omega)} + \|(\mathbf{F}_6, \mathbf{F}_8)\|_{W_q^1(\Omega_-)}.\end{aligned}$$

Since $\theta_+ = \lambda^{-1}(f_+ - \gamma_{2+}\mathrm{div}\,\mathbf{u}_+)$ in (28), the following theorem follows immediately from Theorem 5 and Lemma 1.

Theorem 6. *Let* $1 < q < \infty$, $0 < \epsilon < \pi/2$, *and* $N < r < \infty$. *Assume that* $r \geq \max(q, q')$, *that* $\Omega_\pm$ *are uniform* $W_r^{2-1/r}$ *domains, and that the weak Dirichlet problem is uniquely solvable on* Ω_- *with exponents* q *and* q'. *Let* $X_q^1(\Omega)$ *and* $\mathcal{X}_q^1(\Omega)$ *be the sets defined by:*

$$\begin{aligned}X_q^1(\Omega) &= \{\mathbf{G}^1 = (\mathbf{G}^0, f_+) \mid \mathbf{G}^0 \in X_q^0(\Omega),\ \ f_+ \in W_q^1(\Omega_+)\}, \\ \mathcal{X}_q^1(\Omega) &= \{\mathbf{F}^1 = (\mathbf{F}^0, \mathbf{F}_{10}) \mid \mathbf{F}^0 \in \mathcal{X}_q^0(\Omega),\ \ \mathbf{F}_{10} \in W_q^1(\Omega_+)\}.\end{aligned}$$

Then, there exist a constant $\lambda_0 > 0$ *and operator families* $\mathcal{A}_\pm^1(\lambda)$ *and* $\mathcal{B}_\pm^1(\lambda)$ *with:*

$$\begin{aligned}&\mathcal{A}_\pm^1(\lambda) \in \mathrm{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q^1(\Omega), W_q^2(\Omega_\pm)^N)),\ \ \mathcal{B}_+^1(\lambda) \in \mathrm{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q^1(\Omega), W_q^1(\Omega_+))), \\ &\mathcal{B}_-^1(\lambda) \in \mathrm{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q^1(\Omega), W_q^1(\Omega_-) + \hat{W}_{q,0}^1(\Omega_-)))\end{aligned}$$

such that $\mathbf{u}_\pm = \mathcal{A}_\pm^1(\lambda)F_\lambda^1\mathbf{G}^1$, $\theta_+ = \mathcal{B}_+^1(\lambda)F_\lambda^1\mathbf{G}^1$, *and* $p_- = \mathcal{B}_-^1(\lambda)F_\lambda^1\mathbf{G}^1$ *are unique solutions to Problem* (28) *for any* $\lambda \in \Gamma_{\epsilon,\lambda_0}$ *and* $\mathbf{G}^1 = (\mathbf{g}_+, \mathbf{g}_-, \mathbf{h}, \mathbf{h}_-, f_-, \mathbf{f}_-, f_+) \in X_q^1(\Omega)$, *and:*

$$\begin{aligned}\mathcal{R}_{\mathcal{L}(\mathcal{X}_q^1(\Omega), W_q^{2-j}(\Omega_\pm)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{A}_\pm^0(\lambda)) \mid \lambda = \gamma + i\tau \in \Gamma_{\epsilon,\lambda_0}\}) &\leq C, \\ \mathcal{R}_{\mathcal{L}(\mathcal{X}_q^1(\Omega), W_q^1(\Omega_+))}(\{(\tau\partial_\tau)^\ell(\lambda^k\mathcal{B}_+^1(\lambda)) \mid \lambda = \gamma + i\tau \in \Gamma_{\epsilon,\lambda_0}\}) &\leq C, \\ \mathcal{R}_{\mathcal{L}(\mathcal{X}_q^1(\Omega), L_q(\Omega_-)^N)}(\{(\tau\partial_\tau)^\ell(\nabla\mathcal{B}_-^0(\lambda)) \mid \lambda = \gamma + i\tau \in \Gamma_{\epsilon,\lambda_0}\}) &\leq C\end{aligned}$$

for $j = 0, 1, 2$, $k = 0, 1$, *and* $\ell = 0, 1$. *Here, we set* $F_\lambda^1\mathbf{G}^1 = (F_\lambda^0\mathbf{G}^0, f_+)$.

Remark 7. *The variable* $\mathbf{F}_{10}$ *corresponds to* f_+, *and we set:*

$$\|\mathbf{G}^1\|_{X_q^1(\Omega)} = \|\mathbf{G}^0\|_{X_q^0(\Omega_+)} + \|f_+\|_{W_q^1(\Omega_+)},\ \ \|\mathbf{F}^1\|_{\mathcal{X}_q^1(\Omega)} = \|\mathbf{F}^0\|_{\mathcal{X}_q^0(\Omega)} + \|\mathbf{F}_{10}\|_{W_q^1(\Omega_+)}.$$

2.2. Reduced Generalized Resolvent Problem

Since the pressure term p_- has no time evolution in (22), we eliminate p_- from (29) and derive a reduced problem. Before this discussion, we consider the resolvent problem for the Laplace operator with non-homogeneous Dirichlet condition of the form:

$$\lambda(w, \varphi)_{\Omega_-} + \rho_{0-}^{-1}(\nabla w, \nabla\varphi)_{\Omega_-} = -(\mathbf{f}, \nabla\varphi)_{\Omega_-} \quad \text{for any } \varphi \in W_{q',0}^1(\Omega_-) \tag{33}$$

subject to $w|_\Gamma = g_1$ and $w|_{\Gamma_-} = g_2$. Here and in the following, we write $(\cdot,\cdot) = (\cdot,\cdot)_{\Omega_-}$ for short. Note that:

$$(w, \nabla\varphi)_{\Omega_-} = -(\lambda^{-1}(\mathbf{f} + \rho_{0-}^{-1}\nabla w), \nabla\varphi)_{\Omega_-} \quad \text{for any } \varphi \in W_{q',0}^1(\Omega_-). \tag{34}$$

We can show the following theorem by using the method in Shibata [13].

Theorem 7. *Let $1 < q < \infty$, $N < r < \infty$, and $0 < \epsilon < \pi/2$. Assume that $r \geq \max(q, q')$ and that Ω_- is a uniform $W_r^{1-1/r}$ domain. Set:*

$$X_q^2(\Omega_-) = \{\mathbf{G}^2 = (\mathbf{f}, g_1, g_2) \mid \mathbf{f} \in L_q(\Omega_-)^N,\ g_1 \in W_q^1(\Omega_-),\ g_2 \in W_q^1(\Omega_-)\},$$
$$\mathcal{X}_q^2(\Omega_-) = \{\mathbf{F}^2 = (\mathbf{F}_2, \mathbf{F}_{11}, \ldots, \mathbf{F}_{14}) \mid \mathbf{F}_2, \mathbf{F}_{11}, \mathbf{F}_{13} \in L_q(\Omega_-)^N,\ \mathbf{F}_{12}, \mathbf{F}_{14} \in W_q^1(\Omega_-)\}.$$

Then, there exist a $\lambda_0 > 0$ and an operator family $\mathfrak{d}(\lambda) \in \mathrm{Hol}(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q^2(\Omega_-), W_q^1(\Omega_-)))$ such that for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ and $\mathbf{G}^2 = (\mathbf{f}, g_1, g_2) \in X_q^2(\Omega_-)$, $w = \mathfrak{d}(\lambda)F_\lambda^2\mathbf{G}^2$ is a unique solution to (33), and:

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q^2(\Omega_-), W_q^{1-k}(\Omega_-))}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathfrak{d}(\lambda)) \mid \lambda = \gamma + i\tau \in \Sigma_{\epsilon,\lambda_0}\}) \leq C \quad (\ell = 0, 1,\ \ k = 0, 1). \tag{35}$$

Here, we set $F_\lambda^2\mathbf{G}^2 = (\mathbf{f}, \lambda^{1/2}g_1, g_1, \lambda^{1/2}g_2, g_2)$.

Remark 8. (i) *$\mathbf{F}_{11}$, $\mathbf{F}_{12}$, $\mathbf{F}_{13}$, and $\mathbf{F}_{14}$ are the corresponding variables to $\lambda^{1/2}g_1$, g_1, $\lambda^{1/2}g_2$ and g_2.*
(ii) *Since $\mathcal{R}$-boundedness implies the usual boundedness, by (34) and (35) we have:*

$$\|\lambda w\|_{W^1_{q',0}(\Omega_-)^*} + \|\lambda^{1/2}w\|_{L_q(\Omega_-)} + \|w\|_{W_q^1(\Omega)} \leq C\|(\mathbf{f}, \lambda^{1/2}g_1, g_1, \lambda^{1/2}g_2, g_2)\|_{L_q(\Omega_-)} \tag{36}$$

with $w = \mathfrak{d}(\lambda)F^2(\mathbf{f}, g_1, g_2)$. Here, $W^1_{q',0}(\Omega_-)^*$ is the dual space of $W^1_{q',0}(\Omega_-)$.

We start our main discussion in this subsection. Given $w_+ \in W_q^1(\Omega_+)$, let $\mathrm{Ext}^-[w_+]$ denote an extension of w_+ to Ω_- such that $\mathrm{Ext}^-[w_+]|_{\Gamma-0} = w_+|_{\Gamma+0}$ and $\|\mathrm{Ext}^-[w_+]\|_{W_q^1(\Omega)} \leq C\|w_+\|_{W_q^1(\Omega_-)}$. Since we can choose some uniform covering of $\Omega_\pm$ (cf. Proposition 4 in Section 5 below), $\mathrm{Ext}^-[w_+]$ is defined by the even extension of w_+ in each local chart. For $\mathbf{u}_\pm \in W_q^2(\Omega_\pm)^N$, we define an operator $K(\mathbf{u}_+, \mathbf{u}_-)$ by $K(\mathbf{u}_+, \mathbf{u}_-) = \tilde{\mathcal{K}}(\mathbf{f}, g_1, g_2)$, where we set $\tilde{\mathcal{K}}(\mathbf{f}, g_1, g_2) = \mathfrak{d}(\lambda)(0, g_1, g_2) + \mathcal{K}(\mathbf{f} - \nabla\mathfrak{d}(\lambda)(0, g_1, g_2))$,

$$\begin{aligned}
\mathbf{f} &= \mathbf{f}(\mathbf{u}_-) = \mathrm{Div}\,\mathbf{S}_-(\mathbf{u}_-) - \nabla\mathrm{div}\,\mathbf{u}_-,\\
g_1 &= g_1(\mathbf{u}_\pm) =< \mathbf{S}_-(\mathbf{u}_-)\mathbf{n}, \mathbf{n} > -\mathrm{div}\,\mathbf{u}_- - < (\mathrm{Ext}^-[\mathbf{S}_+(\mathbf{u}_+)] + \delta\gamma_{3+}\mathrm{Ext}^-[\mathrm{div}\,\mathbf{u}_+]\mathbf{I})\mathbf{n}, \mathbf{n} >,\\
g_2 &= g_2(\mathbf{u}_-) =< \mathbf{S}_-(\mathbf{u}_-)\mathbf{n}_-, \mathbf{n}_- > -\mathrm{div}\,\mathbf{u}_-,
\end{aligned} \tag{37}$$

and $\mathcal{K}$ is the operator defined in Remark 2. Note that $K(\mathbf{u}_+, \mathbf{u}_-) \in W_q^1(\Omega_-) + \hat{W}^1_{q,0}(\Omega_-)$ and satisfies the variational equation:

$$(\nabla K(\mathbf{u}_+, \mathbf{u}_-), \nabla\varphi)_{\Omega_-} = (\rho_{0-}^{-1}(\mathrm{Div}\,\mathbf{S}_-(\mathbf{u}_-) - \nabla\mathrm{div}\,\mathbf{u}_-), \nabla\varphi)_{\Omega_-} \quad \text{for any } \varphi \in \hat{W}^1_{q'}(\Omega)_- \tag{38}$$

subject to:

$$\begin{aligned}
K(\mathbf{u}_+, \mathbf{u}_-)|_{\Gamma-0} &= (< \mathbf{S}_-(\mathbf{u}_-)\mathbf{n}, \mathbf{n} > -\mathrm{div}\,\mathbf{u}_-)|_{\Gamma-0} - < (\mathbf{S}_+(\mathbf{u}_+) + \delta\gamma_{3+}\mathrm{div}\,\mathbf{u}_+\mathbf{I})\mathbf{n}, \mathbf{n} > |_{\Gamma+0},\\
K(\mathbf{u}_+, \mathbf{u}_-)|_{\Gamma_-} &= (< \mathbf{S}_-(\mathbf{u}_-)\mathbf{n}_-, \mathbf{n}_- > -\mathrm{div}\,\mathbf{u}_-)|_{\Gamma_-},
\end{aligned} \tag{39}$$

and the estimate:

$$\|\nabla K(\mathbf{u}_+, \mathbf{u}_-)\|_{L_q(\Omega_-)} \leq C(\|\nabla\mathbf{u}_+\|_{W_q^1(\Omega_+)} + \|\nabla\mathbf{u}_-\|_{W_q^1(\Omega_-)}). \tag{40}$$

The reduced generalized resolvent problem (RGRP) is the following:

$$\begin{cases}
\lambda\mathbf{u}_+ - \gamma_{0+}^{-1}(\mathrm{Div}\,\mathbf{S}_+(\mathbf{u}_+) + \delta\nabla(\gamma_{3+}\mathrm{div}\,\mathbf{u}_+)) = \mathbf{g}_+ & \text{in } \Omega_+,\\
\lambda\mathbf{u}_- - \rho_{0-}^{-1}(\mathrm{Div}\,\mathbf{S}_-(\mathbf{u}_-) - \nabla K(\mathbf{u}_+, \mathbf{u}_-)) = \mathbf{g}_- & \text{in } \Omega_-,\\
(\mathbf{S}_+(\mathbf{u}_+) + \delta\gamma_{3+}\mathrm{div}\,\mathbf{u}_+\mathbf{I})\mathbf{n}|_{\Gamma+0} - (\mathbf{S}_-(\mathbf{u}_-) - K(\mathbf{u}_+, \mathbf{u}_-)\mathbf{I})\mathbf{n}|_{\Gamma-0} = \mathbf{h}|_\Gamma, &\\
(\mathbf{S}_-(\mathbf{u}_-) - K(\mathbf{u}_+, \mathbf{u}_-)\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = \mathbf{h}_-|_{\Gamma_-}, &\\
\mathbf{u}_+|_{\Gamma+0} = \mathbf{u}_-|_{\Gamma-0}, \quad \mathbf{u}_+|_{\Gamma_+} = 0. &
\end{cases} \tag{41}$$

Using $\mathcal{T}_{\mathbf{z}}[\mathbf{w}]$ defined in (14), we can write the interface condition and free boundary condition in (41) as follows:

$$\begin{aligned} \mathbf{h}|_\Gamma &= \mathcal{T}_{\mathbf{n}}[\mathbf{S}_+(\mathbf{u}_+)\mathbf{n}]|_{\Gamma+0} - \mathcal{T}_{\mathbf{n}}[\mathbf{S}_-(\mathbf{u}_-)\mathbf{n}]|_{\Gamma-0} - (\mathrm{div}\,\mathbf{u}_-|_{\Gamma-0})\mathbf{n}, \\ \mathbf{h}_-|_{\Gamma_-} &= \mathcal{T}_{\mathbf{n}_-}[\mathbf{S}_-(\mathbf{u}_-)\mathbf{n}_-]|_{\Gamma_-} + (\mathrm{div}\,\mathbf{u}_-|_{\Gamma_-})\mathbf{n}_-. \end{aligned} \tag{42}$$

We say that $(\mathbf{u}_+, \mathbf{u}_-)$ is a solution to (41) with $(\mathbf{g}_+, \mathbf{g}_-, \mathbf{h}, \mathbf{h}_-)$ if $\mathbf{u}_\pm \in W^2_q(\Omega_\pm)^N$ and $\mathbf{u}_\pm$ satisfies Equation (41). Furthermore, we say that $(\mathbf{u}_+, \mathbf{u}_-, p_-)$ is a solution to (29) with $(\mathbf{g}_+, \mathbf{g}_-, f_- = \mathrm{div}\,\mathbf{f}_-, \mathbf{h}, \mathbf{h}_-)$ if $\mathbf{u}_\pm \in W^2_q(\Omega_\pm)^N$, $p_- \in W^1_q(\Omega) + \hat{W}^1_{q,0}(\Omega)$ and $\mathbf{u}_\pm$ and p_- satisfy Equation (29). In this subsection, we show the equivalence of the solutions between (29) and (41).

Assertion 1. If (29) is solvable, then so is (41).

In fact, we define $f_- \in W^1_q(\Omega_-)$ by $f_- = \mathfrak{d}(\lambda)F^2_\lambda(\mathbf{g}_-, -h, h_-)$ with $\mathbf{g}_- \in L_q(\Omega_-)^N$, $h =< \mathbf{h}, \mathbf{n} >$ and $h_- =< \mathbf{h}_-, \mathbf{n}_- >$. Notice that $f_- = \mathrm{div}\,\mathbf{f}_-$ with $\mathbf{f}_- = \lambda^{-1}(\mathbf{g}_- + \rho_{0-}^{-1}\nabla f_-)$. Let $(\mathbf{u}_+, \mathbf{u}_-, p_-)$ be a solution to (29) with $(\mathbf{g}_+, \mathbf{g}_-, f_- = \mathrm{div}\,\mathbf{f}_-, \mathbf{h}, \mathbf{h}_-)$. In particular, $\mathrm{div}\,\mathbf{u}_- = f_- = \mathrm{div}\,(\lambda^{-1}(\mathbf{g}_- + \rho_{0-}^{-1}\nabla f_-))$, namely $\mathrm{div}\,\mathbf{u} = f_-$ and $\lambda\mathbf{u} - (\mathbf{g}_- + \rho_{0-}^{-1}\nabla f_-) \in J_q(\Omega_-)$. From the second equation of (29), it follows that for any $\varphi \in \hat{W}^1_{q',0}(\Omega_-)$:

$$\begin{aligned} &(\mathbf{g}_-, \nabla\varphi)_{\Omega_-} \\ &= \lambda(\mathbf{u}_-, \nabla\varphi) - (\rho_{0-}^{-1}\nabla\mathrm{div}\,\mathbf{u}_-, \nabla\varphi)_{\Omega_-} - (\rho_{0-}^{-1}(\mathrm{Div}\,\mathbf{S}_-(\mathbf{u}_-) - \nabla\mathrm{div}\,\mathbf{u}_-), \nabla\varphi)_{\Omega_-} + (\rho_{0-}^{-1}\nabla p_-, \nabla\varphi)_{\Omega_-} \\ &= (\mathbf{g}_- + \rho_{0-}^{-1}\nabla f_-, \nabla\varphi)_{\Omega_-} - (\rho_{0-}^{-1}\nabla f_-, \nabla\varphi)_{\Omega_-} + (\rho_{0-}^{-1}\nabla(p_- - K(\mathbf{u}_+, \mathbf{u}_-)), \nabla\varphi)_{\Omega_-}, \end{aligned}$$

which yields that $(\rho_{0-}^{-1}\nabla(p_- - K(\mathbf{u}_+, \mathbf{u}_-)), \nabla\varphi)_{\Omega_-} = 0$ for any $\varphi \in \hat{W}^1_{q',0}(\Omega_-)$. Moreover,

$$\begin{aligned} p_- - K(\mathbf{u}_+, \mathbf{u}_-) &=< \mathbf{h}, \mathbf{n} > + \mathrm{div}\,\mathbf{u}_- =< \mathbf{h}, \mathbf{n} > + f_- = 0 && \text{on } \Gamma, \\ p_- - K(\mathbf{u}_+, \mathbf{u}_-) &= - < \mathbf{h}_-, \mathbf{n}_- > + \mathrm{div}\,\mathbf{u}_- = - < \mathbf{h}_-, \mathbf{n}_- > + f_- = 0 && \text{on } \Gamma_-. \end{aligned}$$

Thus, the uniqueness yields that $p_- = K(\mathbf{u}_+, \mathbf{u}_-)$, and so, $(\mathbf{u}_+, \mathbf{u}_-)$ is a solution to (41) with $(\mathbf{g}_+, \mathbf{g}_-, \mathbf{h}, \mathbf{h}_-)$.

Assertion 2. If (41) is solvable, then so is (29).

In fact, given $\mathbf{g}_- \in L_q(\Omega_-)^N$, $\mathbf{h} \in W^1_q(\Omega)$, and $\mathbf{h}_- \in W^1_q(\Omega_-)^N$, we define $\hat{p}_1$ by $\hat{p}_1 = \hat{p}_2 + \mathcal{K}(\mathbf{g}_- - \nabla\hat{p}_2)$ with $\hat{p}_2 = \mathfrak{d}(\lambda)F^2_\lambda(0, h, -h_-)$. Next, given $f_- = \mathrm{div}\,\mathbf{f}_-$, we define $\hat{p}_3$ by:

$$\hat{p}_3 = \mathfrak{d}(\lambda)F^2_\lambda(0, f_-, f_-) + \mathcal{K}(-\mathbf{F} - \nabla\mathfrak{d}(\lambda)F^2_\lambda(0, f_-, f_-))$$

with $\mathbf{F} = \lambda\mathbf{f}_- - \rho_{0-}^{-1}\nabla f_-$. Let $(\mathbf{u}_+, \mathbf{u}_-)$ be a solution to equations:

$$\begin{cases} \lambda\mathbf{u}_+ - \gamma_{0+}^{-1}(\mathrm{Div}\,\mathbf{S}_+(\mathbf{u}_+) + \delta\nabla(\gamma_{3+}\mathrm{div}\,\mathbf{u}_+)) = \mathbf{g}_+ & \text{in } \Omega_+, \\ \lambda\mathbf{u}_- - \rho_{0-}^{-1}(\mathrm{Div}\,\mathbf{S}_-(\mathbf{u}_-) - \nabla K(\mathbf{u}_+, \mathbf{u}_-)) = \mathbf{g}_- - \rho_{0-}^{-1}\nabla(\hat{p}_1 + \hat{p}_3) & \text{in } \Omega_-, \\ (\mathbf{S}_+(\mathbf{u}_+) + \delta\gamma_{3+}\mathrm{div}\,\mathbf{u}_+\mathbf{I})\mathbf{n}|_{\Gamma+0} - (\mathbf{S}_-(\mathbf{u}_-) - K(\mathbf{u}_+, \mathbf{u}_-)\mathbf{I})\mathbf{n}|_{\Gamma-0} = (\mathcal{T}_{\mathbf{n}}[\mathbf{h}] - f_-\mathbf{n})|_\Gamma & \\ (\mathbf{S}_-(\mathbf{u}_-) - K(\mathbf{u}_+, \mathbf{u}_-)\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = (\mathcal{T}_{\mathbf{n}_-}[\mathbf{h}_-] + f_-\mathbf{n}_-)|_{\Gamma_-}, & \\ \mathbf{u}_+|_{\Gamma+0} = \mathbf{u}_-|_{\Gamma-0}, \quad \mathbf{u}_+|_{\Gamma_+} = 0. & \end{cases} \tag{43}$$

Setting $p_- = K(\mathbf{u}_+, \mathbf{u}_-) + \hat{p}_1 + \hat{p}_3$, we see that $(\mathbf{u}_+, \mathbf{u}_-, p_-)$ is a solution to (29) with $(\mathbf{g}_+, \mathbf{g}_-, f_- = \mathrm{div}\,\mathbf{f}_-, \mathbf{h}, \mathbf{h}_-)$. In fact, our task is to prove that $\mathrm{div}\,\mathbf{u}_- = f_- = \mathrm{div}\,\mathbf{f}_-$. Notice that $\rho_{0-}^{-1}(\nabla\hat{p}_1, \nabla\varphi)_{\Omega_-} = (\mathbf{g}_-, \nabla\varphi)_{\Omega_-}$ and $\rho_{0-}^{-1}(\nabla\hat{p}_3, \nabla\varphi)_{\Omega_-} = -(\lambda\mathbf{f}_- - \rho_{0-}^{-1}\nabla f_-, \nabla\varphi)_{\Omega_-}$ for any $\varphi \in \hat{W}^1_{q',0}(\Omega_-)$. Thus, by (38):

$$(\lambda\mathbf{f}_- - \rho_{0-}^{-1}\nabla f_-, \nabla\varphi)_{\Omega_-} = (\mathbf{g}_- - \rho_{0-}^{-1}\nabla(\hat{p}_1 + \hat{p}_3), \nabla\varphi)_{\Omega_-} = \lambda(\mathbf{u}_-, \nabla\varphi)_{\Omega_-} - \rho_{0-}^{-1}(\nabla\mathrm{div}\,\mathbf{u}_-, \nabla\varphi)_{\Omega_-}$$

for any $\varphi \in \hat{W}^1_{q',0}(\Omega_-)$, which yields that:

$$\lambda(\mathbf{u} - \mathbf{f}_-, \nabla\varphi)_{\Omega_-} - \rho_{0-}^{-1}(\nabla(\operatorname{div}\mathbf{u}_- - f_-), \nabla\varphi)_{\Omega_-} = 0 \quad \text{for any } \varphi \in \hat{W}^1_{q',0}(\Omega_-). \tag{44}$$

Taking $\varphi \in W^1_{q',0}(\Omega_-) \subset \hat{W}^1_{q',0}(\Omega_-)$ in (44), using the divergence theorem of Gauss, and noticing that $\operatorname{div}\mathbf{f}_- = f_-$ give that:

$$\lambda(\operatorname{div}\mathbf{u}_- - f_-, \varphi)_{\Omega_-} + \rho_{0-}^{-1}(\nabla(\operatorname{div}\mathbf{u}_- - f_-), \nabla\varphi)_{\Omega_-} = 0 \quad \text{for any } \varphi \in W^1_{q',0}(\Omega_-).$$

Moreover, from the third equation in (42) and (39), it follows that:

$$\begin{aligned} f_-|_\Gamma =& < \mathcal{T}_\mathbf{n}[\mathbf{h}], \mathbf{n} > -\{< (\mathbf{S}_+(\mathbf{u}_+) + \delta\gamma_{3+}\operatorname{div}\mathbf{u}_+\mathbf{I})\mathbf{n}|_{\Gamma+0}, \mathbf{n} > - < \mathbf{S}_-(\mathbf{u}_-)\mathbf{n}|_{\Gamma-0}\mathbf{n}, \mathbf{n} >\} \\ & - K(\mathbf{u}_+, \mathbf{u}_-)|_\Gamma \\ =& \operatorname{div}\mathbf{u}_-|_\Gamma, \\ f_-|_{\Gamma_-} =& - < \mathcal{T}_{\mathbf{n}_-}[\mathbf{h}_-], \mathbf{n}_- > + < (S_-(\mathbf{u}_-) - K(\mathbf{u}_+, \mathbf{u}_-)\mathbf{I})\mathbf{n}_-|_{\Gamma_-}, \mathbf{n}_- >= \operatorname{div}\mathbf{u}_-|_{\Gamma_-}. \end{aligned}$$

Thus, the uniqueness yields that $\operatorname{div}\mathbf{u}_- = f_-$ in Ω_-. Inserting this fact into (44) and using the fact that $\lambda \neq 0$, we have $\mathbf{u} - \mathbf{f}_- \in J_q(\Omega)$, which shows that $\operatorname{div}\mathbf{u} = f_- = \operatorname{div}\mathbf{f}_-$.

Noting that $\hat{p}_1 = h =< \mathbf{h}, \mathbf{n} >$ and $\hat{p}_3 = f_-$ on Γ and that $\hat{p}_1 = h_- =< \mathbf{h}, \mathbf{n}_- >$ and $\hat{p}_3 = -f_-$ on Γ_-, we have:

$$\begin{aligned} &(\mathbf{S}_+(\mathbf{u}_+) + \delta\gamma_{3+}\operatorname{div}\mathbf{u}_+\mathbf{I})\mathbf{n}|_{\Gamma+0} - (\mathbf{S}_-(\mathbf{u}_-) - (K(\mathbf{u}_+, \mathbf{u}_-) + \hat{p}_1 + \hat{p}_3)\mathbf{I})\mathbf{n}|_{\Gamma-0} \\ &\quad = \mathcal{T}_\mathbf{n}[\mathbf{h}] - f_-\mathbf{n} + < \mathbf{h}, \mathbf{n} > \mathbf{n} + f_-\mathbf{n} = \mathbf{h} \quad \text{on } \Gamma, \\ &(\mathbf{S}_-(\mathbf{u}_-) - (K(\mathbf{u}_+, \mathbf{u}_-) + \hat{p}_1 + \hat{p}_3)\mathbf{I})\mathbf{n}_- = \mathcal{T}_{\mathbf{n}_-}[\mathbf{h}_-] + f_-\mathbf{n}_- + < \mathbf{h}_-, \mathbf{n}_- > \mathbf{n}_- - f_-\mathbf{n}_- = \mathbf{h}_- \quad \text{on } \Gamma_-. \end{aligned}$$

Thus, $(\mathbf{u}_+, \mathbf{u}_-, p_-)$ is a solution of Equation (29) with $(\mathbf{g}_+, \mathbf{g}_-, f_- = \operatorname{div}\mathbf{f}_-, \mathbf{h}, \mathbf{h}_-)$.

2.3. Existence of $\mathcal{R}$-Bounded Solution Operators for Problem (41)

The following theorem is concerned with the existence of $\mathcal{R}$-bounded solution operators to Problem (41).

