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Local Well-Posedness for Free Boundary Problem of Viscous Incompressible Magnetohydrodynamics

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Abstract: In this paper, we consider the motion of incompressible magnetohydrodynamics (MHD) with resistivity in a domain bounded by a free surface. An electromagnetic field generated by some currents in an external domain keeps an MHD flow in a bounded domain. On the free surface, free boundary conditions for MHD flow and transmission conditions for electromagnetic fields are imposed. We proved the local well-posedness in the general setting of domains from a mathematical point of view. The solutions are obtained in an anisotropic space $H^1_p((0,T),H^1_q) \cap L_p((0,T),H^3_q)$ for the velocity field and in an anisotropic space $H^1_p((0,T),L_q) \cap L_p((0,T),H^2_q)$ for the magnetic fields with $2 < p < \infty$, $N < q < \infty$ and $2/p + N/q < 1$. To prove our main result, we used the L_p-L_q maximal regularity theorem for the Stokes equations with free boundary conditions and for the magnetic field equations with transmission conditions, which have been obtained by Frolova and the second author.

Keywords: free boundary problem; transmission condition; magnetohydorodynamics; local well-posedness; L_p-L_q maximal regularity

MSC: 35K59; 76W05

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1. Introduction

In this paper, we prove the local well-posedness of a free boundary problem for the viscous non-homogeneous incompressible magnetohydrodynamics. The problem here is formulated as follows: Let Ω_+ be a domain in the N-dimensional Euclidean space $\mathbb{R}^N$ ($N \geq 2$), and let Γ be the boundary of Ω_+. Let Ω_- be also a domain in $\mathbb{R}^N$ whose boundary is Γ and S_-. We assume that $\Omega_+ \cap \Omega_- = \emptyset$. Throughout the paper, we assume that $\Omega_\pm$ are uniform C^2 domains, that the weak Dirichlet problem is uniquely solvable in Ω_+ (The definition of uniform C^2 domains and the weak Dirichlet problem will be given in Section 3 below.) and that $\operatorname{dist}(\Gamma, S_-) \geq 2d_-$ with some positive constants d_-, where $\operatorname{dist}(A,B)$ denotes the distance of any two subsets A and B of $\mathbb{R}^N$ defined by setting $\operatorname{dist}(A,B) = \inf\{|x-y| \mid x \in A, y \in B\}$. Let $\Omega = \Omega_+ \cup \Gamma \cup \Omega_-$ and $\dot\Omega = \Omega_+ \cup \Omega_-$. The boundary of Ω is S_-. We may consider the case that S_- is an empty set, and in this case $\Omega = \mathbb{R}^N$. Physically, we consider the case where Ω_+ is filled by a non-homogeneous incompressible magnetohydrodynamic (MHD) fluid and Ω_- is filled by an insulating gas. We consider a motion of an MHD fluid in a time dependent domain Ω_{t+} whose boundary is Γ_t subject to an electomagnetic field generated in a domain $\Omega_{t-} = \Omega \setminus (\Omega_{t+} \cup \Gamma_t)$ by some currents located on a fixed boundary S_- of Ω_{t-}. Let $\mathbf{n}_t$ be the unit outer normal to Γ_t oriented from Ω_{t+} into Ω_{t-}, and let $\mathbf{n}_-$ be respective the unit outer normals to S_-. Given any functions, $v_\pm$, defined on $\Omega_{t\pm}$, v is defined by $v(x) = v_\pm(x)$ for $x \in \Omega_{t\pm}$ for $t \geq 0$, where $\Omega_{0\pm} = \Omega_\pm$. Moreover, what $v = v_\pm$ denotes that $v(x) = v_+(x)$ for $x \in \Omega_{t+}$ and $v(x) = v_{t-}(x)$ for $x \in \Omega_{t-}$. Let

$$[[v]](x_0) = \lim_{\substack{x\to x_0 \\ x\in\Omega_{t+}}} v_+(x) - \lim_{\substack{x\to x_0 \\ x\in\Omega_{t-}}} v_-(x)$$

for every point $x_0 \in \Gamma_t$, which is the jump quantity of v across Γ.

The purpose of this paper is to prove the local well-posedness of the free boundary problem formulated by the set of the following equations:

$$\begin{aligned}
\rho(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) - \mathrm{Div}\,(\mathbf{T}(\mathbf{v},\mathfrak{p}) + \mathbf{T}_M(\mathbf{H}_+)) = 0, \quad \mathrm{div}\,\mathbf{v} = 0 \quad & \text{in } \Omega_+^T,\\
\mu_+\partial_t\mathbf{H}_+ + \mathrm{Div}\,\{\alpha_+^{-1}\mathrm{curl}\,\mathbf{H}_+ - \mu_+(\mathbf{v}\otimes\mathbf{H}_+ - \mathbf{H}_+\otimes\mathbf{v})\} = 0, \quad \mathrm{div}\,\mathbf{H}_+ = 0 \quad & \text{in } \Omega_+^T,\\
\mu_-\partial_t\mathbf{H}_- + \mathrm{Div}\,\{\alpha_-^{-1}\mathrm{curl}\,\mathbf{H}_-\} = 0, \quad \mathrm{div}\,\mathbf{H}_- = 0 \quad & \text{in } \Omega_-^T,\\
(\mathbf{T}(\mathbf{v},\mathfrak{p}) + \mathbf{T}_M(\mathbf{H}_+))\mathbf{n}_t = 0, \quad V_{\Gamma_t} = \mathbf{v}\cdot\mathbf{n}_t \quad & \text{on } \Gamma^T,\\
[[(\alpha^{-1}\mathrm{curl}\,\mathbf{H})\mathbf{n}_t]] - \mu_+(\mathbf{v}\otimes\mathbf{H}_+ - \mathbf{H}_+\otimes\mathbf{v})\mathbf{n}_t = 0 \quad & \text{on } \Gamma^T,\\
[[\mu\mathbf{H}\cdot\mathbf{n}_t]] = 0, \quad [[\mathbf{H} - <\mathbf{H},\mathbf{n}_t> \mathbf{n}_t]] = 0 \quad & \text{on } \Gamma^T,\\
\mathbf{n}_-\cdot\mathbf{H}_- = 0, \quad (\mathrm{curl}\,\mathbf{H}_-)\mathbf{n}_- = 0 \quad & \text{on } S_-^T,\\
(\mathbf{v},\mathbf{H}_+)|_{t=0} = (\mathbf{v}_0,\mathbf{H}_{0+}) \quad \text{in } \Omega_+, \quad \mathbf{H}_-|_{t=0} = \mathbf{H}_{0-} \quad & \text{in } \Omega_-.
\end{aligned} \tag{1}$$

Here,

$$\Omega_\pm^T = \bigcup_{0<t<T} \Omega_{t\pm}\times\{t\}, \quad \Gamma^T = \bigcup_{0<t<T}\Gamma_t\times\{t\}, \quad S_-^T = S_-\times(0,T);$$

$\mathbf{v} = (v_1(x,t),\ldots,v_N(x,t))^\top$ is the velocity vector field, where $M^\top$ stands for the transposed M, $\mathfrak{p} = \mathfrak{p}(x,t)$ the pressure fields, and $\mathbf{H} = \mathbf{H}_\pm = (H_{\pm 1}(x,t),\ldots,H_{\pm N}(x,t))^\top$ the magnetic vector field. The $\mathbf{v}$, $\mathfrak{p}$, and $\mathbf{H}$ are unknows, while $\mathbf{v}_0$ and $\mathbf{H}_0$ are prescribed N-component vectors of functions. As for the remaining symbols, $\mathbf{T}(\mathbf{v},\mathfrak{p}) = \nu\mathbf{D}(\mathbf{v}) - \mathfrak{p}\mathbf{I}$ is the viscous stress tensor, $\mathbf{D}(\mathbf{v}) = \nabla\mathbf{v} + (\nabla\mathbf{v})^\top$ is the doubled deformation tensor whose (i,j)th component is $\partial_j v_i + \partial_i v_j$ with $\partial_i = \partial/\partial x_i$, $\mathbf{I}$ the $N\times N$ unit matrix, $\mathbf{T}_M(\mathbf{H}_+) = \mu_+(\mathbf{H}_+\otimes\mathbf{H}_+ - \frac{1}{2}|\mathbf{H}_+|^2\mathbf{I})$ the magnetic stress tensor, $\mathrm{curl}\,\mathbf{v} = (\nabla\mathbf{v})^\top - \nabla\mathbf{v}$ the doubled rotation tensor whose (i,j)th component is $\partial_j v_i - \partial_i v_j$, V_{Γ_t} the velocity of the evolution of Γ_t in the direction of $\mathbf{n}_t$. Moreover, ρ, $\mu_\pm$, ν, and $\alpha_\pm$ are positive constants describing respective the mass density, the magnetic permeability, the kinematic viscosity, and the conductivity. Finally, for any matrix field $\mathbf{K}$ with (i,j)th component K_{ij}, the quantity $\mathrm{Div}\,K$ is an N-vector of functions with the ith component $\sum_{j=1}^N \partial_j K_{ij}$, and for any N-vectors of functions $\mathbf{u} = (u_1,\ldots,u_N)^\top$ and $\mathbf{w} = (w_1,\ldots,w_N)^\top$, $\mathrm{div}\,\mathbf{u} = \sum_{j=1}^N \partial_j u_j$, $\mathbf{u}\cdot\nabla\mathbf{w}$ is an N-vector of functions with the ith component $\sum_{j=1}^N u_j\partial_j w_i$, and $\mathbf{u}\otimes\mathbf{w}$ an $N\times N$ matrix with the (i,j)th component $u_i w_j$. We notice that in the three dimensional case

$$\begin{aligned}
&\Delta\mathbf{v} = -\mathrm{Div}\,\mathrm{curl}\,\mathbf{v} + \nabla\mathrm{div}\,\mathbf{v}, \quad \mathrm{Div}\,(\mathbf{v}\otimes\mathbf{H} - \mathbf{H}\otimes\mathbf{v}) = \mathbf{v}\mathrm{div}\,\mathbf{H} - \mathbf{H}\mathrm{div}\,\mathbf{v} + \mathbf{H}\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{H},\\
&\mathrm{rot}\,\mathrm{rot}\,\mathbf{H} = \mathrm{Div}\,\mathrm{curl}\,\mathbf{H}, \quad \mathrm{rot}\,(\mathbf{v}\times\mathbf{H}) = \mathrm{Div}\,(\mathbf{v}\otimes\mathbf{H} - \mathbf{H}\otimes\mathbf{v}),
\end{aligned} \tag{2}$$

where $\times$ is the exterior product. In particular, in the three dimensional case, the set of equations for the magnetic vector field in Equation (1) is written by

$$\begin{aligned}
\mu_+\partial_t\mathbf{H}_+ + \mathrm{rot}\,(\alpha_+^{-1}\mathrm{rot}\,\mathbf{H}_+ - \mu_+\mathbf{v}\times\mathbf{H}_+) = 0, \quad \mathrm{div}\,\mathbf{H}_+ = 0 \quad & \text{in } \Omega_+^T,\\
\mu_-\partial_t\mathbf{H}_- + \mathrm{rot}\,(\alpha_-^{-1}\mathrm{rot}\,\mathbf{H}_-) = 0, \quad \mathrm{div}\,\mathbf{H}_- = 0 \quad & \text{in } \Omega_-^T,\\
[[\mathbf{n}_t\times(\alpha^{-1}\mathrm{rot}\,\mathbf{H})]] - \mathbf{n}_t\times(\mu_+\mathbf{v}\times\mathbf{H}_+) = 0, \quad [[\mu\mathbf{H}\cdot\mathbf{n}_t]] = 0, \quad [[\mathbf{H} - <\mathbf{H},\mathbf{n}_t> \mathbf{n}_t]] = 0 \quad & \text{on } \Gamma^T.
\end{aligned}$$

This is a standard description and so the set of equations for the magnetic field in Equation (1) is the N-dimensional mathematical description of equations for the magnetic vector field with transmission conditions.

In Equation (1), there is one equation for the magnetic fields $\mathbf{H}_\pm$ too many, so that in this paper instead of (1), we consider the following equations:

$$\begin{aligned}
\rho(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) - \mathrm{Div}\,(\mathbf{T}(\mathbf{v},\mathfrak{p}) + \mathbf{T}_M(\mathbf{H}_+)) = 0, \quad \mathrm{div}\,\mathbf{v} = 0 &\quad \text{in } \Omega_+^T,\\
\mu_+\partial_t\mathbf{H}_+ - \alpha_+^{-1}\Delta\mathbf{H}_+ - \mathrm{Div}\,\mu_+(\mathbf{v}\otimes\mathbf{H}_+ - \mathbf{H}_+\otimes\mathbf{v}) = 0 &\quad \text{in } \Omega_+^T,\\
\mu_-\partial_t\mathbf{H}_- - \alpha_-^{-1}\Delta\mathbf{H}_- = 0 &\quad \text{in } \Omega_-^T,\\
(\mathbf{T}(\mathbf{v},\mathfrak{p}) + \mathbf{T}_M(\mathbf{H}_+))\mathbf{n}_t = 0, \quad V_{\Gamma_t} = \mathbf{v}\cdot\mathbf{n}_t &\quad \text{on } \Gamma^T,\\
[[(\alpha^{-1}\mathrm{curl}\,\mathbf{H})\mathbf{n}_t]] - \mu_+(\mathbf{v}\otimes\mathbf{H}_+ - \mathbf{H}_+\otimes\mathbf{v})\mathbf{n}_t = 0, \quad [[\mu\mathrm{div}\,\mathbf{H}]] = 0 &\quad \text{on } \Gamma^T,\\
[[\mu\mathbf{H}\cdot\mathbf{n}_t]] = 0, \quad [[\mathbf{H} - <\mathbf{H},\mathbf{n}_t>\mathbf{n}_t]] = 0 &\quad \text{on } \Gamma^T,\\
\mathbf{n}_-\cdot\mathbf{H}_- = 0, \quad (\mathrm{curl}\,\mathbf{H}_-)\mathbf{n}_- = 0 &\quad \text{on } S_-^T,\\
(\mathbf{v},\mathbf{H}_+)|_{t=0} = (\mathbf{v}_0,\mathbf{H}_{0+}) \quad \text{in } \Omega_+, \quad \mathbf{H}_{t-}|_{t=0} = \mathbf{H}_{0-} &\quad \text{in } \Omega_-.
\end{aligned} \tag{3}$$

Namely, two equations: $\mathrm{div}\,\mathbf{H}_\pm = 0$ in $\Omega_\pm^T$ are replaced with one transmission condition: $[[\mu\mathrm{div}\,\mathbf{H}]] = 0$ on Γ^T. Employing the same argument as in Frolova and Shibata ([1], Appendix), we see that in Equation (3) if $\mathrm{div}\,\mathbf{H} = 0$ initially, then $\mathrm{div}\,\mathbf{H} = 0$ in $\dot\Omega$ follows automatically for any $t > 0$ as long as solutions exist. Thus, the local well-posedness of Equation (1) follows from that of Equation (3) provided that the initial data $\mathbf{H}_{0\pm}$ satisfy the divergence zero condition: $\mathrm{div}\,\mathbf{H}_{0\pm} = 0$. This paper is devoted to proving the local well-posedness of Equation (3) in the maximal L_p-L_q regularity framework.

The MHD equations can be found in [2,3]. The solvability of MHD equations was first obtained by Ladyzhenskaya and Solonnikov [4]. The initial-boundary value problem for MHD equations with non-slip conditions for the velocity vector field and perfect wall conditions for the magnetic vector field was studied by Sermange and Temam [5] in a bounded domain and by Yamaguchi [6] in an exterior domain. In their studies [5,6], the boundary is fixed. On the other hand, in the field of engineering, when a thermonuclear reaction is caused artificially, a high-temperature plasma is sometimes subjected to a magnetic field and held in the air, and the boundary of the fluid at this time is a free one. From this point of view, the free boundary problem for MHD equations is important. The local well-posedness for free boundary problem for MHD equations was first proved by Padula and Solonnikov [7] in the case where Ω_{t+} is a bounded domain surrounded by a vacuum area, Ω_{t-}. In [7], the solution was obtained in Sobolev-Slobodetskii spaces in the L_2 framework of fractional order greater than 2. Later on, the global well-posedness was proved by Frolova [8] and Solonnikov and Frolova [9]. Moreover, the L_p approach to the same problem was done by Solonnikov [10,11]. When Ω_{t+} is a bounded domain, which is surrounded by an electromagnetic field generated in a domain, Ω_{t-}, Kacprzyk proved the local well-posedness in [12] and global well-posedness in [13]. In [12,13], the solution was also obtained in Sobolev-Slobodetskii spaces in the L_2 framework of fractional order greater than 2.