Theorem 8. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$ and $N < r < \infty$. Assume that $r \geq \max(q, q')$, that $\Omega_\pm$ are uniform $W^{2-1/r}_r$ domains, and the weak Dirichlet problem is uniquely solvable in Ω_- with exponents q and q'. Let $X_q(\Omega)$ and $\mathcal{X}_q(\Omega)$ be the sets defined by:*

$$\begin{aligned} X_q(\Omega) &= \{\mathbf{G} = (\mathbf{g}_+, \mathbf{g}_-, \mathbf{h}, \mathbf{h}_-) \mid \mathbf{g}_\pm \in L_q(\Omega_\pm)^N,\ \mathbf{h} \in W^1_q(\Omega)^N,\ \mathbf{h}_- \in W^1_q(\Omega_-)^N\}, \\ \mathcal{X}_q(\Omega) &= \{\mathbf{F} = (\mathbf{F}_1, \ldots, \mathbf{F}_6) \mid \mathbf{F}_1 \in L_q(\Omega_+)^N,\ \mathbf{F}_2, \mathbf{F}_5 \in L_q(\Omega_-)^N,\ \mathbf{F}_3 \in L_q(\Omega)^N, \\ &\qquad \mathbf{F}_4 \in W^1_q(\Omega)^N,\ \mathbf{F}_6 \in W^1_q(\Omega_-)^N\}. \end{aligned}$$

Then, there exist a constant $\lambda_0 > 0$ and operator families $\mathcal{S}_\pm(\lambda) \in \operatorname{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega), W^2_q(\Omega_\pm)^N))$ such that for any $\lambda \in \Gamma_{\epsilon,\lambda_0}$ and $\mathbf{G} = (\mathbf{g}_+, \mathbf{g}_-, \mathbf{h}, \mathbf{h}_-) \in X_q(\Omega)$, $\mathbf{u}_\pm = \mathcal{S}_\pm(\lambda)F_\lambda\mathbf{G}$ is a unique solution to (41) and:

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), W^{2-j}q(\Omega_\pm)^{\tilde{N}})}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{S}_\pm(\lambda)) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \leq C,$$

for $\ell = 0, 1$ and $j = 0, 1, 2$, where we set $F_\lambda\mathbf{G} = (\mathbf{g}_+, \mathbf{g}_-, \lambda^{1/2}\mathbf{h}, \nabla\mathbf{h}, \lambda^{1/2}\mathbf{h}_-, \nabla\mathbf{h}_-)$ and $G_\lambda\mathbf{u} = (\lambda\mathbf{u}, \gamma\mathbf{u}, \lambda^{1/2}\nabla\mathbf{u}, \nabla^2\mathbf{u})$.

Remark 9. *For any subdomain $G \subset \Omega$, we set:*

$$\begin{aligned} \|\mathbf{G}\|_{X_q(G)} &= \|\mathbf{g}_+\|_{L_q(\Omega_+\cap G)} + \|\mathbf{g}_-\|_{L_q(\Omega_-\cap G)} + \|\mathbf{h}\|_{W^1_q(G)} + \|\mathbf{h}_-\|_{W^1_q(\Omega_-\cap G)}, \\ \|\mathbf{F}\|_{\mathcal{X}_q(G)} &= \|\mathbf{F}_1\|_{L_q(\Omega_+\cap G)} + \|\mathbf{F}_3\|_{L_q(G)} + \|\mathbf{F}_4\|_{W^1_q(G)} + \|(\mathbf{F}_2, \mathbf{F}_5)\|_{L_q(\Omega_-\cap G)} + \|\mathbf{F}_6\|_{W^1_q(\Omega_-\cap G)}. \end{aligned}$$

Obviously, according to Assertion 2 in Section 2.2, by Theorem 8, Lemma 1, and Lemma 2 we have Theorem 6. Thus, we shall prove Theorem 8 only.

2.4. The Uniqueness of Solutions to Problem (41)

Assuming the existence of solutions to Problem (41) with exponent q', we prove the uniqueness of solutions to (41). Namely, we prove the following lemma.

Lemma 3. *Let $1 < q < \infty$ and $N < r < \infty$. Assume that $r \geq \max(q, q')$, that $\Omega_\pm$ are uniform $W^{2-1/r}_r$ domains, and that the weak Dirichlet problem is uniquely solvable on Ω_- with exponents q and q'. If there exists a $\lambda_0 > 0$ such that Problem (41) is solvable with exponent q' for any $\lambda \in \Gamma_{\epsilon,\lambda_0}$, then the uniqueness for (41) with exponent q is valid for any $\lambda \in \Gamma_{\epsilon,\lambda_0}$.*

Remark 10. (i) *The reason why we assume that $r \geq \max(q, q')$ is that we use the existence of solutions to the dual problem to prove the uniqueness.*
(ii) *The uniqueness means that if $(\mathbf{u}_+, \mathbf{u}_-)$ is a solution to (41) with $(0, 0, 0, 0)$, then $\mathbf{u}_\pm = 0$.*

Before proving Lemma 3, first we prove that if $(\mathbf{u}_+, \mathbf{u}_-)$ is a solution to (41) with $(\mathbf{g}_+, \mathbf{g}_-, 0, 0)$ and if $\mathbf{g}_- \in J_q(\Omega_-)$, then $\mathbf{u}_- \in J_q(\Omega_-)$, as well. In fact, for any $\varphi \in \hat W^1_{q',0}(\Omega_-)$, we have:

$$\begin{aligned} 0 &= (\mathbf{g}_-, \nabla\varphi)_{\Omega_-} = (\lambda\mathbf{u}_- - \rho_{0-}^{-1}(\mathrm{Div}\,\mathbf{S}_-(\mathbf{u}_-) - \nabla K(\mathbf{u}_+, \mathbf{u}_-)), \nabla\varphi)_{\Omega_-} \\ &= \lambda(\mathbf{u}_-, \nabla\varphi)_{\Omega_-} - \rho_{0-}^{-1}(\nabla \mathrm{div}\,\mathbf{u}_-, \nabla\varphi)_{\Omega_-}. \end{aligned} \tag{45}$$

Choosing $\psi \in W^1_{q',0}(\Omega_-) \subset \hat W^1_{q',0}(\Omega_-)$, we have:

$$\lambda(\mathrm{div}\,\mathbf{u}_-, \psi)_{\Omega_-} + \rho_{0-}^{-1}(\nabla \mathrm{div}\,\mathbf{u}_-, \nabla\psi)_{\Omega_-} = 0 \quad \text{for any } \psi \in W^1_{q',0}(\Omega_-). \tag{46}$$

In addition, we have:

$$\begin{aligned} &0 =< (\mathbf{S}_+(\mathbf{u}_+) + \delta\gamma_{3+}\mathrm{div}\,\mathbf{u}_+)\mathbf{n}|_{\Gamma+0}, \mathbf{n} > - < (\mathbf{S}_-(\mathbf{u}_-) - K(\mathbf{u}_+, \mathbf{u}_-))\mathbf{n}|_{\Gamma-0}, \mathbf{n} >= \mathrm{div}\,\mathbf{u}_- \quad \text{on } \Gamma, \\ &0 =< (\mathbf{S}_-(\mathbf{u}_-) - K(\mathbf{u}_+, \mathbf{u}_-)\mathbf{I})\mathbf{n}_-, \mathbf{n}_- >= \mathrm{div}\,\mathbf{u}_- \quad \text{on } \Gamma_-. \end{aligned}$$

Thus, the uniqueness guaranteed by Theorem 7 implies that $\mathrm{div}\,\mathbf{u}_- = 0$, which inserted into (45) yields that $\mathbf{u}_- \in J_q(\Omega_-)$.

Secondly, for any $\mathbf{u}_\pm \in W^2_q(\Omega_\pm)^N$ and $\mathbf{v}_\pm \in W^2_{q'}(\Omega_\pm)^N$ with $\mathbf{u}_- \in J_q(\Omega_-)$ and $\mathbf{v}_- \in J_{q'}(\Omega_-)$:

$$\begin{aligned} &(-(\mathrm{Div}\,\mathbf{S}_+(\mathbf{u}_+) + \delta\nabla(\gamma_{3+}\mathrm{div}\,\mathbf{u}_+)), \mathbf{v}_+)_{\Omega_+} + (-(\mathrm{Div}\,\mathbf{S}_-(\mathbf{u}_-) - \nabla K(\mathbf{u}_+, \mathbf{u}_-)), \mathbf{v}_-)_{\Omega_-} \\ &- \{(\mathbf{u}_+, -(\mathrm{Div}\,\mathbf{S}_+(\mathbf{v}_+) + \delta\nabla(\gamma_{3+}\mathrm{div}\,\mathbf{v}_+))_{\Omega_+} + (\mathbf{u}_-, -(\mathrm{Div}\,\mathbf{S}_-(\mathbf{v}_-) - \nabla K(\mathbf{v}_+, \mathbf{v}_-))_{\Omega_-}\} \\ &= -A + B \end{aligned} \tag{47}$$

with:

$$\begin{aligned} A &= ((\mathbf{S}_+(\mathbf{u}_+) + \delta\gamma_{3+}\mathrm{div}\,\mathbf{u}_+)\mathbf{n}|_{\Gamma+0} - ((\mathbf{S}_-(\mathbf{u}_-) - K(\mathbf{u}_+, \mathbf{u}_-)\mathbf{I})\mathbf{n}|_{\Gamma-0}, \mathbf{v}_-)_\Gamma \\ &\quad + ((\mathbf{S}_-(\mathbf{u}_-) - K(\mathbf{u}_+, \mathbf{u}_-)\mathbf{I})\mathbf{n}_-, \mathbf{v}_-)_{\Gamma_-}, \\ B &= (\mathbf{u}_+, (\mathbf{S}_+(\mathbf{v}_+) + \delta\gamma_{3+}\mathrm{div}\,\mathbf{v}_+)\mathbf{n}|_{\Gamma+0} - (\mathbf{u}_-, (\mathbf{S}_-(\mathbf{v}_-) - K(\mathbf{v}_+, \mathbf{v}_-)\mathbf{I})\mathbf{n}|_{\Gamma-0})_\Gamma \\ &\quad + (\mathbf{u}_-, (\mathbf{S}_-(\mathbf{v}_-) - K(\mathbf{v}_+, \mathbf{v}_-)\mathbf{I})\mathbf{n}_-)_{\Gamma_-} \end{aligned}$$

provided that $\mathbf{w}_+|_{\Gamma+0} = \mathbf{w}_-|_{\Gamma-0}$ with $\mathbf{w} = \mathbf{u}$ and $\mathbf{v}$, where for $G = \Gamma$ and Γ_-, we set $(a, b)_G = \int_G a(x)b(x)\,d\sigma$, $d\sigma$ being the surface element on G. In fact, setting $K(\mathbf{u}_+, \mathbf{u}_-) = p_1 + p_2 \in W^1_q(\Omega_-) + \hat W^1_{q,0}(\Omega_-)$, by the divergence theorem of Gauss, we have:

$$\begin{aligned} &(-(\mathrm{Div}\,\mathbf{S}_+(\mathbf{u}_+) + \delta\nabla(\gamma_{3+}\mathrm{div}\,\mathbf{u}_+)), \mathbf{v}_+)_{\Omega_+} + (-(\mathrm{Div}\,\mathbf{S}_-(\mathbf{u}_-) - \nabla K(\mathbf{u}_+, \mathbf{u}_-)), \mathbf{v}_-)_{\Omega_-} \\ &= A + (\nabla p_2, \mathbf{v}_-)_{\Omega_-} + C \end{aligned}$$

with:

$$C = \sum_{\ell=+,-} \frac{\mu_\ell}{2}(\mathbf{D}(\mathbf{u}_\ell), \mathbf{D}(\mathbf{v}_\ell))_{\Omega_\ell} - (\delta\gamma_{3+}\mathrm{div}\,\mathbf{u}_+, \mathrm{div}\,\mathbf{v}_+)_{\Omega_+} + (\nu_+ - \mu_+)(\mathrm{div}\,\mathbf{u}_+, \mathrm{div}\,\mathbf{v}_+)_{\Omega_+},$$

because $K(\mathbf{u}_+, \mathbf{u}_-)|_{\Gamma-0} = p_1|_{\Gamma-0}$ as follows from $p_2|_{\Gamma-0} = 0$. Analogously, we have:

$$\begin{aligned}&(\mathbf{u}_+, -(\mathrm{Div}\,\mathbf{S}_+(\mathbf{v}_+) + \delta\nabla(\gamma_{3+}\mathrm{div}\,\mathbf{v}_+)))_{\Omega_+} + (\mathbf{u}_-, -(\mathrm{Div}\,\mathbf{S}_-(\mathbf{v}_-) - \nabla K(\mathbf{v}_+, \mathbf{v}_-)))_{\Omega_-}\\ &\quad = B + (\mathbf{u}_-, \nabla p_2^*)_{\Omega_-} + C\end{aligned}$$

with $K(\mathbf{v}_+, \mathbf{v}_-) = p_1^* + p_2^* \in W^1_{q'}(\Omega_-) + \hat{W}^1_{q',0}(\Omega_-)$. Since $\mathbf{u}_- \in J_q(\Omega_-)$ and $\mathbf{v}_- \in J_{q'}(\Omega_-)$, we have $(\nabla p_2, \mathbf{v}_-)_{\Omega_-} = (\mathbf{u}_-, \nabla p_2^*)_{\Omega_-} = 0$, so that we have (47).

Proof of Lemma 3. Let $(\mathbf{u}_+, \mathbf{u}_-)$ satisfy (41) with $(0, 0, 0, 0)$, that is let $(\mathbf{u}_+, \mathbf{u}_-)$ satisfy the homogeneous equation. In particular, $\mathbf{u}_- \in J_q(\Omega_-)$. Let $\mathbf{f}_+$ and $\mathbf{f}_-$ be any vectors of functions in $C_0^\infty(\Omega_\pm)^N$. We define ψ by $\psi = \mathcal{K}(\mathbf{f}_-) \in \hat{W}^1_{q',0}(\Omega_-)$, and then, $\mathbf{f}_- - \nabla\psi \in J_{q'}(\Omega_-)$. Let $(\mathbf{v}_+, \mathbf{v}_-)$ be a solution to (41) with $(\mathbf{f}_+, \mathbf{f}_- - \nabla\psi, 0, 0)$. Since $\mathbf{f}_- - \nabla\psi \in J_{q'}(\Omega_-)$, $\mathbf{v}_- \in J_{q'}(\Omega_-)$, so that by (47) and the fact that $(\mathbf{u}_-, \nabla\psi)_{\Omega_-} = 0$, we have:

$$0 = (\gamma_{0+}\mathbf{u}_+, \mathbf{f}_+)_{\Omega_+} + (\rho_{0-}\mathbf{u}_-, \mathbf{f}_-)_{\Omega_-}.$$

Since $\mathbf{f}_\pm$ are chosen arbitrarily, we have $\mathbf{u}_\pm = 0$, which completes the proof of Lemma 3. □

3. Model Problems

In this section, we consider a model problem for the incompressible-compressible viscous fluid in $\mathbb{R}^N$. In what follows, we set:

$$\mathbb{R}^N_\pm = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid \pm x_N > 0\}, \quad \mathbb{R}^N_0 = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N = 0\},$$

and $\mathbf{n}_0 = (0, \ldots, 0, 1)$. Before stating the main results of this section, we notice that the following two variational problems are uniquely solvable:

$$\rho_{0-}^{-1}(\nabla v, \nabla\varphi)_{\mathbb{R}^N_-} = (\mathbf{f}, \nabla\varphi)_{\mathbb{R}^N_-} \quad \text{for any } \varphi \in \hat{W}^1_{q',0}(\mathbb{R}^N_-), \tag{48}$$

$$\lambda(w, \nabla\varphi) + \rho_{0-}^{-1}(\nabla w, \nabla\varphi)_{\mathbb{R}^N_-} = (\mathbf{g}, \nabla\varphi)_{\mathbb{R}^N_-} \quad \text{for any } \varphi \in W^1_{q',0}(\mathbb{R}^N_-) \tag{49}$$

subject to $w|_{\mathbb{R}^N_0} = g$. More precisely, let $1 < q < \infty$. As is well known, for any $\mathbf{f} \in L_q(\mathbb{R}^N_-)^N$, Problem (48) admits a unique solution $v \in \hat{W}^1_{q,0}(\mathbb{R}^N_-)$ possessing the estimate: $\|\nabla v\|_{L_q(\mathbb{R}^N_-)} \leq C\|\mathbf{f}\|_{L_q(\mathbb{R}^N_-)}$. We define an operator $\mathcal{P}$ acting on $\mathbf{f}$ by setting $v = \mathcal{P}\mathbf{f}$.

Moreover, for any $\mathbf{g} \in L_q(\mathbb{R}^N_-)^N$, $g \in H^1_q(\mathbb{R}^N_-)$, and $\lambda \in \Sigma_\epsilon$, Problem (49) admits a unique solution $w \in W^1_q(\mathbb{R}^N_-)$ possessing the estimate: $|\lambda|^{1/2}\|w\|_{L_q(\mathbb{R}^N_-)} + \|\nabla w\|_{L_q(\mathbb{R}^N_-)} \leq C\|\mathbf{g}\|_{L_q(\mathbb{R}^N_-)}$, where C is independent of λ. This assertion is also known (cf. [13]). In particular, we have $w = -\mathrm{div}\,\lambda^{-1}(\mathbf{g} - \rho_{0-}^{-1}\nabla w)$.

In this section, assuming that γ_{0+} and γ_{3+} are positive constants such that:

$$\rho_{0+}/2 \leq \gamma_{0+} \leq 2\rho_{0+}, \quad 0 \leq \gamma_{3+} \leq (\rho_{2+})^2,$$

we consider the following interface problem in $\mathbb{R}^N$:

$$\begin{aligned}\lambda\mathbf{u}_+ - \gamma_{0+}^{-1}\mathrm{Div}\,\mathbf{T}_+(\mathbf{u}_+) = \mathbf{g}_+ \quad &\text{in } \mathbb{R}^N_+,\\ \lambda\mathbf{u}_- - \rho_{0-}^{-1}\mathrm{Div}\,\mathbf{T}_-(\mathbf{u}_-, K_I^0(\mathbf{u}_+, \mathbf{u}_-)) = \mathbf{g}_- \quad &\text{in } \mathbb{R}^N_-,\\ \mathbf{T}_+(\mathbf{u}_+)\mathbf{n}_0|_{x_N=0+} - \mathbf{T}_-(\mathbf{u}_-, K_I^0(\mathbf{u}_+, \mathbf{u}_-))\mathbf{n}_0|_{x_N=0-} = \mathbf{h}|_{x_N=0}, \quad &\mathbf{u}_+|_{x_N=0+} = \mathbf{u}_-|_{x_N=0-}.\end{aligned} \tag{50}$$

Here, $\mathbf{g}_\pm \in L_q(\mathbb{R}^N_\pm)$ and $\mathbf{h} \in W^1_q(\mathbb{R}^N)$ are prescribed functions, and for notational simplicity, we set:

$$\mathbf{T}_+(\mathbf{u}_+) = \mathbf{S}_+(\mathbf{u}_+) + \delta\gamma_{3+}\mathrm{div}\,\mathbf{u}_+\mathbf{I}, \quad \mathbf{T}_-(\mathbf{u}_-, p) = \mathbf{S}_-(\mathbf{u}_-) - p\mathbf{I}. \tag{51}$$

Moreover, $v = K_I^0(\mathbf{u}_+, \mathbf{u}_-)$ is a unique solution to the variational problem:

$$(\nabla v, \nabla\varphi)_{\mathbb{R}^N_-} = (\mathrm{Div}\,\mathbf{S}_-(\mathbf{u}_-) - \nabla\mathrm{div}\,\mathbf{u}_-, \nabla\varphi)_{\mathbb{R}^N_-} \quad \text{for any } \varphi \in \hat{W}^1_{q',0}(\mathbb{R}^N_-) \tag{52}$$

subject to $v = g_1$ on $\mathbb{R}^N_0 = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N = 0\}$ with:

$$g_1 =< \mathbf{S}_-(\mathbf{u}_-)\mathbf{n}_0, \mathbf{n}_0 > |_{x_N=0-} - \mathrm{div}\, \mathbf{u}_-|_{x_N=0-} - < \mathbf{T}_+(\mathbf{u}_+)\mathbf{n}_0, \mathbf{n}_0 > |_{x_N=0+}.$$

We prove the following theorem.

Theorem 9. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$, $\lambda_0 > 0$. Let $X^3_q(\mathbb{R}^N)$ and $\mathcal{X}^3_q(\mathbb{R}^N)$ be the sets defined by:*

$$X^3_q(\mathbb{R}^N) = \{\mathbf{G}^3 = (\mathbf{g}_+, \mathbf{g}_-, \mathbf{h}) \mid \mathbf{g}_\pm \in L_q(\mathbb{R}^N_\pm)^N,\ \ \mathbf{h} \in W^1_q(\mathbb{R}^N)\},$$
$$\mathcal{X}^3_q(\mathbb{R}^N) = \{\mathbf{F}^3 = (\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4) \mid \mathbf{F}_1 \in L_q(\mathbb{R}^N_+)^N, \mathbf{F}_2 \in L_q(\mathbb{R}^N_-)^N, \mathbf{F}_3 \in L_q(\mathbb{R}^N)^N, \mathbf{F}_4 \in L_q(\mathbb{R}^N)^{N^2}\}.$$

Then, there exist operator families $\mathcal{E}^0_\pm(\lambda) \in \mathrm{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}^3_q(\mathbb{R}^N), W^2_q(\mathbb{R}^N_\pm)^N))$ such that for any $\lambda \in \Gamma_{\epsilon,\lambda_0}$ and $\mathbf{G}^3 = (\mathbf{g}_+, \mathbf{g}_-, \mathbf{h}) \in X^3_q(\mathbb{R}^N)$, $\mathbf{u}_\pm = \mathcal{E}_\pm(\lambda)F^3_\lambda\mathbf{G}^3$ is a unique solution to (50), and:

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}^3_q(\mathbb{R}^N), W^{2-j}_q(\mathbb{R}^N_\pm)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{E}_\pm(\lambda)) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \le r_b$$

for $\ell = 0, 1$ and $j = 0, 1, 2$ with some constant r_b depending on ϵ, q, λ_0, δ_0, $\mu_\pm$, ν_+, $\rho_{0\pm}$, ρ_{2+}, and N. Here, we set $F^3_\lambda\mathbf{G}^3 = (\mathbf{g}_+, \mathbf{g}_-, \lambda^{1/2}\mathbf{h}, \nabla\mathbf{h})$.

Remark 11. *We set:*

$$\begin{aligned}\|\mathbf{G}^3\|_{X_q(\mathbb{R}^N)} &= \|\mathbf{g}_+\|_{L_q(\mathbb{R}^N_+)} + \|\mathbf{g}_-\|_{L_q(\mathbb{R}^N_-)} + \|\mathbf{h}\|_{W^1_q(\mathbb{R}^N)},\\ \|\mathbf{F}^3\|_{\mathcal{X}_q(\mathbb{R}^N)} &= \|\mathbf{F}_1\|_{L_q(\mathbb{R}^N_+)} + \|\mathbf{F}_2\|_{L_q(\mathbb{R}^N_-)} + \|(\mathbf{F}_3, \mathbf{F}_4)\|_{L_q(\mathbb{R}^N)}.\end{aligned} \tag{53}$$

According to Assertion 1 in Section 2.2, we consider the following system of equations:

$$\begin{aligned}\lambda\mathbf{v}_+ - \gamma_{0+}^{-1}\mathrm{Div}\, \mathbf{T}_+(\mathbf{v}_+) &= \mathbf{g}_+ && \text{in } \mathbb{R}^N_+,\\ \lambda\mathbf{v}_- - \rho_{0-}^{-1}\mathrm{Div}\, \mathbf{T}_-(\mathbf{v}_-, q_-) &= \mathbf{g}_- && \text{in } \mathbb{R}^N_-,\\ \mathrm{div}\, \mathbf{v}_- = f_- &= \mathrm{div}\, \mathbf{f}_- && \text{in } \mathbb{R}^N_-,\\ \mathbf{T}_+(\mathbf{v}_+)\mathbf{n}_0|_{x_N=0+} - \mathbf{T}_-(\mathbf{v}_-, q_-)\mathbf{n}_0|_{x_N=0-} &= \mathbf{h},\\ \mathbf{v}_+|_{x_N=0+} &= \mathbf{v}_-|_{x_N=0-}.\end{aligned} \tag{54}$$

Then, Theorem 9 follows from the following theorem, because (49) is uniquely solvable.

Theorem 10. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$, $\lambda_0 > 0$. Let $Y^0_q(\mathbb{R}^N)$ and $\mathcal{Y}^0_q(\mathbb{R}^N)$ be the sets defined by:*

$$\begin{aligned}Y^0_q(\mathbb{R}^N) = \{\mathcal{G}^0 = (\mathbf{g}_+, \mathbf{g}_-, \mathbf{h}, f_-, \mathbf{f}_-) \mid \mathbf{g}_\pm \in L_q(\mathbb{R}^N_\pm)^N,\ \ \mathbf{h} \in W^1_q(\mathbb{R}^N),\ \ f_- \in W^1_q(\mathbb{R}^N_-),\\ \mathbf{f}_- \in L_q(\mathbb{R}^N_-)^N,\ \ f_- = \mathrm{div}\, \mathbf{f}_-\},\\ \mathcal{Y}^0_q(\mathbb{R}^N) = \{\mathcal{F}^0 = (\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4, \mathbf{F}_7, \mathbf{F}_8, \mathbf{F}_9) \mid \mathbf{F}_1 \in L_q(\mathbb{R}^N_+)^N, \mathbf{F}_2, \mathbf{F}_9 \in L_q(\mathbb{R}^N_-)^N,\\ \mathbf{F}_3 \in L_q(\mathbb{R}^N)^N, \mathbf{F}_4 \in W^1_q(\mathbb{R}^N), \mathbf{F}_7 \in L_q(\mathbb{R}^N_-), \mathbf{F}_9 \in W^1_q(\mathbb{R}^N_-)\}.\end{aligned}$$

Then, there exist operator families $\mathcal{A}^1_\pm(\lambda)$ and $\mathcal{B}^1_-(\lambda)$ with:

$$\mathcal{A}^1_\pm(\lambda) \in \mathrm{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}^0_q(\mathbb{R}^N), W^2_q(\mathbb{R}^N_\pm)^N)),\quad \mathcal{B}^1_-(\lambda) \in \mathrm{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}^0_q(\mathbb{R}^N), W^1_q(\mathbb{R}^N_\pm) + \hat{W}^1_{q,0}(\mathbb{R}^N_-)))$$

such that for any $\lambda \in \Gamma_{\epsilon,\lambda_0}$ *and* $\mathcal{G}^0 = (\mathbf{g}_+, \mathbf{g}_-, \mathbf{h}, f_-, \mathbf{f}_-) \in Y_q^0(\mathbb{R}^N)$, $\mathbf{v}_\pm = \mathcal{A}^1_\pm(\lambda)\mathcal{F}^0_\lambda\mathcal{G}^0$ *and* $p_- = \mathcal{B}^0(\lambda)\mathcal{F}^0_\lambda\mathcal{G}^0$ *are unique solutions to* (54), *and:*

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q^0(\mathbb{R}^N), W_q^{2-j}(\mathbb{R}^N_\pm)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{A}^1_\pm(\lambda)) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \le r_b,$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q^0(\mathbb{R}^N), L_q(\mathbb{R}^N_-)^N)}(\{(\tau\partial_\tau)^\ell(\nabla\mathcal{B}^1_-(\lambda)) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \le r_b$$

for $\ell = 0, 1$ *and* $j = 0, 1, 2$ *with some constant* r_b *depending on* δ, $\rho_\pm$, ϵ, λ_0, *and* q. *Here, we set* $\mathcal{F}^0_\lambda\mathcal{G}^0 = (\mathbf{g}_+, \mathbf{g}_-, \lambda^{1/2}\mathbf{h}, \mathbf{h}, \lambda^{1/2}f_-, f_-, \lambda\mathbf{f}_-)$.

Remark 12. *We set:*

$$\begin{aligned}\|\mathcal{G}^0\|_{X_q(\mathbb{R}^N)} &= \|\mathbf{g}_+\|_{L_q(\mathbb{R}^N_+)} + \|\mathbf{g}_-\|_{L_q(\mathbb{R}^N_-)} + \|\mathbf{h}\|_{W^1_q(\mathbb{R}^N)} + \|f_-\|_{W^1_q(\mathbb{R}^N_-)} + \|\mathbf{f}_-\|_{L_q(\mathbb{R}^N_-)};\\ \|\mathcal{F}^0\|_{\mathcal{X}_q(\mathbb{R}^N)} &= \|\mathbf{F}_1\|_{L_q(\mathbb{R}^N_+)} + \|(\mathbf{F}_2, \mathbf{F}_9)\|_{L_q(\mathbb{R}^N_-)} + \|\mathbf{F}_3\|_{L_q(\mathbb{R}^N)} + \|\mathbf{F}_4\|_{W^1_q(\mathbb{R}^N)}\\ &\quad + \|\mathbf{F}_7\|_{L_q(\mathbb{R}_-)} + \|\mathbf{F}_8\|_{W^1_q(\mathbb{R}^N_-)}\end{aligned} \tag{55}$$

To prove Theorem 10, we first reduce the problem to the case where $f_- = \mathrm{div}\,\mathbf{f}_- = 0$ and $\mathbf{g}_\pm = 0$. Concerning the incompressible part, we consider the following equations:

$$\begin{aligned}\lambda\mathbf{w}_- - \rho_{0-}^{-1}\mathrm{Div}\,(\mathbf{S}_-(\mathbf{w}_-) - p_-\mathbf{I}) = \mathbf{g}_- &\quad \text{in } \mathbb{R}^N_-,\\ \mathrm{div}\,\mathbf{w}_- = f_- = \mathrm{div}\,\mathbf{f}_- &\quad \text{in } \mathbb{R}^N_-,\\ p_-|_{\mathbb{R}^N_0} = 0, \quad w_j|_{\mathbb{R}^N_0} = 0, \quad D_N w_N|_{\mathbb{R}^N_0} = f_-, &\end{aligned} \tag{56}$$

where $\mathbf{w}_- = {}^T(w_1, \ldots, w_N)$. We start with proving that for any $\mathbf{w}_- \in W^2_q(\mathbb{R}^N_-)^N$ and $\varphi \in \hat{W}^1_{q',0}(\mathbb{R}^N_-)$:

$$(\Delta\mathbf{w}_-, \nabla\varphi)_{\mathbb{R}^N_-} = (\nabla\mathrm{div}\,\mathbf{w}_-, \nabla\varphi)_{\mathbb{R}^N_-}. \tag{57}$$

Since $C^\infty_0(\mathbb{R}^N_-)$ is not dense in $\hat{W}^1_{q',0}(\mathbb{R}^N_-)$ in general (cf. Shibata [6]), we give a proof below. To prove (57), we use an inequality:

$$\|x_N^{-1}\varphi\|_{L_q(\mathbb{R}^N_-)} \le C\|\nabla\varphi\|_{L_q(\mathbb{R}^N_-)} \tag{58}$$

for any $\varphi \in \hat{W}^1_{q,0}(\mathbb{R}^N_-)$ and $1 < q < \infty$. In fact, representing $\varphi(x', x_N) = -\int_{-x_N}^0 (D_N\varphi)(x', s)\,ds$ with $x' = (x_1, \ldots, x_{N-1})$ and using the Hardy inequality, we have:

$$\begin{aligned}\|x_N^{-1}\varphi\|^q_{L_q(\mathbb{R}^N_-)} &\le \int_{\mathbb{R}^{N-1}}\int_0^\infty \Big(x_N^{-1}\int_0^{x_N}|\varphi(x', -s)|\,ds\Big)^q\,dx_N dx'\\ &\le C_q\int_{\mathbb{R}^{N-1}}\int_0^\infty |(D_N\varphi)(x', -s)|^q\,dsdx',\end{aligned}$$

which yields (58). To prove (57), we take $\psi(x_N) \in C^\infty(\mathbb{R})$, which equals one for $|x_N| < 1$ and zero for $|x_N| > 2$, and set $\psi_R(x_N) = \psi(x_N/R)$. For any $v \in L_q(\mathbb{R}^N_-)^N$ and $\varphi \in \hat{W}^1_{q',0}(\mathbb{R}^N_-)$,

$$(\mathbf{v}, \nabla\varphi)_{\mathbb{R}^N_-} = \lim_{R\to\infty}(\mathbf{v}, \nabla(\psi_R\varphi))_{\mathbb{R}^N_-}. \tag{59}$$

In fact, by (58):

$$|(\mathbf{v}, (D_N\psi_R)\varphi)_{\mathbb{R}^N_-}| \le C\|\mathbf{v}\|_{L_q(\mathbb{R}^{N-1}\times\{-2R\le x_N\le -R\})}\|\nabla\varphi\|_{L_{q'}(\mathbb{R}^N_-)} \to 0$$

as $R \to \infty$, which yields (59). We now prove (57). Notice that $\psi_R\varphi \in W^1_{q',0}(\mathbb{R}^N_-)$. Since $C^\infty_0(\mathbb{R}^N_-)$ is dense in $W^1_{q',0}(\mathbb{R}^N_-)$, we take a sequence $\{\omega_j\}_{j=1}^\infty$ of $C^\infty_0(\mathbb{R}^N_-)$ such that $\|\omega_j - \psi_R\varphi\|_{W^1_q(\mathbb{R}^N_-)} \to 0$ as $j \to \infty$. Then, by (59),

$$
\begin{aligned}
(\Delta\mathbf{w}_-, \nabla\varphi)_{\mathbb{R}^N_-} &= \lim_{R\to\infty}\big(\lim_{j\to\infty}(\Delta\mathbf{w}_-, \nabla\omega_j)_{\mathbb{R}^N_-}\big)\\
&= \lim_{R\to\infty}\big(\lim_{j\to\infty}(\nabla\mathrm{div}\,\mathbf{w}_-, \nabla\omega_j)_{\mathbb{R}^N_-}\big) = \lim_{R\to\infty}(\nabla\mathrm{div}\,\mathbf{w}_-, \nabla(\psi_R\varphi))_{\mathbb{R}^N_-} = (\nabla\mathrm{div}\,\mathbf{w}_-, \nabla\varphi)_{\mathbb{R}^N_-},
\end{aligned}
$$

which shows (57).

We now consider equations:

$$
\begin{aligned}
\lambda\mathbf{w}_- - \rho_{0-}^{-1}\mathrm{Div}\,(\mathbf{S}_-(\mathbf{w}_-) - p_-\mathbf{I}) = \mathbf{g}_-, \quad \mathrm{div}\,\mathbf{w}_- = f_- = \mathrm{div}\,\mathbf{f}_- &\quad \text{in } \mathbb{R}^N_-,\\
w_j|_{\mathbb{R}^N_0} = 0, \quad D_N w_N|_{\mathbb{R}^N_0} = f_- &
\end{aligned}
\tag{60}
$$

for $j = 1, \ldots, N-1$, where $\mathbf{w}_- = {}^T(w_1, \ldots, w_N)$. Noticing that $\mathrm{Div}\,\mathbf{S}_-(\mathbf{w}_-) = \Delta\mathbf{w}_- + \nabla\mathrm{div}\,\mathbf{w}_-$ and using (57), for any $\varphi \in \hat{W}^1_{q',0}(\mathbb{R}^N_-)$, we have:

$$
\begin{aligned}
(\mathbf{g}_-, \nabla\varphi)_{\mathbb{R}^N_-} &= \lambda(\mathbf{w}_-, \nabla\varphi)_{\mathbb{R}^N_-} - 2\rho_{0-}^{-1}(\nabla\mathrm{div}\,\mathbf{w}_-, \nabla\varphi)_{\mathbb{R}^N_-} + \rho_{0-}^{-1}(\nabla p_-, \nabla\varphi)_{\mathbb{R}^N_-}\\
&= \lambda(\mathbf{f}_-, \nabla\varphi)_{\mathbb{R}^N_-} - 2\rho_{0-}^{-1}(\nabla f_-, \nabla\varphi)_{\mathbb{R}^N_-} + \rho_{0-}^{-1}(\nabla p_-, \nabla\varphi))_{\mathbb{R}^N_-}.
\end{aligned}
$$

Thus, we have $p_- = \rho_{0-}\mathcal{P}(\mathbf{g}_- - \lambda\mathbf{f}_- + 2\rho_{0-}^{-1}\nabla f_-)$, and so, the first equation in (60) is reduced to equations:

$$
\begin{aligned}
\lambda\mathbf{w}_- - \rho_{0-}^{-1}\Delta\mathbf{w}_- = \mathbf{g}_- - \nabla\mathcal{P}(\mathbf{g}_- - \lambda\mathbf{f}_- + 2\rho_{0-}^{-1}\nabla f_-) + \rho_{0-}^{-1}\nabla f_- &\quad \text{in } \mathbb{R}^N_-,\\
\mathrm{div}\,\mathbf{w}_- = f_- = \mathrm{div}\,\mathbf{f}_- &\quad \text{in } \mathbb{R}^N_-,\\
w_j|_{\mathbb{R}^N_0} = 0, \quad D_N w_N|_{\mathbb{R}^N_0} = f_- &
\end{aligned}
\tag{61}
$$

for $j = 1, \ldots, N-1$. The first equations and third equations in (61) become the following equations:

$$
\lambda w_j - \rho_{0-}^{-1}\Delta w_j = \tilde{g}_j \quad \text{in } \mathbb{R}^N_-, \quad w_j|_{\mathbb{R}^N_0} = 0, \tag{62}
$$

$$
\lambda w_N - \rho_{0-}^{-1}\Delta w_N = \tilde{g}_N \quad \text{in } \mathbb{R}^N_-, \quad D_N w_N|_{\mathbb{R}^N_0} = f_-, \tag{63}
$$

where $\tilde{g}_j$ denotes the jth component of N-vector, $\tilde{\mathbf{g}} := \mathbf{g}_- - \nabla\mathcal{P}(\mathbf{g}_- - \lambda\mathbf{f}_- + 2\rho_{0-}^{-1}\nabla f_-) + \rho_{0-}^{-1}\nabla f_-$. We use the following theorem, which was proven in [13].

Proposition 1. *Let* $1 < q < \infty$, $0 < \epsilon < \pi/2$ *and* $\lambda_0 > 0$. *Then, the following two assertions hold:* (1) *There exists an operator family* $\mathcal{S}_d(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(L_q(\mathbb{R}^N_-), W^2_q(\mathbb{R}^N_-)))$ *such that for any* $\lambda \in \Sigma_{\epsilon,\lambda_0}$ *and* $\tilde{g}_j \in L_q(\mathbb{R}^N_-)$, $w_j = \mathcal{S}_d(\lambda)\tilde{g}_j$ $(j = 1, \ldots, N-1)$ *are unique solutions of Equation* (62), *and:*

$$
\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_-), W^{2-j}_q(\mathbb{R}^N_-))}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{S}_d(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r_b
$$

for $\ell = 0, 1$, *and* $j = 0, 1, 2$ *with some constant* r_b *depending on* λ_0.

(2) *Let*

$$
\begin{aligned}
Y^1_q(\mathbb{R}^N_-) &= \{(\mathbf{g}_-, f_-) \mid \mathbf{g} \in L_q(\mathbb{R}^N_-)^N, \quad f_- \in W^1_q(\mathbb{R}^N_-)\},\\
\mathcal{Y}^1_q(\mathbb{R}^N_-) &= \{(\mathbf{F}_2, \mathbf{F}_7, \mathbf{F}_8) \mid \mathbf{F}_2 \in L_q(\mathbb{R}^N_-)^N, \quad \mathbf{F}_7 \in L_q(\mathbb{R}^N_-), \quad \mathbf{F}_8 \in W^1_q(\mathbb{R}^N_-)\}.
\end{aligned}
$$

Then, there exists an operator family $\mathcal{S}_n(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}^1_q(\mathbb{R}^N_-), W^2_q(\mathbb{R}^N_-)))$ such that for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ and $(\tilde{g}_N, f_-) \in Y^1_q(\mathbb{R}^N_-)$, $w_N = \mathcal{S}_n(\lambda)(\tilde{g}_N, \lambda^{1/2} f_-, f_-)$ is a unique solution of Equation (63), and:

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}^1_q(\mathbb{R}^N_-), W^{2-j}_q(\mathbb{R}^N_-))}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{S}_n(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq r_b$$

for $\ell = 0, 1$, and $j = 0, 1, 2$ with some constant r_b depending on λ_0.

Finally, we prove that $\mathrm{div}\,\mathbf{w}_- = f_- = \mathrm{div}\,\mathbf{f}_-$ with $\mathbf{w}_- = {}^T(w_1, \ldots, w_N)$. In fact, for any $\varphi \in \hat{W}^1_{q',0}(\mathbb{R}^N_-)$, by (57), (62), and (63) we have:

$$\begin{aligned}(\mathbf{g}_- - \nabla\mathcal{P}(\mathbf{g}_- - \lambda\mathbf{f}_- + 2\rho_{0-}^{-1}\nabla f_-) + \rho_{0-}^{-1}\nabla f_-, \nabla\varphi)_{\mathbb{R}^N_-} &= (\lambda\mathbf{w}_- - \rho_{0-}^{-1}\Delta\mathbf{w}_-, \nabla\varphi)_{\mathbb{R}^N_-} \\ &= \lambda(\mathbf{w}_-, \nabla\varphi)_{\mathbb{R}^N_-} - \rho_{0-}^{-1}(\nabla\mathrm{div}\,\mathbf{w}_-, \nabla\varphi)_{\mathbb{R}^N_-},\end{aligned}$$

which yields that:

$$\lambda(\mathbf{w}_- - \mathbf{f}_-, \nabla\varphi)_{\mathbb{R}^N_-} - \rho_{0-}^{-1}(\nabla(\mathrm{div}\,\mathbf{w}_- - f_-), \nabla\varphi)_{\mathbb{R}^N_-} = 0 \tag{64}$$

for any $\varphi \in \hat{W}^1_{q',0}(\mathbb{R}^N_-)$. By the divergence theorem of Gauss and the assumption that $f_- = \mathrm{div}\,\mathbf{f}_-$, we have:

$$\lambda(\mathrm{div}\,\mathbf{w}_- - f_-, \varphi)_{\mathbb{R}^N_-} + \rho_{0-}^{-1}(\nabla(\mathrm{div}\,\mathbf{w}_- - f_-), \nabla\varphi)_{\mathbb{R}^N_-} = 0$$

for any $\varphi \in W^1_{q',0}(\mathbb{R}^N_-)$. Since $\mathrm{div}\,\mathbf{w}_-|_{x_N=0} = D_N w_N|_{x_N=0} = f_-|_{x_N=0}$, therefore the uniqueness yields that $\mathrm{div}\,\mathbf{w}_- = f_-$. Thus, by (64), we have $\mathbf{w}_- - \mathbf{f}_- \in J_q(\mathbb{R}^N_-)$, which shows that $\mathbf{w}_-$ and p satisfy (56).

Summing up, we proved the following proposition.

Proposition 2. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$, and $\lambda_0 > 0$. Let:*

$$\begin{aligned}Y^2_q(\mathbb{R}^N_-) &= \{(\mathbf{g}_-, f_-, \mathbf{f}_-) \mid \mathbf{g}, \mathbf{f}_- \in L_q(\mathbb{R}^N_-)^N, \quad f_- \in W^1_q(\mathbb{R}^N_-), \quad f_- = \mathrm{div}\,\mathbf{f}_-\}, \\ \mathcal{Y}^1_q(\mathbb{R}^N_-) &= \{(\mathbf{F}_2, \mathbf{F}_7, \mathbf{F}_8, \mathbf{F}_9) \mid \mathbf{F}_2, \mathbf{F}_9 \in L_q(\mathbb{R}^N_-)^N, \quad \mathbf{F}_7 \in L_q(\mathbb{R}^N_-), \quad \mathbf{F}_8 \in W^1_q(\mathbb{R}^N_-)\}.\end{aligned}$$

Then, there exists an operator family $\mathcal{T}^1_-(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}^2_q(\mathbb{R}^N_-), W^2_q(\mathbb{R}^N_-)^N))$ such that for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ and $(\tilde{g}_N, f_-, \mathbf{f}_-) \in Y^2_q(\mathbb{R}^N_-)$, $\mathbf{w}_- = \mathcal{T}^1_-(\lambda)(\mathbf{g}_-, \lambda^{1/2} f_-, f_-, \lambda\mathbf{f}_-)$ is a unique solution of Equation (56), and:

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}^2_q(\mathbb{R}^N_-), W^{2-j}_q(\mathbb{R}^N_-)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{T}^1_-(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq r_b$$

for $\ell = 0, 1$, and $j = 0, 1, 2$ with some constant r_b depending on λ_0.

Concerning the compressible part, we consider the equations:

$$\lambda\mathbf{w}_+ - \gamma_{0+}^{-1}\mathrm{Div}\,\mathbf{T}_+(\mathbf{w}_+) = \mathbf{g}_+ \quad \text{in } \mathbb{R}^N_+, \quad \gamma_{0+}^{-1}\mathbf{T}_+(\mathbf{w}_+)\mathbf{n}_0|_{\mathbb{R}^N_0} = 0. \tag{65}$$

We know the following theorem, which was proven by Götz and Shibata [14].

Proposition 3. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$, $\delta_0 > 0$, and $\lambda_0 > 0$. Then, there exists an operator family $\mathcal{T}^1_+(\lambda) \in \mathrm{Hol}\,(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(L_q(\mathbb{R}^N_+)^N, W^2_q(\mathbb{R}^N_+)^N))$ such that for any $\mathbf{g}_+$ and $\lambda \in \Gamma_{\epsilon,\lambda_0}$, $\mathbf{w}_+ = \mathcal{T}^1_+(\lambda)\mathbf{g}_+$ is a unique solution to Problem (65), and:*

$$\mathcal{R}_{\mathcal{L}_q(\mathbb{R}^N_+)^N, W^{2-j}_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{T}^1_+(\lambda)) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \leq r_b$$

for $\ell = 0, 1$, $j = 0, 1, 2$ *with some constant* r_b *depending on* ϵ, λ_0, ρ_{+0}, δ_0, q, *and* N.

We now set $\mathbf{v}_\pm = \mathbf{w}_\pm + \mathbf{u}_\pm$ and $q_- = p_- + \theta$ in Equation (54), and then, the equations for $\mathbf{u}_\pm$ and θ are the following:

$$\begin{aligned} \lambda\mathbf{u}_+ - \gamma_{0+}^{-1}\mathrm{Div}\,\mathbf{T}_+(\mathbf{u}_+) = 0 \qquad & \text{in } \mathbb{R}^N_+, \\ \lambda\mathbf{u}_- - \rho_{0-}^{-1}\mathrm{Div}\,\mathbf{T}_-(\mathbf{u}_-, \theta) = 0, \quad \mathrm{div}\,\mathbf{u}_- = 0 \qquad & \text{in } \mathbb{R}^N_-, \\ \mathbf{T}_+(\mathbf{u}_+)\mathbf{n}_0|_{x_N=0+} - \mathbf{T}_-(\mathbf{u}_-, \theta)\mathbf{n}_0|_{x_N=0-} = \mathbf{h}, & \\ \mathbf{u}_+|_{x_N=0+} - \mathbf{u}_-|_{x_N=0-} = \mathbf{k}. & \end{aligned} \tag{66}$$

Concerning Equation (66), we know the following theorem, which was proven by Kubo, Shibata, and Soga [15].

Theorem 11. *Let* $1 < q < \infty$, $0 < \epsilon < \pi/2$, *and* $\lambda_0 > 0$. *Let:*

$$Y_q^3(\mathbb{R}^N) = \{(\mathbf{h}, \mathbf{k}) \mid \mathbf{h} \in W_q^1(\mathbb{R}^N), \quad \mathbf{k} \in W_q^2(\mathbb{R}^N)\},$$
$$\mathcal{Y}_q^3(\mathbb{R}^N) = \{(\mathbf{F}_3, \mathbf{F}_4, \mathbf{F}_{10}, \mathbf{F}_{11}, \mathbf{F}_{12}) \mid \mathbf{F}_3, \mathbf{F}_{10} \in L_q(\mathbb{R}^N)^N, \quad \mathbf{F}_4, \mathbf{F}_{11} \in W_q^1(\mathbb{R}^N)^N, \quad \mathbf{F}_{12} \in W_q^2(\mathbb{R}^N)\}.$$

Then, there exist operator families $\mathcal{T}^2_\pm(\lambda)$ *and* $\mathcal{Q}_-(\lambda)$ *with:*

$$\mathcal{T}^2_\pm(\lambda) \in \mathrm{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}_q^3(\mathbb{R}^N), W_q^2(\mathbb{R}^N_\pm)^N)), \;\; \mathcal{Q}^0_-(\lambda) \in \mathrm{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}_q^2(\mathbb{R}^N), W_q^1(\mathbb{R}^N_-) + \hat{W}^1_{q,0}(\mathbb{R}^N_-))),$$

such that for any $\lambda \in \Gamma_{\epsilon,\lambda_0}$ *and* $\mathbf{G}^3 = (\mathbf{h}, \mathbf{k}) \in Y_q^3(\mathbb{R}^N)$, $\mathbf{u}_\pm = \mathcal{T}^2_\pm(\lambda)F^3_\lambda\mathbf{G}^3$ *and* $\theta = \mathcal{Q}_-(\lambda)F^3_\lambda\mathbf{G}^3$ *are unique solutions to* (66), *where* $F^3_\lambda\mathbf{G}^3 = (\lambda^{1/2}\mathbf{h}, \mathbf{h}, \lambda\mathbf{k}, \lambda^{1/2}\mathbf{k}, \mathbf{k})$,

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q^3(\mathbb{R}^N), W_q^{2-j}(\mathbb{R}^N_\pm)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{T}^2_\pm(\lambda)) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \le r_b,$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q^3(\mathbb{R}^N), L_q(\mathbb{R}^N_-)^N)}(\{(\tau\partial_\tau)^\ell(\nabla\mathcal{Q}_-(\lambda)) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \le r_b$$

for $\ell = 0, 1$ *and* $j = 0, 1, 2$ *with some constant* r_b *depending on* ϵ, λ_0, $\rho_{\pm 0}$, δ_0, q, *and* N.

Remark 13. $\mathbf{F}_3$, $\mathbf{F}_4$, $\mathbf{F}_{10}$, $\mathbf{F}_{11}$, *and* $\mathbf{F}_{12}$ *are the corresponding variables to* $\lambda^{1/2}\mathbf{h}$, $\mathbf{h}$, $\lambda\mathbf{k}$, $\lambda^{1/2}\mathbf{k}$ *and* $\mathbf{k}$. *We set:*

$$\|(\mathbf{F}_3, \mathbf{F}_4, \mathbf{F}_{10}, \mathbf{F}_{11}, \mathbf{F}_{12})\|_{\mathcal{Y}_q^3(\mathbb{R}^N)} = \|(\mathbf{F}_3, \mathbf{F}_{10})\|_{L_q(\mathbb{R}^N)} + \|(\mathbf{F}_4, \mathbf{F}_{11})\|_{W_q^1(\mathbb{R}^N)} + \|\mathbf{F}_{12}\|_{W_q^2(\mathbb{R}^N)}.$$

Combining Proposition 2 and Proposition 3 with Lemma 1 and Lemma 2, we have Theorem 10. This completes the proof of Theorem 9.

4. Several Problems in Bent Spaces

Let $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ be a bijection of the C^1 class, and let Φ^{-1} be its inverse map. Writing $\nabla\Phi = \mathcal{A} + B(x)$ and $\nabla\Phi^{-1} = \mathcal{A}_- + B_-(x)$, we assume that $\mathcal{A}$ and $\mathcal{A}_{-1}$ are orthonormal matrices with constant coefficients and $B(x)$ and $B_{-1}(x)$ are matrices of functions in $W^1_{r,\mathrm{loc}}(\mathbb{R}^N)$ with $N < r < \infty$ such that:

$$\|(B, B_{-1})\|_{L_\infty(\mathbb{R}^N)} \le M_1, \quad \|\nabla(B, B_{-1})\|_{L_r(\mathbb{R}^N)} \le M_2. \tag{67}$$

We will choose M_1 small enough eventually, and so we may assume that $0 < M_1 < 1 \le M_2$. We set $D_\pm = \Phi(\mathbb{R}^N_\pm)$ and $S = \Phi(\mathbb{R}^N_0)$, and we denote the unit outward normal to S pointing from D_- to D_+ by $\mathbf{n}_+$. Since S is represented by $\Phi_{-1,N}(y) = 0$ with $\Phi^{-1} = (\Phi_{-1,1}, \ldots, \Phi_{-1,N})$, we have:

$$\mathbf{n}_+ = \frac{\nabla\Phi_{-1,N}}{|\nabla\Phi_{-1,N}|} = \frac{(\mathcal{A}_{N1} + B_{N1}, \ldots, \mathcal{A}_{NN} + B_{NN})}{(\sum_{i=1}^N(\mathcal{A}_{Ni} + B_{Ni})^2)^{1/2}}, \tag{68}$$

where we set $\mathcal{A}_{-1} = (\mathcal{A}_{ij})$ and $B_{-1} = (B_{ij})$. Notice that $\mathbf{n}_+$ is defined on the whole $\mathbb{R}^N$. By (67) with small M_1,

$$\Big\{\sum_{i=1}^N (\mathcal{A}_{Ni} + B_{Ni})^2\Big\}^{-1/2} = 1 + b_0 \tag{69}$$

with $b_0 \in W^1_{r,\mathrm{loc}}(\mathbb{R}^N)$ possessing the estimate: $\|b_0\|_{L_\infty(\mathbb{R}^N)} \leq C_N M_1$ and $\|\nabla b_0\|_{L_r(\mathbb{R}^N)} \leq CM_2$. Let $\gamma_{0+}(x)$ and $\gamma_{3+}(x)$ be real-valued functions defined on $\mathbb{R}^N$ satisfying the following conditions:

$$\begin{aligned}&\rho_{0+}/2 \leq \gamma_{0+} \leq 2\rho_{0+}, \ \ 0 \leq \gamma_{3+} \leq (\rho_{2+})^2 \ \ (x \in D_+),\\ &\|\gamma_{\ell+} - \hat\gamma_{\ell+}\|_{L_\infty(D_+)} \leq M_1, \ \ \|\nabla\gamma_{\ell+}\|_{L_r(D_+)} \leq C_{M_2}\end{aligned} \tag{70}$$

for $\ell = 0$ and 3, where $\hat\gamma_{\ell+}$ ($\ell = 0, 3$) are some constants with $\rho_{0+}/2 < \hat\gamma_{0+} < 2\rho_{0+}$ and $0 \leq \hat\gamma_{3+} \leq (\rho_{2+})^2$.

First, we consider the following problem:

$$\begin{aligned}\lambda\mathbf{u}_+ - \gamma_{0+}^{-1}\mathrm{Div}\,\mathbf{T}_+(\mathbf{u}_+) = \mathbf{g}_+ \quad &\text{in } D_+,\\ \lambda\mathbf{u}_- - \rho_{0-}^{-1}\mathrm{Div}\,\mathbf{T}_-(\mathbf{u}_-, K_I(\mathbf{u}_+,\mathbf{u}_-)) = \mathbf{g}_- \quad &\text{in } D_-,\\ \mathbf{T}_+(\mathbf{u}_+)\mathbf{n}_+|_{S+0} - \mathbf{T}_-(\mathbf{u}_-, K_I(\mathbf{u}_+,\mathbf{u}_-))\mathbf{n}_+|_{S-0} = \mathbf{h}|_S, \ \ &\mathbf{u}_+|_{S+0} = \mathbf{u}_-|_{S-0}.\end{aligned} \tag{71}$$

Moreover, $v = K_I(\mathbf{u}_+, \mathbf{u}_-)$ is a solution to the weak Dirichlet problem:

$$(\nabla v, \nabla\varphi)_{D_-} = (\mathrm{Div}\,\mathbf{S}_-(\mathbf{u}_-) - \nabla\mathrm{div}\,\mathbf{u}_-, \nabla\varphi)_{D_-} \quad \text{for any } \varphi \in \hat W^1_{q',0}(D_-) \tag{72}$$

subject to $v =< \mathbf{S}_-(\mathbf{u}_-)\mathbf{n}_+, \mathbf{n}_+ > |_{S-0} - \mathrm{div}\,\mathbf{u}_-|_{S-0} - < (\mathbf{S}_+(\mathbf{u}_+) + \delta\gamma_{3+}\mathrm{div}\,\mathbf{u}_+\mathbf{I})\mathbf{n}_+, \mathbf{n}_+ > |_{S+0}$. We have the following theorem.

Theorem 12. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$, and $r \geq \max(q, q')$. Let $X^3_q(D)$ and $\mathcal{X}^3_q(D)$ be sets defined by replacing $\mathbb{R}^N$ and $\mathbb{R}^N_\pm$ by D and $D_\pm$, respectively, in Theorem 9. Then, there exist constants $M_1 \in (0,1)$, $\lambda_0 = \lambda_{M_2} \geq 1$ and operator families $\mathcal{E}^1_\pm(\lambda) \in \mathrm{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}^3_q(D), W^2_q(D_\pm)^N))$ such that for any $\lambda \in \Gamma_{\epsilon,\lambda_0}$ and $\mathbf{G}^3 = (\mathbf{g}_+, \mathbf{g}_-, \mathbf{h}) \in X^3_q(D)$, $\mathbf{u}_\pm = \mathcal{E}^1_\pm(\lambda)F^3_\lambda\mathbf{G}_3$ is a unique solution to (71), and:*

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}^3_q(D), W^{2-j}_q(D_\pm)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{E}^1_\pm(\lambda)) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \leq C_{M_2} \quad (\ell = 0,1, j = 0,1,2).$$

Remark 14. *Here and in the following, M_1 depends on ϵ, q, $\mu_\pm$, ν_+, $\rho_{0\pm}$, ρ_{2+}, but is independent of M_2. In addition, constants denoted by λ_{M_2} and C_{M_2} depend on M_2, ϵ, q, $\mu_\pm$, ν_+, $\rho_{0\pm}$, ρ_{2+}, and N, but we mention only dependence on M_2.*

Proof. The idea of the proof here follows Shibata [16] and von Below, Enomoto, and Shibata [17]. Using the change of variable: $x = \Phi^{-1}(y)$ with $y \in D$ and $x \in \mathbb{R}^N$ and the change of unknown functions: $\mathbf{v}_\pm = \mathcal{A}_{-1}\mathbf{u}_\pm \circ \Phi$, writing $\gamma_{0+}^{-1} = \hat\gamma_{0+}^{-1} + (\gamma_{0+}^{-1} - \hat\gamma_{0+}^{-1})$ and $\gamma_{3+} = \hat\gamma_{3+} + (\gamma_{3+} - \hat\gamma_{3+})$, and setting $p = K_I(\mathbf{u}_+, \mathbf{u}_-)$, we see that Problem (71) is transferred to the following equivalent problem:

$$\begin{aligned}\lambda\mathbf{v}_+ - \hat\gamma_{0+}^{-1}[\mathrm{Div}\,\mathbf{S}_+(\mathbf{v}_+) + \delta\hat\gamma_{3+}\nabla\mathrm{div}\,\mathbf{v}_+] - \mathcal{F}^1_+(\mathbf{v}_+) = \mathbf{G}_+ \quad &\text{in } \mathbb{R}^N_+,\\ \lambda\mathbf{v}_- - \rho_{0-}^{-1}[\mathrm{Div}\,\mathbf{S}_-(\mathbf{v}_-) - \nabla p] - \mathcal{F}^1_-(\mathbf{v}_-) + \mathcal{P}^1\nabla p = \mathbf{G}_- \quad &\text{in } \mathbb{R}^N_-\end{aligned} \tag{73}$$

subject to the interface condition: $\mathbf{v}_+|_{x_N=0+} = \mathbf{v}_-|_{x_N=0-}$ and:

$$\{(\mathbf{S}_+(\mathbf{v}_+) + \delta\hat\gamma_{3+}\mathrm{div}\,\mathbf{v}_+\mathbf{I})\mathbf{n}_0 + \mathcal{F}^2_+(\mathbf{v}_+)\}|_{x_N=0+} - \{(\mathbf{S}_-(\mathbf{v}_-) - p\mathbf{I})\mathbf{n}_0 + \mathcal{F}^2_-(\mathbf{v}_-)\}|_{x_N=0-} = \mathbf{H}.$$

p satisfies the following variational equation:

$$(\nabla p, \nabla\varphi) + (\mathcal{P}^2\nabla p, \nabla\varphi) = (\mathrm{Div}\,\mathbf{S}_-(\mathbf{v}_-) - \nabla\mathrm{div}\,\mathbf{v}_- + \mathcal{F}^3_-(\mathbf{v}_-), \nabla\varphi) \ \text{ for any } \varphi \in \hat W^1_{q',0}(\mathbb{R}^N_-) \tag{74}$$

subject to:

$$\begin{aligned} p|_{x_N=0-} &= \{< \mathbf{S}_-(\mathbf{v}_-)\mathbf{n}_0, \mathbf{n}_0 > -\mathrm{div}\,\mathbf{v}_- + \mathcal{F}_-^4(\mathbf{v}_-)\}|_{x_N=0-} \\ &\quad - \{< (\mathbf{S}_+(\mathbf{v}_+) + \delta\hat{\gamma}_{3+}\mathrm{div}\,\mathbf{v}_+\mathbf{I})\mathbf{n}_0, \mathbf{n}_0 > + \mathcal{F}_+^3(\mathbf{v}_+)\}|_{x_N=0+}. \end{aligned}$$

Here, we write $(\cdot,\cdot) = (\cdot,\cdot)_{\mathbb{R}^N_-}$ for short, and $\mathbf{G}_\pm = \mathcal{A}_{-1}\mathbf{g}_\pm \circ \Phi$, $\mathbf{H} = \mathcal{A}_{-1}\mathbf{h} \circ \Phi$, and $\mathcal{F}^i_\pm(\mathbf{v}_\pm)$ are the vector of functions of the forms:

$$\mathcal{F}^1_\pm(\mathbf{v}_\pm) = \mathcal{R}^1_\pm\nabla^2\mathbf{v}_\pm + \mathcal{S}_\pm\nabla\mathbf{v}_\pm, \;\; \mathcal{F}^i_+(\mathbf{v}_+) = \mathcal{R}^i_+\nabla\mathbf{v}_+, \quad \mathcal{F}^j_-(\mathbf{v}_-) = \mathcal{R}^j_-\nabla\mathbf{v}_- \tag{75}$$

for $i = 2,3$ and $j = 2,3,4$. In view of (67)–(70), we can assume that $\mathcal{R}^i_\pm$, $\mathcal{S}_\pm$, and $\mathcal{P}^i$ possesses the following estimate:

$$\|(\mathcal{R}^i_+, \mathcal{R}^j_-, \mathcal{P}^k)\|_{L_\infty(\mathbb{R}^N)} \leq CM_1, \quad \|(\nabla\mathcal{R}^i_+, \nabla\mathcal{R}^j_-, \nabla\mathcal{P}^k, \mathcal{S}_\pm)\|_{L_r(\mathbb{R}^N)} \leq C_{M_2} \tag{76}$$

for $i = 1,2,3$, $j = 1,2,3,4$, and $k = 1,2$. Following Shibata ([16] Section 4), we treat the $\mathbb{R}^N_-$ side as follows: Let $v = K^0_I(\mathbf{v}_+, \mathbf{v}_-)$ be a function defined in (52), which satisfies the estimate:

$$\|\nabla K^0_I(\mathbf{v}_+, \mathbf{v}_-)\|_{L_q(\mathbb{R}^N_-)} \leq C(\|\nabla\mathbf{v}_+\|_{W^1_q(\mathbb{R}^N_+)} + \|\nabla\mathbf{v}_-\|_{W^1_q(\mathbb{R}^N_-)}). \tag{77}$$

Setting $p = K^0_I(\mathbf{v}_+, \mathbf{v}_-) + p_1$, we see that p_1 satisfies the variational equation:

$$(\nabla p_1, \nabla\varphi) + (\mathcal{P}^2\nabla p_1, \nabla\varphi) = (\mathcal{F}^3_-(\mathbf{v}_-) - \mathcal{P}^2\nabla K^0_I(\mathbf{v}_+, \mathbf{v}_-), \nabla\varphi) \quad \text{for any } \varphi \in W^1_{q',0}(\mathbb{R}^N_-) \tag{78}$$

subject to:

$$p_1|_{x_N=0-} = \mathcal{F}^4_-(\mathbf{v}_-)|_{x_N=0-} - \mathcal{F}^3_+(\mathbf{v}_+)|_{x_N=0+}. \tag{79}$$

Since $\|\mathcal{P}^2\|_{L_\infty(\mathbb{R}^N)}$ is small enough, we can show the following lemma by the small perturbation from the weak Dirichlet problem in $\mathbb{R}^N_-$.