Recently, the L_p-L_q maximal regularity theorem for the initial boundary value problem of the system of parabolic equations with non-homogeneous boundary conditions has been studied by using $\mathcal{R}$-solver in [14] and references therein and by using H^∞ calculus in [15] and references therein. They are completely different approaches. In particular, Shibata [16,17] proved the L_p-L_q maximal regularity for the Stokes equations with non-homogeneous free boundary conditions by $\mathcal{R}$-solver theory and Frolova and Shibata [1] proved it for linearized equations for the magnetic vecor fields with transmission conditions on the interface and perfect wall conditions on the fixed boundary arising in the study of two phase problems for the MHD flows also by using the $\mathcal{R}$-solver. The results in [1,16,17] enable us to prove the local well-posedness for Equation (3) in the L_p-L_q maximal regularity class.

Aside from dynamical boundary conditions on Γ_t, a kinematic condition, $V_{\Gamma_t} = \mathbf{v}\cdot\mathbf{n}_t$, is satisfied on Γ_t, which represents Γ_t as a set of points $x = x(\xi, t)$ for $\xi \in \Gamma$, where $x(\xi, t)$ is the solution of the Cauchy problem:

$$\frac{dx}{dt} = \mathbf{v}(x,t), \quad x|_{t=0} = \xi. \tag{4}$$

This expresses the fact that the free surface Γ_t consists for all $t > 0$ of the same fluid particles, which do not leave it and are not incident on it from inside Ω_{t+}. Problem (3) can be written as an initial-boundary problem with transmission conditions on Γ if we go over the Euler coordinates $x \in \dot{\Omega}_t = \Omega_{t+} \cup \Omega_{t-}$ to the Lagrange coordinates $\xi \in \dot{\Omega} = \Omega_+ \cup \Omega_-$ connected with x by (4). Since the velocity field, $\mathbf{u}_+(\xi,t) = \mathbf{v}(x,t)$, is given only in Ω_+, we extend it to $\mathbf{u}_-$ defined on Ω_- in such a way that

$$\begin{aligned}
&\lim_{\substack{\xi\to\xi_0\\ \xi\in\Omega_+}} \partial_\xi^\alpha \mathbf{u}_+(\xi,t) = \lim_{\substack{\xi\to\xi_0\\ \xi\in\Omega_-}} \partial_\xi^\alpha \mathbf{u}_-(\xi,t) \quad \text{for } |\alpha| \le 3 \text{ and } (\xi_0,t) \in \Gamma \times (0,T),\\
&\|\mathbf{u}_-(\cdot,t)\|_{H^i_q(\Omega_-)} \le C_q \|\mathbf{u}_+(\cdot,t)\|_{H^i_q(\Omega_+)} \quad \text{for } i = 0,1,2,3 \text{ and } t \in (0,T).
\end{aligned} \tag{5}$$

Let $\varphi(\xi)$ be a $C^\infty(\mathbb{R}^N)$ function, which equals 1 when $\operatorname{dist}(\xi, S_-) \ge 2d_-$ and equals 0 when $\operatorname{dist}(\xi, S_-) \le d_-$. The connection between Euler coordinates x and Lagrangian coordinates ξ is defined by setting

$$x = \xi + \varphi(\xi)\int_0^t \mathbf{u}(\xi,s)\,ds = X_{\mathbf{u}}(\xi,t). \tag{6}$$

Define $\mathfrak{q}_+(\xi,t) := \mathfrak{p}(x,t)$ and $\tilde{\mathbf{H}}(\xi,t) = \tilde{\mathbf{H}}_\pm(\xi,t) = \mathbf{H}_\pm(x,t)$. Problem (3) is transformed by (6) to the following equations:

$$\begin{aligned}
\rho\partial_t\mathbf{u}_+ - \operatorname{Div}\mathbf{T}(\mathbf{u}_+,\mathfrak{q}_+) &= \mathbf{N}_1(\mathbf{u}_+,\tilde{\mathbf{H}}_+) && \text{in } \Omega_+ \times (0,T),\\
\operatorname{div}\mathbf{u}_+ = \mathbf{N}_2(\mathbf{u}_+) &= \operatorname{div}\mathbf{N}_3(\mathbf{u}_+) && \text{in } \Omega_+ \times (0,T),\\
T(\mathbf{u}_+,\mathfrak{q}_+)\mathbf{n} &= \mathbf{N}_4(\mathbf{u}_+,\tilde{\mathbf{H}}_+) && \text{on } \Gamma \times (0,T),\\
\mu\partial_t\tilde{\mathbf{H}} - \alpha^{-1}\Delta\tilde{\mathbf{H}} &= \mathbf{N}_5(\mathbf{u},\tilde{\mathbf{H}}) && \text{in } \dot{\Omega} \times (0,T),\\
[[\alpha^{-1}\operatorname{curl}\tilde{\mathbf{H}}]]\mathbf{n} &= \mathbf{N}_6(\mathbf{u},\tilde{\mathbf{H}}) && \text{on } \Gamma \times (0,T),\\
[[\mu\operatorname{div}\tilde{\mathbf{H}}]] &= N_7(\mathbf{u},\tilde{\mathbf{H}}) && \text{on } \Gamma \times (0,T),\\
[[\mu\tilde{\mathbf{H}}\cdot\mathbf{n}]] &= N_8(\mathbf{u},\tilde{\mathbf{H}}) && \text{on } \Gamma \times (0,T),\\
[[\tilde{\mathbf{H}}_\tau]] &= \mathbf{N}_9(\mathbf{u},\tilde{\mathbf{H}}) && \text{on } \Gamma \times (0,T),\\
\mathbf{n}_-\cdot\tilde{\mathbf{H}}_- = 0, \quad (\operatorname{curl}\tilde{\mathbf{H}}_-)\mathbf{n}_- &= 0 && \text{on } S_- \times (0,T),\\
\mathbf{u}_+|_{t=0} = \mathbf{u}_{0+} \quad \text{in } \Omega_+, \quad \tilde{\mathbf{H}}|_{t=0} &= \tilde{\mathbf{H}}_0 && \text{in } \dot{\Omega}.
\end{aligned} \tag{7}$$

Here, $\mathbf{n}$ is the unit outer normal to Γ oriented from Ω_+ into Ω_-, $\mathbf{d}_\tau = \mathbf{d} - <\mathbf{d},\mathbf{n}>\mathbf{n}$ for any N-vector $\mathbf{d}$, and the $\mathbf{N}_1(\mathbf{u}_+,\tilde{\mathbf{H}}_+),\dots,\mathbf{N}_9(\mathbf{u},\tilde{\mathbf{H}})$ are nonlinear terms defined in Section 2 below.

Our main result is the following theorem.

Theorem 1. *Let $1 < p, q < \infty$ and $B \ge 1$. Assume that $2/p + N/q < 1$, that Ω_+ is a uniform C^3 domain and $\Omega = \Omega_+ \cup \Gamma \cup \Omega_-$ a uniform C^2 domain, and that the weak problem is uniquely solvable in Ω_+ for q and $q' = q/(q-1)$. Let initial data $\mathbf{u}_{0+}$ and $\mathbf{H}_{0\pm}$ with*

$$\mathbf{u}_{0+} \in B^{3-2/p}_{q,p}(\Omega_+), \quad \tilde{\mathbf{H}}_{0\pm} \in B^{2(1-1/p)}_{q,p}(\Omega_\pm)$$

satisfy the conditions:

$$\|\mathbf{u}_{0+}\|_{B^{3-2/p}_{q,p}(\Omega_+)} + \|\tilde{\mathbf{H}}_0\|_{B^{2(1-1/p)}_{q}(\dot{\Omega})} \le B \tag{8}$$

and compatibility conditions

$$
\begin{aligned}
\operatorname{div} \mathbf{u}_{0+} &= 0 && \text{in } \Omega_+, \\
((\nu \mathbf{D}(\mathbf{u}_{0+}) + \mathbf{T}_M(\tilde{\mathbf{H}}_{0+})\mathbf{n})_\tau &= 0 && \text{on } \Gamma, \\
([[\alpha^{-1}\operatorname{curl} \tilde{\mathbf{H}}_0]] - \mu_+(\mathbf{u}_{0+} \otimes \tilde{\mathbf{H}}_{0+} - \tilde{\mathbf{H}}_{0+} \otimes \mathbf{u}_{0+}))\mathbf{n} &= 0 && \text{on } \Gamma, \\
[[\mu \operatorname{div} \tilde{\mathbf{H}}_0]] = 0, \quad [[\mu \tilde{\mathbf{H}}_0 \cdot \mathbf{n}]] = 0, \quad [[(\tilde{\mathbf{H}}_0)_\tau]] &= 0 && \text{on } \Gamma, \\
\mathbf{n}_- \cdot \tilde{\mathbf{H}}_{0-} = 0, \quad (\operatorname{curl} \tilde{\mathbf{H}}_{0-})\mathbf{n}_- &= 0 && \text{on } S_-.
\end{aligned} \tag{9}
$$

Then, the there exists a time $T > 0$ for which problem (7) admits unique solutions $\mathbf{u}_+$ and $\tilde{\mathbf{H}}_\pm$ with

$$
\begin{aligned}
\mathbf{u}_+ &\in L_p((0,T), H^3_q(\Omega_+)^N) \cap H^1_p((0,T), H^1_q(\Omega_+)^N), \\
\tilde{\mathbf{H}}_\pm &\in L_p((0,T), H^2_q(\Omega_\pm)^N) \cap H^1_p((0,T), L_q(\Omega_\pm)^N)
\end{aligned}
$$

possessing the estimate:

$$
\begin{aligned}
&\|\mathbf{u}_+\|_{L_p((0,T),H^3_q(\Omega_+))} + \|\partial_t \mathbf{u}_+\|_{L_p((0,T),H^1_q(\Omega_+))} \\
&\quad + \|\tilde{\mathbf{H}}\|_{L_p((0,T),H^2_q(\dot{\Omega}))} + \|\partial_t \tilde{\mathbf{H}}\|_{L_p((0,T),L_q(\dot{\Omega}))} \leq f(B)
\end{aligned}
$$

with some polynomial $f(B)$ with respect to B.

Remark 1. *As was mentioned after Equation (3), if we assume that $\operatorname{div} \tilde{\mathbf{H}}_{0\pm} = 0$ in $\Omega_\pm$ in addition, then $\operatorname{div} \mathbf{H}_\pm = 0$ in $\Omega^T_\pm$, and so $\mathbf{v}$ and $\mathbf{H}$ are solutions of Equation (1). Thus, we obtain the local well-posedness of Equation (1) from Theorem 1.*

Finally, we explain some symbols used throughout the paper.

Notation We denote the set of all natural numbers, real numbers, complex numbers by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$, respectively, and set $\mathbb{N}_0$ $\mathbb{N} \cup \{0\}$. For any multi-index $\kappa = (\kappa_1, \ldots, \kappa_N)$, $\kappa_j \in \mathbb{N}_0$, we set $\partial^\kappa_x = \partial^{\kappa_1}_1 \cdots \partial^{\kappa_N}_N$ and $|\kappa| = \sum_{j=1}^N \kappa_j$. For a scalar function, f, and an N-vector of functions, $\mathbf{g} = (g_1, \ldots, g_N)^\top$, we set $\nabla^n f = \{\partial^\kappa_x f \mid |\kappa| = n\}$ and $\nabla^n \mathbf{g} = \{\partial^\kappa_x g_j \mid |\kappa| = n, j = 1, \ldots, N\}$. In particular, $\nabla^0 f = f$, $\nabla^0 \mathbf{g} = \mathbf{g}$, $\nabla^1 f = \nabla f$, and $\nabla^1 \mathbf{g} = \nabla \mathbf{g}$. For notational convention, $\nabla \mathbf{g}$ and $\nabla^2 \mathbf{g}$ are sometimes considered as N^2 and N^3 column vectors, respectively, in the following way:

$$
\begin{aligned}
\nabla \mathbf{g} &= (\partial_1 g_1, \ldots, \partial_N g_1, \ldots, \partial_1 g_N, \ldots, \partial_N g_N)^\top, \\
\nabla^2 \mathbf{g} &= (\cdots, \partial_1\partial_1 g_\ell, \ldots, \partial_i\partial_j g_\ell, \ldots, \partial_N\partial_N g_\ell, \cdots)^\top
\end{aligned}
$$

for $\ell = 1, \ldots, N$, and $1 \leq i \leq j \leq N$.

For $1 \leq q \leq \infty$, $m \in \mathbb{N}$, $s \in \mathbb{R}$, and any domain $D \subset \mathbb{R}^N$, we denote the standard Lebesgue space, Sobolev space, and Besov space by $L_q(D)$, $H^m_q(D)$, and $B^s_{q,p}(D)$ respectively, while $\|\cdot\|_{L_q(D)}$, $\|\cdot\|_{H^m_q(D)}$, and $\|\cdot\|_{B^s_{q,p}(D)}$ denote their norms. We write $W^s_q(D) = B^s_{q,q}(D)$ and $H^0_q(D) = L_q(D)$. What $f = f_\pm$ means that $f(x) = f_\pm(x)$ for $x \in \Omega_\pm$. For $\mathcal{H} \in \{L_q, H^m_q, B^s_{q,p}\}$, the function spaces $\mathcal{H}(\dot{\Omega})$ ($\dot{\Omega} = \Omega_+ \cup \Omega_-$) and their norms are defined by setting

$$
\mathcal{H}(\dot{\Omega}) = \{f = f_\pm \mid f_\pm \in \mathcal{H}(\Omega_\pm)\}, \quad \|f\|_{\mathcal{H}(\dot{D})} = \|f_+\|_{\mathcal{H}(D_+)} + \|f_-\|_{\mathcal{H}(D_-)}.
$$

For any Banach space X, $\|\cdot\|_X$ being its norm, X^d denotes the d product space defined by $\{x = (x_1, \ldots, x_d) \mid x_i \in X\}$, while the norm of X^d is simply written by $\|\cdot\|_X$, which is defined by setting $\|x\|_X = \sum_{j=1}^d \|x_j\|_X$. For any time interval (a,b), $L_p((a,b), X)$ and $H^m_p((a,b), X)$ denote respective the standard X-valued Lebesgue space and X-valued Sobolev space, while $\|\cdot\|_{L_p((a,b),X)}$ and $\|\cdot\|_{H^m_p((a,b),X)}$ denote their norms. Let $\mathcal{F}$ and $\mathcal{F}^{-1}$

be respectively the Fourier transform and the Fourier inverse transform. Let $H^s_p(\mathbb{R}, X)$, $s > 0$, be the Bessel potential space of order s defined by

$$H^s_p(\mathbb{R}, X) = \{f \in \mathcal{S}'(\mathbb{R}, X) \mid \|f\|_{H^s_p(\mathbb{R},X)} = \|\mathcal{F}^{-1}[(1 + |\tau|^2)^{s/2}\mathcal{F}[f](\tau)]\|_{L_p(\mathbb{R},X)} < \infty\}.$$

where $\mathcal{S}'$ denotes the set of all X-valued tempered distributions on $\mathbb{R}$.

Let $\mathbf{a} \cdot \mathbf{b} =< \mathbf{a}, \mathbf{b} >= \sum_{j=1}^N a_j b_j$ for any N-vectors $\mathbf{a} = (a_1, \ldots, a_N)$ and $\mathbf{b} = (b_1, \ldots, b_N)$. For any N-vector $\mathbf{a}$, let $\mathbf{a}_\tau := \mathbf{a}- < \mathbf{a}, \mathbf{n} > \mathbf{n}$. For any two $N \times N$-matrices $\mathbf{A} = (A_{ij})$ and $\mathbf{B} = (B_{ij})$, the quantity $\mathbf{A} : \mathbf{B}$ is defined by $\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^N A_{ij}B_{ji}$. For any domain G with boundary ∂G, we set

$$(\mathbf{u}, \mathbf{v})_G = \int_G \mathbf{u}(x) \cdot \overline{\mathbf{v}(x)}\, dx, \quad (\mathbf{u}, \mathbf{v})_{\partial G} = \int_{\partial G} \mathbf{u}(x) \cdot \overline{\mathbf{v}(x)}\, d\sigma,$$

where $\overline{\mathbf{v}(x)}$ is the complex conjugate of $\mathbf{v}(x)$ and $d\sigma$ denotes the surface element of ∂G. Given $1 < q < \infty$, let $q' = q/(q-1)$. Throughout the paper, the letter C denotes generic constants and $C_{a,b,\cdots}$ the constant which depends on a, b, $\cdots$. The values of constants C, $C_{a,b,\cdots}$ may be changed from line to line.

When we describe nonlinear terms $\mathbf{N}_1(\mathbf{u}_+, \tilde{\mathbf{H}}_+), \ldots, \mathbf{N}_9(\mathbf{u}, \tilde{\mathbf{H}})$ in (7), we use the following notational conventions. Let $\mathbf{u}_i$ $(i = 1, \ldots, m)$ be n_i-vectors whose jth component is u_{ij}, and then $\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_m$ denotes an $n = \prod_{i=1}^m n_i$ vector whose $(j_1, \ldots, j_m)$th component is $\Pi_{i=1}^m u_{ij_i}$ and the set $\{(j_1, \ldots, j_m) \mid 1 \le j_i \le n_i,\ \ i = 1, \ldots, m\}$ is rearranged as $\{k \mid k = 1, 2, \ldots, n\}$ and k is the corresponding number to some $(j_1, \ldots, j_m)$. For example, $\mathbf{u} \otimes \nabla\mathbf{v}$ is an $N + N^2$ vector whose (i, j, k) component is $u_i\partial_j v_k$ and $\mathbf{u} \otimes \nabla\mathbf{v} \otimes \nabla^2\mathbf{w}$ is an $N + N^2 + N^3$ vector whose (i, j, k, ℓ, m, n) component is $u_i(\partial_j v_k)\partial_\ell\partial_m w_n$. Here, the sets $\{(i, j, k) \mid 1 \le i, j, k \le N\}$ and $\{(i, j, k, \ell, m, n) \mid 1 \le i, j, k, \ell, m, n \le N\}$ are rearranged as $\{k \mid 1 \le k \le N + N^2\}$ and $\{k \mid 1 \le k \le N + N^2 + N^3\}$, respectively. Let $\mathbf{u}^\ell_i$ $(i = 1, \ldots, m_\ell, \ell = 1, \ldots, n)$ be n^ℓ_i- vectors, let $\mathbf{A}^\ell$ be $n^\ell \times N$ matrices, where $n^\ell = \prod_{i=1}^{m_\ell} n^\ell_i$, and set $\mathbf{A} = \{\mathbf{A}^1, \ldots, \mathbf{A}^m\}$. And then, we write

$$\mathbf{A}(\mathbf{u}^1_1 \otimes \cdots \otimes \mathbf{u}^1_{m_1}, \ldots, \mathbf{u}^m_1 \otimes \cdots \otimes \mathbf{u}^m_{n_m}) = \sum_{\ell=1}^n \mathbf{A}^\ell \mathbf{u}^\ell_1 \otimes \cdots \otimes \mathbf{u}^\ell_{m_\ell}.$$

When there are two sets of matrices $\mathbf{A} = \{\mathbf{A}^1, \ldots, \mathbf{A}^n\}$ and $\mathbf{B} = \{\mathbf{B}^1, \ldots, \mathbf{B}^n\}$, we write

$$(\mathbf{A} - \mathbf{B})(\mathbf{u}^1_1 \otimes \cdots \otimes \mathbf{u}^1_{m_1}, \ldots, \mathbf{u}^m_1 \otimes \cdots \otimes \mathbf{u}^m_{n_m}) = \sum_{\ell=1}^n (\mathbf{A}^\ell - \mathbf{B}^\ell)\mathbf{u}^\ell_1 \otimes \cdots \otimes \mathbf{u}^\ell_{m_\ell}.$$

2. Derivation of Nonlinear Terms

Let $\mathbf{u}_+(\xi, t)$ be the velocity field with respect to the Lagrange coordinates $\xi \in \Omega_+$, and let $\mathbf{u}_-(\xi, t)$ be the extension of $\mathbf{u}_+$ to $\xi \in \Omega_-$ satisfying the conditions given in (5). Let

$$\psi_{\mathbf{u}}(\xi, t) = \varphi(\xi) \int_0^t \mathbf{u}(\xi, s)\, ds$$

and we consider the correspondence: $x = \xi + \psi_{\mathbf{u}}(\xi, t)$ for $\xi \in \Omega$, which has been already given in (6). Let $\delta \in (0, 1)$ be a small constant and we assume that

$$\sup_{t\in(0,T)} \|\psi_{\mathbf{u}}(\cdot, t)\|_{H^1_\infty(\Omega)} \le \delta. \tag{10}$$

Then, the correspondence: $x = \xi + \psi_{\mathbf{u}}(\xi, t)$ is one to one. Since $\psi_{\mathbf{u}}(\xi, t) = 0$ when $\mathrm{dist}\,(\xi, S_\pm) \le d_0$, if $\mathbf{u}$ satisfies the regularity condition:

$$\mathbf{u}(\xi, t) \in H^1_p((0, T), H^1_q(\Omega)) \cap L_p((0, T), H^3_q(\Omega)), \tag{11}$$

then the correspondence $x = \xi + \psi_{\mathbf{u}}(\xi, t)$ is a bijection from Ω onto Ω, and so we set

$$\Omega_{t\pm} = \{x = \xi + \psi_{\mathbf{u}}(\xi, t) \mid \xi \in \Omega_{\pm}\}, \quad \Gamma_t = \{x = \xi + \psi_{\mathbf{u}}(\xi, t) \mid \xi \in \Gamma\}. \tag{12}$$

In the following, for notational simplicity we set

$$\Psi_{\mathbf{u}} = \int_0^t \nabla(\varphi(\xi)\mathbf{u}(\xi, s))\,ds, \tag{13}$$

where $\nabla = (\partial/\partial\xi_1, \ldots, \partial/\partial\xi_N)$. The Jacobi matrix of the correspondence: $x = \xi + \psi_{\mathbf{u}}(\xi, t)$ is

$$\frac{\partial x}{\partial \xi} = \mathbf{I} + \Psi_{\mathbf{u}}. \tag{14}$$

Notice that

$$\|\Psi_{\mathbf{u}}\|_{L_\infty((0,T), L_\infty(\Omega))} \leq \delta \tag{15}$$

as follows from the assumption (10). Thus, we have

$$\frac{\partial \xi}{\partial x} = \left(\frac{\partial x}{\partial \xi}\right)^{-1} = \mathbf{I} + \sum_{k=1}^{\infty}(-\Psi(\xi, t))^k = \mathbf{I} + \mathbf{V}_0(\Psi_{\mathbf{u}}). \tag{16}$$

Here and in the following, we set

$$\mathbf{V}_0(\mathbf{k}) = \sum_{k=1}^{\infty}(-\mathbf{k})^k \tag{17}$$

for $|\mathbf{k}| \leq \delta(< 1)$. The $\mathbf{k} = (\mathbf{k}_1, \ldots, \mathbf{k}_N) \in \mathbb{R}^{N^2}$ with $\mathbf{k}_i = (k_{i1}, \ldots, k_{iN}) \in \mathbb{R}^N$ denotes the independent variables corresponding to $\Psi_{\mathbf{u}} = (\Psi_{1\mathbf{u}}, \ldots, \Psi_{N\mathbf{u}})$ with $\Psi_{\ell\mathbf{u}} = \int_0^t \nabla(\varphi(\xi)u_\ell(\xi, s))\,ds$. The $\mathbf{V}_0(\mathbf{k})$ is a matrix of analytic functions defined on $|\mathbf{k}| \leq \delta$ with $\mathbf{V}_0(0) = 0$. Using this symbol, we have

$$\nabla_x = (\mathbf{I} + \mathbf{V}_0(\Psi_{\mathbf{u}}))\nabla_\xi, \quad \frac{\partial}{\partial x_i} = \frac{\partial}{\partial \xi_i} + \sum_{j=1}^{N} V_{0ij}(\Psi_{\mathbf{u}})\frac{\partial}{\partial \xi_j}, \tag{18}$$

where V_{0ij} is the (i, j)th component of the $N \times N$ matrix $\mathbf{V}_0$.

For any N-vector of functions, $\mathbf{w}(x, t) = (w_1(x, t), \ldots, w_N(x, t))^\top$, we set $\tilde{\mathbf{w}}(\xi, t) = \mathbf{w}(x, t)$ and $\mathbf{v}(x, t) = \mathbf{u}(\xi, t)$. Then, by (18)

$$\partial_t \mathbf{w}(x, t) + \mathbf{v}(x, t) \cdot \nabla \mathbf{w}(x, t) = \partial_t \tilde{\mathbf{w}}(\xi, t) + (1 - \varphi(\xi))\mathbf{u}(\xi, t) \cdot ((\mathbf{I} + \mathbf{V}_0(\Psi_{\mathbf{u}}))\nabla \tilde{\mathbf{w}}(\xi, t)). \tag{19}$$

By (18),

$$\mathbf{D}(\mathbf{w}) = \mathbf{D}(\tilde{\mathbf{w}}) + \mathcal{D}(\Psi_{\mathbf{u}})\nabla \tilde{\mathbf{w}}, \quad \operatorname{curl} \mathbf{w} \;= \operatorname{curl} \tilde{\mathbf{w}} + \mathcal{C}(\Psi_{\mathbf{u}})\nabla \tilde{\mathbf{w}}, \tag{20}$$

with

$$\mathcal{D}(\Psi_{\mathbf{u}})\nabla \tilde{\mathbf{w}} = \mathbf{V}_0(\Psi_{\mathbf{u}})\nabla \tilde{\mathbf{w}} + (\mathbf{V}_0(\Psi_{\mathbf{u}})\nabla \tilde{\mathbf{w}})^\top, \quad \mathcal{C}(\Psi_{\mathbf{u}})\nabla \tilde{\mathbf{w}} = \mathbf{V}_0(\Psi_{\mathbf{u}})\nabla \tilde{\mathbf{w}} - (\mathbf{V}_0(\Psi_{\mathbf{u}})\nabla \tilde{\mathbf{w}})^\top.$$

By (18),

$$\operatorname{div} \mathbf{w} = \operatorname{div} \tilde{\mathbf{w}} + V_0(\Psi_{\mathbf{u}}) : \nabla \tilde{\mathbf{w}}, \tag{21}$$

with $V_0(\Psi_{\mathbf{u}}) : \nabla \tilde{\mathbf{w}} = \sum_{i,j=1}^N V_{0ij}(\Psi_{\mathbf{u}})\frac{\partial \tilde{w}_i}{\partial \xi_j}$. Analogously, for any $N \times N$ matrix of functions, $A = (A_{ij})$, we set $\tilde{A}(\xi, t) = A(x, t)$ and $\tilde{A}_{ij}(\xi, t) = A_{ij}(x, t)$. Let $\mathbf{V}_0(\Psi_{\mathbf{u}}) : \nabla \tilde{A}$ be an N-vector of functions whose i-th component is $\sum_{j,k=1}^N V_{0jk}(\Psi_{\mathbf{u}})\frac{\partial \tilde{A}_{ij}}{\partial \xi_k}$, and then

$$\operatorname{Div} A = \operatorname{Div} \tilde{A} + \mathbf{V}_0(\Psi_{\mathbf{u}}) : \nabla \tilde{A}. \tag{22}$$

On the other hand, using the dual form, we see that

$$\operatorname{div}\mathbf{w} = \operatorname{div}((\mathbf{I}+\mathbf{V}_0(\Psi_\mathbf{u})^\top)\tilde{\mathbf{w}}) = \operatorname{div}\tilde{\mathbf{w}} + \operatorname{div}(\mathbf{V}_0(\Psi_\mathbf{u})^\top\tilde{\mathbf{w}}). \tag{23}$$

Let $\mathfrak{q}(\xi,t) = \mathfrak{p}(x,t)$, $\mathbf{u}_+(\xi,t) = \mathbf{v}(x,t)$, and $\tilde{\mathbf{H}}_\pm = \mathbf{H}(x,t)$. By (19), (20), and (22), we see that the first equation in Equation (3) is transformed to

$$\begin{aligned}&\rho\partial_t\mathbf{u}_+ + \rho(1-\varphi)\mathbf{u}_+\cdot(\mathbf{I}+\mathbf{V}_0(\Psi_\mathbf{u}))\nabla\mathbf{u}_+ - \operatorname{Div}(\nu\mathbf{D}(\mathbf{u}_+) + \nu\mathcal{D}(\Psi_\mathbf{u})\nabla\mathbf{u}_+)\\ &-\mathbf{V}_0(\Psi_\mathbf{u}):\nabla(\nu\mathbf{D}(\mathbf{u}_+)+\nu\mathcal{D}(\Psi_\mathbf{u})\nabla\mathbf{u}_+) + (\mathbf{I}+\mathbf{V}_0(\Psi_\mathbf{u}))\nabla\mathfrak{q} - \operatorname{Div}\mathbf{T}_M(\tilde{\mathbf{H}}_+) - \mathbf{V}_0(\Psi_\mathbf{u}):\nabla\mathbf{T}_M(\tilde{\mathbf{H}}_+) = 0.\end{aligned}$$

By (17) and (16), $(\mathbf{I}+\Psi_\mathbf{u})(\mathbf{I}+\mathbf{V}_0(\Psi_\mathbf{u})) = \mathbf{I}$, and so we have

$$\rho\partial_t\mathbf{u}_+ - \operatorname{Div}\mathbf{T}(\mathbf{u}_+,\mathfrak{q}) = \mathbf{N}_1(\mathbf{u}_+,\mathbf{H}_+),$$

with

$$\begin{aligned}\mathbf{N}_1(\mathbf{u}_+,\tilde{\mathbf{H}}_+) &= -\Psi_\mathbf{u}\{\rho\partial_t\mathbf{u}_+ - \operatorname{Div}(\nu\mathbf{D}(\mathbf{u}_+))\}\\ &+\Big(\mathbf{I}+\Psi_\mathbf{u})\{-\rho(1-\varphi)\mathbf{u}_+\cdot(\mathbf{I}+\mathbf{V}_0(\Psi_\mathbf{u}))\nabla\mathbf{u}_+ + \nu\operatorname{Div}(\mathcal{D}(\Psi_\mathbf{u})\nabla\mathbf{u}_+)\\ &+\mathbf{V}_0(\Psi_\mathbf{u}):\nabla(\nu\mathbf{D}(\mathbf{u}_+)+\nu\mathcal{D}(\Psi_\mathbf{u})\nabla\mathbf{u}_+) + \operatorname{Div}\mathbf{T}_M(\tilde{\mathbf{H}}_+) + \mathbf{V}_0(\Psi_\mathbf{u}):\nabla\mathbf{T}_M(\tilde{\mathbf{H}}_+)\}.\end{aligned}$$

using the notational convention given in Notation, we may write

$$\begin{aligned}&\mathbf{N}_1(\mathbf{u}_+,\tilde{\mathbf{H}}_+)\\ &=\mathcal{A}_1(\Psi_\mathbf{u})(\Psi_\mathbf{u}\otimes(\partial_t\mathbf{u}_+,\nabla^2\mathbf{u}_+),\mathbf{u}_+\otimes\nabla\mathbf{u}_+,\nabla\Psi_\mathbf{u}\otimes\nabla\mathbf{u}_+,\tilde{\mathbf{H}}_+\otimes\nabla\tilde{\mathbf{H}}_+),\end{aligned} \tag{24}$$

where $\mathcal{A}_1(\mathbf{k})$ is a set of matrices of smooth functions defined for $|\mathbf{k}|\le\delta$. Combining (21) and (22), we see that the condition $\operatorname{div}\mathbf{v} = 0$ in Ω_t is transformed to

$$\operatorname{div}\mathbf{u}_+ = -\mathbf{V}_0(\Psi_\mathbf{u}):\nabla\mathbf{u}_+ = -\operatorname{div}(\mathbf{V}_0(\Psi_\mathbf{u})^\top\mathbf{u}_+).$$