Lemma 4. *Let $1 < q < \infty$. Then, there exist a constant $M_1 \in (0,1)$ and an operator Ψ with:*

$$\Psi \in \mathcal{L}(L_q(\mathbb{R}^N_-)^{2N}, W^1_q(\mathbb{R}^N_-) + \hat{W}^1_{q,0}(\mathbb{R}^N_-))$$

such that for any $\mathbf{f} \in L_q(\mathbb{R}^N_-)^N$ and $g \in \hat{W}^1_{q,0}(\mathbb{R}^N_-)$, $\theta = \Psi(\mathbf{f}, \nabla g)$ is a unique solution to the variational problem:

$$(\nabla\theta, \nabla\varphi) + (\mathcal{P}^2\nabla\theta, \nabla\varphi) = (\mathbf{f}, \nabla\varphi) \quad \textit{for any } \varphi \in \hat{W}^1_{q',0}(\mathbb{R}^N_-) \tag{80}$$

subject to $\theta|_{x_N=0-} = g|_{x_N=0}$.

By Lemma 4, $p_1 = p_1(\mathbf{v}_+, \mathbf{v}_-) = \Psi(\mathbf{f}, \nabla g)$ with $\mathbf{f} = \mathcal{F}^3_-(\mathbf{v}_-) - \mathcal{P}^2\nabla K^0_I(\mathbf{v}_+, \mathbf{v}_-)$ and $g = \mathcal{F}^4_-(\mathbf{v}_-) - \mathcal{F}^3_+(\mathbf{v}_+)$. Inserting $p = K^0_I(\mathbf{v}_+, \mathbf{v}_-) + p_1(\mathbf{v}_+, \mathbf{v}_-)$ into (73), we have:

$$\begin{aligned} &\lambda\mathbf{v}_+ - \hat{\gamma}^{-1}_{0+}[\mathrm{Div}\,\mathbf{S}_+(\mathbf{v}_+) - \delta\gamma^{-1}_{3+}\nabla\mathrm{div}\,\mathbf{v}_+] - \mathcal{F}^1_+(\mathbf{v}_+) = \mathbf{G}_+ \quad \text{in } \mathbb{R}^N_+, \\ &\lambda\mathbf{v}_- - \rho^{-1}_{0-}[\mathrm{Div}\,\mathbf{S}_-(\mathbf{v}_-) - \nabla K^0_I(\mathbf{v}_+, \mathbf{v}_-)] - \mathcal{F}^1_-(\mathbf{v}_-) \\ &\qquad + \rho^{-1}_{0-}[\mathcal{P}^1\nabla K^0_I(\mathbf{v}_+, \mathbf{v}_-) + (I + \mathcal{P}^1)\nabla p_1(\mathbf{v}_+, \mathbf{v}_-)] = \mathbf{G}_- \quad \text{in } \mathbb{R}^N_- \end{aligned} \tag{81}$$

subject to the interface conditions $\mathbf{v}_+|_{x_N=0+} = \mathbf{v}_-|_{x_N=0-}$ and:

$$\begin{aligned} &\{(\mathbf{S}_+(\mathbf{v}_+) + \delta\hat{\gamma}_{3+}\mathrm{div}\,\mathbf{v}_+\mathbf{I})\mathbf{n}_0 + \mathcal{F}^2_+(\mathbf{v}_+) + \mathcal{F}^3_+(\mathbf{v}_+)\mathbf{n}_0\}|_{x_N=0+} \\ &\quad - \{(\mathbf{S}_-(\mathbf{v}_-) - (K^0_I(\mathbf{v}_+, \mathbf{v}_-) + \mathcal{F}^4_-(\mathbf{v}_-)\mathbf{I})\mathbf{n}_0 + \mathcal{F}^2_-(\mathbf{v}_-)\}|_{x_N=0-} = \mathbf{H}. \end{aligned}$$

Here, we used (78).

To solve (81) for any right members $\mathbf{G}^3 = (\mathbf{g}_+, \mathbf{g}_-, \mathbf{h}) \in X_q^3(\mathbb{R}^N)$, we set $\mathbf{v}_\pm = \mathcal{E}_\pm(\lambda)F_\lambda^3\mathbf{G}^3$ in (81), where $\mathcal{E}_\pm(\lambda)$ are operators given in Theorem 9, and then, we have:

$$\begin{aligned}
&\lambda\mathbf{v}_+ - \hat{\gamma}_{0+}^{-1}[\mathrm{Div}\,\mathbf{S}_+(\mathbf{v}_+) - \delta\gamma_{3+}^{-1}\nabla\mathrm{div}\,\mathbf{v}_+] - \mathcal{F}_+^1(\mathbf{v}_+) = \mathbf{G}_+ - \mathbf{F}_+^1(\lambda)\mathbf{G}^3 \quad \text{in } \mathbb{R}_+^N,\\
&\lambda\mathbf{v}_- - \rho_{0-}^{-1}[\mathrm{Div}\,\mathbf{S}_-(\mathbf{v}_-) - \nabla K_I^0(\mathbf{v}_+, \mathbf{v}_-)] - \mathcal{F}_-^1(\mathbf{v}_-)\\
&\quad + \rho_{0-}^{-1}[\mathcal{P}^1\nabla K_I^0(\mathbf{v}_+, \mathbf{v}_-) + (I + \mathcal{P}^1)\nabla p_1(\mathbf{v}_+, \mathbf{v}_-)] = \mathbf{G}_- - \mathbf{F}_-^1(\lambda)\mathbf{G}^3 \quad \text{in } \mathbb{R}_-^N
\end{aligned} \tag{82}$$

subject to the interface conditions $\mathbf{v}_+|_{x_N=0+} = \mathbf{v}_-|_{x_N=0-}$ and:

$$\begin{aligned}
&\{(\mathbf{S}_+(\mathbf{v}_+) + \delta\hat{\gamma}_{3+}\mathrm{div}\,\mathbf{v}_+\mathbf{I})\mathbf{n}_0 + \mathcal{F}_+^2(\mathbf{v}_+) + \mathcal{F}_+^3(\mathbf{v}_+)\mathbf{n}_0\}|_{x_N=0+}\\
&\quad - \{(\mathbf{S}_-(\mathbf{v}_-) - (K_I^0(\mathbf{v}_+, \mathbf{v}_-) + \mathcal{F}_-^4(\mathbf{v}_-)\mathbf{I})\mathbf{n}_0 + \mathcal{F}_-^2(\mathbf{v}_-)\}|_{x_N=0-} = \mathbf{H} - \mathbf{F}^2(\lambda)\mathbf{G}^3.
\end{aligned}$$

Here, we set:

$$\begin{aligned}
\mathbf{F}_+^1(\lambda)\mathbf{G}^3 &= \mathcal{F}_+^1(\mathcal{E}_+(\lambda)F_\lambda^3\mathbf{G}^3),\\
\mathbf{F}_-^1(\lambda)\mathbf{G}^3 &= \mathcal{F}_-^1(\mathcal{E}_-(\lambda)F_\lambda^3\mathbf{G}^3)\\
&\quad - \rho_{0-}^{-1}[\mathcal{P}^1\nabla K_I^0(\mathcal{E}_+(\lambda)F_\lambda^3\mathbf{G}^3, \mathcal{E}_-(\lambda)F_\lambda^3\mathbf{G}^3) + (I + \mathcal{P}^1)\nabla p_1(\mathcal{E}_+(\lambda)F_\lambda^3\mathbf{G}^3, \mathcal{E}_-(\lambda)F_\lambda^3\mathbf{G}^3)],\\
\mathbf{F}^2(\lambda)\mathbf{G}^3 &= -\mathrm{Ext}^-[\mathcal{F}_+^2(\mathcal{E}_+(\lambda)F_\lambda^3\mathbf{G}^3) + \mathcal{F}_+^3(\mathcal{E}_+(\lambda)F_\lambda^3\mathbf{G}^3)\mathbf{n}_0]\\
&\quad + \mathrm{Ext}^+[\mathcal{F}_-^4(\mathcal{E}_-(\lambda)F_\lambda^3\mathbf{G}^3)\mathbf{I}\mathbf{n}_0 - \mathcal{F}_-^2(\mathcal{E}_-(\lambda)F_\lambda^3\mathbf{G}^3)],
\end{aligned}$$

and $\mathrm{Ext}^\pm[f_\mp]$ denote the even extension of functions $f_\mp$ defined on $\mathbb{R}_\mp^N$ to $\mathbb{R}^N$. Note that:

$$\|\nabla^\ell \mathrm{Ext}^\pm[f_\mp]\|_{L_q(\mathbb{R}^N)} \le 2\|f_\mp\|_{L_q(\mathbb{R}_\mp^N)} \quad (\ell = 0, 1) \tag{83}$$

with $\nabla^0 f_\mp = f_\mp$. Let us define the corresponding $\mathcal{R}$- bounded operators $\mathcal{R}_\pm^1(\lambda)$ and $\mathcal{R}^2(\lambda)$ by:

$$\begin{aligned}
\mathcal{R}_+^1(\lambda)\mathbf{F}^3 &= \mathcal{F}_+^1(\mathcal{E}_+(\lambda)\mathbf{F}^3),\\
\mathcal{R}_-^1(\lambda)\mathbf{F}^3 &= \mathcal{F}_-^1(\mathcal{E}_-(\lambda)\mathbf{F}^3) - \rho_{0-}^{-1}[\mathcal{P}^1\nabla K_I^0(\mathcal{E}_+(\lambda)\mathbf{F}^3, \mathcal{E}_-(\lambda)\mathbf{F}^3) + (I + \mathcal{P}^1)\nabla p_1(\mathcal{E}_+(\lambda)\mathbf{F}^3, \mathcal{E}_-(\lambda)\mathbf{F}^3)],\\
\mathcal{R}^2(\lambda)\mathbf{F}^3 &= -\mathrm{Ext}^-[\mathcal{F}_+^2(\mathcal{E}_+(\lambda)\mathbf{F}^3) + \mathcal{F}_+^3(\mathcal{E}_+(\lambda)\mathbf{F}^3)\mathbf{n}_0] - \mathrm{Ext}^+[\mathcal{F}_-^4(\mathcal{E}_-(\lambda)\mathbf{F}^3)\mathbf{n}_0 - \mathcal{F}_-^2(\mathcal{E}_-(\lambda)\mathbf{F}^3)].
\end{aligned}$$

Set $\mathbf{R}(\lambda)\mathbf{G}^3 = (\mathbf{F}_+^1(\lambda)\mathbf{G}^3, \mathbf{F}_-^1(\lambda)\mathbf{G}^3, \mathbf{F}^2(\lambda)\mathbf{G}^3)$ and $\mathcal{R}(\lambda)\mathbf{F}^3 = (\mathcal{R}_+^1(\lambda)\mathbf{F}^3, \mathcal{R}_-^1(\lambda)\mathbf{F}^3, \mathcal{R}^2(\lambda)\,\mathbf{F}^3)$. Obviously,

$$\mathbf{R}(\lambda)\mathbf{G}^3 = \mathcal{R}(\lambda)F_\lambda^3\mathbf{G}^3. \tag{84}$$

To obtain:

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q^3(\mathbb{R}^N))}(\{(\tau\partial_\tau)^\ell\mathcal{R}(\lambda) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \le C(\sigma + M_1 + C_{\sigma,M_2}\lambda_0^{-1/2}) \quad (\ell = 0, 1), \tag{85}$$

we use the following lemma (cf. Shibata ([4] Lemma 2.4)).

Lemma 5. *Let $D = \mathbb{R}^N$ or $\mathbb{R}_\pm^N$. Let $1 < q \le r < \infty$ and $N < r < \infty$. Then, there exists a constant $C_{N,q,r}$ such that for any $\sigma > 0$, $a \in L_r(D)$ and $b \in W_q^1(D)$, it holds that:*

$$\|ab\|_{L_q(D)} \le \sigma\|\nabla b\|_{L_q(D)} + C_{N,q,r}\sigma^{-\frac{N}{r-N}}\|a\|_{L_r(D)}^{\frac{r}{r-N}}\|b\|_{L_q(D)}.$$

To prove (85), for example, we treat $\mathrm{Ext}^-[\mathcal{F}_+^2(\mathcal{E}_+(\lambda)\mathbf{F}^3)]$. Recalling (75) and using (83), (76), Lemma 5, Lemma 2, Theorem 9, and (55), we have:

$$\int_0^1 \| \sum_{k=1}^n r_k(u)\mathrm{Ext}^-[\mathcal{F}_+^2(\mathcal{E}_+(\lambda_k)\mathbf{F}^3)]\|_{L_q(\mathbb{R}^N)}^q\,du = \int_0^1 \left\| \nabla \mathrm{Ext}^-\left[\sum_{k=1}^n r_k(u)\mathcal{F}_+^2(\mathcal{E}_+(\lambda_k)\mathbf{F}^3)\right]\right\|_{L_q(\mathbb{R}^N)}^q du$$
$$\leq 2^q \int_0^1 \left\| \mathcal{F}_+^2\left(\sum_{k=1}^n r_k(u)\mathcal{E}_+(\lambda_k)\mathbf{F}^3\right)\right\|_{L_q(\mathbb{R}^N_+)}^q du$$
$$\leq C_q \int_0^1 \left\{\sigma \| \sum_{k=1}^n r_k(u)\nabla^2\mathcal{E}_+(\lambda_k)\mathbf{F}^3\|_{L_q(\mathbb{R}^N_+)} + C_{\sigma,M_2}\| \sum_{k=1}^n r_k(u)\nabla\mathcal{E}_+(\lambda_k)\mathbf{F}^3\|_{L_q(\mathbb{R}^N_+)}\right\}^q du$$
$$\leq C_q\sigma^q \int_0^1 \| \sum_{k=1}^n r_k(u)\nabla^2\mathcal{E}_+(\lambda_k)\mathbf{F}^3\|_{L_q(\mathbb{R}^N_+)}^q du$$
$$+ C_q(C_{\sigma,M_2}\lambda_0^{-1/2})^q \int_0^1 \| \sum_{k=1}^n r_k(u)\lambda_k^{1/2}\nabla\mathcal{E}_+(\lambda_k)\mathbf{F}^3\|_{L_q(\mathbb{R}^N_+)}^q du$$
$$\leq C_q(\sigma + C_{\sigma,M_2}\lambda_0^{-1/2})^q \int_0^1 \| \sum_{k=1}^n r_k(u)\mathbf{F}^3\|_{\mathcal{X}_q^3(\mathbb{R}^N)}\,du.$$

Analogously, we can estimate the $\mathcal{R}$-bound of any other terms, and therefore, we have (85).

Recalling (55) and $F_\lambda^3\mathbf{G}^3 = (\mathbf{g}_+, \mathbf{g}_-, \lambda^{1/2}\mathbf{h}, \nabla\mathbf{h})$, we see that $\|F_\lambda^3\mathbf{G}^3\|_{\mathcal{X}_q^3(\mathbb{R}^N)}$ gives equivalent norms of $X_q^3(\mathbb{R}^N)$. By (84) and (85), we have:

$$\|\mathbf{R}(\lambda)\mathbf{G}^3\|_{\mathcal{X}_q^3(\mathbb{R}^N)} \leq C(\sigma + M_1 + C_{\sigma,M_2}\lambda_0^{-1/2})\|F_\lambda^3\mathbf{G}^3\|_{\mathcal{X}_q^3(\mathbb{R}^N)}$$

for any $\mathbf{G}^3 = (\mathbf{g}_+, \mathbf{g}_-, \mathbf{h}) \in X_q^3(\mathbb{R}^N)$. Thus, choosing σ and M_1 so small and λ_0 so large that $C(\sigma + M_1 + C_{\sigma,M_2}\lambda^{-1/2}) \leq 1/2$, we have:

$$\|\mathbf{R}(\lambda)\|_{\mathcal{L}(X_q(\mathbb{R}^N))} \leq 1/2 \quad \text{for any } \lambda \in \Gamma_{\epsilon,\lambda_0},$$

and therefore, $(I - \mathbf{R}(\lambda))^{-1}$ exists in $\mathcal{L}(X_q^3(\mathbb{R}^N))$. If we set $\mathbf{v}_\pm = \mathcal{E}_\pm(\lambda)F_\lambda^3(I-\mathbf{R}(\lambda))^{-1}\mathbf{G}^3$, with $\mathbf{G} = (\mathbf{G}_+, \mathbf{G}_-, \mathbf{H})$, then in view of (82), $\mathbf{v}_\pm$ solve (81). Moreover, using (84), we have $F_\lambda^3(I-\mathbf{R}(\lambda))^{-1} = (I - F_\lambda^3\mathcal{R}(\lambda))^{-1}F_\lambda^3$, and so, defining operators $\tilde{\mathcal{E}}_\pm(\lambda)$ by $\tilde{\mathcal{E}}_\pm(\lambda) = \mathcal{E}_\pm(\lambda)(I - F_\lambda^3\mathcal{R}(\lambda))^{-1}$ and using (85) and Theorem 9, we see that $\mathbf{v}_\pm = \tilde{\mathcal{E}}_\pm(\lambda)F_\lambda^3\mathbf{G}^3$ with $\mathbf{G}^3 = (\mathbf{G}_+, \mathbf{G}_-, \mathbf{H})$ is a unique solution to (81), and:

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q^3(\mathbb{R}^N), L_q(\mathbb{R}^N_\pm)^{\tilde{N}})}(\{(\tau\partial_\tau)^\ell G_\lambda\tilde{\mathcal{E}}_\pm(\lambda) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \leq C \quad (\ell = 0, 1).$$

Since $\mathbf{u}_\pm = (\mathcal{A}^{-1})_{-1}[\mathbf{v}_\pm \circ \Phi^{-1}]$ is a unique solution to (71), we have Theorem 12 by the pullback. □

Next, for the compressible part, we consider the following two problems.

$$\lambda\mathbf{v}_+ - \gamma_{0+}^{-1}(\mathrm{Div}\,\mathbf{S}_+(\mathbf{v}_+) + \delta\nabla(\gamma_{3+}\mathrm{div}\,\mathbf{v}_+)) = \mathbf{g}_+ \quad \text{in } D_+, \quad \mathbf{v}_+|_S = 0; \tag{86}$$
$$\lambda\mathbf{v}_+ - \gamma_{0+}^{-1}(\mathrm{Div}\,\mathbf{S}_+(\mathbf{v}_+) + \delta\nabla(\gamma_{3+}\mathrm{div}\,\mathbf{v}_+)) = \mathbf{g}_+ \quad \text{in } \mathbb{R}^N. \tag{87}$$

Since we know the existence of $\mathcal{R}$-bounded solution operators in $\mathbb{R}^N_+$ and $\mathbb{R}^N$ (cf. Enomoto and Shibata [18]), in a similar fashion to the proof of Theorem 9, we can prove the following theorem (cf. von Below, Enomoto and Shibata [17]).

Theorem 13. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$, and $r \geq \max(q, q')$. Then, there exist constants $M_1 \in (0,1)$ and $\lambda_0 = \lambda_{M_2} \geq 1$ such that the following two assertions hold:*

(1) *There exists an operator family* $\mathcal{E}_{D+}(\lambda) \in \mathrm{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(L_q(D_+)^N, W^2_q(D_+)^N))$ *such that for any* $\lambda \in \Gamma_{\epsilon,\lambda_0}$ *and* $\mathbf{g}_+ \in L_q(D_+)^N$, $\mathbf{v}_+ = \mathcal{E}_{D+}(\lambda)\mathbf{g}_+$ *is a unique solution to* (86), *and:*

$$\mathcal{R}_{\mathcal{L}(L_q(D_+)^N, W^{2-j}_q(D_+)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{E}_{D+}(\lambda)) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \leq C \quad (\ell = 0,1, j = 0,1,2).$$

(2) *There exists an operator family* $\mathcal{E}_{0+}(\lambda) \in \mathrm{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(L_q(\mathbb{R}^N)^N, W^2_q(\mathbb{R}^N)^N))$ *such that for any* $\lambda \in \Gamma_{\epsilon,\lambda_0}$ *and* $\mathbf{g}_+ \in L_q(\mathbb{R}^N)^N$, $\mathbf{v}_+ = \mathcal{E}_{0+}(\lambda)\mathbf{g}_+$ *is a unique solution to* (87), *and:*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^N, W^{2-j}_q(\mathbb{R}^N)^N)}(\{(\tau\partial_\tau)^\ell\mathcal{E}_{0+}(\lambda) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \leq C \quad (\ell = 0,1, j = 0,1,2).$$

Finally, for the incompressible part, we consider the following two problems:

$$\lambda\mathbf{v}_- - \rho_{0-}^{-1}(\mathrm{Div}\,\mathbf{S}_-(\mathbf{v}_-) - \nabla K_F(\mathbf{v}_-)) = \mathbf{g}_- \quad \text{in } D_-, \quad (S_-(\mathbf{v}_-) - K_F(\mathbf{v}_-)\mathbf{I})\mathbf{n}_+|_S = \mathbf{h}_-|_S, \tag{88}$$

$$\lambda\mathbf{v}_- - \rho_{0-}^{-1}(\mathrm{Div}\,\mathbf{S}_-(\mathbf{v}_-) - \nabla K_0(\mathbf{v}_-)) = \mathbf{g}_-, \quad \text{in } \mathbb{R}^N, \tag{89}$$

where $K_F(\mathbf{v}_-)$ and $K_0(\mathbf{v}_-)$ are unique solutions to the following variational problems:

$$(\nabla K_F(\mathbf{v}_-), \nabla\varphi)_{D_-} = (\mathbf{S}_-(\mathbf{v}_-) - \nabla\mathrm{div}\,\mathbf{v}_-, \nabla\varphi)_{D_-} \quad \text{for any } \varphi \in \hat{W}^1_{q',0}(D_-)$$

subject to $K_F(\mathbf{v}_-) =< \mathbf{S}_-(\mathbf{v}_-)\mathbf{n}_+, \mathbf{n}_+ > -\mathrm{div}\,\mathbf{v}_-$ on S, and:

$$(\nabla K_0(\mathbf{v}_-), \nabla\varphi)_{\mathbb{R}^N} = (\mathbf{S}_-(\mathbf{v}_-) - \nabla\mathrm{div}\,\mathbf{v}_-, \nabla\varphi)_{\mathbb{R}^N} \quad \text{for any } \varphi \in \hat{W}^1_{q',0}(\mathbb{R}^N),$$

respectively. Since we know the existence of $\mathcal{R}$-bounded solution operators in $\mathbb{R}^N_+$ and $\mathbb{R}^N$ (cf. Shibata and Shimizu [19]), in a similar fashion to the proof of Theorem 9, we can prove the following theorem (cf. Shibata [16]).

Theorem 14. *Let* $1 < q < \infty$ *and* $0 < \epsilon < \pi/2$. *Then, there exist constants* $M_1 \in (0,1)$ *and* $\lambda_0 = \lambda_{M_2} \geq 1$ *such that the following two assertions hold.*
(1) *Let* $X^5_q(D_-)$ *and* $\mathcal{X}^5_q(D_-)$ *be sets defined by:*

$$X^5_q(D_-) = \{(\mathbf{g}_-, \mathbf{h}_-) \mid \mathbf{g}_- \in L_q(D_-)^N,\ \mathbf{h}_- \in W^1_q(D_-)^N\},$$
$$\mathcal{X}^5_q(D_-) = \{\mathbf{F}^5 = (\mathbf{F}_2, \mathbf{F}_5, \mathbf{F}_6) \mid \mathbf{F}_2, \mathbf{F}_5 \in L_q(D_-)^N,\ \mathbf{F}_6 \in L_q(D_-)^{N^2}\}.$$

Then, there exists an operator family $\mathcal{E}_{D-}(\lambda) \in \mathrm{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(X^5_q(D_-), W^2_q(D_-)^N))$ *such that for any* $\lambda \in \Gamma_{\epsilon,\lambda_0}$ *and* $\mathbf{G}^5 = (\mathbf{g}_-, \mathbf{h}_-) \in X^5_q(D_-)$, $\mathbf{v}_- = \mathcal{E}_{D-}(\lambda)F^5_\lambda\mathbf{G}^5$ *is a unique solution to* (88), *and:*

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}^5_q(D_-)^N, W^{2-j}_q(D_-)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{E}_{D-}(\lambda)) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \leq C \quad (\ell = 0,1, j = 0,1,2).$$

Here, $F^5\mathbf{G}^5 = (\mathbf{g}_-, \lambda^{1/2}\mathbf{h}_-, \nabla\mathbf{h}_-)$.
(2) *There exists an operator family* $\mathcal{E}_{0-}(\lambda) \in \mathrm{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(L_q(\mathbb{R}^N)^N, W^2_q(\mathbb{R}^N)^N))$ *such that for any* $\lambda \in \Gamma_{\epsilon,\lambda_0}$ *and* $\mathbf{g}_- \in L_q(\mathbb{R}^N)^N$, $\mathbf{v}_- = \mathcal{E}_{0-}(\lambda)\mathbf{g}_-$ *is a unique solution to* (89), *and:*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^N, W^{2-j}_q(\mathbb{R}^N)^N)}(\{(\tau\partial_\tau)^\ell\mathcal{E}_{0-}(\lambda) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \leq C \quad (\ell = 0,1, j = 0,1,2).$$

5. A Proof of Theorem 8

5.1. Some Preparations for the Proof of Theorem 8

We first give several properties of the uniform $W^{2-1/r}_r$ domain in the following proposition.