Thus, we set $\mathbf{N}_2(\mathbf{u}_+) = -\mathbf{V}_0(\mathbf{k}):\nabla\mathbf{u}_+$ and $\mathbf{N}_3(\mathbf{u}_+) = -\mathbf{V}_0(\mathbf{k})^\top\mathbf{u}_+$, and then by using notational convenience defined in Notation, we may write

$$\mathbf{N}_2(\mathbf{u}_+) = \mathcal{A}_2(\Psi_\mathbf{u})\Psi_\mathbf{u}\otimes\nabla\mathbf{u}_+,\quad \mathbf{N}_3(\mathbf{u}_+) = \mathcal{A}_3(\Psi_\mathbf{u})\Psi_\mathbf{u}\otimes\mathbf{u}_+, \tag{25}$$

where $\mathcal{A}_i(\mathbf{k})$ $(i=2,3)$ are matrices of smooth functions defined for $|\mathbf{k}|\le\delta$. By (18),

$$\Delta\mathbf{H}_\pm = \nabla\cdot\nabla\mathbf{H}_\pm = \Delta\tilde{\mathbf{H}}_\pm + \nabla(\mathbf{V}_0(\Psi_\mathbf{u})\nabla\tilde{\mathbf{H}}_\pm) + \mathbf{V}_0(\Psi_\mathbf{u})\nabla((\mathbf{I}+\mathbf{V}_0(\Psi_\mathbf{u}))\nabla\tilde{\mathbf{H}}_\pm).$$

and so noting (19) we see that the second and third equations in (3) are transformed to

$$\begin{aligned}\mu_+\partial_t\tilde{\mathbf{H}}_+ - \alpha_+^{-1}\Delta\tilde{\mathbf{H}}_+ &= \mathbf{N}_{5+}(\mathbf{u},\tilde{\mathbf{H}}) &&\text{in } \Omega_+\times(0,T),\\ \mu_-\partial_t\tilde{\mathbf{H}}_- - \alpha_-^{-1}\Delta\tilde{\mathbf{H}}_- &= \mathbf{N}_{5-}(\mathbf{u},\tilde{\mathbf{H}}) &&\text{in } \Omega_-\times(0,T),\end{aligned}$$

with

$$\begin{aligned}\mathbf{N}_{5+}(\mathbf{u},\tilde{\mathbf{H}}) &= \mu_+\varphi\mathbf{u}\cdot(\mathbf{I}+\mathbf{V}_0(\Psi_\mathbf{u}))\nabla\tilde{\mathbf{H}}_+\\ &+\alpha_+^{-1}\{\operatorname{Div}(\mathbf{V}_0(\Psi_\mathbf{u})\nabla\tilde{\mathbf{H}}_+) + \mathbf{V}_0(\Psi_\mathbf{u})\nabla((\mathbf{I}+\mathbf{V}_0(\Psi_\mathbf{u}))\nabla\tilde{\mathbf{H}}_+)\}\\ &+\operatorname{Div}(\mu_+(\mathbf{u}_+\otimes\tilde{\mathbf{H}}_+ - \tilde{\mathbf{H}}_+\otimes\mathbf{u}_+)) + \mathbf{V}_0(\Psi_\mathbf{u}):\nabla(\mu_+(\mathbf{u}_+\otimes\tilde{\mathbf{H}}_+ - \tilde{\mathbf{H}}_+\otimes\mathbf{u}_+));\\ \mathbf{N}_{5-}(\mathbf{u},\tilde{\mathbf{H}}) &= \mu_-\varphi\mathbf{u}\cdot(\mathbf{I}+\mathbf{V}_0(\Psi_\mathbf{u}))\nabla\tilde{\mathbf{H}}_-\\ &+\alpha_-^{-1}\{\operatorname{Div}(\mathbf{V}_0(\Psi_\mathbf{u})\nabla\tilde{\mathbf{H}}_-) + \mathbf{V}_0(\Psi_\mathbf{u})\nabla((\mathbf{I}+\mathbf{V}_0(\Psi_\mathbf{u}))\nabla\tilde{\mathbf{H}}_-)\}.\end{aligned}$$

Thus, using the notational convention given in Notation, we may write

$$\mathbf{N}_{5\pm}(\mathbf{u},\tilde{\mathbf{H}}) = \mathcal{A}_{5\pm}(\Psi_\mathbf{u})(\tilde{\mathbf{H}}_\pm\otimes\nabla\tilde{\mathbf{H}}_\pm,\Psi_\mathbf{u}\otimes\nabla^2\tilde{\mathbf{H}}_\pm,\nabla\Psi_\mathbf{u}\otimes\nabla\tilde{\mathbf{H}}_\pm,\delta_\pm\nabla\mathbf{u}_\pm\otimes\tilde{\mathbf{H}}_\pm,\mathbf{u}_\pm\otimes\nabla\tilde{\mathbf{H}}_\pm), \tag{26}$$

where $\delta_+ = 1$ and $\delta_- = 0$, where $\mathcal{A}_{5\pm}(\mathbf{k})$ are two sets of matrices of smooth functions defined for $|\mathbf{k}| \leq \delta$. In particular, we have the fourth equation in (7).

We now consider the transmission conditions. The unit outer normal, $\mathbf{n}_t$, to the Γ_t is represented by

$$\mathbf{n}_t = \frac{(\mathbf{I} + \mathbf{V}_0(\Psi_\mathbf{u}))^\top \mathbf{n}}{|(\mathbf{I} + \mathbf{V}_0(\Psi_\mathbf{u}))^\top \mathbf{n}|}.$$

Choosing $\delta > 0$ small enough, we may write

$$\mathbf{n}_t = (\mathbf{I} + \mathbf{V}_\mathbf{n}(\Psi_\mathbf{u}))\mathbf{n}, \tag{27}$$

where $\mathbf{V}_\mathbf{n}(\mathbf{k})$ is a matrix of smooth functions defined on $|\mathbf{k}| \leq \delta$ such that $\mathbf{V}_\mathbf{n}(0) = 0$. By (20)

$$\begin{aligned}&(\mathbf{T}(\mathbf{v}, \mathfrak{p}) + \mathbf{T}_M(\mathbf{H}_+))\mathbf{n}_t \\ &\quad = \nu(\mathbf{D}(\mathbf{u}_+) + \mathcal{D}(\Psi_\mathbf{u})\nabla\mathbf{u})(\mathbf{I} + \mathbf{V}_\mathbf{n}(\Psi_\mathbf{u}))\mathbf{n} - (\mathbf{I} + \mathbf{V}_\mathbf{n}(\Psi_\mathbf{u}))\mathfrak{q}\mathbf{n} + \mathbf{T}_M(\tilde{\mathbf{H}}_+)(\mathbf{I} + \mathbf{V}_\mathbf{n}(\Psi_\mathbf{u}))\mathbf{n} = 0.\end{aligned}$$

Choosing $\delta > 0$ small if necessary, we may assume that $(\mathbf{I} + \mathbf{V}_\mathbf{n}(\mathbf{k}))^{-1}$ exists and we may write $(\mathbf{I} + \mathbf{V}_\mathbf{n}(\mathbf{k}))^{-1} = \mathbf{I} + \mathbf{V}_{\mathbf{n},-1}(\mathbf{k})$, where $\mathbf{V}_{\mathbf{n},-1}(\mathbf{k})$ is a matrix of smooth functions defined on $|\mathbf{k}| \leq \delta$ such that $\mathbf{V}_{\mathbf{n},-1}(0) = 0$. Thus, setting

$$\begin{aligned}\mathbf{N}_4(\mathbf{u}_+, \tilde{\mathbf{H}}_+) = -\{&\nu\mathbf{D}(\mathbf{u}_+)\mathbf{V}_\mathbf{n}(\Psi_\mathbf{u}))\mathbf{n} + \mathbf{V}_{\mathbf{n},-1}(\Psi_\mathbf{u})\nu\mathbf{D}(\mathbf{u}_+)(\mathbf{I} + \mathbf{V}_\mathbf{n}(\Psi_\mathbf{u})\mathbf{n}) \\ &+ (\mathbf{I} + \mathbf{V}_{\mathbf{n},-1}(\Psi_\mathbf{u}))(\nu\mathcal{D}(\Psi_\mathbf{u})\nabla\mathbf{u}_+ + \mathbf{T}_M(\tilde{\mathbf{H}}_+))(\mathbf{I} + \mathbf{V}_\mathbf{n}(\Psi_\mathbf{u}))\mathbf{n}\},\end{aligned}$$

we have

$$\mathbf{T}(\mathbf{u}_+, \mathfrak{q}) = \mathbf{N}_4(\mathbf{u}_+, \tilde{\mathbf{H}}_+) \quad \text{on } \Gamma \times (0, T).$$

Using the notational convention defined in Notation, we may write

$$\mathbf{N}_4(\mathbf{u}_+, \tilde{\mathbf{H}}_+) = \mathcal{A}_4(\Psi_\mathbf{u})(\Psi_\mathbf{u} \otimes \nabla\mathbf{u}_+, \tilde{\mathbf{H}}_+ \otimes \tilde{\mathbf{H}}_+), \tag{28}$$

where $\mathcal{A}_4(\mathbf{k})$ is a set of matrices of functions consisting of products of elements of $\mathbf{n}$ and smooth functions defined for $|\mathbf{k}| \leq \delta$.

By (20) and (27),

$$\begin{aligned}&[[(\alpha^{-1}\mathrm{curl}\,\mathbf{H})\mathbf{n}_t]] - \mu_+(\mathbf{v} \otimes \mathbf{H}_+ - \mathbf{H}_+ \otimes \mathbf{v})\mathbf{n}_t \\ &\quad = [[(\alpha^{-1}(\mathrm{curl}\,\tilde{\mathbf{H}} + \mathcal{C}(\Psi_\mathbf{u})\nabla\tilde{\mathbf{H}})(\mathbf{I} + \mathbf{V}_\mathbf{n}(\Psi_\mathbf{u}))\mathbf{n}]] - \mu_+(\mathbf{u}_+ \otimes \tilde{\mathbf{H}}_+ - \tilde{\mathbf{H}}_+ \otimes \mathbf{u}_+)(\mathbf{I} + \mathbf{V}_\mathbf{n}(\Psi_\mathbf{u}))\mathbf{n} = 0.\end{aligned}$$

Thus, setting

$$\begin{aligned}\mathbf{N}_6(\mathbf{u}, \tilde{\mathbf{H}}) = -\{&[[(\alpha^{-1}\mathrm{curl}\,\tilde{\mathbf{H}})\mathbf{V}_\mathbf{n}(\Psi_\mathbf{u})\mathbf{n}]] + [[(\alpha^{-1}\mathcal{C}(\Psi_\mathbf{u})\nabla\tilde{\mathbf{H}})(\mathbf{I} + \mathbf{V}_\mathbf{n}(\Psi_\mathbf{u}))\mathbf{n}]]\} \\ &+ \mu_+(\mathbf{u}_+ \otimes \tilde{\mathbf{H}}_+ - \tilde{\mathbf{H}}_+ \otimes \mathbf{u}_+)(\mathbf{I} + \mathbf{V}_\mathbf{n}(\Psi_\mathbf{u}))\mathbf{n},\end{aligned}$$

we have

$$[[(\alpha^{-1}\mathrm{curl}\,\tilde{\mathbf{H}})\mathbf{n}]] = \mathbf{N}_6(\mathbf{u}, \tilde{\mathbf{H}}) \quad \text{on } \Gamma \times (0, T).$$

Using the notational convention defined in Notation and noting that $[[\Psi_\mathbf{u}]] = 0$ on Γ as follows from (5), we may write

$$\mathbf{N}_6(\mathbf{u}, \tilde{\mathbf{H}}) = \mathcal{A}_{61}(\Psi_\mathbf{u})[[\alpha^{-1}\nabla\tilde{\mathbf{H}}]] + (\mathcal{B} + \mathcal{A}_{62}(\Psi_\mathbf{u}))\mathbf{u}_+ \otimes \tilde{\mathbf{H}}_+, \tag{29}$$

where $\mathcal{A}_{61}(\mathbf{k})$ and $\mathcal{A}_{62}(\mathbf{k})$ are a matrix and a set of matrices of functions consisting of products of elements of $\mathbf{n}$ and smooth functions defined for $|\mathbf{k}| \leq \delta$, and $\mathcal{B}$ is a set of matrices of functions such that $\mathcal{B}\mathbf{u}_+ \otimes \tilde{\mathbf{H}}_+ = \mu_+(\mathbf{u}_+ \otimes \tilde{\mathbf{H}}_+ - \tilde{\mathbf{H}}_+ \otimes \mathbf{u}_+)\mathbf{n}$. In particular,

$$\|\mathcal{A}_{6i}(\Psi_\mathbf{u})\|_{H^1_\infty(\Omega)} \leq C\|\Psi_\mathbf{u}\|_{H^1_\infty(\Omega)}. \tag{30}$$

By (23),

$$[[\mu \mathrm{div}\, \mathbf{H}]] = [[\mu \mathrm{div}\, \tilde{\mathbf{H}} + \mu \mathbf{V}_0(\Psi_{\mathbf{u}}) : \nabla \tilde{\mathbf{H}}]] = 0,$$

and so setting

$$\mathbf{N}_7(\mathbf{u}, \tilde{\mathbf{H}}) = -[[\mu \mathbf{V}_0(\Psi_{\mathbf{u}}) : \nabla \tilde{\mathbf{H}}]] = \mathcal{A}_7(\Psi_{\mathbf{u}})[[\mu \nabla \tilde{\mathbf{H}}]] \tag{31}$$

we have

$$[[\mu \mathrm{div}\, \tilde{\mathbf{H}}]] = \mathbf{N}_7(\mathbf{u}, \tilde{\mathbf{H}}) \quad \text{on } \Gamma \times (0, T),$$

where $\mathcal{A}_7(\mathbf{k})$ is a matrix of functions consisting of products of elements of $\mathbf{n}$ and smooth functions defined for $|\mathbf{k}| \leq \delta$. Notice that

$$\|\mathcal{A}_7(\Psi_{\mathbf{u}})\|_{H^1_\infty(\Omega)} \leq C \|\Psi_{\mathbf{u}}\|_{H^1_\infty(\Omega)}. \tag{32}$$

By (27), we have

$$[[\mu \mathbf{H} \cdot \mathbf{n}_t]] = [[\mu \tilde{\mathbf{H}}(\mathbf{I} + \mathbf{V}_{\mathbf{n}}(\Psi_{\mathbf{u}}))\mathbf{n}]] = 0,$$

and so, setting

$$\mathbf{N}_8(\mathbf{u}, \tilde{\mathbf{H}}) = -[[\mu \tilde{\mathbf{H}}]] \mathbf{V}_{\mathbf{n}}(\Psi_{\mathbf{u}})\mathbf{n}, \tag{33}$$

we have

$$[[\mu \tilde{\mathbf{H}} \cdot \mathbf{n}]] = \mathbf{N}_8(\mathbf{u}, \tilde{\mathbf{H}}) \quad \text{on } \Gamma \times (0, T).$$

Finally, by (27)

$$\begin{aligned}[[\mathbf{H} - < \mathbf{H}, \mathbf{n}_t > \mathbf{n}_t]] &= [[\tilde{\mathbf{H}} - < \tilde{\mathbf{H}}, (\mathbf{I} + \mathbf{V}_{\mathbf{n}}(\Psi_{\mathbf{u}}))\mathbf{n} > (\mathbf{I} + \mathbf{V}_{\mathbf{n}}(\Psi_{\mathbf{u}}))\mathbf{n}]] \\ &= [[\tilde{\mathbf{H}}_\tau]] - [[< \tilde{\mathbf{H}}, \mathbf{n} > \mathbf{V}_{\mathbf{n}}(\Psi_{\mathbf{u}})\mathbf{n}]] - [[< \tilde{\mathbf{H}}, \mathbf{V}_{\mathbf{n}}(\Psi_{\mathbf{u}})\mathbf{n} > (\mathbf{I} + \mathbf{V}_{\mathbf{n}}(\Psi_{\mathbf{u}}))\mathbf{n}]],\end{aligned}$$

and so, setting

$$\mathbf{N}_9(\mathbf{u}, \tilde{\mathbf{H}}) = < [[\tilde{\mathbf{H}}]], \mathbf{n} > \mathbf{V}_{\mathbf{n}}(\Psi_{\mathbf{u}})\mathbf{n} + < [[\tilde{\mathbf{H}}]], \mathbf{V}_{\mathbf{n}}(\Psi_{\mathbf{u}})\mathbf{n} > (\mathbf{I} + \mathbf{V}_{\mathbf{n}}(\Psi_{\mathbf{u}}))\mathbf{n}, \tag{34}$$

we have

$$[[\tilde{\mathbf{H}}_\tau]] = \mathbf{N}_9(\mathbf{u}, \tilde{\mathbf{H}}) \quad \text{on } \Gamma \times (0, T).$$

For notational simplicity, we set

$$(\mathbf{N}_8(\mathbf{u}, \tilde{\mathbf{H}}), \mathbf{N}_9(\mathbf{u}, \tilde{\mathbf{H}}) = \mathcal{A}_8(\Psi_{\mathbf{u}})[[\tilde{\mathbf{H}}]], \tag{35}$$

where $\mathcal{A}_8(\mathbf{k})$ is a set of matrices of functions consisting of products of elements of $\mathbf{n}$ and smooth functions defined for $|\mathbf{k}| \leq \delta$. Notice that

$$\|\mathcal{A}_8(\Psi_{\mathbf{u}})\|_{H^1_\infty(\Omega)} \leq C \|\Psi_{\mathbf{u}}\|_{H^1_\infty(\Omega)}. \tag{36}$$

3. Linear Theory

Since the coupling of the velocity field and the magnetic field in (7) is semilinear, the linearized equations are decoupled. Namely, we consider the two linearlized equations: one is the Stokes equations with free boundary conditions on Γ, and another is a system of heat equations with transmission conditions on Γ and the perfect wall conditions on S_-. Recall that $\dot{\Omega} = \Omega_+ \cup \Omega_-$ and $\Omega = \dot{\Omega} \cup \Gamma$.