Proposition 4. *Let $N < r < \infty$, and let $\Omega_\pm$ be uniform $W_r^{2-1/r}$ domains in $\mathbb{R}^N$. Let M_1 be the number given in (67). Then, there exist constants $M_2 > 0$, $0 < d^i < 1$ $(i = 1, \ldots, 5)$, at most countably many N-vectors of functions $\Phi_j^i \in W_r^2(\mathbb{R}^N)^N$ $(i = 1, \ldots, 5,\ j \in \mathbb{N})$, and points $x_j^1 \in \Gamma$, $x_j^2 \in \Gamma_+$, $x_j^3 \in \Gamma_-$, $x_j^4 \in \Omega_+$ and $x_j^5 \in \Omega_-$ such that the following assertions hold:*

(i) *The maps: $\mathbb{R}^N \ni x \mapsto \Phi_j^i(x) \in \mathbb{R}^N$ $(i = 1,2,3,\ j \in \mathbb{N})$ are bijective such that $\nabla\Phi_j^i = \mathcal{A}_j^i + B_j^i$, $\nabla(\Phi_j^i)^{-1} = \mathcal{A}_{j,-1}^i + B_{j,-1}^i$, where $\mathcal{A}_j^i$ and $\mathcal{A}_{j,-1}^i$ are $N \times N$ constant orthonormal matrices, and B_j^i and $B_{j,-1}^i$ are $N \times N$ matrices of $W_r^1(\mathbb{R}^N)$ functions that satisfy the conditions: $\|(B_j^i, B_{j,-1}^i)\|_{L_\infty(\mathbb{R}^N)} \le M_1$ and $\|\nabla(B_j^i, B_{j,-1}^i)\|_{L_r(\mathbb{R}^N)} \le M_2$.*

(ii) *$\Omega = \left\{\bigcup_{i=1,2,3}\bigcup_{j=1}^\infty(\Phi_j^i(H_i) \cap B_{d^i}(x_j^i))\right\} \cup \left\{\bigcup_{i=4,5}\bigcup_{j=1}^\infty B_{d^i}(x_j^i)\right\}$ with $H_1 = \mathbb{R}^N$, $H_2 = \mathbb{R}^N_+$ and $H_3 = \mathbb{R}^N_-$, $\Phi_j^1(\mathbb{R}^N) \cap B_{d^1}(x_j^1) = \Omega \cap B_{d^1}(x_j^1)$, $\Phi_j^2(\mathbb{R}^N_+) \cap B_{d^2}(x_j^2) = \Omega_+ \cap B_{d^2}(x_j^2)$, $\Phi_j^3(\mathbb{R}^N_-) \cap B_{d^3}(x_j^3) = \Omega_- \cap B_{d^3}(x_j^3)$, $B_{d^4}(x_j^4) \subset \Omega_+$, $B_{d^5}(x_j^5) \subset \Omega_-$, and $\Phi_j^i(\mathbb{R}^N_0) \cap B_{d^i}(x_j^i) = \Gamma_i \cap B_{d^i}(x_j^i)$ $(i = 1,2,3)$. Here and in the following, we set $\Gamma_1 = \Gamma$, $\Gamma_2 = \Gamma_+$, and $\Gamma_3 = \Gamma_-$ for notational convenience.*

(iii) *There exist C^∞ functions ζ_j^i and $\tilde\zeta_j^i$ $(i = 1, \ldots, 5,\ j \in \mathbb{N})$ such that $\|(\zeta_j^i, \tilde\zeta_j^i)\|_{W^2_\infty(\mathbb{R}^N)} \le c_0$, $0 \le \zeta_j^i, \tilde\zeta_j^i \le 1$, $\operatorname{supp}\zeta_j^i, \operatorname{supp}\tilde\zeta_j^i \subset B_{d^i}(x_j^i)$, $\tilde\zeta_j^i = 1$ on $\operatorname{supp}\zeta_j^i$, $\sum_{i=1,\ldots,5}\sum_{j=1}^\infty \zeta_j^i = 1$ on $\overline{\Omega}$, $\sum_{j=1}^\infty \zeta_j^i = 1$ on Γ_i $(i = 1,2,3)$.*

(iv) *There exists a natural number $L \ge 2$ such that any $L+1$ distinct sets of $\{B_{d^i}(x_j^i) \mid i = 1, \ldots, 5,\ j \in \mathbb{N}\}$ have an empty intersection.*

Proof. For a detailed proof, we refer to Enomoto and Shibata ([18] Appendix). □

In the following, choosing M_2 larger if necessary, we may assume that $\|\nabla\gamma_{k+}\|_{L_r(B_{d^i}(x_j^i)\cap\Omega_+)} \le M_2$ $(k = 0,3, i = 1,3,5, j \in \mathbb{N})$, which is a weaker assumption than the last condition in (23). Since functions in W_r^1 are Hölder continuous of order α with $0 < \alpha < 1 - N/r$, as follows from Sobolev's imbedding theorem, we have $|\gamma_{k+}(x) - \gamma_{k+}(x_j^i)| \le C\|\gamma_{k+}\|_{W_r^1(B_{d^i}(x_j^i))}|x - x_j^i|^\alpha$ for any $x \in B_{d^i}(x_j^i)$ $(k = 0,3)$ with some constant C independent of j, and so choosing $d^i > 0$ smaller and more points x_j^i suitably, we may assume that $|\gamma_{k+}(x) - \gamma_{k+}(x_j^i)| \le M_1$ for $x \in B_{d^i}(x_j^i)$ $(k = 0,3, i = 1,3,5, j \in \mathbb{N})$. Here and in the following, constants denoted by C are independent of $j \in \mathbb{N}$. In addition, in view of (68), we may assume that each unit outward normal $\mathbf{n}_j^i$ to $\Phi_j^i(\mathbb{R}^N_0)$ $(i = 1,3,4, j \in \mathbb{N})$ is defined on $\mathbb{R}^N$ and satisfies the conditions: $\|\mathbf{n}_j^i\|_{L_\infty(\mathbb{R}^N)} = 1$ and $\|\nabla\mathbf{n}_j^i\|_{L_r(\mathbb{R}^N)} \le CM_2$. Note that $\mathbf{n} = \mathbf{n}_j^1$ on $B_{d^1}(x_j^1) \cap \Gamma$ and $\mathbf{n}_- = \mathbf{n}_j^4$ on $B_{d^4}(x_j^4) \cap \Gamma_-$.

Summing up, from now on, we may assume that:

$$
\begin{aligned}
&\|\gamma_{k+} - \gamma_{k+}(x_j^i)\|_{L_\infty(\Omega\cap B_{d^i}(x_j^i))} \le CM_1, \ \|\nabla\gamma_{k+}\|_{L_r(B_{d^i}(x_j^i))} \le M_2 \ (k = 0,3), \\
&\|\nabla\mathbf{n}\|_{L_r(B_{d^1}(x_j^1)\cap\Omega)} \le M_2, \ \|\nabla\mathbf{n}_-\|_{L_r(B_{d^4}(x_j^4)\cap\Omega)} \le M_2,
\end{aligned} \tag{90}
$$

and that both $\mathbf{n}$ and $\mathbf{n}_-$ are defined on $\mathbb{R}^N$ with $\|\mathbf{n}\|_{L_\infty(\Omega)} = 1$ and $\|\mathbf{n}_-\|_{L_\infty(\Omega_-)} = 1$, respectively.

Next, we prepare two lemmas used to construct a parametrix.

Lemma 6. *Let X be a Banach space and X^* its dual space, while $\|\cdot\|_X$, $\|\cdot\|_{X^*}$ and $<\cdot,\cdot>$ are the norm of X, the norm of X^*, and the duality of X and X^*, respectively. Let $n \in \mathbb{N}$, and for $i = 1, \ldots, n$, let $a_i \in \mathbb{C}$, let $\{f_j^{(i)}\}_{j=1}^\infty$ be sequences in X^*. Let $\{g_j^{(i)}\}_{j=1}^\infty$ and $\{h_j\}_{j=1}^\infty$ be sequences of positive numbers. Assume that there exist maps $\mathcal{N}_j : X \to [0,\infty)$ such that:*

$$
| < f_j^{(i)}, \varphi > | \le M_3 g_j^i \mathcal{N}_j(\varphi) \ \ (i = 1, \ldots, n), \ \ | < \sum_{i=1}^n a_i f_j^{(i)}, \varphi > | \le M_3 h_j \mathcal{N}_j(\varphi)
$$

for any $\varphi \in L_{q'}(D)$ *with some constant* M_3 *independent of* $j \in \mathbb{N}$. *If:*

$$\sum_{j=1}^{\infty}(g_j^{(i)})^q < \infty, \quad \sum_{j=1}^{\infty}(h_j)^q < \infty, \quad \sum_{j=1}^{\infty}\mathcal{N}_j(\varphi)^{q'} \leq M_4^{q'}\|\varphi\|_X^{q'},$$

then the infinite sum $f^{(i)} = \sum_{j=1}^{\infty} f_j^{(i)}$ *exists in the strong topology of* X^* *and:*

$$\|f^{(i)}\|_{X^*} \leq M_3 M_4 \left(\sum_{j=1}^{\infty}(g_j^{(i)})^q\right)^{1/q}, \quad \|\sum_{i=1}^{n} a_i f^{(i)}\|_{X^*} \leq M_3 M_4 \left(\sum_{j=1}^{\infty}(h_j)^q\right)^{1/q}.$$

Lemma 7. *Let* D *be a domain in* $\mathbb{R}^N$, *and assume that there exists at most countably many covering* $\{B_j\}_{j=1}^{\infty}$ *such that* $D \subset \bigcup_{j=1}^{\infty} B_j$ *and* $\{B_j\}_{j=1}^{\infty}$ *has a finite intersection property of order* L, *that is any* $L+1$ *distinct sets of* $\{B_j\}_{j=1}^{\infty}$ *have an empty intersection. Let* $1 < q < \infty$. *Then, the following assertions hold.*

(i) *There exists a constant* $C_{q,L}$ *such that:*

$$\Big(\sum_{j=1}^{\infty}\|f\|_{L_q(D\cap B_j)}^q\Big)^{1/q} \leq C_{q,L}\|f\|_{L_q(D)} \quad \text{for any } f \in L_q(D).$$

(ii) *Let* $m \in \mathbb{N}_0$. *Let* $\{f_j\}_{j=1}^{\infty}$ *be a sequence in* $W_q^m(D)$, *and let* $\{g_j^{(\ell)}\}_{j=1}^{\infty}$ $(\ell = 0, 1, \ldots, m)$ *be sequences of positive numbers. Assume that:*

$$\sum_{j=1}^{\infty}(g_j^{(\ell)})^q < \infty, \quad |(\nabla^\ell f_j, \varphi)_D| \leq M_3 g_j^\ell \|\varphi\|_{L_{q'}(D\cap B_j)} \text{ for any } \varphi \in L_{q'}(D) \text{ and } \ell = 0, 1, \ldots, m$$

with some constant M_3 *independent of* $j \in \mathbb{N}$. *Then,* $f = \sum_{j=1}^{\infty} f_j$ *exists in the strong topology of* $W_q^m(D)$ *and:*

$$\|\nabla^\ell f\|_{L_q(D)} \leq C_{q,L} M_3 (\sum_{j=1}^{\infty}(g_j^{(\ell)})^q)^{1/q}.$$

Remark 15. *To prove Lemma 6, we consider the difference of finite sum* $\sum_{j=1}^{N} f_j^{(i)}$ *and use the Hölder inequality for the sequence. The assertion* (i) *of Lemma 7 follows immediately from the property of the Lebesgue measure and suitable decomposition of covering sets* $\{B_j\}_{j=1}^{\infty}$, *and the assertion* (ii) *of Lemma 7 follows from Lemma 6 and Lemma 7* (i).

5.2. Local Solutions

In the following, we write $B_{d^i}(x_j^i) = B_j^i$ $(i = 1, \ldots, 5)$, $\mathcal{H}_j^1 = \Phi_j^1(\mathbb{R}^N)$, $\mathcal{H}_{\pm j}^1 = \Phi_j^1(\mathbb{R}_\pm^N)$, $\mathcal{H}_j^2 = \Phi_j^2(\mathbb{R}_+^N)$, $\mathcal{H}_j^3 = \Phi_j^3(\mathbb{R}_-^N)$, $\mathcal{H}^4 = \mathcal{H}^5 = \mathbb{R}^N$, $\Gamma_j^1 = \Phi_j^1(\mathbb{R}_0^N)$, $\Gamma_j^2 = \Phi_j^2(\mathbb{R}_0^N)$, and $\Gamma_j^3 = \Phi_j^3(\mathbb{R}_0^N)$ for short. $\mathbf{n}_j^1$ denote the unit outward normals to Γ_j^1 pointing from $\mathcal{H}_{-j}^1$ to $\mathcal{H}_{+j}^1$, and $\mathbf{n}_j^3$ denote the unit outward normals to Γ_j^3 for $j \in \mathbb{N}$. In view of (90), we define the functions γ_{jk}^i by:

$$\gamma_{jk}^i(x) = (\gamma_{k+}(x) - \gamma_{k+}(x_j^i))\tilde{\zeta}_j^i(x) + \gamma_{k+}^i(x_j^i)$$

for $k = 0, 3$, $i = 1, 2, 4$, and $j \in \mathbb{N}$. Noting that $0 \leq \tilde{\zeta}_j^i \leq 1$ and $\|\nabla\zeta_j^i\|_{L_\infty(\mathbb{R}^N)} \leq c_0$, by (90) and (23):

$$\begin{aligned}&\rho_{0+}/2 \leq \gamma_{j0}^i(x) \leq 2\rho_{0+}, \ \ 0 \leq \gamma_{j3}^i(x) \leq (\rho_{2+})^2 \ \ (x \in \{\mathcal{H}_{+j}^1, \mathcal{H}_j^2, \mathcal{H}_j^4\}),\\ &\|\gamma_{jk}^i(\cdot) - \gamma_{jk}^i(x_j^i)\|_{L_\infty(\mathcal{H}_j^i)} \leq CM_1, \ \ \|\nabla\gamma_{jk}^i\|_{L_r(\mathcal{H}_j^i)} \leq M_2\end{aligned} \tag{91}$$

for $k = 0, 3$, $i = 1, 2, 4$, and $j \in \mathbb{N}$. In addition, we have:

$$\gamma^i_{jk}(x) = \gamma_{k+}(x) \quad (x \in \operatorname{supp}\zeta^i_j, k = 0,3, i = 1,2,4, j \in \mathbb{N}), \tag{92}$$

because $\tilde{\zeta}^i_j = 1$ on $\operatorname{supp}\zeta^i_j$. For $\mathbf{G} = (\mathbf{g}_+, \mathbf{g}_-, \mathbf{h}, \mathbf{h}_-) \in X_q(\Omega)$, we consider the equations:

$$\begin{aligned} \lambda\mathbf{v}^1_{+j} - (\gamma^1_{j0})^{-1}[\operatorname{Div}\mathbf{S}_+(\mathbf{v}^1_{+j}) + \delta\nabla(\gamma^1_{3j}\operatorname{div}\mathbf{v}^1_{+j})] &= \tilde{\zeta}^1_j\mathbf{g}_+ \quad &&\text{in } \mathcal{H}^1_{+j}, \\ \lambda\mathbf{v}^1_{-j} - \rho_{0-}^{-1}[\operatorname{Div}\mathbf{S}_-(\mathbf{v}^1_{-j}) - \nabla K^1_{-j}(\mathbf{v}^1_{+j}, \mathbf{v}^1_{-j})] &= \tilde{\zeta}^1_j\mathbf{g}_- \quad &&\text{in } \mathcal{H}^1_{-j}, \\ (\mathbf{S}_+(\mathbf{v}^1_{+j}) + \delta\gamma^1_{j3}\operatorname{div}\mathbf{v}^1_{+j}\mathbf{I})\mathbf{n}^1_j|_{\Gamma^1_j+0} - (\mathbf{S}_-(\mathbf{v}^1_{-j}) - K^1_{-j}(\mathbf{v}^1_{+j}, \mathbf{v}^1_{-j})\mathbf{I})\mathbf{n}^1_j|_{\Gamma^1_j-0} &= \tilde{\zeta}^1_j\mathbf{h}|_{\Gamma^1_j}, \\ \mathbf{v}^1_{+j}|_{\Gamma^1_j+0} &= \mathbf{v}^1_{-j}|_{\Gamma^1_j-0}. \end{aligned} \tag{93}$$

Here, $v = K^1_{-j}(\mathbf{v}^1_{+j}, \mathbf{v}^1_{-j})$ is a unique solution to the variational problem:

$$(\nabla v, \nabla\varphi)_{\mathcal{H}^1_{-j}} = \rho_{0-}^{-1}(\operatorname{Div}\mathbf{S}_-(\mathbf{v}^1_{-j}) - \nabla\operatorname{div}\mathbf{v}^1_{-j}, \nabla\varphi)_{\mathcal{H}^1_{-j}} \quad \text{for any } \varphi \in \hat{W}^1_{q',0}(\mathcal{H}^1_{-j}) \tag{94}$$

subject to $v|_{\Gamma^1_j} = (< \mathbf{S}_-(\mathbf{v}^1_{-j})\mathbf{n}^1_j, \mathbf{n}^1_j > -\operatorname{div}\mathbf{v}^1_{-j})|_{\Gamma^1_j-0} - < (\mathbf{S}_+(\mathbf{v}^1_{+j}) - \delta\gamma^1_{j3}\operatorname{div}\mathbf{v}^1_{+j}\mathbf{I})\mathbf{n}^1_j, \mathbf{n}^1_j > |_{\Gamma^1_j+0}$. Here and in the following, $X_q(\Omega)$ and $\mathcal{X}_q(\Omega)$ denote the spaces defined in Theorem 8 in Section 2.3.

Moreover, we consider the following four problems:

$$\begin{aligned} \lambda\mathbf{v}^2_j - (\gamma^1_{j0})^{-1}[\operatorname{Div}\mathbf{S}_+(\mathbf{v}^2_j) + \delta\nabla(\gamma^1_{3j}\operatorname{div}\mathbf{v}^2_j)] &= \tilde{\zeta}^2_j\mathbf{g}_+ \quad \text{in } \mathcal{H}^2_j, \quad \mathbf{v}^2_j|_{\Gamma^2_j} = 0, \\ \lambda\mathbf{v}^4_j - (\gamma^1_{j0})^{-1}[\operatorname{Div}\mathbf{S}_+(\mathbf{v}^4_j) + \delta\nabla(\gamma^1_{3j}\operatorname{div}\mathbf{v}^4_j)] &= \tilde{\zeta}^4_j\mathbf{g}_+ \quad \text{in } \mathcal{H}^4_j, \\ \lambda\mathbf{v}^3_j - \rho_{0-}^{-1}[\operatorname{Div}\mathbf{S}_-(\mathbf{v}^3_j) - \nabla K^3_j(\mathbf{v}^3_j)] &= \tilde{\zeta}^3_j\mathbf{g}_- \quad \text{in } \mathcal{H}^3_j, \quad (\mathbf{S}_-(\mathbf{v}^3_j) - K^3_j(\mathbf{v}^3_j))\mathbf{n}^3_j|_{\Gamma^3_j} = \tilde{\zeta}^3_j\mathbf{h}_-, \\ \lambda\mathbf{v}^5_j - \rho_{0-}^{-1}[\operatorname{Div}\mathbf{S}_-(\mathbf{v}^5_j) - \nabla K^5_j(\mathbf{v}^5_j)] &= \tilde{\zeta}^5_j\mathbf{g}_- \quad \text{in } \mathcal{H}^5_j. \end{aligned} \tag{95}$$

Here, $K^3_j(\mathbf{v}^3_j)$ and $K^5_j(\mathbf{v}^5_j)$ are unique solutions to the variational problem:

$$(\nabla K^3_j(\mathbf{v}^3_j), \nabla\varphi)_{\mathcal{H}^3_j} = \rho_{0-}^{-1}(\operatorname{Div}\mathbf{S}_-(\mathbf{v}^3_j) - \nabla\operatorname{div}\mathbf{v}^3_j, \nabla\varphi)_{\mathcal{H}^4_j} \quad \text{for any } \varphi \in W^1_{q',0}(\mathcal{H}^4_j) \tag{96}$$

subject to $K^4_j(\mathbf{v}^4_j)|_{\Gamma^4_j} = (< \mathbf{S}_-(\mathbf{v}^4_-)\mathbf{n}^4_j, \mathbf{n}^4_j > -\mathbf{v}^4_j)|_{\Gamma^4}$, and the variational problem:

$$(\nabla K^5_j(\mathbf{v}^5_j), \nabla\varphi)_{\mathcal{H}^5_j} = \rho_{0-}^{-1}(\operatorname{Div}\mathbf{S}_-(\mathbf{v}^5_j) - \nabla\operatorname{div}\mathbf{v}^5_j, \nabla\varphi)_{\mathcal{H}^5_j} \quad \text{for any } \varphi \in W^1_{q',0}(\mathcal{H}^5_j). \tag{97}$$

Here and in the following, λ_0 and general constants denoted by C are independent of $i = 1, \ldots, 5$ and $j \in \mathbb{N}$. By Theorem 12 in Section 4, there exist operator families $\mathcal{T}^1_{\pm j}(\lambda) \in \operatorname{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}^3_q(\mathcal{H}^1_{\pm j}), W^2_q(\mathcal{H}^1_{\pm j})^N))$ such that for any $\lambda \in \Gamma_{\epsilon,\lambda_0}$, $\mathbf{v}^1_{\pm j} = \mathcal{T}^1_{\pm j}(\lambda)F^1_\lambda\mathcal{G}^1_j$ are unique solutions to the problem in (93), and:

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}^3_q(\mathcal{H}^1_{\pm j}), W^{2-m}_q(\mathcal{H}^1_{\pm j})^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{m/2}\mathcal{T}^1_{\pm j}(\lambda)) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \leq C \tag{98}$$

for $\ell = 0, 1$ and $m = 0, 1, 2$. Moreover, by Theorem 13 and Theorem 14 in Section 4, there exist operator families $\mathcal{T}^k_j(\lambda) \in \operatorname{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}^3_q(\mathcal{H}^k_j), W^{2-m}_q(\mathcal{H}^k_j)))$ such that for any $\lambda \in \Gamma_{\epsilon,\lambda_0}$ $v^k_j = \mathcal{T}^k_j(\lambda)F^k_\lambda\mathcal{G}^k_j$ are unique solutions of the problems in (95) ($k = 2, 3, 4, 5$), and:

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}^3_q(\mathcal{H}^k_j), W^{2-m}_q(\mathcal{H}^k_j)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{m/2}\mathcal{T}^k_j(\lambda)) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \leq C \tag{99}$$

for $\ell = 0, 1$ and $m = 0, 1, 2$. Here and in the following, we set $\mathcal{X}^3_q(\mathcal{H}^2_j) = X^3_q(\mathcal{H}^2_j) = L_q(\mathcal{H}^2_j)^N$, $\mathcal{X}^3_q(\mathcal{H}^3_j) = \mathcal{X}^5_q(\mathcal{H}^3_j)$, $X^3_q(\mathcal{H}^3_j) = X^5_q(\mathcal{H}^3_j)$ (cf. X^5_q and $\mathcal{X}^5_q$ were given in Theorem 14), $\mathcal{X}^4_q(\mathcal{H}^4_j) = X^4_q(\mathcal{H}^4_j) = L_q(\mathcal{H}^4_j)^N$, $\mathcal{H}^5_q(\mathcal{H}^5_j) = X^5_q(\mathcal{H}^5_j) = L_q(\mathcal{H}^5_j)^N$. Moreover, we set

$F^1_\lambda \mathcal{G}^1_j = \tilde\zeta^1_j(\mathbf{g}_+, \mathbf{g}_-, \lambda^{1/2}\mathbf{h}, \mathbf{h}))$, $F^2_\lambda \mathcal{G}^2_j = \tilde\zeta^2_j \mathbf{g}_+$, $F^3_\lambda \mathcal{G}^3_j = \tilde\zeta^3_j(\mathbf{g}_-, \lambda^{1/2}\mathbf{h}_-, \mathbf{h}_-)$, $F^4_\lambda \mathcal{G}^4_j = \tilde\zeta^4_j \mathbf{g}_+$, and $F^5_\lambda \mathcal{G}^5_j = \tilde\zeta^5_j \mathbf{g}_-$. Since $\mathcal{R}$-boundedness implies the usual boundedness, by (98) and (99),

$$\begin{aligned}
&\sum_\pm (|\lambda| \|\mathbf{v}^1_{\pm j}\|_{L_q(\mathcal{H}^1_{\pm j})} + |\lambda|^{1/2} \|\mathbf{v}^1_{\pm j}\|_{W^1_q(\mathcal{H}^1_{\pm j})} + \|\mathbf{v}^1_{\pm j}\|_{W^2_q(\mathcal{H}^1_{\pm j})}) \\
&\qquad \le C(\sum_\pm \|\tilde\zeta^1_j \mathbf{g}_\pm\|_{L_q(\mathcal{H}^1_j)} + |\lambda|^{1/2}\|\tilde\zeta^1_j \mathbf{h}\|_{L_q(\mathbb{R}^N)} + \|\tilde\zeta^1_j \mathbf{h}\|_{W^1_q(\mathbb{R}^N)}), \\
&|\lambda| \|\mathbf{v}^m_j\|_{L_q(\mathcal{H}^m_j)} + |\lambda|^{1/2}\|\mathbf{v}^m_j\|_{W^1_q(\mathcal{H}^m_j)} + \|\mathbf{v}^m_j\|_{W^2_q(\mathcal{H}^m_j)} \le C\|\tilde\zeta^m_j \mathbf{g}_+\|_{L_q(\mathcal{H}^m_j)} \quad (m = 2,4,5), \\
&|\lambda| \|\mathbf{v}^3_j\|_{L_q(\mathcal{H}^3_j)} + |\lambda|^{1/2}\|\mathbf{v}^3_j\|_{W^1_q(\mathcal{H}^3_j)} + \|\mathbf{v}^3_j\|_{W^2_q(\mathcal{H}^3_j)} \\
&\qquad \le C(\|\tilde\zeta^3_j \mathbf{g}_+\|_{L_q(\mathcal{H}^3_j)} + |\lambda|^{1/2}\|\tilde\zeta^3_j \mathbf{h}_-\|_{L_q(\mathcal{H}^3_j)} + \|\tilde\zeta^3_j \mathbf{h}_-\|_{W^1_q(\mathcal{H}^3_j)}).
\end{aligned} \tag{100}$$

for any $\lambda \in \Gamma_{\epsilon,\lambda_0}$, because $|\lambda| \ge \lambda_0 \ge 1$.

5.3. Construction of Parametrices

For $\mathbf{G} = (\mathbf{g}_+, \mathbf{g}_-, \mathbf{h}, \mathbf{h}_-) \in X_q(\Omega)$, we define parametrices $\mathbf{U}_\pm(\lambda)$ by:

$$\mathbf{U}_+(\lambda)\mathbf{G} = \sum_{j=1}^\infty \zeta^1_j \mathbf{v}^1_{+j} + \sum_{i=2,4}\sum_{j=1}^\infty \zeta^i_j \mathbf{v}^i_j, \quad \mathbf{U}_-(\lambda)\mathbf{G} = \sum_{j=1}^\infty \zeta^1_j \mathbf{v}^1_{-j} + \sum_{i=3,5}\sum_{j=1}^\infty \zeta^i_j \mathbf{v}^i_j \tag{101}$$

Set $G_\lambda \mathbf{v} = (\lambda \mathbf{v}, \lambda^{1/2}\bar\nabla \mathbf{v}, \bar\nabla^2 \mathbf{v})$, where $\bar\nabla \mathbf{v} = (\nabla \mathbf{v}, \mathbf{v})$ and $\bar\nabla^2 \mathbf{v} = (\nabla^2 \mathbf{v}, \nabla \mathbf{v}, \mathbf{v})$, and $B^i_j = B_{d^i}(x^i_j)$ for notational simplicity. By (100),

$$\begin{aligned}
&|(G_\lambda(\zeta^1_j \mathbf{v}^1_{\pm j}), \varphi_\pm)_{\Omega_\pm}| \le C(\|\mathbf{g}_\pm\|_{L_q(\Omega_\pm \cap B^1_j)} + |\lambda|^{1/2}\|\mathbf{h}\|_{L_q(\Omega \cap B^1_j)} + \|\mathbf{h}\|_{W^1_q(\Omega \cap B^1_j)})\|\varphi\|_{L_{q'}(\Omega_\pm \cap B^i_j)}, \\
&|(G_\lambda(\zeta^i_j \mathbf{v}^i_j), \varphi_+)_{\Omega_+}| \le C\|\mathbf{g}_+\|_{L_q(\Omega_+ \cap B^i_j)}\|\varphi\|_{L_{q'}(\Omega_+ \cap B^i_j)} \quad (i = 2,4), \\
&|(G_\lambda(\zeta^3_j \mathbf{v}^3_j), \varphi_-)_{\Omega_-}| \le C(\|\mathbf{g}_-\|_{L_q(\Omega_- \cap B^3_j)} + |\lambda|^{1/2}\|\mathbf{h}_-\|_{L_q(\Omega_- \cap B^3_j)} + \|\mathbf{h}_-\|_{W^1_q(\Omega_- \cap B^3_j)})\|\varphi\|_{L_{q'}(\Omega_- \cap B^3_j)} \\
&|(G_\lambda(\zeta^5_j \mathbf{v}^5_j), \varphi_-)_{\Omega_-}| \le C\|\mathbf{g}_-\|_{L_q(\Omega_- \cap B^5_j)}\|\varphi\|_{L_{q'}(\Omega_- \cap B^5_j)},
\end{aligned}$$

for any $\varphi \in L_{q'}(\Omega_\pm)^N$, and so, by Lemma 6 and Lemma 7, the infinite sums in (101) exist in the strong topology of $W^2_q(\Omega_\pm)^N$ and $\mathbf{U}_\pm(\lambda) \in \mathrm{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(X_q(\Omega), W^2_q(\Omega_\pm)^N))$. By (37), (42), (92), (93), (95)–(97), and (101), setting $\mathbf{w}_\pm = \mathbf{U}_\pm(\lambda)\mathbf{G}$, we have:

$$\begin{aligned}
\lambda \mathbf{w}_+ - \gamma_{0+}^{-1}(\mathrm{Div}\, \mathbf{S}_+(\mathbf{w}_+) + \delta\nabla(\gamma_{3+}\mathrm{div}\, \mathbf{w}_+)) &= \mathbf{g}_+ - \mathbf{R}^1_+(\lambda)\mathbf{G} && \text{in } \Omega_+, \\
\lambda \mathbf{w}_- - \rho_{0-}^{-1}(\mathrm{Div}\, \mathbf{S}_-(\mathbf{w}_-) - \nabla K(\mathbf{w}_+, \mathbf{w}_-)) &= \mathbf{g}_- - (\mathbf{R}^1_-(\lambda)\mathbf{G} - L(\lambda)\mathbf{G}) && \text{in } \Omega_+, \\
(\mathbf{S}_+(\mathbf{w}_+) + \delta\gamma_{3+}\mathrm{div}\, \mathbf{w}_+\mathbf{I})\mathbf{n}|_{\Gamma+0} - (\mathbf{S}_-(\mathbf{w}_-) - K(\mathbf{w}_+, \mathbf{w}_-)\mathbf{I})\mathbf{n}|_{\Gamma-0} &= \mathbf{h} - \mathbf{R}^2(\lambda)\mathbf{G}, \\
\mathbf{w}_+|_{\Gamma+0} = \mathbf{w}_-|_{\Gamma-0}, \quad (\mathbf{S}_-(\mathbf{w}_-) - K(\mathbf{w}_+, \mathbf{w}_-)\mathbf{I})\mathbf{n}_-|_{\Gamma_-} &= \mathbf{h}_- - \mathbf{R}^3(\lambda)\mathbf{G},
\end{aligned} \tag{102}$$

where we set:

$$
\begin{aligned}
\mathbf{R}^1_+(\lambda)\mathbf{G} &= \sum_{j=1}^\infty \gamma_{0+}^{-1}[\mathrm{Div}\,\mathbf{S}_+(\zeta^1_j\mathbf{v}^1_{+j}) - \zeta^1_j\mathrm{Div}\,\mathbf{S}_+(\mathbf{v}^1_{+j}) + \delta\{\nabla(\gamma_{3+}\mathrm{div}\,(\zeta^1_j\mathbf{v}^1_{+j})) - \zeta^1_j\nabla(\gamma_{3+}\mathrm{div}\,\mathbf{v}^1_{+j})\}] \\
&\quad + \sum_{i=2,4}\sum_{j=1}^\infty \gamma_{0+}^{-1}[\mathrm{Div}\,\mathbf{S}_+(\zeta^i_j\mathbf{v}^i_j) - \zeta^i_j\mathrm{Div}\,\mathbf{S}_+(\mathbf{v}^i_j) + \delta\{\nabla(\gamma_{3+}\mathrm{div}\,(\zeta^i_j\mathbf{v}^i_j)) - \zeta^i_j\nabla(\gamma_{3+}\mathrm{div}\,\mathbf{v}^i_j)\}] \\
\mathbf{R}^1_-(\lambda)\mathbf{G} &= \sum_{j=1}^\infty \rho_{0-}^{-1}[\mathrm{Div}\,\mathbf{S}_-(\zeta^1_j\mathbf{v}^1_{-j}) - \zeta^1_j\mathrm{Div}\,\mathbf{S}_-(\mathbf{v}^1_{-j})] + \sum_{i=3,5}\sum_{j=1}^\infty \rho_{0-}^{-1}[\mathrm{Div}\,\mathbf{S}_-(\zeta^i_j\mathbf{v}^i_j) - \zeta^i_j\mathrm{Div}\,\mathbf{S}_-(\mathbf{v}^1_j)] \\
L(\lambda)\mathbf{G} &= \nabla K(\mathbf{U}_+(\lambda)\mathbf{G},\mathbf{U}_-(\lambda)\mathbf{G}) - \sum_{j=1}^\infty\{\zeta^1_j\nabla K^1_j(\mathbf{v}^1_{+j},\mathbf{v}^1_{-j}) - \zeta^3_j\nabla K^3_j(\mathbf{v}^3_j) + \zeta^5_j\nabla K^5_j(\mathbf{v}^5_j)\} \qquad (103) \\
\mathbf{R}^2(\lambda)\mathbf{G} &= -\sum_{j=1}^\infty\{\mathcal{T}_{\mathbf{n}^1_j}[\mathrm{Ext}^-[\mathbf{S}_+(\zeta^1_j\mathbf{v}^1_{+j}) - \zeta^1_j\mathbf{S}_+(\mathbf{v}^1_{+j})]\mathbf{n}^1_j] - \mathcal{T}_{\mathbf{n}^1_j}[\mathrm{Ext}^+[\mathbf{S}_-(\zeta^1_j\mathbf{v}^1_{-j}) - \zeta^1_j\mathbf{S}_-(\mathbf{v}^1_{-j})]\mathbf{n}^1_j]\} \\
&\quad - \sum_{j=1}^\infty\{\mathrm{div}\,(\zeta^1_j\mathbf{v}^1_{-j}) - \zeta^1_j\mathrm{div}\,\mathbf{v}^1_{-j}\}, \\
\mathbf{R}^3(\lambda)\mathbf{G} &= -\sum_{j=1}^\infty \mathcal{T}_{\mathbf{n}^3_j}[(\mathbf{S}_-(\zeta^3_j\mathbf{v}^3_j) - \zeta^3_j\mathbf{S}_-(\mathbf{v}^3_j))\mathbf{n}^3_j] + \sum_{j=1}^\infty\{\mathrm{div}\,(\zeta^3_j\mathbf{v}^3_j) - \zeta^3_j\mathrm{div}\,\mathbf{v}^3_j\}.
\end{aligned}
$$

Finally, we construct $\mathcal{R}$-bounded solution operators that represent $\mathbf{U}_\pm(\lambda)\mathbf{G}$. For $\mathbf{F} = (\mathbf{F}_1,\ldots,\mathbf{F}_6) \in \mathcal{X}_q(\Omega)$, we set:

$$
\begin{aligned}
&\mathcal{U}^1_{\pm j}(\lambda)\mathbf{F} = \mathcal{T}^1_{\pm j}(\lambda)\tilde\zeta^1_j(\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_3,\mathbf{F}_4), \qquad \mathcal{U}^i_j(\lambda)\mathbf{F} = \mathcal{T}^i_j(\lambda)\tilde\zeta^i_j\mathbf{F}_1 \quad (i=2,4), \\
&\mathcal{U}^3_j(\lambda)\mathbf{F} = \mathcal{T}^3_j(\lambda)\tilde\zeta^3_j(\mathbf{F}_2,\mathbf{F}_5,\mathbf{F}_6), \qquad \mathcal{U}^5_j(\lambda)\mathbf{F} = \mathcal{T}^5_j(\lambda)\tilde\zeta^5_j\mathbf{F}_2.
\end{aligned} \qquad (104)
$$

Obviously,

$$
\mathbf{v}^i_j = \mathcal{U}^i_j(\lambda)F_\lambda\mathbf{G} \quad (\mathbf{G} = (\mathbf{g}_+,\mathbf{g}_-,\mathbf{h},\mathbf{h}_-) \in X_q(\Omega)), \qquad (105)
$$

where $F_\lambda\mathbf{G} = (\mathbf{g}_+,\mathbf{g}_-,\lambda^{1/2}\mathbf{h},\mathbf{h},\lambda^{1/2}\mathbf{h}_-,\mathbf{h}_-)$. By (98), we have:

$$
\int_0^1 \|\sum_{k=1}^n r_k(u)G_{\lambda_k}\mathcal{U}^i_j(\lambda_k)\mathbf{F}_k\|^q_{L_q(\Omega)}\,du \le C\int_0^1 \|\sum_{k=1}^n r_k(u)\mathbf{F}_k\|^q_{\mathcal{X}_q(\Omega\cap B^i_j)}\,du \qquad (106)
$$

for any $n \in \mathbb{N}$, $\{\lambda_k\}_{k=1}^n \subset \Gamma_{\epsilon,\lambda_0}$ and $\{\mathbf{F}_k\}_{k=1}^n \subset \mathcal{X}_q(\Omega)$, where $\{r_k(u)\}_{k=1}^n$ are the same as in Definition 3. By (98), Lemma 6, and Lemma 7, $\mathcal{U}^1_\pm(\lambda)\mathbf{F} = \sum_{j=1}^\infty \zeta^i_j\mathcal{U}^1_{\pm j}(\lambda)\mathbf{F}$ exists in the strong topology of $W^2_q(\Omega_\pm)^N$. $\mathcal{U}^i(\lambda)\mathbf{F} = \sum_{j=1}^\infty \zeta^i_j\mathcal{U}^i_j(\lambda)\mathbf{F}$ exist in the strong topology of $W^2_q(\Omega_\pm)^N$ with $\Omega_i = \Omega_+$ for $i = 2,4$ and $\Omega_i = \Omega_-$ for $i = 3,5$, and:

$$
\begin{aligned}
&\|\sum_{k=1}^n a_kG_{\lambda_k}\mathcal{U}^1_\pm(\lambda_k)\mathbf{F}_k\|^q_{L_q(\Omega_\pm)} \le C_{q',L}\sum_{j=1}^\infty\|\sum_{k=1}^n a_k\mathbf{F}_k\|^q_{\mathcal{X}_q(\Omega)}, \\
&\|\sum_{k=1}^n a_kG_{\lambda_k}\mathcal{U}^i(\lambda_k)\mathbf{F}_k\|^q_{L_q(\Omega_i)} \le C_{q',L}\sum_{j=1}^\infty\|\sum_{k=1}^n a_k\mathbf{F}_k\|^q_{\mathcal{X}_q(\Omega)} \quad (i=2,\ldots,5)
\end{aligned}
$$

for any complex numbers a_k, $\lambda_k \in \Gamma_{\epsilon,\lambda_0}$, and $\mathbf{F}_k \in \mathcal{X}_q(\Omega)$ ($k = 1,\ldots,n, n \in \mathbb{N}$). Setting $\mathcal{U}_+(\lambda)\mathbf{F} = \mathcal{U}^1_+(\lambda)\mathbf{F} + \sum_{i=3,5}\mathcal{U}^i(\lambda)\mathbf{F}$ and $\mathcal{U}_-(\lambda)\mathbf{F} = \mathcal{U}^1_-(\lambda)\mathbf{F} + \sum_{i=2,4}\mathcal{U}^i(\lambda)\mathbf{F}$, by the facts that $\mathbf{v}^i_j = \mathcal{U}^i_j(\lambda)F_\lambda\mathbf{G}$ and (106), we have:

$$
\begin{aligned}
&\mathcal{U}_\pm(\lambda) \in \mathrm{Hol}(\Gamma_{\epsilon,\lambda_0},\mathcal{L}(\mathcal{X}_q(\Omega),W^2_q(\Omega_\pm)^N)), \quad \mathcal{U}_\pm(\lambda)F_\lambda\mathbf{G} = \mathbf{U}_\pm(\lambda)\mathbf{G} \quad (\mathbf{G} \in X_q(\Omega)), \\
&\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega),W^{2-j}_q(\Omega_\pm)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{U}_\pm(\lambda)) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \le C \quad (\ell = 0,1, j = 0,1,2).
\end{aligned} \qquad (107)
$$

5.4. Estimates of the Remainder Terms

We introduce the operators that represent $\mathcal{R}^1_+(\lambda)$, $\mathcal{R}^1_-(\lambda)$, $\mathcal{R}^2(\lambda)$, and $\mathcal{R}^3(\lambda)$ as follows:

$$\begin{aligned}
\mathcal{R}^1_+(\lambda)\mathbf{F} &= \sum_{j=1}^\infty \gamma_{0+}^{-1}[\mathrm{Div}\,\mathbf{S}_+(\zeta^1_j\mathcal{U}^1_{+j}(\lambda)\mathbf{F}) - \zeta^1_j\mathrm{Div}\,\mathbf{S}_+(\mathcal{U}^1_{+j}(\lambda)\mathbf{F})\\
&\quad + \delta\{\nabla(\gamma_{3+}\mathrm{div}\,(\zeta^1_j\mathcal{U}^1_{+j}(\lambda)\mathbf{F})) - \zeta^1_j\nabla(\gamma_{3+}\mathrm{div}\,(\mathcal{U}^1_{+j}(\lambda)\mathbf{F}))\}]\\
&\quad + \sum_{i=2,4}\sum_{j=1}^\infty \gamma_{0+}^{-1}[\mathrm{Div}\,\mathbf{S}_+(\zeta^i_j\mathcal{U}^i_j(\lambda)\mathbf{F}) - \zeta^i_j\mathrm{Div}\,S_+(\mathcal{U}^i_j(\lambda)\mathbf{F})\\
&\quad + \delta\{\nabla(\gamma_{3+}\mathrm{div}\,(\zeta^i_j\mathcal{U}^i_j(\lambda)\mathbf{F})) - \zeta^i_j\nabla(\gamma_{3+}\mathrm{div}\,(\mathcal{U}^i_j(\lambda)\mathbf{F})\}];\\
\mathcal{R}^1_-(\lambda)\mathbf{F} &= \sum_{j=1}^\infty \rho_{0-}^{-1}[\mathrm{Div}\,\mathbf{S}_-(\zeta^1_j\mathcal{U}^1_{-j}(\lambda)\mathbf{F}) - \zeta^1_j\mathrm{Div}\,\mathbf{S}_-(\mathcal{U}^1_{-j}(\lambda)\mathbf{F})]\\
&\quad + \sum_{i=3,5}\sum_{j=1}^\infty \rho_{0-}^{-1}[\mathrm{Div}\,\mathbf{S}_-(\zeta^i_j\mathcal{U}^i_j(\lambda)\mathbf{F}) - \zeta^i_j\mathrm{Div}\,\mathbf{S}_-(\mathcal{U}^i_j(\lambda)\mathbf{F})] + \mathcal{L}(\lambda)\mathbf{F};\\
\mathcal{L}(\lambda)\mathbf{F} &= \nabla K(\mathcal{U}_+(\lambda)\mathbf{F},\mathcal{U}_-(\lambda)\mathbf{F}) - \sum_{j=1}^\infty \zeta^1_j\nabla K^1_j(\mathcal{U}^1_{+j}(\lambda)\mathbf{F},\mathcal{U}^1_{-j}(\lambda)\mathbf{F})\\
&\quad - \sum_{j=1}^\infty \zeta^3_j\nabla K^3_j(\mathcal{U}^3_j(\lambda)\mathbf{F}) - \sum_{j=1}^\infty \zeta^5_j\nabla K^5_j(\mathcal{U}^5_j(\lambda)\mathbf{F});\\
\mathcal{R}^2(\lambda)\mathbf{F} &= -\sum_{j=1}^\infty[\mathcal{T}_{\mathbf{n}^1_j}[\mathrm{Ext}^-[\mathbf{S}_+(\zeta^1_j\mathcal{U}^1_{+j}(\lambda)\mathbf{F}) - \zeta^1_j\mathbf{S}_+(\mathcal{U}^1_{+j}(\lambda)\mathbf{F})]\mathbf{n}^1_j\\
&\quad - \mathcal{T}_{\mathbf{n}^1_j}[\mathrm{Ext}^+[\mathbf{S}_-(\zeta^1_j\mathcal{U}^1_{-j}(\lambda)\mathbf{F}) - \zeta^1_j\mathbf{S}_-(\mathcal{U}^1_{-j}(\lambda)\mathbf{F})]\mathbf{n}^1_j] - \{\mathrm{div}\,(\zeta^1_j\mathcal{U}^1_{-j}(\lambda)\mathbf{F}) - \zeta^1_j\mathrm{div}\,(\mathcal{U}^1_{-j}(\lambda)\mathbf{F})\}],\\
\mathcal{R}^3(\lambda)\mathbf{F} &= -\sum_{j=1}^\infty[\mathcal{T}_{\mathbf{n}^3_j}[(\mathbf{S}_-(\zeta^3_j\mathcal{U}^3_j(\lambda)\mathbf{F}) - \zeta^3_j\mathbf{S}_-(\mathcal{U}^3_j(\lambda)\mathbf{F}))\mathbf{n}^3_j] + \mathrm{div}\,(\zeta^3_j\mathcal{U}^3_j(\lambda)\mathbf{F}) - \zeta^3_j\mathrm{div}\,\mathcal{U}^3_j(\lambda)\mathbf{F}]
\end{aligned} \tag{108}$$

for any $\mathbf{F} \in \mathcal{X}_q(\Omega)$. Setting:

$$\mathcal{R}(\lambda)\mathbf{F} = (\mathcal{R}^1_+(\lambda)\mathbf{F}, \mathcal{R}^1_-(\lambda)\mathbf{F} + \mathcal{L}(\lambda)\mathbf{F}, \mathcal{R}^2(\lambda)\mathbf{F}, \mathcal{R}^3(\lambda)\mathbf{F}),$$
$$\mathbf{R}(\lambda)\mathbf{G} = (\mathbf{R}^1_+(\lambda)\mathbf{G}, \mathbf{R}^1_-(\lambda)\mathbf{G} + L(\lambda)\mathbf{G}, \mathbf{R}^2(\lambda)\mathbf{G}, \mathbf{R}^3(\lambda)\mathbf{G}),$$

we have:

$$F_\lambda\mathcal{R}(\lambda)F_\lambda\mathbf{G} = F_\lambda\mathbf{R}(\lambda)\mathbf{G}, \tag{109}$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega))}(\{(\tau\partial_\tau)^\ell F_\lambda\mathcal{R}(\lambda) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \leq C\sigma + C_\sigma\lambda_0^{-1/2} \quad (\ell = 0, 1) \tag{110}$$

for any $\sigma > 0$ with some constant C_σ depending on σ. If we prove (110), then, choosing $\sigma > 0$ so small and $\lambda_0 \geq 1$ so large that $C\sigma + C_\sigma\lambda_0^{-1/2} \leq 1/2$, we see that $I - F_\lambda\mathcal{R}(\lambda)$ exists in $\mathrm{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega)))$. Thus, in view of (107), (109), (110), and (102), we see easily that $\mathcal{S}_\pm(\lambda) = \mathcal{U}_\pm(\lambda)(I - F_\lambda\mathcal{R}(\lambda))^{-1}$ has a required $\mathcal{R}$-bounded solution operator to (41), which completes the proof of Theorem 8.

Thus, we prove (110) in the following. By direct use of Lemma 6, Lemma 7, Lemma 1, Lemma 2, Remark 9, (98), and (104), we can estimate $\mathcal{R}^1_\pm(\lambda)$, $\mathcal{R}^2(\lambda)$, and $\mathcal{R}^3(\lambda)$, except for $\mathcal{L}(\lambda)$. In fact, for example,

$$|(\mathrm{Div}\,\mathbf{S}_\pm(\zeta^1_j\mathcal{U}^1_{\pm j}(\lambda)\mathbf{F}) - \zeta^1_j\mathrm{Div}\,\mathbf{S}_\pm(\mathcal{U}^1_{\pm j}(\lambda)\mathbf{F}), \varphi)_{\Omega_\pm}| \leq C\|\mathcal{U}^1_{\pm j}(\lambda)\mathbf{F}\|_{W^1_q(\mathcal{H}^1_{\pm j})}\|\varphi\|_{L_{q'}(\Omega_\pm \cap B^1_j)}$$

for any $\varphi \in L_{q'}(\Omega_\pm)$, and so, there exists an operator family $\mathcal{R}^{11}_\pm(\lambda) \in \mathrm{Hol}(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega_\pm)))$ such that $\mathcal{R}^{11}_\pm(\lambda)\mathbf{F} = \sum_{j=1}^\infty\{\mathrm{Div}\,\mathbf{S}_\pm(\zeta^1_j\mathcal{U}^1_{\pm j}(\lambda)\mathbf{F}) - \zeta^1_\pm\mathrm{Div}\,\mathbf{S}_\pm(\mathcal{U}^i_{\pm j}(\lambda)\mathbf{F})\}$ exists in the strong topology of $L_q(\Omega_\pm)$ and $\|\mathcal{R}^{11}_\pm(\lambda)\mathbf{F}\|^q_{L_q(\Omega_\pm)} \leq C\sum_{j=1}^\infty \|\mathcal{U}^1_{\pm j}(\lambda)\mathbf{F}\|^q_{W^1_q(\mathcal{H}^1_{\pm j})}$. By (98) and the monotone convergence theorem, for any $n \in \mathbb{N}$, $\{\lambda_k\}_{k=1}^n \subset \Gamma_{\epsilon,\lambda_0}$, and $\{\mathbf{F}_k\}_{k=1}^n \subset \mathcal{X}_q(\Omega)$:

$$\begin{aligned}
&\int_0^1 \|\sum_{k=1}^n r_k(u)\mathcal{R}^{11}_\pm(\lambda_k)\mathbf{F}_k\|^q_{L_q(\Omega_\pm)}\,du \leq C\sum_{j=1}^\infty \int_0^1 \|\sum_{k=1}^n r_k(u)\mathcal{U}^1_{\pm j}(\lambda_k)\mathbf{F}_k\|^q_{W^1_q(\mathcal{H}^1_{\pm j})}du\\
&\leq C\sum_{j=1}^\infty\left\{\lambda_0^{-\frac{q}{2}}\int_0^1 \|\sum_{k=1}^n r_k(u)\lambda_k^{\frac12}\nabla\mathcal{U}^1_{\pm j}(\lambda_k)\mathbf{F}_k\|^q_{L_q(\mathcal{H}^1_{\pm j})}\,du + \lambda_0^{-q}\int_0^1 \|\sum_{k=1}^n r_k(u)\lambda_k\mathcal{U}^1_{\pm j}(\lambda_k)\mathbf{F}_k\|^q_{L_q(\mathcal{H}^1_{\pm j})}\,du\right\}\\
&\leq C\lambda_0^{-\frac{q}{2}}\sum_{j=1}^\infty\int_0^1 \|\sum_{k=1}^n r_k(u)\mathbf{F}_k\|^q_{\mathcal{X}_q(\Omega\cap B^i_j)}\,du \leq C\lambda_0^{-\frac{q}{2}}\int_0^1 \|\sum_{k=1}^n r_k(u)\mathbf{F}_k\|^q_{\mathcal{X}_q(\Omega)}\,du,
\end{aligned}$$

from which it follows that $\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega))}(\{\mathcal{R}^{11}_\pm(\lambda) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \leq C\lambda_0^{-1/2}$. The other terms except for $\mathcal{L}(\lambda)$ can be estimated in the same manner. Namely, we have:

$$\begin{aligned}
&\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega),L_q(\Omega_\pm)^N)}(\{(\tau\partial_\tau)^\ell\mathcal{R}^1_\pm(\lambda) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \leq C\lambda_0^{-1/2},\\
&\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega),L_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{1/2}\mathcal{R}^2(\lambda)) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \leq C\lambda_0^{-1/2},\\
&\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega),W^1_q(\Omega))}(\{(\tau\partial_\tau)^\ell\mathcal{R}^2(\lambda) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \leq C\lambda_0^{-1/2},\\
&\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega),L_q(\Omega_-)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{1/2}\mathcal{R}^3(\lambda)) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \leq C\lambda_0^{-1/2},\\
&\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega),W^1_q(\Omega_-))}(\{(\tau\partial_\tau)^\ell\mathcal{R}^3(\lambda) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \leq C\lambda_0^{-1/2}.
\end{aligned}$$

Next, we estimate $\mathcal{L}(\lambda)$. We use the following two lemmas due to Shibata [4].

Lemma 8. *Let $1 < q < \infty$. Then, there exists a constant c independent of $j \in \mathbb{N}$ such that:*

$$\begin{aligned}
\|\psi\|_{W^1_q(\Omega_-\cap B^i_j)} &\leq c\|\nabla\psi\|_{L_q(\Omega_-\cap B^i_j)} \qquad \text{for any } \psi \in \hat{W}^1_{q,0}(\Omega_-),\ (i = 1, 3),\\
\|\psi - c_j(\psi)\|_{W^1_q(\Omega_-\cap B^5_j)} &\leq c\|\nabla\psi\|_{L_q(\Omega_-\cap B^5_j)} \qquad \text{for any } \psi \in \hat{W}^1_{q,0}(\Omega_-),
\end{aligned}$$

where $c_j(\psi)$ are suitable constants depending on ψ.

Lemma 9. *Let $1 < q < \infty$. Then, there exists a constant c independent of $j \in \mathbb{N}$ such that:*

$$\begin{aligned}
\|K^1_j(\mathbf{u}_+, \mathbf{u}_-)\|_{L_q(\mathcal{H}^1_{-j})} &\leq c\sum_\pm(\|\nabla\mathbf{u}_\pm\|_{L_q(\mathcal{H}^1_{\pm j}\cap B^1_j)} + \|\nabla\mathbf{u}_\pm\|^{1-1/q}_{L_q(\mathcal{H}^1_{\pm j})}\|\nabla^2\mathbf{u}_\pm\|^{1/q}_{L_q(\mathcal{H}^1_{\pm j})});\\
\|K^i_j(\mathbf{v})\|_{L_q(\mathcal{H}^i_j\cap B^i_j)} &\leq c(\|\nabla\mathbf{v}\|_{L_q(\mathcal{H}^i_j)} + \delta_i\|\nabla\mathbf{v}\|^{1-1/q}_{L_q(\mathcal{H}^i_j)}\|\nabla^2\mathbf{v}\|^{1/q}_{L_q(\mathcal{H}^i_j)})
\end{aligned}$$

for any $\mathbf{u}_\pm \in W^2_q(\mathcal{H}^1_{\pm j})$ and $\mathbf{v} \in W^2_q(\mathcal{H}^i_j)$ $(i = 3, 5)$, where δ_i are symbols defined by $\delta_3 = 1$ and $\delta_5 = 0$.

Lemma 10. *Let $1 < q < \infty$. Then,*

$$\|v\|_{L_q(\Gamma^i_j)} \leq C_q(\|v\|_{L_q(\Omega_-)} + \|\nabla v\|^{1/q}_{L_q(\Omega_-)}\|v\|^{1-1/q}_{L_q(\Omega_-)})$$

for $i = 1, 3$ and $j \in \mathbb{N}$, where C_q is a constant independent of $j \in \mathbb{N}$.

To estimate $\mathcal{L}(\lambda)$, we write $\mathcal{L}(\lambda)\mathbf{F} = \nabla\mathcal{L}^1(\lambda)\mathbf{F} + \mathcal{L}^2(\lambda)\mathbf{F}$ with:

$$\begin{aligned}
\mathcal{L}^1(\lambda)\mathbf{F} &= K(\mathcal{U}_+(\lambda)\mathbf{F}, \mathcal{U}_-(\lambda)\mathbf{F}) - \sum_{j=1}^{\infty} \zeta_j^1 K_j^1(\mathcal{U}_{+j}^1(\lambda)\mathbf{F}, \mathcal{U}_{-j}^1(\lambda)\mathbf{F}) \\
&\quad - \sum_{j=1}^{\infty} \zeta_j^3 K_q^3(\mathcal{U}_j^3(\lambda)\mathbf{F}) - \sum_{j=1}^{\infty} \zeta_j^5 K_j^5(\mathcal{U}_j^5(\lambda)\mathbf{F}), \\
\mathcal{L}^2(\lambda)\mathbf{F} &= \sum_{j=1}^{\infty} (\nabla\zeta_j^1) K_j^1(\mathcal{U}_{+j}^1(\lambda)\mathbf{F}, \mathcal{U}_{-j}^1(\lambda)\mathbf{F}) + \sum_{j=1}^{\infty} (\nabla\zeta_j^3) K_j^3(\mathcal{U}_j^3(\lambda)\mathbf{F}) + \sum_{j=1}^{\infty} (\nabla\zeta_j^5) K_j^5(\mathcal{U}_j^5(\lambda)\mathbf{F}).
\end{aligned}$$

By (98), (99), Lemma 9, Lemma 6, and Lemma 7, we have:

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega_-)^N))}(\{(\tau\partial_\tau)^\ell \mathcal{L}^2(\lambda) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \leq C\sigma + C_\sigma \lambda_0^{-1/2}.$$

Denoting the duality of $\hat{W}^{-1}_{q',0}(\Omega_-)^*$ and $\hat{W}^1_{q',0}(\Omega_-)$ by $< \cdot, \cdot >$ and using (38) and (94), we define an operator $\mathbf{L}(\lambda)$ acting on $\mathbf{F} \in \mathcal{X}_q(\Omega)$ by the following formulas: For any $\varphi \in \hat{W}^1_{q',0}(\Omega_-)$:

$$(\nabla\mathcal{L}^1(\lambda)\mathbf{F}, \nabla\varphi)_{\Omega_-} = < \mathbf{L}(\lambda)\mathbf{F}, \varphi > \tag{111}$$

with:

$$\begin{aligned}
< \mathbf{L}(\lambda)\mathbf{F}, \varphi > &= \sum_{j=1}^{\infty} \rho_{0-}^{-1} (\mathrm{Div}\, \mathbf{S}_-(\zeta_j^1 \mathcal{U}_{-j}^1(\lambda)\mathbf{F}) - \zeta_j^1 \mathrm{Div}\, \mathbf{S}_-(\mathcal{U}_{-j}^1(\lambda)\mathbf{F}), \nabla\varphi)_{\mathcal{H}_{-j}^1} \\
&\quad - \sum_{j=1}^{\infty} \rho_{0-}^{-1} (\nabla \mathrm{div}\, (\zeta_j^1 \mathcal{U}_{-j}^1(\lambda)\mathbf{F}) - \zeta_j^1 \nabla \mathrm{div}\, \mathcal{U}_{-j}^1(\lambda)\mathbf{F}, \nabla\varphi)_{\mathcal{H}_{-j}^1} \\
&\quad + \sum_{j=1}^{\infty} \rho_{0-}^{-1} (K_j^1(\mathcal{U}_{+j}^1(\lambda)\mathbf{F}, \mathcal{U}_{-j}^1(\lambda)\mathbf{F})\mathbf{n}_j^1, (\nabla\zeta_j^1)\varphi)_{\Gamma_j^1} \\
&\quad - 2\sum_{j=1}^{\infty} \rho_{0-}^{-1} (K_j^1(\mathcal{U}_{+j}^1(\lambda)\mathbf{F}, \mathcal{U}_{-j}^1(\lambda)\mathbf{F})(\nabla\zeta_j^1), \nabla\varphi)_{\mathcal{H}_{-j}^1} \\
&\quad - \sum_{j=1}^{\infty} \rho_{0-}^{-1} (K_j^1(\mathcal{U}_{+j}^1(\lambda)\mathbf{F}, \mathcal{U}_{-j}^1(\lambda)\mathbf{F})(\Delta\zeta_j^1), \varphi)_{\mathcal{H}_{-j}^1} \\
&\quad + \sum_{j=1}^{\infty} \rho_{0-}^{-1} (\mathrm{Div}\, \mathbf{S}_-(\zeta_j^3 \mathcal{U}_j^3(\lambda)\mathbf{F}) - \zeta_j^3 \mathrm{Div}\, \mathbf{S}_-(\mathcal{U}_j^3(\lambda)\mathbf{F}), \nabla\varphi)_{\mathcal{H}_j^3} \\
&\quad - \sum_{j=1}^{\infty} \rho_{0-}^{-1} (\nabla \mathrm{div}\, (\zeta_j^3 \mathcal{U}_j^3(\lambda)\mathbf{F}) - \zeta_j^3 \nabla \mathrm{div}\, \mathcal{U}_j^3(\lambda)\mathbf{F}, \nabla\varphi)_{\mathcal{H}_j^3} \\
&\quad + \sum_{j=1}^{\infty} \rho_{0-}^{-1} (K_j^3(\mathcal{U}_j^3(\lambda)\mathbf{F})\mathbf{n}_j^3, (\nabla\zeta_j^3)\varphi)_{\Gamma_j^3} - 2\sum_{j=1}^{\infty} \rho_{0-}^{-1} (K_j^3(\mathcal{U}_j^3(\lambda)\mathbf{F})(\nabla\zeta_j^3), \nabla\varphi)_{\mathcal{H}_j^3} \\
&\quad - \sum_{j=1}^{\infty} \rho_{0-}^{-1} (K_j^3(\mathcal{U}_j^3(\lambda)\mathbf{F})(\Delta\zeta_j^3), \varphi)_{\mathcal{H}_j^3} \\
&\quad + \sum_{j=1}^{\infty} \rho_{0-}^{-1} (\mathrm{Div}\, \mathbf{S}_-(\zeta_j^5 \mathcal{U}_j^5(\lambda)\mathbf{F}) - \zeta_j^5 \mathrm{Div}\, \mathbf{S}_-(\mathcal{U}_j^5(\lambda)\mathbf{F}), \nabla\varphi)_{\mathcal{H}_j^5} \\
&\quad - \sum_{j=1}^{\infty} \rho_{0-}^{-1} (\nabla \mathrm{div}\, (\zeta_j^3 \mathcal{U}_j^3(\lambda)\mathbf{F}) - \zeta_j^5 \nabla \mathrm{div}\, \mathcal{U}_j^5(\lambda)\mathbf{F}, \nabla\varphi)_{\mathcal{H}_j^5} \\
&\quad - 2\sum_{j=1}^{\infty} \rho_{0-}^{-1} (K_j^5(\mathcal{U}_j^5(\lambda)\mathbf{F})(\nabla\zeta_j^3), \nabla\varphi)_{\mathcal{H}_j^3} - \sum_{j=1}^{\infty} \rho_{0-}^{-1} (K_j^5(\mathcal{U}_j^5(\lambda)\mathbf{F})(\Delta\zeta_j^5), \varphi - c_j(\varphi)_{\mathcal{H}_j^5}
\end{aligned}$$

By (98), (99), Lemma 8, Lemma 9, and Lemma 10, we have:

$$\|\mathbf{L}(\lambda)\mathbf{F}\|_{\hat{W}^1_{q',0}(\Omega_-)^*} \le C\sigma(\sum_{\pm}\sum_{j=1}^{\infty}\|\mathcal{U}^1_{\pm j}(\lambda)\mathbf{F}\|^q_{W^2_q(\Omega_\pm\cap B^1_j)} + \sum_{j=1}^{\infty}\|\mathcal{U}^3_j(\lambda)\mathbf{F}\|^q_{W^2_q(\Omega_-\cap B^3_j)})$$
$$+ C_\sigma(\sum_{\pm}\sum_{j=1}^{\infty}\|\mathcal{U}^1_{\pm j}(\lambda)\mathbf{F}\|^q_{W^1_q(\Omega_\pm\cap B^1_j)} + \sum_{j=1}^{\infty}\|\mathcal{U}^3_j(\lambda)\mathbf{F}\|^q_{W^1_q(\Omega_-\cap B^3_j)}),$$

which, combined with (98) and (99), yields that:

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega),\hat{W}^1_{q',0}(\Omega_-)^*)}(\{(\tau\partial_\tau)^\ell\mathbf{L}(\lambda) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \le C\sigma + C_\sigma\lambda_0^{-1/2}.$$

Identifying $\hat{W}^1_{q,0}(\Omega_-) = \{\nabla\varphi \mid \varphi \in \hat{W}^1_{q',0}(\Omega_-)\} \subset L_{q'}(\Omega_-)^N$, by the Hahn–Banach theorem, there exists an operator family $\mathbf{M}(\lambda) \in \mathrm{Hol}\,(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega_-), L_q(\Omega_-)^N))$ such that $(\mathbf{M}(\lambda)\mathbf{F}, \nabla\varphi)_{\Omega_-} =< \mathbf{L}(\lambda)\mathbf{F}, \varphi >$ and:

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega),L_q(\Omega_-)^N)}(\{(\tau\partial_\tau)^\ell\mathbf{M}(\lambda) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \le C\sigma + C_\sigma\lambda_0^{-1/2}.$$

Moreover, by (39), (94), and (96), there exists an operator family $\mathbf{m}(\lambda) \in \mathrm{Hol}\,(\Gamma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega), W^1_q(\Omega_-)))$ such that $\mathbf{m}(\lambda)\mathbf{F} = \mathcal{L}^1(\lambda)\mathbf{F}$ on Γ and Γ_-, and:

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega),W^1_q(\Omega_-))}(\{(\tau\partial_\tau)^\ell\mathbf{m}(\lambda) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \le C\sigma + C_\sigma\lambda_0^{-1/2}.$$

Since the weak Dirichlet problem is assumed to be uniquely solvable, we have $\mathcal{L}^1(\lambda)\mathbf{F} = \mathbf{m}(\lambda)\mathbf{F} + \mathcal{K}(\mathbf{M}(\lambda)\mathbf{F} - \nabla\mathbf{m}(\lambda)\mathbf{F})$, which yields that:

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega),L_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell(\nabla\mathcal{L}^1(\lambda)) \mid \lambda \in \Gamma_{\epsilon,\lambda_0}\}) \le C\sigma + C_\sigma\lambda_0^{-1/2}.$$

Therefore, we have (110), and so, the proof of Theorem 8 is complete.