3.1. The Stokes Equations with Free Boundary Conditions

This subsection is devoted to presenting the L_p-L_q maximal regularity theorem for the Stokes equations with free boundary conditions. The problem considered here is formulated by the following equations:

$$\begin{aligned} \rho\partial_t \mathbf{v} - \mathrm{Div}\,\mathbf{T}(\mathbf{v},\mathfrak{q}) = \mathbf{f}_1 &\quad \text{in } \Omega_+ \times (0,T), \\ \mathrm{div}\,\mathbf{v} = g = \mathrm{div}\,\mathbf{g} &\quad \text{in } \Omega_+ \times (0,T), \\ \mathbf{T}(\mathbf{v},\mathfrak{q})\mathbf{n} = \mathbf{h} &\quad \text{on } \Gamma \times (0,T), \\ \mathbf{v}|_{t=0} = \mathbf{u}_{0+} &\quad \text{in } \Omega_+. \end{aligned} \tag{37}$$

To state assumptions for Equation (37), we make two definitions.

Definition 1. *Let Ω_+ be a domain given in the introduction. We say that Ω_+ is a uniform C^3 domain, if there exist positive constants a_1, a_2, and A such that the following assertion holds: For any $x_0 = (x_{01}, \ldots, x_{0N}) \in \Gamma$ there exist a coordinate number j and a C^3 function $h(x')$ defined on $B'_{a_1}(x'_0)$ such that $\|h\|_{H^k_\infty(B'_{a_1}(x'_0))} \leq A$ for $k \leq 3$ and*

$$\begin{aligned} \Omega_+ \cap B_{a_2}(x_0) &= \{x \in \mathbb{R}^N \mid x_j > h(x') \ \ (x' \in B'_{a_1}(x'_0))\} \cap B_{a_2}(x_0), \\ \Gamma \cap B_{a_2}(x_0) &= \{x \in \mathbb{R}^N \mid x_j = h(x') \ \ (x' \in B'_{a_1}(x'_0))\} \cap B_{a_2}(x_0). \end{aligned}$$

Here, we have set

$$\begin{aligned} y' &= (y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_N) \ \ (y \in \{x, x_0\}), \\ B'_{a_1}(x'_0) &= \{x' \in \mathbb{R}^{N-1} \mid |x' - x'_0| < a_1\}, \\ B_{a_2}(x_0) &= \{x \in \mathbb{R}^N \mid |x - x_0| < a_2\}. \end{aligned}$$

Let $\hat{H}^1_{q,0}(\Omega_+)$ be an homogeneous Sobolev space defined by letting

$$\hat{H}^1_{q,0}(\Omega_+) = \{\varphi \in L_{q,\mathrm{loc}}(\Omega_+) \mid \nabla\varphi \in L_q(\Omega_+)^N, \quad \varphi|_\Gamma = 0\}. \tag{38}$$

Let $1 < q < \infty$. The variational equation:

$$(\nabla u, \nabla\varphi)_\Omega = (\mathbf{f}, \nabla\varphi)_\Omega \quad \text{for all } \varphi \in \hat{H}^1_{q',0}(\Omega_+) \tag{39}$$

is called the weak Dirichlet problem, where $q' = q/(q-1)$.

Definition 2. *We say that the weak Dirichlet problem (39) is uniquely solvable for an index q if for any $\mathbf{f} \in L_q(\Omega)^N$, problem (39) admits a unique solution $u \in \hat{H}^1_{q,0}(\Omega)$ possessing the estimate: $\|\nabla u\|_{L_q(\Omega)} \leq C\|\mathbf{f}\|_{L_q(\Omega)}$.*

We say that $\mathbf{u} \in L_q(\Omega_+)$ is solenoidal if $\mathbf{u}$ satisfies

$$(\mathbf{u}, \nabla\varphi)_{\Omega_+} = 0 \quad \text{for any } \varphi \in \hat{H}^1_{q',0}(\Omega_+). \tag{40}$$

Let $J_q(\Omega_+)$ be the set of all solenoidal vector of functions.

In this paper, we assume that

(1) Ω_+ is a uniform C^3 domain.
(2) The weak Dirichlet problem is uniquely solvable in Ω_+ for indices $q \in (1, \infty)$ and $q' = q/(q-1)$.

By assumption (2), we see that $L_q(\Omega_+)^N = J_q(\Omega_+) \oplus G_q(\Omega_+)$, where $G_q(\Omega_+) = \{\nabla\varphi \mid \varphi \in \hat{H}^1_{q,0}(\Omega_+)\}$ and the symbol $\oplus$ here denotes the direct sum of $J_q(\Omega_+)$ and $G_q(\Omega_+)$.

Theorem 2. *Let $1 < p, q < \infty$ with $2/p + N/q \neq 1$, and $T > 0$. Let $\mathbf{u}_{0+} \in B^{3-2/p}_{q,p}(\dot\Omega)$ and let $\mathbf{f}$, g, $\mathbf{g}$, $\mathbf{h}$ be functions appearing in Equation (37) satisfying the following conditions:*

$$\mathbf{f} \in L_p((0,T), H^1_q(\Omega_+)^N), \quad g \in L_p((0,T), H^2_q(\Omega_+)) \cap H^1_p((0,T), L_q(\Omega_+)),$$
$$\mathbf{g} \in H^1_p((0,T), H^1_q(\Omega_+)^N), \quad \mathbf{h} \in L_p((0,T), H^2_q(\Omega_+)^N) \cap H^1_p((0,T), L_q(\Omega_+)^N).$$

Assume that $\mathbf{u}_0$, g and $\mathbf{h}$ satisfy the following compatibility conditions:

$$\begin{aligned} \operatorname{div} \mathbf{u}_{0+} &= g|_{t=0} && \text{on } \dot\Omega, \quad \mathbf{u}_{0+} - \mathbf{g}|_{t=0} \in J_q(\Omega_+), \\ (\nu \mathbf{D}(\mathbf{u}_{0+})\mathbf{n})_\tau &= \mathbf{h}_\tau|_{t=0} && \text{on } \Gamma, \quad \text{provided } 2/p + 1/q < 1, \end{aligned} \tag{41}$$

where $\mathbf{d}_\tau = \mathbf{d} - < \mathbf{d}, \mathbf{n} > \mathbf{n}$. Then, problem (37) admit unique solutions $\mathbf{v}$ and $\mathfrak{q}$ with

$$\mathbf{v} \in L_p((0,T), H^3_q(\Omega_+)^N) \cap H^1_p((0,T), H^1_q(\Omega_+)^N), \quad \mathfrak{q} \in L_p((0,T), H^1_q(\Omega_+) + \hat H^1_{q,0}(\Omega_+)),$$

and $\nabla^2 \mathfrak{q} \in L_p((0,T), L_q(\Omega_+)^{N^2})$ possessing the estimates:

$$\|\partial_t \mathbf{v}\|_{L_p((0,T), H^1_q(\Omega_+))} + \|\mathbf{v}\|_{L_p((0,T), H^3_q(\Omega_+))} \leq C e^{\gamma_1 T}\{\|\mathbf{u}_0\|_{B^{3-2/p}_{q,p}(\Omega_+)} + F_v(\mathbf{f}, g, \mathbf{g}, \mathbf{h})\}$$

with

$$F_v(\mathbf{f}, g, \mathbf{g}, \mathbf{h}) = \|\mathbf{f}\|_{L_p((0,T), H^1_q(\Omega_+))} + \|(g, \mathbf{h})\|_{L_p(\mathbb{R}, H^2_q(\Omega_+))} + \|\mathbf{g}\|_{H^1_p(\mathbb{R}, H^1_q(\Omega_+))} + \|(g, \mathbf{h})\|_{H^1_p(\mathbb{R}, L_q(\Omega_+))}$$

for some positive constants C and γ_1 are independent of T.

Remark 2. (1) *Theorem 2 has been proved by Shibata [16] in the standard case where*

$$\mathbf{v} \in H^1_p((0,T), L_q(\Omega_+)^N) \cap L_p((0,T), H^2_q(\Omega_+)^N).$$

But, in Theorem 2 one more additional regularity is stated, which is necessary for our approach to prove the well-posedness of Equation (3). The idea of proving how to obtain third order regularity of the fluid vector field will be given in Appendix A below.
(2) *The uniqueness holds in the following sense. Let $\mathbf{v}$ and $\mathfrak{q}$ with*

$$\mathbf{v} \in L_p((0,T), H^2_q(\Omega_+)^N) \cap H^1_p((0,T), L_q(\Omega_+)^N), \quad \mathfrak{q} \in L_p((0,T), H^1_q(\Omega) + \hat H^1_{q,0}(\Omega_+))$$

satisfy the homogeneous equations:

$$\begin{aligned} \rho \partial_t \mathbf{v} - \operatorname{Div} \mathbf{T}(\mathbf{v}, \mathfrak{q}) = 0, \quad \operatorname{div} \mathbf{v} &= 0 && \text{in } \Omega_+ \times (0,T), \\ \mathbf{T}(\mathbf{v}, \mathfrak{q})\mathbf{n} &= 0 && \text{on } \Gamma \times (0,T), \\ \mathbf{v}|_{t=0} &= 0 && \text{in } \Omega_+, \end{aligned}$$

then $\mathbf{v} = 0$ and $\mathfrak{q} = 0$.

3.2. Two Phase Problem for the Linear Electro-Magnetic Vector Field Equations

This subsection is devoted to presenting the L_p-L_q maximal regularity due to Frolova and Shibata [1] for the linear electro-magnetic vector field equations. The problem is formulated by a set of the following equations:

$$\begin{aligned} \mu \partial_t \mathbf{H} - \alpha^{-1} \Delta \mathbf{H} &= \mathbf{f} && \text{in } \dot\Omega \times (0,T), \\ [[\alpha^{-1} \operatorname{curl} \mathbf{H}]]\mathbf{n} = \mathbf{h}', \quad [[\mu \operatorname{div} \mathbf{H}]] &= h_N && \text{on } \Gamma \times (0,T), \\ [[\mathbf{H} - < \mathbf{H}, \mathbf{n} > \mathbf{n}]] = \mathbf{k}', \quad [[\mu \mathbf{H} \cdot \mathbf{n}]] &= k_N && \text{on } \Gamma \times (0,T), \\ \mathbf{n}_- \cdot \mathbf{H}_- = 0, \quad (\operatorname{curl} \mathbf{H}_-)\mathbf{n}_- &= 0 && \text{on } S_- \times (0,T), \\ \mathbf{H}|_{t=0} &= \mathbf{H}_0 && \text{in } \dot\Omega. \end{aligned} \tag{42}$$

To state the main result, we make a definition.

Definition 3. *Let $\Omega = \Omega_+ \cup \Gamma \cup \Omega_-$ be a domain given in the introduction. We say that Ω is a uniform C^2 domain with interface Γ if there exist positive constants a_1, a_2, and A such that the following assertion holds: For any $x_0 = (x_{01}, \ldots, x_{0N}) \in \Gamma$ there exist a coordinate number j and a C^2 function $h(x')$ defined on $B'_{a_1}(x'_0)$ such that $\|h\|_{H^k_\infty(B'_{a_1}(x'_0))} \leq A$ for $k \leq 2$ and*

$$\Gamma \cap B_{a_2}(x_0) = \{x \in \mathbb{R}^N \mid x_j = h(x') \ \ (x' \in B'_{a_1}(x'_0))\} \cap B_{a_2}(x_0),$$
$$\Omega_\pm \cap B_{a_2}(x_0) = \{x \in \mathbb{R}^N \mid \pm x_j > h(x') \ \ (x' \in B_{a_1}(x'_0))\} \cap B_{a_2}(x_0),$$

and for any $x_0 = (x_{01}, \ldots, x_{0N}) \in S_-$ there exists a coordinate number j and a C^2 function $h(x')$ defined on $B'_{a_1}(x'_0)$ such that $\|h\|_{H^k_\infty(B'_{a_1}(x'_0))} \leq A$ for $k \leq 2$ and

$$\Omega \cap B_{a_2}(x_0) = \{x \in \mathbb{R}^N \mid x_j > h(x') \ \ (x' \in B_{a_1}(x'_0))\} \cap B_{a_2}(x_0),$$
$$S_- \cap B_{a_2}(x_0) = \{x \in \mathbb{R}^N \mid x_j = h(x') \ \ (x' \in B'_{a_1}(x'_0))\} \cap B_{a_2}(x_0).$$

Here, we have set

$$y' = (y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_N) \ \ (y \in \{x, x_0\}),$$
$$B'_{a_1}(x'_0) = \{x' \in \mathbb{R}^{N-1} \mid |x' - x'_0| < a_1\},$$
$$B_{a_2}(x_0) = \{x \in \mathbb{R}^N \mid |x - x_0| < a_2\}.$$

Theorem 3. *Let $1 < p, q < \infty$, $2/p + N/q \neq 1, 2$, and $T > 0$. Assume that Ω is a uniform C^2 domain with interface Γ. Then, there exists a γ_2 such that the following assertion holds: Let $\mathbf{H}_0 \in B^{2(1-1/p)}_{q,p}(\dot\Omega)$ and let $\mathbf{f} \in L_p((0,T), L_q(\dot\Omega)^N)$, and let $\tilde{\mathbf{h}} = (\tilde{\mathbf{h}}', \tilde{h}_N)$, and $\tilde{\mathbf{k}} = (\tilde{\mathbf{k}}', \tilde{k}_N)$ be functions such that $\tilde{\mathbf{h}}' = \mathbf{h}'$, $\tilde{h}_N = h_N$, $\tilde{\mathbf{k}}' = \mathbf{k}$, and $\tilde{k}_N = k_N$ for $t \in (0,T)$, where $\mathbf{h}'$, h_N, $\mathbf{k}'$, and k_N are functions given in the right side of (42), and the following conditions hold:*

$$e^{-\gamma t}\tilde{\mathbf{h}} \in L_p(\mathbb{R}, H^1_q(\Omega)^N) \cap H^{1/2}_p(\mathbb{R}, L_q(\Omega)^N), \quad e^{-\gamma t}\tilde{\mathbf{k}} \in L_p(\mathbb{R}, H^2_q(\Omega)^N) \cap H^1_p(\mathbb{R}, L_q(\Omega)^N)$$

for any $\gamma \geq \gamma_2$. Moreover, we assume that $\mathbf{H}_0$, $\mathbf{h}$ and $\mathbf{k}$ satisfy the following compatibility conditions:

$$[[\alpha^{-1}\mathrm{curl}\,\mathbf{H}_0]]\mathbf{n} = \mathbf{h}'|_{t=0}, \quad [[\mu \mathrm{div}\,\mathbf{H}_0]] = h_N|_{t=0} \quad \text{on } \Gamma, \quad (\mathrm{curl}\,\mathbf{H}_{0-})\mathbf{n}_- = 0 \quad \text{on } S_- \tag{43}$$

provided $2/p + N/q < 1$;

$$[[\mathbf{H}_0 - <\mathbf{H}_0, \mathbf{n}> \mathbf{n}]] = \mathbf{k}'|_{t=0}, \quad [[\mu \mathbf{H}_0 \cdot \mathbf{n}]] = k_N|_{t=0} \quad \text{on } \Gamma, \quad \mathbf{n}_- \cdot \mathbf{H}_{0-} = 0 \quad \text{on } S_- \tag{44}$$

provided $2/p + N/q < 2$. Then, problem (42) admits a unique solution $\mathbf{H}$ with

$$\mathbf{H} \in L_p((0,T), H^2_q(\dot\Omega)^N) \cap H^1_p((0,T), L_q(\dot\Omega)^N)$$

possessing the estimate:

$$\|\partial_t \mathbf{H}\|_{L_p((0,T),L_q(\dot\Omega))} + \|\mathbf{H}\|_{L_p((0,T),H^2_q(\dot\Omega))} \leq Ce^{\gamma T}\{\|\mathbf{H}_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)} + F_H(\mathbf{f}, \tilde{\mathbf{h}}, \tilde{\mathbf{k}})\},$$

with

$$F_H(\mathbf{f}, \tilde{\mathbf{h}}, \tilde{\mathbf{k}}) = \|\mathbf{f}\|_{L_p(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\tilde{\mathbf{h}}\|_{L_p(\mathbb{R},H^1_q(\Omega))} + \|e^{-\gamma t}\tilde{\mathbf{h}}\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))}$$
$$+ \gamma^{1/2}\|e^{-\gamma t}\tilde{\mathbf{h}}\|_{L_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma t}\tilde{\mathbf{k}}\|_{L_p(\mathbb{R},H^2_q(\Omega))} + \|e^{-\gamma t}\partial_t\tilde{\mathbf{k}}\|_{L_p(\mathbb{R},L_q(\Omega))}\}$$

for any $\gamma \geq \gamma_2$ with some constant $C > 0$ independent of γ.