5.5. *A Proof of Theorem 3*

Instead of Problem (22), we consider:

$$\begin{cases} \gamma_{0+}\partial_t\mathbf{u}_+ - \mathrm{Div}\,(\mathbf{S}_+(\mathbf{u}_+) - \delta\nabla(\gamma_{3+}\mathbf{u}_+) = \mathbf{g}_+ & \text{in } \Omega_+ \times (0,\infty), \\ \rho_{0-}\partial_t\mathbf{u}_- - \mathrm{Div}\,\mathbf{S}_-(\mathbf{u}_-) + \nabla p_- = \mathbf{g}_- & \text{in } \Omega_- \times (0,\infty), \\ \mathrm{div}\,\mathbf{u}_- = f_- = \mathrm{div}\,\mathbf{f}_- & \text{in } \Omega_- \times (0,\infty), \\ (\mathbf{S}_+(\mathbf{u}_+) + \delta\gamma_{3+}\mathrm{div}\,\mathbf{u}_+\mathbf{I})\mathbf{n}|_{\Gamma_t+0} - (\mathbf{S}_-(\mathbf{u}_-) - p_-\mathbf{I})\mathbf{n}|_{\Gamma_t-0} = \mathbf{h} & \text{for } t > 0, \\ \mathbf{u}_+|_{\Gamma_t+0} - \mathbf{u}_-|_{\Gamma_t-0} = 0 & \text{for } t > 0, \\ \mathbf{u}_+|_{\Gamma_+} = 0, \quad (\mathbf{S}_-(\mathbf{u}_-) - p_-\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = \mathbf{h}_- & \text{for } t > 0, \\ \mathbf{u}_+|_{t=0} = \mathbf{u}_{0+} \text{ in } \Omega_+, \quad \mathbf{u}_-|_{t=0} = \mathbf{u}_{0-} & \text{in } \Omega_-. \end{cases} \tag{112}$$

We first consider the generation of the C^0 analytic semigroup associated with the following equations:

$$\begin{cases} \partial_t\mathfrak{p}_+ + \gamma_{2+}\mathrm{div}\,\mathbf{w}_+ = 0 & \text{in } \Omega_+ \times (0,\infty), \\ \gamma_{0+}\partial_t\mathbf{w}_+ - \mathrm{Div}\,(\mathbf{S}_+(\mathbf{w}_+) + \nabla(\gamma_{1+}\mathfrak{p}_+) = 0 & \text{in } \Omega_+ \times (0,\infty), \\ \rho_{0-}\partial_t\mathbf{w}_- - \mathrm{Div}\,\mathbf{S}_-(\mathbf{w}_-) + \nabla K(\mathbf{w}_+,\mathbf{w}_-) = 0 & \text{in } \Omega_- \times (0,\infty), \\ (\mathbf{S}_+(\mathbf{w}_+) - \gamma_{1+}\mathfrak{p}_+\mathbf{I})\mathbf{n}|_{\Gamma_t+0} - (\mathbf{S}_-(\mathbf{w}_-) - K(\mathbf{w}_+,\mathbf{w}_-)\mathbf{I})\mathbf{n}|_{\Gamma_t-0} = 0 & \text{for } t > 0, \\ \mathbf{w}_+|_{\Gamma_t+0} - \mathbf{w}_-|_{\Gamma_t-0} = 0 & \text{for } t > 0, \\ \mathbf{w}_+|_{\Gamma_+} = 0, \quad (\mathbf{S}_-(\mathbf{w}_-) - K(\mathbf{w}_+,\mathbf{w}_-)\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = 0 & \text{for } t > 0, \\ (\mathfrak{p}_+,\mathbf{w}_+)|_{t=0} = (\mathfrak{p}_{0,+},\mathbf{w}_{0+}) \text{ in } \Omega_+, \quad \mathbf{w}_-|_{t=0} = \mathbf{w}_{0-} & \text{in } \Omega_-. \end{cases} \tag{113}$$

In view of Theorem 8, let $\mathbf{u}_\pm = \mathcal{S}_\pm(\lambda)(\mathbf{g}_+, \mathbf{g}_-, 0, 0, 0, 0)$, and set $\theta_+ = \lambda^{-1}(f_+ - \gamma_{2+}\mathrm{div}\,\mathbf{u}_+)$, then $\mathbf{u}_\pm$ and θ_+ are unique solutions of Equation (28) with $p_- = K(\mathbf{u}_+, \mathbf{u}_-)$ and $f_- = \mathbf{f}_- = \mathbf{h} = \mathbf{h}_- = 0$ and possess the estimates:

$$|\lambda|\|\theta_+\|_{W^1_q(\Omega_+)} + \sum_\pm(|\lambda|\|\mathbf{u}_\pm\|_{L_q(\Omega_\pm)} + \|\mathbf{u}_\pm\|_{W^2_q(\Omega_\pm)}) \leq C(\|f_+\|_{W^1_q(\Omega_+)} + \|\mathbf{g}_+\|_{L_q(\Omega_+)} + \|\mathbf{g}_-\|_{L_q(\Omega_-)}) \tag{114}$$

for any $\lambda \in \Gamma_{\epsilon,\lambda_0}$. Here, using the same argument as in Assertion 2 in Sect.2, we see that $\mathbf{u}_- \in J_q(\Omega_-)$, and so, $\mathrm{div}\,\mathbf{u}_- = 0$ in (28). Set:

$$\begin{aligned} X_q(\Omega) &= \{(\theta_+, \mathbf{u}_+, \mathbf{u}_-) \mid \theta_+ \in W^1_q(\Omega_+),\ \ \mathbf{u}_+ \in L_q(\Omega_+),\ \ \mathbf{u}_- \in J_q(\Omega_-)\}, \\ Y_q(\Omega) &= \{(\theta_+, \mathbf{u}_+, \mathbf{u}_-) \in X_q(\Omega) \mid (\mathbf{S}_+(\mathbf{u}_+) - \gamma_{1+}\theta_+\mathbf{I})\mathbf{n}|_{\Gamma+0} - (\mathbf{S}_-(\mathbf{u}_-) - K(\mathbf{u}_+, \mathbf{u}_-)\mathbf{I})\mathbf{n}|_{\Gamma-0} = 0, \\ &\quad (\mathbf{S}_-(\mathbf{u}_-) - K(\mathbf{u}_+, \mathbf{u}_-)\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = 0, \quad \mathbf{u}_+|_{\Gamma+0} = \mathbf{u}_-|_{\Gamma-0}, \quad \mathbf{u}_+|_{\Gamma_+} = 0\}. \end{aligned}$$

Then, Problem (113) generates a C^0 analytic semigroup on $X_q(\Omega)$. Let $\mathcal{D}_q(\Omega) = (X_q, Y_q)_{1-1/p,p}$, where $(\cdot,\cdot)_{1-1/p,p}$ denotes a real interpolation functor. By a standard real interpolation method (trace method), we see that Problem (113) admits unique solutions $\mathfrak{p}_+$ and $\mathbf{w}_\pm$ with:

$$\mathfrak{p}_+ \in W^1_p((0,\infty), W^1_q(\Omega_+)),\ \ \mathbf{w}_\pm \in W^1_p((0,\infty), L_q(\Omega_\pm)^N) \cap L_p((0,\infty), W^2_q(\Omega_\pm)^N), \tag{115}$$

possessing the estimate:

$$\begin{aligned} &\|e^{-\gamma t}\mathfrak{p}_+\|_{L_p((0,\infty),W^1_q(\Omega))} + \sum_\pm \|e^{-\gamma t}\mathbf{w}_\pm\|_{L_p((0,\infty),L_q(\Omega_\pm))} + \|e^{-\gamma t}\nabla K(\mathbf{w}_+, \mathbf{w}_-)\|_{L_p((0,\infty),L_q(\Omega_-))} \\ &\quad \leq C(\|\mathfrak{p}_{0+}\|_{L_p(\mathbb{R},W^1_q(\Omega_+))} + \sum_\pm \|\mathbf{w}_{0\pm}\|_{B^{2(1-1/p)}_{q,p}(\Omega_\pm)}) \end{aligned} \tag{116}$$

for any $\gamma > \lambda_0$. Moreover, $\mathbf{w}_- \in J_q(\Omega_-)$ for any $t \in (0,\infty)$. The uniqueness follows from Duhamel's principle.

We now consider equations:

$$\begin{cases} \partial_t \mathfrak{q}_+ + \gamma_{2+}\mathrm{div}\,\mathbf{v}_+ = f_+ & \text{in } \Omega_+ \times \mathbb{R}, \\ \gamma_{0+}\partial_t\mathbf{v}_+ - \mathrm{Div}\,\mathbf{S}_+(\mathbf{v}_+) + \nabla(\gamma_{1+}\mathfrak{q}_+) = \mathbf{g}_+ & \text{in } \Omega_+ \times \mathbb{R}, \\ \rho_{0-}\partial_t\mathbf{v}_- - \mathrm{Div}\,\mathbf{S}_-(\mathbf{v}_-) + \nabla\mathfrak{q}_- = \mathbf{g}_- & \text{in } \Omega_- \times \mathbb{R}, \\ \mathrm{div}\,\mathbf{v}_- = f_- = \mathrm{div}\,\mathbf{f}_- & \text{in } \Omega_- \times \mathbb{R}, \\ (\mathbf{S}_+(\mathbf{v}_+) - \gamma_{1+}\mathfrak{q}_+\mathbf{I})\mathbf{n}|_{\Gamma_t+0} - (\mathbf{S}_-(\mathbf{v}_-) - \mathfrak{q}_-\mathbf{I})\mathbf{n}|_{\Gamma_t-0} = \mathbf{h}|_\Gamma & \text{for } t \in \mathbb{R}, \\ \mathbf{v}_+|_{\Gamma_t+0} - \mathbf{v}_-|_{\Gamma_t-0} = 0 & \text{for } t \in \mathbb{R}, \\ \mathbf{v}_+|_{\Gamma_+} = 0, \quad (\mathbf{S}_-(\mathbf{v}_-) - \mathfrak{q}_-\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = \mathbf{h}_- & \text{for } t \in \mathbb{R}. \end{cases} \tag{117}$$

Applying the Laplace transform to (117) and setting $\hat{\mathbf{v}}_\pm = \mathcal{L}[\mathbf{v}_\pm](\lambda)$ and $\hat{\mathfrak{q}}_\pm = \mathcal{L}[\mathfrak{q}_\pm](\lambda)$, we have:

$$\begin{cases} \lambda\hat{\mathfrak{q}}_+ + \gamma_{2+}\mathrm{div}\,\hat{\mathbf{v}}_+ = \hat{f}_+ & \text{in } \Omega_+, \\ \lambda\hat{\mathbf{v}}_+ - \gamma_{0+}^{-1}(\mathrm{Div}\,\mathbf{S}_+(\hat{\mathbf{v}}_+) - \nabla(\gamma_{1+}\hat{\mathfrak{q}}_+)) = \hat{\mathbf{g}}_+ & \text{in } \Omega_+, \\ \lambda\hat{\mathbf{v}}_- - \rho_{0-}^{-1}(\mathrm{Div}\,\mathbf{S}_-(\hat{\mathbf{v}}_-) - \nabla\hat{\mathfrak{q}}_-) = \hat{\mathbf{g}}_- & \text{in } \Omega_-, \\ \mathrm{div}\,\hat{\mathbf{v}}_- = \hat{f}_- = \mathrm{div}\,\hat{\mathbf{f}}_- & \text{in } \Omega_-, \\ (\mathbf{S}_+(\hat{\mathbf{v}}_+) - \gamma_{1+}\hat{\mathfrak{q}}_+\mathbf{I})\mathbf{n}|_{\Gamma+0} - (\mathbf{S}_-(\hat{\mathbf{v}}_-) - \hat{\mathfrak{q}}_-\mathbf{I})\mathbf{n}|_{\Gamma-0} = \hat{\mathbf{h}}|_\Gamma, & \\ \hat{\mathbf{v}}_+|_{\Gamma+0} - \hat{\mathbf{v}}_-|_{\Gamma-0} = 0, & \\ \hat{\mathbf{v}}_+|_{\Gamma_+} = 0, \quad (\mathbf{S}_-(\hat{\mathbf{v}}_-) - \hat{\mathfrak{q}}_-\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = \hat{\mathbf{h}}_-|_{\Gamma_-}. & \end{cases} \tag{118}$$

Applying Theorem 5 yields that $\hat{\mathbf{v}}_\pm = \mathcal{A}^0_\pm(\lambda)F^0_\lambda\mathbf{G}^0$, $\hat{\mathfrak{q}}_+ = \lambda^{-1}(\hat{f}_+ - \gamma_{2+}\mathrm{div}\,\hat{\mathbf{v}}_+)$, and $\hat{\mathfrak{q}}_- = \mathcal{B}^0_-(\lambda)F^0_\lambda\mathbf{G}^0$ satisfy Equation (118), and so, $\mathbf{v}_\pm = \mathcal{L}^{-1}[\hat{\mathbf{v}}_\pm]$ and $\mathfrak{q}_\pm = \mathcal{L}^{-1}[\hat{\mathfrak{q}}_\pm]$ satisfy Equation (118). Moreover, applying Theorem 4 yields that:

$$\begin{aligned}&\|e^{-\gamma t}\mathfrak{q}_+\|_{W^1_p(\mathbb{R},W^1_q(\Omega_+))}+\sum_{\pm}(\|e^{-\gamma t}\partial_t\mathbf{v}_\pm\|_{L_p(\mathbb{R},L_q(\Omega_\pm))}+\|e^{-\gamma t}\mathbf{v}_\pm\|_{L_p(\mathbb{R},W^2_q(\Omega_\pm))})+\|\nabla\mathfrak{q}_-\|_{L_p(\mathbb{R},L_q(\Omega_-))}\\&\quad\leq C\{\sum_{\pm}\|e^{-\gamma t}\mathbf{g}_\pm\|_{L_p(\mathbb{R},L_q(\Omega_\pm))}+\|e^{-\gamma t}\Lambda^{1/2}_\gamma(f_-,\mathbf{h}_-)\|_{L_p(\mathbb{R},L_q(\Omega_-))}+\|e^{-\gamma t}(f_-,\mathbf{h}_-)\|_{L_p(\mathbb{R},W^1_q(\Omega_-))}\\&\qquad+\|e^{-\gamma t}\Lambda^{1/2}_\gamma\mathbf{h}\|_{L_p(\mathbb{R},L_q(\Omega))}+\|e^{-\gamma t}\mathbf{h}\|_{L_p(\mathbb{R},W^1_q(\Omega))}\}\quad\text{for any }\gamma>\lambda_0.\end{aligned}\tag{119}$$

Here, we may assume that $\lambda_0\geq 1$.

To prove Theorem 3, setting $\theta_+=\mathfrak{q}_++\mathfrak{p}_+$, $\mathbf{u}_\pm=\mathbf{v}_\pm+\mathbf{w}_\pm$ and $p_-=\mathfrak{q}_-+\theta_-$ in (22), we see that $\mathfrak{p}_+$, $\mathbf{w}_\pm$, and $\theta_-=K(\mathbf{u}_+,\mathbf{u}_-)$ satisfy Equation (113) with $\mathfrak{p}_{0+}=\theta_{0+}-\mathfrak{q}_+|_{t=0}$ and $\mathbf{w}_{0\pm}=\mathbf{u}_{0\pm}-\mathbf{v}_\pm|_{t=0}$. By compatibility conditions (25) and the assumption that $2/p+N/q\neq 1,2$, we see that $(\theta_{0+}-\mathfrak{q}_+|_{t=0},\mathbf{u}_{0+}-\mathbf{v}_+|_{t=0},\mathbf{u}_{0-}-\mathbf{v}_-|_{t=0})\in(X_q,Y_q)_{1-1/p,p}$, and so, Problem (113) admits unique solutions $\mathfrak{p}_+$ and $\mathbf{w}_\pm$ satisfying (115) and (116). By the real interpolation theorem, we have:

$$\begin{aligned}\|\mathfrak{p}_+|_{t=0}\|_{W^1_q(\Omega_+)}&\leq C_\gamma\|e^{-\gamma t}\mathfrak{p}_+\|_{W^1_p((0,\infty),W^1_q(\Omega_+))},\\\|\mathbf{v}_\pm|_{t=0}\|_{B^{2(1-1/p)}_{q,p}(\Omega_\pm)}&\leq C(\|e^{-\gamma t}\partial_t\mathbf{v}_\pm\|_{L_p((0,\infty),L_q(\Omega_\pm))}+\|e^{-\gamma t}\mathbf{v}_\pm\|_{L_p((0,\infty),W^2_q(\Omega_\pm))}),\end{aligned}$$

which, combined with (119), yields the existence part of Theorem 3, because $\mathbf{w}_-\in J_q(\Omega_-)$ for any $t>0$. The uniqueness follows from Duhamel's principle or the existence theorem of dual problems (cf. ([20] Section 3.5.10)). This completes the proof of Theorem 3.

6. A Proof of Theorem 1

In what follows, we assume that $2<p<\infty$, $N<q<\infty$, $2/p+N/q<1$, that $\Omega_\pm$ are uniform $W^{2-1/q}_q$ domains in $\mathbb{R}^N$ ($N\geq 2$), and that the weak Dirichlet problem is uniquely solvable in Ω_-. By Sobolev's imbedding theorem, we have:

$$W^1_q(\Omega_\pm)\subset L_\infty(\Omega_\pm),\quad \|\prod_{j=1}^m f_j\|_{W^1_q(\Omega_\pm)}\leq C\prod_{j=1}^m\|f_j\|_{W^1_q(\Omega_\pm)}.\tag{120}$$

Let $\theta_{0+}\in W^1_q(\Omega_+)$ and $\mathbf{u}_{0\pm}\in B^{2(1-1/p)}_{q,p}(\Omega_\pm)$ be initial data satisfying the compatibility condition (20), range condition (21), and $\|\theta_{0+}\|_{W^1_q(\Omega)}+\|\mathbf{v}_{0+}\|_{B^{2(1-1/p)}_{q,p}(\Omega_+)}+\|\mathbf{v}_{0-}\|_{B^{2(1-1/p)}_{q,p}(\Omega_+)}\leq R_1$.

To prove Theorem 1, we follow the argument due to Shibata and Shimizu ([21] Section 2). Let Π_+ and $\mathbf{Z}_\pm$ be solutions to linear problem:

$$\begin{cases}\partial_t\Pi_++(\rho_{0+}+\theta_{0+})\operatorname{div}\mathbf{Z}_+=0&\text{in }\Omega_+,\\(\rho_{0+}+\theta_{0+})\partial_t\mathbf{Z}_+-\operatorname{Div}\mathbf{S}_+(\mathbf{Z}_+)+\nabla(p'(\rho_{0+}+\theta_{0+})\Pi_+)=\mathbf{g}_+&\text{in }\Omega_+,\\\rho_{0-}\partial_t\mathbf{Z}_--\operatorname{Div}\mathbf{S}_-(\mathbf{Z}_-)+\nabla p_-=0&\text{in }\Omega_-,\\\operatorname{div}\mathbf{Z}_-=0&\text{in }\Omega_-,\\(\mathbf{S}_+(\mathbf{Z}_+)-(p'(\rho_{0+}+\theta_{0+})\Pi_+\mathbf{I})\mathbf{n}|_{\Gamma+0}-(\mathbf{S}_-(\mathbf{Z}_-)-p_-\mathbf{I})\mathbf{n}|_{\Gamma-0}=\mathbf{h},\\\mathbf{Z}_+|_{\Gamma+0}-\mathbf{Z}_-|_{\Gamma-0}=0,\quad\mathbf{Z}_+|_{\Gamma_+}=0,\quad(\mathbf{S}_-(\mathbf{Z}_-)-p_-\mathbf{I})\mathbf{n}_-|_{\Gamma_-}=0\end{cases}\tag{121}$$

for any $t>0$ subject to the initial condition: $(\Pi_+,\mathbf{Z}_+)|_{t=0}=(0,\mathbf{v}_{0+})$ in Ω_+ and $\mathbf{Z}_-|_{t=0}=\mathbf{v}_{0-}$ in Ω_- with some pressure term p_-, where $\mathbf{g}_+=-p'(\rho_{0+}+\theta_{0+})\nabla\theta_0$ and $\mathbf{h}=-(p(\rho_{0+}+\theta_{0+})-p(\rho_{0+}))\mathbf{n}$. Since $\mathbf{v}_{0\pm}$ satisfy the compatibility condition (20), by Theorem 3, we know the unique existence of Π_+ and $\mathbf{Z}_\pm$ with:

$$\Pi_+\in W^1_{p,\gamma_0}(\mathbb{R}_+,W^1_q(\Omega_+)),\quad\mathbf{Z}_\pm\in W^1_{p,\gamma_0}(\mathbb{R}_+,L_q(\Omega_\pm))\cap L_{p,\gamma_0}(\mathbb{R}_+,W^2_q(\Omega_\pm))\tag{122}$$

with large γ_0 depending on R_1 possessing the estimate:

$$
\begin{aligned}
&\|e^{-\gamma t}\Pi_+\|_{W^1_p(\mathbb{R}_+,W^1_q(\Omega_+))} + \sum_{\ell=+,-}\{\|e^{-\gamma t}\mathbf{Z}_\ell\|_{W^1_p(\mathbb{R}_+,L_q(\Omega_\ell))} + \|e^{-\gamma t}\mathbf{Z}_\ell\|_{L_p(\mathbb{R}_+,W^2_q(\Omega_\ell))}\} \\
&\leq C_{R_1}\sum_{\ell=+,-}\|\mathbf{v}_{0\ell}\|_{B^{2(1-1/p)}_{q,p}(\Omega_\ell)} \leq C_{R_1}R_1
\end{aligned} \tag{123}
$$

for any $\gamma \geq \gamma_0$. In the following, γ is fixed such as $\gamma \geq \gamma_0$. Let Π^0_+ be the zero extension of Π_+ to $t < 0$ and $\tilde{\mathbf{Z}}^e_\pm$ be the even extension of $\mathbf{Z}_\pm$ to $t < 0$, that is:

$$
\Pi^0_+(x,t) = \begin{cases} \Pi_+(x,t) & (t \geq 0), \\ 0 & (t < 0), \end{cases} \quad \tilde{\mathbf{Z}}^e_\pm(x,t) = \begin{cases} \mathbf{Z}_\pm(x,t) & (t \geq 0), \\ \mathbf{Z}_\pm(x,-t) & (t < 0). \end{cases}
$$

Let $\psi(t)$ be a function in $C^\infty(\mathbb{R})$ such that $\psi(t) = 1$ for $t > -1/2$ and $\psi(t) = 0$ for $t < -1$, and set $\mathbf{Z}^e_\pm = \psi\tilde{\mathbf{Z}}^e_\pm$. By (123):

$$
\|e^{-\gamma t}\Pi^0_+\|_{W^1_p(\mathbb{R},W^1_q(\Omega_+))} + \sum_{\ell=+,-}\{\|e^{-\gamma t}\mathbf{Z}^e_\ell\|_{W^1_p(\mathbb{R},L_q(\Omega_\ell))} + \|e^{-\gamma t}\mathbf{Z}^e_\ell\|_{L_p(\mathbb{R},W^2_q(\Omega_\ell))}\} \leq C_{R_1}R_1. \tag{124}
$$

We look for a solution to (11) of the form: $\theta_+ = \Pi^0_+ + \rho_+$ and $\mathbf{u}_\pm = \mathbf{Z}^e_\pm + \mathbf{v}_\pm$, so that ρ_+ and $\mathbf{v}_\pm$ enjoy the equations:

$$
\begin{aligned}
\partial_t\rho_+ + (\rho_{0+} + \theta_{0+})\mathrm{div}\,\mathbf{v}_+ &= F_+(\Pi^0_+ + \rho_+, \mathbf{Z}^e_+ + \mathbf{v}_+) && \text{in } \Omega_+, \\
(\rho_{0+} + \theta_{0+})\partial_t\mathbf{v}_+ - \mathrm{Div}\,\mathbf{S}_+(\mathbf{v}_+) + \nabla(p'(\rho_{0+} + \theta_{0+})\rho_+) &= \mathbf{G}_+(\Pi^0_+ + \rho_+, \mathbf{Z}^e_+ + \mathbf{v}_+) && \text{in } \Omega_+, \\
\rho_{0-}\partial_t\mathbf{v}_- - \mathrm{Div}\,\mathbf{S}_-(\mathbf{v}_-) + \nabla p_- &= \mathbf{G}_-(\mathbf{Z}^e_- + \mathbf{v}_-) && \text{in } \Omega_-, \\
\mathrm{div}\,\mathbf{v}_- = F_-(\mathbf{Z}^e_- + \mathbf{v}_-) &= \mathrm{div}\,\mathbf{F}_-(\mathbf{Z}^e + \mathbf{v}_-) && \text{in } \Omega_-, \\
(\mathbf{S}_+(\mathbf{v}_+) - (p'(\rho_{0+} + \theta_{0+})\rho_+\mathbf{I})\mathbf{n}|_{\Gamma+0} - (\mathbf{S}_-(\mathbf{v}_-) - p_-\mathbf{I})\mathbf{n}|_{\Gamma-0} &= \mathbf{H}(\Pi^0_+ + \rho_+, \mathbf{Z}^e_\pm + \mathbf{v}_\pm)|_\Gamma, \\
\mathbf{v}_+|_{\Gamma+0} = \mathbf{v}_-|_{\Gamma-0}, \quad \mathbf{v}_+|_{\Gamma_+} = 0, \quad (\mathbf{S}_-(\mathbf{v}_-) - p_-\mathbf{I})\mathbf{n}_-|_{\Gamma_-} &= \mathbf{H}_-(\mathbf{Z}^e_- + \mathbf{v}_-)|_{\Gamma_-}
\end{aligned} \tag{125}
$$

for $0 < t < T$ subject to the initial condition: $(\rho_+, \mathbf{v}_+)|_{t=0} = (0,0)$ in Ω_+ and $\mathbf{v}_-|_{t=0} = 0$ in Ω_- with some pressure term p_-. We solve (125) by the contraction mapping principle. For this purpose, we introduce an underlying space $\mathcal{I}_{R,T}$ defined by:

$$
\begin{aligned}
\mathcal{I}_{R,T} = \{&(\rho_+, \mathbf{v}_+, \mathbf{v}_-) \mid (\rho_+, \mathbf{v}_+)|_{t=0} = (0,0) \ \text{ in } \Omega_+, \quad \mathbf{v}_-|_{t=0} = 0 \ \text{ in } \Omega_-, \\
&\rho_+ \in W^1_q((0,T), W^1_q(\Omega_+)), \mathbf{v}_\pm \in W^1_p((0,T), L_q(\Omega_\pm)^N) \cap L_p((0,T), W^2_q(\Omega_\pm)^N), \\
&\|\rho_+\|_{W^1_p((0,T),W^1_q(\Omega_+))} + \sum_{\ell=+,-}(\|\mathbf{v}_\ell\|_{W^1_q((0,T),L_q(\Omega_\ell))} + \|\mathbf{v}_\ell\|_{L_p((0,T),W^2_q(\Omega_\ell))}) \leq R\}.
\end{aligned} \tag{126}
$$

We choose $T > 0$ so small eventually that we may assume that $0 < T < 1$. We choose $R > 0$ large enough that $C_{R_1}R_1 \leq R$ in (124) in such a way that:

$$
\|e^{-\gamma t}\Pi^0_+\|_{W^1_p(\mathbb{R},W^1_q(\Omega_+))} + \sum_{\ell=+,-}\{\|e^{-\gamma t}\mathbf{Z}^e_\ell\|_{W^1_p(\mathbb{R},L_q(\Omega_\ell))} + \|e^{-\gamma t}\mathbf{Z}^e_\ell\|_{L_p(\mathbb{R},W^2_q(\Omega_\ell))}\} \leq R. \tag{127}
$$

In the following, C denotes a generic constant depending on R_1, but we do not mention this dependence. For any function f defined on $(0,T)$ with $f(x,0) = 0$, f^0 denotes the zero extension of f to $t < 0$, and we define $E[f](x,t)$ by $E[f](x,t) = f^0(x,t)$ for $t \leq T$ and $E[f](x,t) = f^0(x, 2T - t)$ for $t > T$. Note that $E[f] = 0$ for $t \notin [0, 2T]$ and that $\partial_t E[f] = \partial_t f$ for $0 < t < T$, $\partial_t E[f](\cdot,t) = -(\partial_t f)(\cdot, 2T - t)$ for $T < t < 2T$, and $\partial_t E[f] = 0$ for $t \notin [0, 2T]$. For $(\kappa_+, \mathbf{w}_+, \mathbf{w}_-) \in \mathcal{I}_{R,T}$, we set:

$$
F_1[\kappa_+] = \Pi^0_+ + E[\kappa_+], \quad F_{2\pm}[\mathbf{w}_\pm] = \mathbf{Z}^e_\pm + E[\mathbf{w}_\pm], \quad F_{3\pm}[\mathbf{w}_\pm] = E\Big[\int_0^t \nabla F_{2\pm}[\mathbf{w}_\pm](\cdot,s)\,ds\Big].
$$

Note that:

$$F_1[\kappa_+] = \Pi^0_+ + \kappa_+, \quad F_{2\pm}[\mathbf{w}_\pm] = \mathbf{Z}^e_\pm + \mathbf{w}_\pm, \quad F_{3\pm}[\mathbf{w}_\pm] = \int_0^t \nabla(\mathbf{Z}^e_\pm + \mathbf{w}_\pm)\, ds \quad \text{when } t \in (0, T). \tag{128}$$

Employing the same argument due to Shibata and Shimizu ([21] Section 2) and using (126) and (127), we have:

$$\|F_1[\kappa_+]\|_{L_\infty(\mathbb{R}, W^1_q(\Omega_+))} \le CRT^{1/p'}, \quad \|F_{3\pm}[\mathbf{w}_\pm]\|_{L_\infty(\mathbb{R}, W^1_q(\Omega_\pm))} \le CRT^{1/p'}, \tag{129}$$

where we used the fact that $F_1[\kappa_+](\cdot, t) = \int_0^t (\partial_s F_1[\kappa])(\cdot, s)\, ds$. In addition, by (126) and (127):

$$\sum_{\ell=+,-} \{\|e^{-\gamma t} F_{2\ell}[\mathbf{w}_\ell]\|_{W^1_p(\mathbb{R}, L_q(\Omega_\ell)} + \|e^{-\gamma t} F_{2\ell}[\mathbf{w}_\ell]\|_{L_p(\mathbb{R}, W^2_q(\Omega_\ell))}\} + \|e^{-\gamma t} F_1[\kappa_+]\|_{W^1_p(\mathbb{R}, W^1_q(\Omega_+))} \le CR. \tag{130}$$

Moreover, we have:

$$\|e^{-\gamma t} \Lambda^{1/2}_\gamma F_{2\pm}[\mathbf{w}_\pm]\|_{L_p(\mathbb{R}, W^1_q(\Omega_\pm))} \le CR, \quad \|e^{-\gamma t} F_{2\pm}[\mathbf{w}_\pm]\|_{L_\infty(\mathbb{R}, W^1_q(\Omega_\pm))} \le CR, \tag{131}$$

$$\|\partial_t F_{3\pm}[\mathbf{w}_\pm]\|_{L_\infty(\mathbb{R}, L_q(\Omega_\pm))} \le CR, \quad \|\partial_t F_{3\pm}[\mathbf{w}_\pm]\|_{L_p(\mathbb{R}, W^1_q(\Omega_\pm))} \le CR. \tag{132}$$

In fact, as was seen in Shibata and Shimizu [22], $L_{p,\gamma}(\mathbb{R}, W^2_q(\Omega_\pm)) \cap W^1_{p,\gamma}(\mathbb{R}, L_q(\Omega_\pm))$ is continuously imbedded into $H^{1/2}_{p,\gamma}(\mathbb{R}, W^1_q(\Omega_\pm))$, and so, we have the first estimate in (131) by (130). Replacing the Fourier multiplier theorem of the Mihlin type [23] by that of Bourgain [12] (cf. Lemma 2) in the paper due to Calderón [24] about the Bessel potential space (cf. Amann [25]), we see that $H^{1/2}_{p,\gamma}(\mathbb{R}, L_q(\Omega_\pm))$ is continuously imbedded into the space $\{v \mid e^{-\gamma t} v \in L_\infty(\mathbb{R}, L_q(\Omega_\pm))\}$ if $p > 2$. Thus, we have:

$$\|e^{-\gamma t} F_{2\pm}[\mathbf{w}_\pm]\|_{L_\infty(\mathbb{R}, W^1_q(\Omega_\pm))} \le C\|e^{-\gamma t} \Lambda^{1/2}_\gamma F_{2\pm}[\mathbf{w}_\pm]\|_{L_p(\mathbb{R}, W^1_q(\Omega_\pm))},$$

and therefore, the second estimate in (131) follows from the first one. Since $\partial_t F_{3\pm}[\mathbf{w}_\pm] = \nabla F_{2\pm}[\mathbf{w}_\pm]$ for $0 \le t \le T$, $\partial_t F_{3\pm}[\mathbf{w}_\pm] = -\nabla F_{2\pm}[\mathbf{w}_\pm](\cdot, 2T - t)$ for $T \le t \le 2T$, and $\partial_t F_{3\pm}[\mathbf{w}_\pm] = 0$ for $t \notin [0, 2T]$, (132) follows from (131) and (130).

We choose $T \in (0, 1)$ so small that:

$$CRT^{1/p'} < \rho_0/4, \quad CRT^{1/p'} < \sigma/2, \tag{133}$$

and therefore, we can define $p(\rho_{0+} + \theta_{0+} + \tau F_1[\kappa_+])$ $(0 \le \tau \le 1)$ and $\mathbf{V}_\ell(F_{3\pm}[\mathbf{w}_\pm])$ $(\ell = 0, D, -1)$. Since $\mathbf{V}_\ell(0) = 0$ $(\ell = 0, D, -1)$, by (129) and (132):

$$\begin{aligned} &\|\mathbf{V}_\ell(F_{3\pm}[\mathbf{w}_\pm])\|_{L_\infty(\mathbb{R}, W^1_q(\Omega_\pm))} \le CRT^{1/p'}, \quad \|\partial_t \mathbf{V}_\ell(F_{3\pm}[\mathbf{w}_\pm])\|_{L_\infty(\mathbb{R}, W^1_q(\Omega_\pm))} \le CR, \\ &\|\partial_t \mathbf{V}_\ell(F_{3\pm}[\mathbf{w}_\pm])\|_{L_\infty(\mathbb{R}, L_q(\Omega_\pm))} \le CR, \quad \|\partial_t \mathbf{V}_\ell(F_{3\pm}[\mathbf{w}_\pm])\|_{L_p(\mathbb{R}, W^1_q(\Omega_\pm))} \le CR \end{aligned} \tag{134}$$

for $\ell = 0, D, -1$.

We define $f_+(\kappa_+, \mathbf{w}_+)$, $\mathbf{g}_+(\kappa_+, \mathbf{w}_+)$, $\mathbf{g}_-(\mathbf{w}_-)$, $f_-(\mathbf{w}_-) = \operatorname{div} \tilde{f}_-(\mathbf{w}_-)$, $\mathbf{h}(\kappa_+, \mathbf{w}_\pm)$, and $\mathbf{h}_-(\mathbf{w}_-)$ by:

$$\begin{aligned} f_+(\kappa_+, \mathbf{w}_+) = &-\{F_1[\kappa_+] \operatorname{div} F_{2+}[\mathbf{w}_+] + (\rho_{0+} + \theta_{0+} + F_1[\kappa_+]) \operatorname{tr}(\mathbf{V}_0(F_{3+}[\mathbf{w}_+]) \nabla F_{2+}[\mathbf{w}_+])\}, \\ \mathbf{g}_+(\kappa_+, \mathbf{g}_+) = &-F_1[\kappa_+] \partial_t F_{2+}[\mathbf{w}_+] + \operatorname{Div}\{\mu_+ \mathbf{V}_D(F_{3+}[\mathbf{w}_+]) \nabla F_{2+}[\mathbf{w}_+] \\ &+ (\nu_+ - \mu_+) \operatorname{tr}(\mathbf{V}_0(F_{3+}[\mathbf{w}_+]) \nabla F_{2+}[\mathbf{w}_+]) \mathbf{I}\} \\ &+ \mathbf{V}_0(F_{3+}[\mathbf{w}_+]) \nabla \{\mu_+(\mathbf{D}(F_{2+}[\mathbf{w}_+]) + \mathbf{V}_D(F_{3+}[\mathbf{w}_+]) \nabla F_{2+}[\mathbf{w}_+]) \\ &+ (\nu_+ - \mu_+)(\operatorname{div} F_{2+}[\mathbf{w}_+] + \operatorname{tr}(\mathbf{V}_0(F_{3+}[\mathbf{w}_+]) \nabla F_{2+}[\mathbf{w}_+]) \mathbf{I}\} \\ &- \nabla\Big(\int_0^1 p''(\rho_{0+} + \theta_{0+} + \tau F_1[\kappa_+])(1 - \tau)\, d\tau\, (F_1[\kappa_+])^2\Big) \\ &- \mathbf{V}_0(F_{3+}[\mathbf{w}_+]) p'(\rho_{0+} + \theta_{0+} + F_1[\kappa_+]) \nabla(\theta_{0+} + F_1[\kappa_+]), \end{aligned}$$

$$\begin{aligned}
\mathbf{g}_-(\mathbf{w}_-) &= -\rho_{0-}\mathbf{V}_{-1}(F_{3-}[\mathbf{w}_-])\partial_t F_{2-}[\mathbf{w}_-] + \mu_-[\mathrm{Div}\,(\mathbf{V}_D(F_{3-}[\mathbf{w}_-])\nabla F_{2-}[\mathbf{w}_-]) \\
&\quad + \mathbf{V}_{-1}(F_{3-}[\mathbf{w}_-])\mathrm{Div}\,\{\mathbf{D}(F_{2-}[\mathbf{w}_-]) + \mathbf{V}_D(F_{3-}[\mathbf{w}_-])\nabla F_{2-}[\mathbf{w}_-]\}], \\
f_-(\mathbf{w}_-) &= -\mathrm{tr}(\mathbf{V}_0(F_{3-}[\mathbf{w}_-])\nabla F_{2-}[\mathbf{w}_-]), \\
\tilde{f}_-(\mathbf{w}_-) &= {}^T\mathbf{V}_0(F_{3-}[\mathbf{w}_-])F_{2-}[\mathbf{w}_-], \\
\mathbf{h}(\kappa_+, \mathbf{w}_\pm) &= \mathbf{h}^1(\mathbf{w}_\pm) + \mathbf{h}^2(\kappa_+), \\
\mathbf{h}_-(\mathbf{w}_-) &= -\mu_-\mathrm{Ext}^+\,[\mathbf{V}_D(F_{3-}[\mathbf{w}_-])\nabla F_{2-}[\mathbf{w}_{2-}] \\
&\quad + \mathbf{V}_{-1}(F_{3-}[\mathbf{w}_-])(\mathbf{D}(F_{2-}[\mathbf{w}_{2-}]) + \mathbf{V}_D(F_{3-}[\mathbf{w}_-])\nabla F_{2-}[\mathbf{w}_-]) \\
&\quad + (\mathbf{I} + \mathbf{V}_-(F_{3-}[\mathbf{w}_-]))(\mathbf{D}(F_{2-}[\mathbf{w}_{2-}]) + \mathbf{V}_D(F_{3-}[\mathbf{w}_-])\nabla F_{2-}[\mathbf{w}_-])\mathbf{V}_0(F_{3-}[\mathbf{w}_-])]\,\mathbf{n}_-,
\end{aligned}$$

where we set:

$$\begin{aligned}
&\mathbf{h}^1(\mathbf{w}_\pm) \\
&= -\mu_+\mathrm{Ext}^-[\mathbf{V}_D(F_{3+}[\mathbf{w}_+])\nabla F_{2+}[\mathbf{w}_+]] - (\nu_+ - \mu_+)\mathrm{Ext}^-[\mathrm{tr}(\mathbf{V}_0[F_{3+}[\mathbf{w}_+])\nabla F_{2+}[\mathbf{w}_+])]\mathbf{n} \\
&\quad - \mu_+\mathrm{Ext}^+[\mathbf{V}_{-1}(F_{3-}[\mathbf{w}_-])]\mathrm{Ext}^-\,[\mathbf{D}(F_{3+}[\mathbf{w}_+]) + \mathbf{V}_D(F_{3+}[\mathbf{w}_+])\nabla F_{2+}[\mathbf{w}_+]]\,\mathbf{n} \\
&\quad - \mu_+(\mathbf{I} + \mathrm{Ext}^+[\mathbf{V}_{-1}(F_{3-}[\mathbf{w}_-])])\mathrm{Ext}^-\,[\mathbf{D}(F_{2+}[\mathbf{w}_+]) + \mathbf{V}_D(F_{3+}[\mathbf{w}_+])\nabla F_{2+}[\mathbf{w}_+])\mathbf{V}_0(F_{3+}[\mathbf{w}_+])]\,\mathbf{n} \\
&\quad + \mu_-\mathrm{Ext}^+\Big[\mathbf{V}_D(F_{3-}[\mathbf{w}_-])\nabla F_{2-}[\mathbf{w}_-] + \mathbf{V}_{-1}(F_{3-}[\mathbf{w}_-])(\mathbf{D}(F_{2-}[\mathbf{w}_-]) + \mathbf{V}_D(F_{3-}[\mathbf{w}_-])\nabla F_{2-}[\mathbf{w}_-]) \\
&\quad + (\mathbf{I} + \mathbf{V}_{-1}(F_{3-}[\mathbf{w}_-]))(\mathbf{D}(F_{2-}[\mathbf{w}_-]) + \mathbf{V}_D(F_{3-}[\mathbf{w}_-])\nabla F_{2-}[\mathbf{w}_-])\mathbf{V}_0(F_{3-}[\mathbf{w}_-])\Big]\,\mathbf{n}
\end{aligned}$$

and:

$$\mathbf{h}^2(\kappa_+) = \mathrm{Ext}^-\Big[\int_0^1 (1-\tau)p''(\rho_{0+} + \theta_{0+} + \tau F_1[\kappa_+])\,d\tau\,(F_1[\kappa_+])^2\Big]\mathbf{n}.$$

By (128), we have:

$$\begin{aligned}
f_+(\kappa_+, \mathbf{w}_+) &= F_+(\Pi_+^0 + \kappa_+, \mathbf{Z}_+^e + \mathbf{w}_+), \quad \mathbf{g}_+(\kappa_+, \mathbf{w}_+) = \mathbf{G}_+(\Pi_+^0 + \kappa_+, \mathbf{Z}_+^e + \mathbf{w}_+), \\
\mathbf{g}_-(\mathbf{w}_-) &= \mathbf{G}_-(\mathbf{Z}_-^e + \mathbf{w}_-), \quad f_-(\mathbf{w}_-) = F_-(\mathbf{Z}_-^e + \mathbf{w}_-), \\
\mathbf{h}(\kappa_+, \mathbf{w}_\pm)|_\Gamma &= \mathbf{H}(\Pi_+^0 + \kappa_+, \mathbf{Z}_\pm^e + \mathbf{w}_\pm), \quad \mathbf{h}_-(\mathbf{w}_-)|_{\Gamma_-} = \mathbf{H}_-(\mathbf{Z}_-^e + \mathbf{w}_-)
\end{aligned} \tag{135}$$

for $0 < t < T$. By (120), (129), (130), and (134), we have:

$$\begin{aligned}
&\|e^{-\gamma t}f_+(\kappa_+, \mathbf{w}_+)\|_{L_p(\mathbb{R}, W_q^1(\Omega_+))} + \|e^{-\gamma t}\mathbf{g}_+(\kappa_+, \mathbf{w}_+)\|_{L_p(\mathbb{R}, L_q(\Omega_+))} + \|e^{-\gamma t}\mathbf{g}_-(\mathbf{w}_-)\|_{L_p(\mathbb{R}, L_q(\Omega_-))} \\
&\quad + \|e^{-\gamma t}\nabla f_-(\mathbf{w}_-)\|_{L_p(\mathbb{R}, L_q(\Omega_-))} + \|e^{-\gamma t}\nabla \mathbf{h}(\kappa_+, \mathbf{w}_\pm)\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{-\gamma t}\nabla \mathbf{h}_-(\mathbf{w}_-)\|_{L_p(\mathbb{R}, L_q(\Omega_-))} \\
&\qquad \le C_R T^{1/p'}
\end{aligned} \tag{136}$$

with some constant C_R depending on R. Since $\partial_t F_{3-}[\mathbf{w}_-] = 0$ for $t \notin [0, T]$, we have:

$$\begin{aligned}
\|e^{-\gamma t}\partial_t \tilde{f}_-(\mathbf{w}_-)\|_{L_p(\mathbb{R}, L_q(\Omega_-))} &\le \|\mathbf{V}_0(F_{3-}[\mathbf{w}_-])\|_{L_\infty(\mathbb{R}, W_q^1(\Omega_-))}\|e^{-\gamma t}F_{2-}[\mathbf{w}_-]\|_{W_p^1(\mathbb{R}, L_q(\Omega_-))} \\
&\quad + T^{1/p}\|\partial_t \mathbf{V}_0(F_{3-}[\mathbf{w}_-])\|_{L_\infty(\mathbb{R}, L_q(\Omega_-))}\|e^{-\gamma t}F_{2-}[\mathbf{w}_-]\|_{L_\infty(\mathbb{R}, W_q^1(\Omega_-))},
\end{aligned}$$

and so, by (134), (131), and (130), we have:

$$\|e^{-\gamma t}\partial_t \tilde{f}_-(\mathbf{w}_-)\|_{L_p(\mathbb{R}, L_q(\Omega_-))} \le C_R T^{1/p}. \tag{137}$$

To estimate $\|e^{-\gamma t}\Lambda_\gamma^{1/2} f_-(\mathbf{w}_-)\|_{L_p(\mathbb{R}, L_q(\Omega_-))}$, we use the following lemma due to Shibata and Shimizu ([21] Lemma 2.6).

Lemma 11. *Let $2 < p < \infty$, $N < q < \infty$, and $0 < T \leq 1$. Let $f \in L_\infty(\mathbb{R}, W^1_q(\Omega_-)) \cap W^1_\infty(\mathbb{R}, L_q(\Omega_-))$ and $g \in H^{1/2}_{p,\gamma}(\mathbb{R}, L_q(\Omega_-)) \cap L_{p,\gamma}(\mathbb{R}, W^1_q(\Omega_-))$. If $\partial_t f \in L_p(\mathbb{R}, W^1_q(\Omega_-))$ and $f(\cdot, t) = 0$ for $t \notin [0, 2T]$, then we have:*

$$\begin{aligned}
&\|e^{-\gamma t}\Lambda^{1/2}_\gamma (fg)\|_{L_p(\mathbb{R}, L_q(\Omega_-))} \\
&\leq C\{\|f\|_{L_\infty(\mathbb{R}, W^1_q(\Omega_-))} + T^{(q-N)/(pq)}\|\partial_t f\|^{1-N/(2q)}_{L_\infty(\mathbb{R}, L_q(\Omega_-))}\|\partial_t f\|^{N/(2q)}_{L_p(\mathbb{R}, W^1_q(\Omega_-))}\} \\
&\qquad \times (\|e^{-\gamma t}\Lambda^{1/2}_\gamma g\|_{L_p(\mathbb{R}, L_q(\Omega_-))} + \|e^{-\gamma t} g\|_{L_p(\mathbb{R}, W^1_q(\Omega_-))}).
\end{aligned}$$

Applying Lemma 11 to $f_-(\mathbf{w}_-)$, we have:

$$\begin{aligned}
&\|e^{-\gamma t}\Lambda^{1/2}_\gamma f_-(\mathbf{w}_-)\|_{L_p(\mathbb{R}, L_q(\Omega_-))} \\
&\leq C\Big\{\|\mathbf{V}_0(F_{3-}[\mathbf{w}_-])\|_{L_\infty(\mathbb{R}, W^1_q(\Omega_-))} \\
&\qquad + T^{(q-N)/(pq)}\|\partial_t \mathbf{V}_0(F_{3-}[\mathbf{w}_-])\|^{1-N/(2q)}_{L_\infty(\mathbb{R}, L_q(\Omega_-))}\|\partial_t \mathbf{V}_0(F_{3-}[\mathbf{w}_-])\|^{N/(2q)}_{L_p(\mathbb{R}, W^1_q(\Omega_-))}\Big\} \\
&\quad \times (\|e^{-\gamma t}\Lambda^{1/2}_\gamma \nabla F_{2-}[\mathbf{w}_-]\|_{L_p(\mathbb{R}, L_q(\Omega_-))} + \|e^{-\gamma t}\nabla F_{2-}[\mathbf{w}_-]\|_{L_p(\mathbb{R}, W^1_q(\Omega_-))}),
\end{aligned}$$

and so, by (130) and (134):

$$\|e^{-\gamma t}\Lambda^{1/2}_\gamma f_-(\mathbf{w}_-)\|_{L_p(\mathbb{R}, L_q(\Omega_-))} \leq C_R(T^{1/p'} + T^{(q-N)/(pq)}). \tag{138}$$

Analogously, we have:

$$\|e^{-\gamma t}\Lambda^{1/2}_\gamma \mathbf{h}^1(\mathbf{w}_\pm)\|_{L_p(\mathbb{R}, L_q(\Omega_\pm))} + \|e^{-\gamma t}\Lambda^{1/2}_\gamma \mathbf{h}_-(\mathbf{w}_-)\|_{L_p(\mathbb{R}, L_q(\Omega_-))} \leq C_R(T^{1/p'} + T^{(q-N)/(pq)}). \tag{139}$$

Since $\|e^{-\gamma t}\Lambda^{1/2}_\gamma \mathbf{h}^2(\kappa_+)\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq C\|e^{-\gamma t}\partial_t \mathbf{h}^2(\kappa_+)\|_{L_p(\mathbb{R}, L_q(\Omega))}$, by (120), (129), and (130), we have:

$$\|e^{-\gamma t}\Lambda^{1/2}_\gamma \mathbf{h}^2(\kappa_+)\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq C_R T^{1/p'}. \tag{140}$$

Let ρ_+ and $\mathbf{v}_\pm$ be solutions to equations:

$$\begin{aligned}
\partial_t\rho_+ + (\rho_{0+} + \theta_{0+})\operatorname{div}\mathbf{v}_+ &= f_+(\kappa_+, \mathbf{w}_+) && \text{in } \Omega_+, \\
(\rho_{0+} + \theta_{0+})\partial_t\mathbf{v}_+ - \operatorname{Div}\mathbf{S}_+(\mathbf{v}_+) + \nabla(p'(\rho_{0+} + \theta_{0+})\rho_+) &= \mathbf{g}_+(\kappa_+, \mathbf{w}_+) && \text{in } \Omega_+, \\
\rho_{0-}\partial_t\mathbf{v}_- - \operatorname{Div}\mathbf{S}_-(\mathbf{v}_-) + \nabla p_- &= \mathbf{g}_-(\mathbf{w}_-) && \text{in } \Omega_-, \\
\operatorname{div}\mathbf{v}_- &= f_-(\mathbf{w}_-) = \operatorname{div}\tilde{f}_-(\mathbf{w}_-) && \text{in } \Omega_-, \\
(\mathbf{S}_+(\mathbf{v}_+) - (p'(\rho_{0+} + \theta_{0+})\rho_+\mathbf{I})\mathbf{n}|_{\Gamma+0} - (\mathbf{S}_-(\mathbf{v}_-) - p_-\mathbf{I})\mathbf{n}|_{\Gamma-0} &= \mathbf{h}(\kappa_+, \mathbf{w}_\pm)|_\Gamma, \\
\mathbf{v}_+|_{\Gamma+0} = \mathbf{v}_-|_{\Gamma-0}, \quad \mathbf{v}_+|_{\Gamma_+} = 0, \quad (\mathbf{S}_-(\mathbf{v}_-) - p_-\mathbf{I})\mathbf{n}_-|_{\Gamma_-} &= \mathbf{h}_-(\mathbf{w}_-)|_{\Gamma_-}
\end{aligned} \tag{141}$$

for $0 < t < T$ subject to the initial condition: $(\rho_+, \mathbf{v}_+)|_{t=0} = (0, 0)$ in Ω_+ and $\mathbf{v}_-|_{t=0} = 0$ in Ω_- with some pressure term p_-. By Theorem 3 and the estimates (136), (137), (139), and (140), we have:

$$\rho_+ \in W^1_{p,\gamma,0}(\mathbb{R}, W^1_q(\Omega_+)), \quad \mathbf{v}_\pm \in W^1_{p,\gamma,0}(\mathbb{R}, L_q(\Omega_\pm)) \cap L_{p,\gamma,0}(\mathbb{R}, W^2_q(\Omega_\pm)). \tag{142}$$

$$\|e^{-\gamma t}\rho\|_{W^1_q(\mathbb{R}, W^1_q(\Omega_+))} + \sum_{\ell=+,-}\|e^{-\gamma t}(\partial_t\mathbf{v}_\ell, \Lambda^{1/2}_\gamma\nabla\mathbf{v}_\ell, \nabla^2\mathbf{v}_\ell)\|_{L_p(|BR, L_q(\Omega_\ell))} \leq C_R T^\omega \tag{143}$$

with some constant C_R depending on R and $\omega = \min(1/p', (q-N)/(pq))$. By (135), ρ_+ and $\mathbf{v}_\pm$ satisfy equations:

$$\begin{aligned}
\partial_t\rho_+ + (\rho_{0+} + \theta_{0+})\operatorname{div}\mathbf{v}_+ &= F_+(\Pi^0_+ + \kappa_+, \mathbf{Z}^e_+ + \mathbf{w}_+) && \text{in } \Omega_+, \\
(\rho_{0+} + \theta_{0+})\partial_t\mathbf{v}_+ - \operatorname{Div}\mathbf{S}_+(\mathbf{v}_+) + \nabla(p'(\rho_{0+} + \theta_{0+})\rho_+) &= \mathbf{G}_+(\Pi^0_+ + \kappa_+, \mathbf{Z}^e_+ + \mathbf{w}_+) && \text{in } \Omega_+, \\
\rho_{0-}\partial_t\mathbf{v}_- - \operatorname{Div}\mathbf{S}_-(\mathbf{v}_-) + \nabla p_- &= \mathbf{G}_-(\mathbf{Z}^e_- + \mathbf{w}_-) && \text{in } \Omega_-,
\end{aligned}$$

$$
\begin{aligned}
\operatorname{div} \mathbf{v}_- &= F_-(\mathbf{Z}^e_- + \mathbf{w}_-) && \text{in } \Omega_-,\\
(\mathbf{S}_+(\mathbf{v}_+) - (p'(\rho_{0+} + \theta_{0+})\rho_+\mathbf{I})\mathbf{n}|_{\Gamma+0} - (\mathbf{S}_-(\mathbf{v}_-) - p_-\mathbf{I})\mathbf{n}|_{\Gamma-0} &= \mathbf{H}(\Pi^0_+ + \kappa_+, \mathbf{Z}^e_\pm + \mathbf{w}_\pm),\\
\mathbf{v}_+|_{\Gamma+0} = \mathbf{v}_-|_{\Gamma-0}, \quad \mathbf{v}_+|_{\Gamma_+} = 0, \quad (\mathbf{S}_-(\mathbf{v}_-) - p_-\mathbf{I})\mathbf{n}_-|_{\Gamma_-} &= \mathbf{H}_-(\mathbf{Z}^e_- + \mathbf{w}_-)
\end{aligned}
$$

for $0 < t < T$ subject to the initial condition: $(\rho_+, \mathbf{v}_+)|_{t=0} = (0, 0)$ in Ω_+ and $\mathbf{v}_-|_{t=0} = 0$ in Ω_- with some pressure term p_-.

Let Φ be a map defined by $\Phi(\kappa_+, \mathbf{w}_\pm) =$, the restriction of $(\rho_+, \mathbf{v}_\pm)$ to the time interval $(0, T)$. Since:

$$
\|\rho_+\|_{W^1_p((0,T),W^1_q(\Omega_+))} + \sum_{\ell=+,-} \{\|\mathbf{v}_\ell\|_{W^1_q((0,T),L_q(\Omega_\ell))} + \|\mathbf{v}_\ell\|_{L_p((0,T),W^2_q(\Omega_\ell))}\} \leq C_\gamma e^{\gamma T} C_R T^\omega
$$

as follows from (143), choosing $T > 0$ so small that $C_\gamma e^{\gamma T} C_R T^\omega \leq R$, we see that Φ is the map from $\mathcal{I}_{R,T}$ into itself. Choosing $T > 0$ smaller if necessary, we can show that Φ is a contraction map on $\mathcal{I}_{R,T}$, and so by the Banach fixed point theorem Φ has a unique fixed point $(\rho_+, \mathbf{v}_+)$ that solves Equation (125) uniquely. This completes the proof of Theorem 1.

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