Remark 3. (1) *Theorem 3 was proved by Froloba and Shibata [18].*

(2) *The uniqueness holds in the following sense. Let* $\mathbf{H}$ *with*

$$\mathbf{H} \in L_p((0,T), H^2_q(\dot\Omega)^N) \cap H^1_p((0,T), L_q(\dot\Omega)^N)$$

satisfy the homogeneous equations:

$$\begin{aligned} \mu\partial_t\mathbf{H} - \alpha^{-1}\Delta\mathbf{H} &= 0 \quad \text{in } \dot\Omega \times (0,T), \\ [[\alpha^{-1}\mathrm{curl}\,\mathbf{H}]]\mathbf{n} = 0, \quad [[\mu\mathrm{div}\,\mathbf{H}]] &= 0 \quad \text{on } \Gamma \times (0,T), \\ [[\mathbf{H} - <\mathbf{H},\mathbf{n}>\mathbf{n}]] = 0, \quad [[\mu\mathbf{H}\cdot\mathbf{n}]] &= 0 \quad \text{on } \Gamma \times (0,T), \\ \mathbf{n}_\pm\cdot\mathbf{H}_\pm = 0, \quad (\mathrm{curl}\,\mathbf{H}_\pm)\mathbf{n}_\pm &= 0 \quad \text{on } S_\pm \times (0,T), \\ \mathbf{H}|_{t=0} &= 0 \quad \text{in } \dot\Omega. \end{aligned} \tag{45}$$

then $\mathbf{H} = 0$ *in* $\dot\Omega \times (0,T)$.

4. Estimate of Non-Linear Terms

Let $\mathbf{u}_+$ and $\mathbf{H}_\pm$ be N-vectors of functions such that

$$\begin{aligned} &\mathbf{u}_+ \in H^1_p((0,T), H^1_q(\Omega_+)^N) \cap L_p((0,T), H^3_q(\Omega_+)^N), \qquad \mathbf{u}_+|_{t=0} = \mathbf{u}_{0+}, \\ &\mathbf{H}_\pm \in H^1_p((0,T), L_q(\Omega_\pm)^N) \cap L_p((0,T), H^2_q(\Omega_\pm)^N), \qquad \mathbf{H}_\pm|_{t=0} = \tilde{\mathbf{H}}_{0\pm}, \end{aligned} \tag{46}$$

and we shall estimate nonlinear terms $\mathbf{N}_1(\mathbf{u}_+,\mathbf{H}_+), \dots, \mathbf{N}_9(\mathbf{u},\mathbf{H})$ appearing in the right side of Equation (7). Here, $\mathbf{w} = \mathbf{w}_+$ for $x \in \Omega_+$ and $\mathbf{w} = \mathbf{w}_-$ for $x \in \Omega_-$ ($\mathbf{w} \in \{\mathbf{u},\mathbf{H}\}$) and $\mathbf{u}_-$ is an extension of $\mathbf{u}_+$ defined in (5). For notational simplicity, we set

$$E^1_T(\mathbf{u}_+) = \|\partial_t\mathbf{u}_+\|_{L_p((0,T),H^1_q(\Omega_+))} + \|\mathbf{u}_+\|_{L_p((0,T),H^3_q(\Omega_+))},$$
$$E^{2,\pm}_T(\mathbf{H}_\pm) = \|\partial_t\mathbf{H}_\pm\|_{L_p((0,T),L_q(\Omega_\pm))} + \|\mathbf{H}_\pm\|_{L_p((0,T),H^2_q(\Omega_\pm))}, \quad E^2_T(\mathbf{H}) = E^{2,+}_T(\mathbf{H}_+) + E^{2,-}_T(\mathbf{H}_-).$$

Moreover, let $\mathbf{u}^i_+$ and $\mathbf{H}^i_\pm$ ($i = 1,2$) be N-vectors of functions such that

$$\begin{aligned} &\mathbf{u}^i_+ \in H^1_p((0,T), H^1_q(\Omega_+)^N) \cap L_p((0,T), H^3_q(\Omega_+)^N), \qquad \mathbf{u}^i_+|_{t=0} = \mathbf{u}_{0+}, \\ &\mathbf{H}^i_\pm \in H^1_p((0,T), L_q(\Omega_\pm)^N) \cap L_p((0,T), H^2_q(\Omega_\pm)^N), \qquad \mathbf{H}^i_\pm|_{t=0} = \tilde{\mathbf{H}}_{0\pm}. \end{aligned} \tag{47}$$

We also consider the differences: $\mathcal{N}_1 = \mathbf{N}_1(\mathbf{u}^1_+,\mathbf{H}^1_+) - \mathbf{N}_1(\mathbf{u}^2_+,\mathbf{H}^2_+), \dots, \mathcal{N}_9 = \mathbf{N}_9(\mathbf{u}^1,\mathbf{H}^1) - \mathbf{N}_9(\mathbf{u}^2,\mathbf{H}^2)$. Here, $\mathbf{w} = \mathbf{w}_+$ for $x \in \Omega_+$ and $\mathbf{w} = \mathbf{w}_-$ for $x \in \Omega_-$ ($\mathbf{w} \in \{\mathbf{u}^1,\mathbf{u}^2,\mathbf{H}^1,\mathbf{H}^2\}$) and $\mathbf{u}^i_-$ is an extension of $\mathbf{u}^i_+$ defined in (5). For notational simplicity, we assume that

$$\|\mathbf{u}_{0+}\|_{B^{3-2/p}_{q,p}(\Omega_+)} + \|\tilde{\mathbf{H}}_{0+}\|_{B^{2(1-1/p)}_{q,p}(\Omega_+)} + \|\tilde{\mathbf{H}}_{0-}\|_{B^{2(1-1/p)}_{q,p}(\Omega_-)} \le B \tag{48}$$

for some constant $B > 0$. In what follows, we assume that $2 < p < \infty$, $N < q < \infty$ and $2/p + N/q < 1$. To estimate nonlinear terms, we use the following inequalities which follows from Sobolev's inequality.

$$\begin{aligned} \|f\|_{L_\infty(\Omega_\pm)} &\le C\|f\|_{H^1_q(\Omega_\pm)}, \\ \|fg\|_{H^1_q(\Omega_\pm)} &\le C\|f\|_{H^1_q(\Omega_\pm)}\|g\|_{H^1_q(\Omega_\pm)}, \\ \|fg\|_{H^2_q(\Omega_\pm)} &\le C(\|f\|_{H^2_q(\Omega_\pm)}\|g\|_{H^1_q(\Omega_\pm)} + \|f\|_{H^1_q(\Omega_\pm)}\|g\|_{H^2_q(\Omega_\pm)}), \\ \|fg\|_{W^{1-1/q}_q(\Gamma)} &\le C\|f\|_{W^{1-1/q}_q(\Gamma)}\|g\|_{W^{1-1/q}_q(\Gamma)}, \\ \|fg\|_{W^{2-1/q}_q(\Gamma)} &\le C(\|f\|_{W^{2-1/q}_q(\Gamma)}\|g\|_{W^{1-1/q}_q(\Gamma)} + \|f\|_{W^{1-1/q}_q(\Gamma)}\|g\|_{W^{2-1/q}_q(\Gamma)}). \end{aligned} \tag{49}$$

By Hölder's inequality and (5), we have

$$\begin{aligned} \|\Psi_{\mathbf{w}}\|_{L_\infty((0,T),H^2_q(\Omega))} &\leq CT^{1/p'}E^1_T(\mathbf{w}_+) \quad \text{for } \mathbf{w} \in \{\mathbf{u}, \mathbf{u}^1, \mathbf{u}^2\}, \\ \|\Psi_{\mathbf{u}^1} - \Psi_{\mathbf{u}^2}\|_{L_\infty((0,T),H^2_q(\Omega))} &\leq CT^{1/p'}E^1_T(\mathbf{u}^1_+ - \mathbf{u}^2_+). \end{aligned} \tag{50}$$

In view of (49) and (50), choosing $T > 0$ so small, we may assume that

$$\|\Psi_{\mathbf{w}}\|_{L_\infty((0,T),L_\infty(\Omega))} \leq \delta \quad \text{for } \mathbf{w} \in \{\mathbf{u}, \mathbf{u}^1, \mathbf{u}^2\}. \tag{51}$$

In the following, for simplicity, choosing $T > 0$ small, we also assume that

$$T^{1/p'}(E^1_T(\mathbf{w}) + B) \leq 1 \quad \text{for } \mathbf{w} \in \{\mathbf{u}_+, \mathbf{u}^1_+, \mathbf{u}^2_+\}. \tag{52}$$

We may assume that the unit outer normal $\mathbf{n}$ to Γ is defined on $\mathbb{R}^N$ and $\|\mathbf{n}\|_{H^2_\infty(\mathbb{R}^N)} < \infty$ because Ω_+ is a uniform C^3 domain. Thus, setting

$$[\mathcal{A}_i(\Psi_{\mathbf{w}})]_{T,2} := \|\mathcal{A}_i(\Psi_{\mathbf{w}})\|_{L_\infty((0,T),L_\infty(\Omega))} + \|\nabla\mathcal{A}_i(\Psi_{\mathbf{w}})\|_{L_\infty((0,T),H^1_q(\Omega))},$$

by (51), (49), (50), and (52) we have

$$[\mathcal{A}_i(\Psi_{\mathbf{w}})]_{T,2} \leq C, \tag{53}$$

with some constant $C > 0$ for $i = 1, \ldots, 9$ and $\mathbf{w} \in \{\mathbf{u}, \mathbf{u}^1, \mathbf{u}^2\}$, where we have set $\mathcal{A}_5(\Psi_{\mathbf{w}}) = (\mathcal{A}_{5+}(\Psi_{\mathbf{w}}), \mathcal{A}_{5-}(\Psi_{\mathbf{w}}))$ and $\mathcal{A}_6(\Psi_{\mathbf{w}}) = (\mathcal{A}_{61}(\Psi_{\mathbf{w}}), \mathcal{A}_{62}(\Psi_{\mathbf{w}}))$.

We first consider $\mathbf{N}_1(\mathbf{u}_+, \mathbf{H}_+)$ and $\mathcal{N}_1 = \mathbf{N}_1(\mathbf{u}^1_+, \mathbf{H}^1_+) - \mathbf{N}_1(\mathbf{u}^2_+, \mathbf{H}^2_+)$. Recall (24). Applying (49) and using (50), we have

$$\begin{aligned} &\|\Psi_{\mathbf{u}} \otimes (\partial_t\mathbf{u}_+, \nabla^2\mathbf{u}_+)\|_{L_p((0,T),H^1_q(\Omega_+))} \leq CT^{1/p'}E^1_T(\mathbf{u}_+)^2, \\ &\|\mathbf{u}_+ \otimes \nabla\mathbf{u}_+\|_{L_p((0,T),H^1_q(\Omega_+))} \leq CT^{1/p}\|\mathbf{u}_+\|^2_{L_\infty((0,T),H^2_q(\Omega_+))}, \\ &\|\nabla\Psi_{\mathbf{u}} \otimes \nabla\mathbf{u}_+\|_{L_p((0,T),H^1_q(\Omega_+))} \leq CT^{1/p'}E^1_T(\mathbf{u}_+)^2, \\ &\|\mathbf{H}_+ \otimes \nabla\mathbf{H}_+\|_{L_p((0,T),H^1_q(\Omega_+))} \leq C\|\mathbf{H}_+\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\mathbf{H}_+\|_{L_p((0,T),H^2_q(\Omega_+))}. \end{aligned} \tag{54}$$

By real interpolation theory, we see that

$$\begin{aligned} &\sup_{t\in(0,T)} \|\mathbf{v}_\pm(\cdot,t)\|_{B^{\ell+2-2/p}_{q,p}(\Omega_\pm)} \\ &\quad \leq C(\|\mathbf{v}|_{t=0}\|_{B^{\ell+2-2/p}_{q,p}(\Omega_\pm)} + \|\partial_t\mathbf{v}_\pm\|_{L_p((0,T),H^\ell_q(\Omega_\pm))} + \|\mathbf{v}_\pm\|_{L_p((0,T),H^{\ell+2}_q(\Omega_\pm))}) \end{aligned} \tag{55}$$

($\ell = 0, 1$). In order to prove this, we make a few preparations. For a X-valued function $f(\cdot, t)$ defined for $t \in (0, T)$, where X is a Banach space, we set

$$e_T[f](\cdot,t) = \begin{cases} 0 & \text{for } t < 0, \\ f(\cdot,t) & \text{for } 0 < t < T, \\ f(\cdot, 2T - t) & \text{for } T < t < 2T, \\ 0 & \text{for } t > 2T. \end{cases} \tag{56}$$

Then, $e_T[f](\cdot,t) = f(\cdot,t)$ for $t \in (0, T)$. If $f|_{t=0} = 0$, then

$$\partial_t e_T[f](\cdot,t) = \begin{cases} 0 & \text{for } t < 0, \\ \partial_t f(\cdot,t) & \text{for } 0 < t < T, \\ -(\partial_t f)(\cdot, 2T - t) & \text{for } T < t < 2T, \\ 0 & \text{for } t > 2T. \end{cases} \tag{57}$$

In particular, we have

$$\begin{aligned}\|e_T[f]\|_{L_p(\mathbb{R},X)} &\leq 2\|f\|_{L_p((0,T),X)},\\ \|\partial_t e_T[f]\|_{L_p(\mathbb{R},X)} &\leq 2\|\partial_t f\|_{L_p((0,T),X)}.\end{aligned} \tag{58}$$

Let $\mathbf{w}_\pm$ be a N-vector of function defined on $\Omega_\pm$ and let $E_\mp[\mathbf{w}_\pm]$ be an extension of $\mathbf{w}_\pm$ to $\Omega_\mp$ for which

$$\begin{aligned}E_\mp[\mathbf{w}_\pm] &= \mathbf{w}_\pm \quad \text{for } x \in \Omega_\pm,\\ \lim_{\substack{x\to x_0\\ x\in\Omega_\mp}} \partial_x^\alpha E_\mp[\mathbf{w}_\pm](x,t) &= \lim_{\substack{x\to x_0\\ x\in\Omega_\pm}} \partial_x^\alpha \mathbf{w}_\pm(x,t) \quad \text{for } |\alpha| \leq 1 \text{ and } x_0 \in \Gamma,\\ \|E_\mp[\mathbf{w}_\pm](\cdot,t)\|_{H^i_q(\Omega)} &\leq C\|\mathbf{w}_\pm(\cdot,t)\|_{H^i_q(\Omega_\pm)} \quad \text{for } i = 0,1,2.\end{aligned} \tag{59}$$

Let $E_{\mathbb{R}^N}[E_\mp[\mathbf{w}_\pm]]$ be an extension of $E_\mp[\mathbf{w}_\pm]$ to $\mathbb{R}^N$ for which

$$\begin{aligned}E_{\mathbb{R}^N}[E_\mp[\mathbf{w}_\pm]] &= E_\mp[\mathbf{w}_\pm] \quad \text{on } \Omega,\\ \|E_{\mathbb{R}^N}[E_\mp[\mathbf{w}_\pm]]\|_{B^{\ell+2(1-1/p)}_{q,p}(\mathbb{R}^N)} &\leq C\|\mathbf{w}_\pm\|_{B^{\ell+2(1-1/p)}_{q,p}(\Omega_\pm)} \quad (\ell = 0,1).\end{aligned} \tag{60}$$

For $\mathbf{v}_0 \in B^{\ell+2(1-1/p)}_{q,p}(\mathbb{R}^N)$, let

$$T(t)\mathbf{v}_0 = e^{(-1+\Delta)t}\mathbf{v}_0 \tag{61}$$

be a C^0 analytic semigroup satisfying the condition: $T(0)\mathbf{v}_0 = \mathbf{v}_0$ and possessing the estimate:

$$\|T(\cdot)\mathbf{v}_0\|_{L_p((0,\infty),H^{\ell+2}_q(\mathbb{R}^N))} + \|\partial_t T(\cdot)\mathbf{v}_0\|_{L_p((0,\infty),H^{\ell}_q(\mathbb{R}^N))} \leq C\|\mathbf{v}_0\|_{B^{\ell+2(1-1/p)}_{q,p}(\mathbb{R}^N)} \tag{62}$$

for $\ell = 0,1,2$. Let $\mathbf{w}_\pm$ be defined on $\Omega_\pm \times (0,T)$ and set $\mathbf{w}_{0\pm} = \mathbf{w}_\pm|_{t=0}$. Let $\psi(t)$ be a function that equals one for $t > -1$ and zero for $t < -2$ and let

$$\mathcal{T}[E_\mp[\mathbf{w}_{0\pm}]](t) = \psi(t)T(|t|)E_{\mathbb{R}^N}[E_\mp[\mathbf{w}_{0\pm}]], \tag{63}$$

Then, by (60) and (62)

$$\begin{aligned}&\mathcal{T}[E_\mp[\mathbf{w}_{0\pm}]](0) = E_\mp[\mathbf{w}_{0\pm}] \quad \text{in } \Omega,\\ &\|\mathcal{T}[E_\mp[\mathbf{w}_{0\pm}]]\|_{L_p(\mathbb{R},H^{2+\ell}_q(\Omega))} + \|\partial_t\mathcal{T}[E_\mp[\mathbf{w}_{0\pm}]]\|_{L_p(\mathbb{R},H^{\ell}_q(\Omega))} \leq C\|\mathbf{w}_0\|_{B^{2(1-1/p)+\ell}_{q,p}(\dot\Omega)}.\end{aligned} \tag{64}$$

Set

$$\mathcal{E}[E_\mp[\mathbf{w}_\pm]] = \mathcal{T}[E_\mp[\mathbf{w}_{0\pm}]] + e_T[E_\mp[\mathbf{w}_\pm] - \mathcal{T}[E_\mp[\mathbf{w}_{0\pm}]]]. \tag{65}$$

Obviously, $\mathcal{E}[E_\mp[\mathbf{w}_\pm]] = E_\mp[\mathbf{w}_\pm]$ for $t \in (0,T)$. Then, (55) is guaranteed by (58), (59) and (64) as follows:

$$\begin{aligned}\sup_{t\in(0,T)} \|\mathbf{v}_\pm(\cdot,t)\|_{B^{\ell+2-2/p}_{q,p}(\Omega_\pm)} &= \sup_{t\in(0,T)} \|\mathcal{E}[E_\mp[\mathbf{v}_\pm]](\cdot,t)\|_{B^{\ell+2-2/p}_{q,p}(\mathbb{R}^N)}\\ &\leq C(\|\mathbf{v}|_{t=0}\|_{B^{\ell+2-2/p}_{q,p}(\Omega_\pm)} + \|\partial_t\mathbf{v}_\pm\|_{L_p((0,T),H^{\ell}_q(\Omega_\pm))} + \|\mathbf{v}_\pm\|_{L_p((0,T),H^{\ell+2}_q(\Omega_\pm))}).\end{aligned}$$

Combining (55) with (48) leads to

$$\begin{aligned}\|\mathbf{w}_+\|_{L_\infty((0,T),H^2_q(\Omega_+))} &\leq C(B + E^1_T(\mathbf{w}_+)) \qquad \text{for } \mathbf{w} \in \{\mathbf{u},\mathbf{u}^1,\mathbf{u}^2\},\\ \|\mathbf{z}_\pm\|_{L_\infty((0,T),H^1_q(\Omega_\pm))} &\leq C(B + E^{2,\pm}_T(\mathbf{z}_\pm)) \qquad \text{for } \mathbf{z} \in \{\mathbf{H},\mathbf{H}^1,\mathbf{H}^2\},\end{aligned} \tag{66}$$

because $B^{\ell+2-2/p}_{q,p}(\Omega_\pm)$ is continuously imbedded into $H^{\ell+1}_q(\Omega_\pm)$ as follows from $2 - 2/p > 1$, that is, $2 < p < \infty$. Moreover,

$$\|\mathbf{H}_+\|_{L_\infty((0,T),H^1_q(\Omega_+))} \leq C(B + T^{s/p'(1+s)}E^{2,+}_T(\mathbf{H}_+)), \tag{67}$$

provided $0 < T < 1$. In fact, we write $\mathbf{H}_+ = \tilde{\mathbf{H}}_{0+} + \mathbf{v}$ with $\mathbf{v} = \mathbf{H}_+ - \tilde{\mathbf{H}}_{0+}$. Since $\mathbf{v}|_{t=0} = 0$, we have

$$\|\mathbf{v}(\cdot,t)\|_{L_q(\Omega_+)} \leq \int_0^t \|\partial_s\mathbf{H}_+(\cdot,s)\|_{L_q(\Omega_+)} \leq T^{1/p'}E^{2,+}_T(\mathbf{H}_+)$$

for $t \in (0,T)$. On the other hand, choosing $s \in (0, 1-2/p)$ yields that $B^{2(1-1/p)}_{q,p}(\Omega_+)$ is continuously imbedded into $W^{1+s}_q(\Omega_+)$, and so by (55)

$$\|\mathbf{v}\|_{W^{1+s}_q(\Omega_+)} \leq C\|\mathbf{v}\|_{B^{2(1-1/p)}_{q,p}(\Omega_+)} \leq C(B + E^{2,+}_T(\mathbf{H}_+)).$$

Since $\|\mathbf{v}\|_{H^1_q(\Omega_+)} \leq C_s\|\mathbf{v}\|^{s/(1+s)}_{L_q(\Omega_+)}\|\mathbf{v}\|^{1/(1+s)}_{W^{1+s}_q(\Omega_+)}$, we have (67) provided $0 < T < 1$.

Combining (54), (53), (66), and (67), we have

$$\begin{aligned} &\|\mathbf{N}_1(\mathbf{u}_+,\mathbf{H}_+)\|_{L_p((0,T),H^1_q(\Omega_+))} \\ &\quad\leq C(T^{1/p'}E^1_T(\mathbf{u})^2 + T^{1/p}(B + E^1_T(\mathbf{u}_+))^2 + (B + T^{s/p'(1+s)}E^{2,+}_T(\mathbf{H}_+))E^{2,+}_T(\mathbf{H}_+)). \end{aligned} \tag{68}$$

We next consider $\mathcal{N}_1 = \mathbf{N}_1(\mathbf{u}^1_+,\mathbf{H}^1_+) - \mathbf{N}_1(\mathbf{u}^2_+,\mathbf{H}^2_+)$, which is represented by $\mathcal{N}_1 = \mathcal{N}_{11} + \mathcal{N}_{12}$ with

$$\begin{aligned} &\mathcal{N}_{11} = (\mathcal{A}_1(\Psi_{\mathbf{u}^1}) - \mathcal{A}_1(\Psi_{\mathbf{u}^2}))(K^1_1, K^1_2, K^1_3, K^1_4), \\ &\mathcal{N}_{12} = \mathcal{A}_1(\Psi_{\mathbf{u}^2})(K^1_1 - K^2_1, K^1_2 - K^2_2, K^1_3 - K^2_3, K^1_4 - K^2_4), \\ &\quad K^i_1 = \Psi_{\mathbf{u}^i} \otimes (\partial_t\mathbf{u}^i_+, \nabla^2\mathbf{u}^i_+), \quad K^i_2 = \mathbf{u}^i_+ \otimes \nabla\mathbf{u}^i_+, \quad K^i_3 = \nabla\Psi_{\mathbf{u}^i} \otimes \nabla\mathbf{u}^i_+, \quad K^i_4 = \mathbf{H}^i_+ \otimes \nabla\mathbf{H}^i_+. \end{aligned}$$

Representing

$$\mathcal{A}_i(\Psi_{\mathbf{u}^1}) - \mathcal{A}_i(\Psi_{\mathbf{u}^2}) = \int_0^1 (d_{\mathbf{k}}\mathcal{A}_i)(\Psi_{\mathbf{u}^2} + \theta(\Psi_{\mathbf{u}^1} - \Psi_{\mathbf{u}^2}))\,d\theta(\Psi_{\mathbf{u}^1} - \Psi_{\mathbf{u}^2}), \tag{69}$$

by (49), (51), (50), and (52), we have

$$\|\mathcal{A}_i(\Psi_{\mathbf{u}^1}) - \mathcal{A}_i(\Psi_{\mathbf{u}^2})\|_{L_\infty((0,T),H^2_q(\Omega_+))} \leq CT^{1/p'}\|\mathbf{u}^1_+ - \mathbf{u}^2_+\|_{L_p((0,T),H^3_q(\Omega_+))} \tag{70}$$

for $i = 1,\ldots,9$. Estimating $\|(K^1_1,\ldots,K^1_4)\|_{L_p((0,T),H^1_q(\Omega_+))}$ in the same manner as in proving (68), we have

$$\|\mathcal{N}_{11}\|_{L_p((0,T),H^1_q(\Omega_+))} \leq CT^{1/p'}(B^2 + E^1_T(\mathbf{u}^1_+)^2 + E^{2,+}_T(\mathbf{H}^1_+)^2)E^1_T(\mathbf{u}^1_+ - \mathbf{u}^2_+), \tag{71}$$

where we have used $0 < T < 1$ and $(B + T^{s/p'(1+s)}E^{2,+}_T(\mathbf{H}_+))E^{2,+}_T(\mathbf{H}_+) \leq (1/2)B^2 + (3/2)E^{2,+}_T(\mathbf{H}_+)^2$. To estimate $\mathcal{N}_{12}$ we use the fact:

$$|f_1f_2 - g_1g_2| \leq |f_1 - g_1||f_2| + |f_2 - g_2||g_1|. \tag{72}$$

Thus, by (49), (50), (52), (66), and (67), we have

$$\begin{aligned}
&\|K_1^1 - K_1^2\|_{L_p((0,T),H_q^1(\Omega_+))} \\
&\quad \leq C(\|\Psi_{\mathbf{u}^1} - \Psi_{\mathbf{u}^2}\|_{L_\infty(0,T),H_q^1(\Omega_+))}\|(\partial_t\mathbf{u}_+^1, \nabla^2\mathbf{u}_+^1)\|_{L_p((0,T),H_q^1(\Omega_+))} \\
&\qquad\qquad + \|\Psi_{\mathbf{u}^2}\|_{L_\infty((0,T),H_q^1(\Omega_+))}E_T^1(\mathbf{u}_+^1 - \mathbf{u}_+^2)) \\
&\quad \leq CT^{1/p'}(E_T^1(\mathbf{u}_+^1) + E_T^2(\mathbf{u}_+^2))E_T^1(\mathbf{u}_+^1 - \mathbf{u}_+^2); \\
&\|K_2^1 - K_2^2\|_{L_p((0,T),H_q^1(\Omega_+))} \leq CT^{1/p}(E_T^1(\mathbf{u}_+^1) + E_T^2(\mathbf{u}_+^2) + B)E_T^1(\mathbf{u}_+^1 - \mathbf{u}_+^2); \\
&\|K_3^1 - K_3^2\|_{L_p((0,T),H_q^1(\Omega_+))} \leq CT^{1/p'}(E_T^1(\mathbf{u}_+^1) + E_T^2(\mathbf{u}_+^2) + B)E_T^1(\mathbf{u}_+^1 - \mathbf{u}_+^2); \\
&\|K_4^1 - K_4^2\|_{L_p((0,T),H_q^1(\Omega_+))} \\
&\quad \leq C(\|\mathbf{H}_+^1 - \mathbf{H}_+^2\|_{L_\infty((0,T),H_q^1(\Omega_+))}E_T^{2,+}(\mathbf{H}_+^1) + \|\mathbf{H}_+^2\|_{L_\infty((0,T),H_q^1(\Omega_+))}E_T^{2,+}(\mathbf{H}_+^1 - \mathbf{H}_+^2)) \\
&\quad \leq C(T^{\frac{s}{p'(1+s)}}E_T^{2,+}(\mathbf{H}_+^1 - \mathbf{H}_+^2)E_T^{2,+}(\mathbf{H}_+^1) + (B + T^{\frac{s}{p'(1+s)}}E_T^{2,+}(\mathbf{H}_+^2))E_T^{2,+}(\mathbf{H}_+^1 - \mathbf{H}_+^2) \\
&\quad \leq C\{B + T^{\frac{s}{p'(1+s)}}(E_T^{2,+}(\mathbf{H}_+^1) + E_T^{2,+}(\mathbf{H}_+^2))\}E_T^{2,+}(\mathbf{H}_+^1 - \mathbf{H}_+^2),
\end{aligned}$$

which, combined with (53) and (71), gives that

$$\begin{aligned}
\|\mathcal{N}_1\|_{L_p((0,T),H_q^1(\Omega_+))} \leq C[&\{T^{1/p'}(B^2 + E_T^1(\mathbf{u}_+^1)^2 + E_T^1(\mathbf{u}_+^2)^2 + E_T^{2,+}(\mathbf{H}_+^1)^2) + \\
&+ (T^{1/p'} + T^{1/p})(E_T^1(\mathbf{u}_+^1) + E_T^1(\mathbf{u}_+^2) + B)\}E_T^1(\mathbf{u}_+^1 - \mathbf{u}_+^2) \\
&+ (B + T^{\frac{s}{p'(1+s)}}(E_T^{2,+}(\mathbf{H}_+^1) + E_T^{2,+}(\mathbf{H}_+^2)))E_T^{2,+}(\mathbf{H}_+^1 - \mathbf{H}_+^2)].
\end{aligned} \tag{73}$$

We now estimate $\mathbf{N}_2$, $\mathbf{N}_3$, $\mathcal{N}_2$ and $\mathcal{N}_3$. By (49), (50), (66), and (53), to the formula of $\mathbf{N}_2(\mathbf{u}_+, \mathbf{H}_+)$ given in (25), we have

$$\begin{aligned}
\|\mathbf{N}_2(\mathbf{u}_+)\|_{L_p((0,T),H_q^2(\Omega_+)} &\leq C[\mathcal{A}_2(\Psi_{\mathbf{u}})]_{T,2}\|\Psi_{\mathbf{u}}\|_{L_\infty((0,T),H_q^2(\Omega_+))}\|\nabla\mathbf{u}_+\|_{L_p((0,T),H_q^2(\Omega_+))} \\
&\leq CT^{1/p'}E_T^1(\mathbf{u}_+)^2.
\end{aligned} \tag{74}$$

By (51), (50), (69), (52), and (66),

$$\begin{aligned}
&\|\partial_t\mathcal{A}_i(\Psi_{\mathbf{w}})\|_{L_\infty((0,T),H_q^1(\Omega))} \leq C\|\mathbf{u}_+\|_{L_\infty((0,T),H_q^2(\Omega_+))} \leq C(B + E_T^1(\mathbf{w}_+)) \quad \text{for } \mathbf{w} \in \{\mathbf{u}, \mathbf{u}^1, \mathbf{u}^2\}; \\
&\|\partial_t(\mathcal{A}_i(\Psi_{\mathbf{u}^1}) - \mathcal{A}_i(\Psi_{\mathbf{u}^2}))\|_{L_\infty((0,T),H_q^1(\Omega))} \leq C(\|\mathbf{u}_+^1 - \mathbf{u}_+^2\|_{L_\infty((0,T),H_q^2(\Omega_+))} \\
&\qquad + (\|\mathbf{u}_+^1\|_{L_\infty((0,T),H_q^2(\Omega_+))} + \|\mathbf{u}_+^2\|_{L_\infty((0,T),H_q^2(\Omega_+))})\|\Psi_{\mathbf{u}^1} - \Psi_{\mathbf{u}^2}\|_{L_\infty((0,T),H_q^1(\Omega_+))}) \\
&\quad \leq CE_T^1(\mathbf{u}_+^1 - \mathbf{u}_+^2)
\end{aligned} \tag{75}$$

By (49), (52), (50), (66), (53), and (75), we have

$$\begin{aligned}
&\|\partial_t\mathbf{N}_2(\mathbf{u}_+)\|_{L_p((0,T),L_q(\Omega_+))} \\
&\qquad \leq C\{\|\partial_t\mathcal{A}_2(\Psi_{\mathbf{u}})\|_{L_\infty((0,T),H_q^1(\Omega))}\|\Psi_{\mathbf{u}}\|_{L_\infty((0,T),H_q^1(\Omega_+))}T^{1/p}\|\nabla\mathbf{u}_+\|_{L_\infty((0,T),L_q(\Omega_+))} \\
&\qquad + T^{1/p}[\mathcal{A}_2(\Psi_{\mathbf{u}})]_{T,2}\|\partial_t\Psi_{\mathbf{u}}\|_{L_\infty((0,T),H_q^1(\Omega_+))}\|\nabla\mathbf{u}_+\|_{L_\infty((0,T),L_q(\Omega_+))} \\
&\qquad + [\mathcal{A}_2(\Psi_{\mathbf{u}})]_{T,2}\|\Psi_{\mathbf{u}}\|_{L_\infty((0,T),H_q^1(\Omega_+))}\|\partial_t\nabla\mathbf{u}_+\|_{L_p((0,T),L_q(\Omega_+))}\} \\
&\qquad \leq C\{T^{1/p'}E_T^1(\mathbf{u}_+)^2 + T^{1/p}(B + E_T^1(\mathbf{u}_+))^2\}; \\
&\|\mathbf{N}_3(\mathbf{u}_+)\|_{L_p((0,T),H_q^1(\Omega_+))} \\
&\qquad \leq CT^{1/p}[\mathcal{A}_3(\Psi_{\mathbf{u}})]_{T,2}\|\Psi_{\mathbf{u}}\|_{L_\infty((0,T),H_q^1(\Omega_+))}\|\mathbf{u}_+\|_{L_\infty((0,T),H_q^1(\Omega_+))}\} \\
&\qquad \leq CT^{1/p}(B + E_T^1(\mathbf{u}_+))^2, \\
&\|\partial_t\mathbf{N}_3(\mathbf{u}_+)\|_{L_p((0,T),H_q^1(\Omega_+))}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\{\|\partial_t \mathcal{A}_3(\Psi_{\mathbf{u}})\|_{L_\infty((0,T),H^1_q(\Omega))}\|\Psi_{\mathbf{u}}\|_{L_\infty((0,T),H^1_q(\Omega_+))}T^p\|\mathbf{u}_+\|_{L_\infty((0,T),H^1_q(\Omega_+))} \\
&\quad + T^{1/p}[\mathcal{A}_3(\Psi_{\mathbf{u}})]_{T,2}\|\partial_t\Psi_{\mathbf{u}}\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\mathbf{u}_+\|_{L_\infty((0,T),H^1_q(\Omega_+))} \\
&\quad + [\mathcal{A}_3(\Psi_{\mathbf{u}})]_{T,2}\|\Psi_{\mathbf{u}}\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\partial_t\mathbf{u}_+\|_{L_p((0,T),H^1_q(\Omega_+))}\} \\
&\leq C\{T^{1/p'}E^1_T(\mathbf{u}_+)^2 + T^{1/p}(B+E^1_T(\mathbf{u}_+))^2\},
\end{aligned}
$$

which, combined with (74), yields that

$$
\begin{aligned}
&\|\mathbf{N}_2(\mathbf{u}_+)\|_{L_p((0,T),H^2_q(\Omega_+))} + \|\partial_t\mathbf{N}_2(\mathbf{u}_+)\|_{L_p((0,T),L_q(\Omega_+))} + \|\mathbf{N}_3(\mathbf{u}_+)\|_{H^1_p((0,T),H^1_q((\Omega_+))} \\
&\quad \leq C\{T^{1/p'}E^1_T(\mathbf{u}_+)^2 + T^{1/p}(B+E^1_T(\mathbf{u}_+))^2\}.
\end{aligned}
\tag{76}
$$

To estimate $\mathcal{N}_2$ and $\mathcal{N}_3$, we write $\mathcal{N}_2 = \mathcal{N}_{21} + \mathcal{N}_{22}$ and $\mathcal{N}_3 = \mathcal{N}_{31} + \mathcal{N}_{32}$ with

$$
\begin{aligned}
&\mathcal{N}_{21} = (\mathcal{A}_2(\Psi_{\mathbf{u}^1}) - \mathcal{A}_2(\Psi_{\mathbf{u}^2}))\Psi_{\mathbf{u}^1}\otimes\nabla\mathbf{u}^1_+, \quad \mathcal{N}_{22} = \mathcal{A}_2(\Psi_{\mathbf{u}^2})(\Psi_{\mathbf{u}^1}\otimes\nabla\mathbf{u}^1_+ - \Psi_{\mathbf{u}^2}\otimes\nabla\mathbf{u}^2_+) \\
&\mathcal{N}_{31} = (\mathcal{A}_3(\Psi_{\mathbf{u}^1}) - \mathcal{A}_3(\Psi_{\mathbf{u}^2}))\Psi_{\mathbf{u}^1}\otimes\mathbf{u}^1_+, \quad \mathcal{N}_{32} = \mathcal{A}_3(\Psi_{\mathbf{u}^2})(\Psi_{\mathbf{u}^1}\otimes\mathbf{u}^1_+ - \Psi_{\mathbf{u}^2}\otimes\mathbf{u}^2_+).
\end{aligned}
$$

Employing the same argument as in proving (76) and using (52), (70) and (75), we have

$$
\begin{aligned}
\|\mathcal{N}_{21}\|_{L_p((0,T),H^2_q(\Omega_+))} &\leq CT^{1/p'}E^1_T(\mathbf{u}^1_+)E^1_T(\mathbf{u}^1_+ - \mathbf{u}^2_+), \\
\|\partial_t\mathcal{N}_{21}\|_{L_p((0,T),L_q(\Omega_+))} &\leq C(T^{1/p'}E^1_T(\mathbf{u}^1_+) + T^{1/p}(B+E^1_T(\mathbf{u}^1_+)))E^1_T(\mathbf{u}^1_+ - \mathbf{u}^2_+); \\
\|\mathcal{N}_{31}\|_{L_p((0,T),H^1_q(\Omega_+))} &\leq CT(B+E^1_T(\mathbf{u}^1_+))E^1_T(\mathbf{u}^1_+ - \mathbf{u}^2_+), \\
\|\partial_t\mathcal{N}_{31}\|_{L_p((0,T),H^1_q(\Omega_+))} &\leq C(T^{1/p'}E^1_T(\mathbf{u}^1_+) + T^{1/p}(B+E^1_T(\mathbf{u}^1_+)))E^1_T(\mathbf{u}^1_+ - \mathbf{u}^2_+).
\end{aligned}
\tag{77}
$$

By (52), (50), (53), (72), (75), and (66), we have

$$
\begin{aligned}
&\|\mathcal{N}_{22}\|_{L_p((0,T),H^2_q(\Omega_+))} \\
&\quad \leq C[\mathcal{A}_2(\Psi_{\mathbf{u}^2})]_{T,2}\{\|\Psi_{\mathbf{u}^1} - \Psi_{\mathbf{u}^2}\|_{L_\infty((0,T),H^2_q(\Omega_+))}\|\nabla\mathbf{u}^1_+\|_{L_p((0,T),H^2_q(\Omega_+))} \\
&\quad + \|\Psi_{\mathbf{u}^2}\|_{L_\infty((0,T),H^2_q(\Omega_+))}\|\nabla(\mathbf{u}^1 - \mathbf{u}^2)\|_{L_p((0,T),H^2_q(\Omega_+))}\} \\
&\quad \leq CT^{1/p'}(E^1_T(\mathbf{u}^1_+) + E^1_T(\mathbf{u}^2_+))E^1_T(\mathbf{u}^1_+ - \mathbf{u}^2_+); \\
&\|\partial_t\mathcal{N}_{22}\|_{L_p((0,T),L_q(\Omega_+))} \\
&\quad \leq C\{\|\partial_t\mathcal{A}_2(\psi_{\mathbf{u}^2})\|_{L_\infty((0,T),H^1_q(\Omega))}(\|\Psi_{\mathbf{u}^1} - \Psi_{\mathbf{u}^2}\|_{L_\infty((0,T),H^1_q(\Omega_+))}T^{1/p}\|\nabla\mathbf{u}^1_+\|_{L_\infty((0,T),L_q(\Omega_+))} \\
&\qquad + T^{1/p}\|\Psi_{\mathbf{u}^2}\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\nabla(\mathbf{u}^1_+ - \mathbf{u}^2_+)\|_{L_\infty((0,T),L_q(\Omega_+))}) \\
&\quad + T^{1/p}[\mathcal{A}_2(\Psi_{\mathbf{u}^2})]_{T,2}(\|\partial_t(\Psi_{\mathbf{u}^1} - \Psi_{\mathbf{u}^2})\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\nabla\mathbf{u}^1_+\|_{L_\infty((0,T),L_q(\Omega_+))} \\
&\qquad + \|\partial_t\Psi_{\mathbf{u}^2}\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\nabla(\mathbf{u}^1_+ - \mathbf{u}^2_+)\|_{L_\infty((0,T),L_q(\Omega_+))}) \\
&\quad + [\mathcal{A}_2(\Psi_{\mathbf{u}^2})]_{T,2}(\|\Psi_{\mathbf{u}^1} - \Psi_{\mathbf{u}^2}\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\partial_t\nabla\mathbf{u}^1_+\|_{L_p((0,T),L_q(\Omega_+))}) \\
&\qquad + \|\Psi_{\mathbf{u}^2}\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\partial_t\nabla(\mathbf{u}^1_+ - \mathbf{u}^2_+)\|_{L_p((0,T),L_q(\Omega_+))})\} \\
&\quad \leq C\{T^{1/p'}(E^1_T(\mathbf{u}^1_+) + E^1_T(\mathbf{u}^2_+)) + T^{1/p}(B + E^1_T(\mathbf{u}^1_+) + E^1_T(\mathbf{u}^2_+))\}E^1_T(\mathbf{u}^1_+ - \mathbf{u}^2_+).
\end{aligned}
\tag{78}
$$

Employing the same argument as in proving the second inequality in (78), we also have

$$
\begin{aligned}
&\|\mathcal{N}_{32}\|_{L_p((0,T),H^1_q(\Omega_+))} \leq CT^{1/p'}(E^1_T(\mathbf{u}^1_+) + E^1_T(\mathbf{u}^2_+))E^1_T(\mathbf{u}^1_+ - \mathbf{u}^2_+), \\
&\|\partial_t\mathcal{N}_{32}\|_{L_p((0,T),H^1_q(\Omega_+))} \\
&\quad \leq C\{T^{1/p'}(E^1_T(\mathbf{u}^1_+) + E^1_T(\mathbf{u}^2_+)) + T^{1/p}(B + E^1_T(\mathbf{u}^1_+) + E^1_T(\mathbf{u}^2_+))\}E^1_T(\mathbf{u}^1_+ - \mathbf{u}^2_+),
\end{aligned}
$$

which, combined with (77) and (78), yields that

$$\begin{aligned}&\|\mathcal{N}_2\|_{L_p((0,T),H^2_q(\Omega_+))}+\|\partial_t\mathcal{N}_2\|_{L_p((0,T),L_q(\Omega_+))}\|\partial_t\mathcal{N}_3\|_{H^1_p((0,T),H^1_q(\Omega_+))}\\&\quad\leq C(T^{1/p'}E^1_T(\mathbf{u}^1_+)+T^{1/p}(B+E^1_T(\mathbf{u}^1_+)))E^1_T(\mathbf{u}^1_+-\mathbf{u}^2_+).\end{aligned}\tag{79}$$

We now estimate $\mathbf{N}_4$ and $\mathcal{N}_4$. Applying (49) to the formula given in (28), we have

$$\begin{aligned}&\|\mathbf{N}_4(\mathbf{u}_+,\mathbf{H}_+)\|_{L_p((0,T),H^2_q(\Omega_+))}\\&\quad\leq C[\mathcal{A}_4(\Psi_{\mathbf{u}})]_{T,2}(\|\Psi_{\mathbf{u}}\|_{L_\infty((0,T),H^2_q(\Omega_+))}\|\nabla\mathbf{u}_+\|_{L_p((0,T),H^2_q(\Omega_+))}\\&\qquad\qquad+\|\mathbf{H}_+\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\mathbf{H}_+\|_{L_p((0,T),H^2_q(\Omega_+))});\\&\|\partial_t\mathbf{N}_4(\mathbf{u}_+,\mathbf{H}_+)\|_{L_p((0,T),L_q(\Omega_+))}\\&\quad\leq C\{\|\partial_t\mathcal{A}_4(\Psi_{\mathbf{u}})\|_{L_\infty((0,T),H^1_q(\Omega_+))}(\|\Psi_{\mathbf{u}}\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\nabla\mathbf{u}_+\|_{L_p((0,T),L_q(\Omega_+))}\\&\qquad\qquad+\|\mathbf{H}_+\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\mathbf{H}_+\|_{L_p((0,T),L_q(\Omega_+))})\\&\quad+T^{1/p}[\mathcal{A}_4(\Psi_{\mathbf{u}})]_{T,2}\|\partial_t\Psi_{\mathbf{u}}\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\nabla\mathbf{u}_+\|_{L_\infty((0,T),L_q(\Omega_+))})\\&\quad+[\mathcal{A}_4(\Psi_{\mathbf{u}})]_{T,2}(\|\Psi_{\mathbf{u}}\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\partial_t\nabla\mathbf{u}_+\|_{L_p((0,T),L_q(\Omega_+))}\\&\qquad\qquad+\|\mathbf{H}_+\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\partial_t\mathbf{H}_+\|_{L_p((0,T),L_q(\Omega_+))})\}.\end{aligned}$$

Thus, by (50), (53), (67), (52), and (75), we have

$$\begin{aligned}\|\mathbf{N}_4(\mathbf{u}_+,\mathbf{H}_+)\|_{L_p((0,T),H^2_q(\Omega_+))}&\leq C(T^{1/p'}E^1_T(\mathbf{u}_+)^2+(B+T^{\frac{s}{p'(1+s)}}E^{2,+}_T(\mathbf{H}_+))E^{2,+}_T(\mathbf{H}_+));\\\|\partial_t\mathbf{N}_4(\mathbf{u}_+,\mathbf{H}_+)\|_{L_p((0,T),L_q(\Omega_+))}&\leq C(T^{1/p'}E^1_T(\mathbf{u}_+)^2+T^{1/p}(B+E^1_T(\mathbf{u}_+))^2\\&\quad+(1+B+E^1_T(\mathbf{u}_+))(B+T^{\frac{s}{p'(1+s)}}E^{2,+}_T(\mathbf{H}_+))E^{2,+}_T(\mathbf{H}_+)).\end{aligned}\tag{80}$$

To estimate $\mathcal{N}_4$, we write $\mathcal{N}_4=\mathcal{N}_{41}+\mathcal{N}_{42}$ with

$$\begin{aligned}\mathcal{N}_{41}&=(\mathcal{A}_4(\Psi_{\mathbf{u}^1})-\mathcal{A}_4(\Psi_{\mathbf{u}^2}))(\Psi_{\mathbf{u}^1}\otimes\nabla\mathbf{u}^1_+,\mathbf{H}^1_+\otimes\mathbf{H}^1_+),\\\mathcal{N}_{42}&=\mathcal{A}_4(\Psi_{\mathbf{u}^2})((\Psi_{\mathbf{u}^1}-\Psi_{\mathbf{u}^2})\otimes\nabla\mathbf{u}^1_++\Psi_{\mathbf{u}^2}\otimes\nabla(\mathbf{u}^1_+-\mathbf{u}^2_+),(\mathbf{H}^1_+-\mathbf{H}^2_+)\otimes\mathbf{H}^1_++\mathbf{H}^2_+\otimes(\mathbf{H}^1_+-\mathbf{H}^2_+)).\end{aligned}$$

By (49), we have

$$\begin{aligned}&\|\mathcal{N}_{41}\|_{L_p((0,T),H^2_q(\Omega_+))}\\&\quad\leq C\|\mathcal{A}_4(\Psi_{\mathbf{u}^1})-\mathcal{A}_4(\Psi_{\mathbf{u}^2})\|_{L_\infty((0,T),H^2_q(\Omega_+))}(\|\Psi_{\mathbf{u}^1}\|_{L_\infty((0,T),H^2_q(\Omega_+))}\|\nabla\mathbf{u}^1_+\|_{L_p((0,T),H^2_q(\Omega_+))}\\&\qquad\qquad+\|\mathbf{H}^1_+\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\mathbf{H}^1_+\|_{L_p((0,T),H^2_q(\Omega_+))});\\&\|\partial_t\mathcal{N}_{41}\|_{L_p((0,T),L_q(\Omega_+))}\\&\quad\leq C\{T^{1/p}\|\partial_t(\mathcal{A}_4(\Psi_{\mathbf{u}^1})-\mathcal{A}_4(\Psi_{\mathbf{u}^2}))\|_{L_\infty((0,T),H^1_q(\Omega_+))}(\|\Psi_{\mathbf{u}^1}\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\nabla\mathbf{u}^1_+\|_{L_\infty((0,T),L_q(\Omega_+))}\\&\qquad\qquad+\|\mathbf{H}^1_+\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\mathbf{H}^1_+\|_{L_\infty((0,T),L_q(\Omega_+))})\\&\quad+\|\mathcal{A}_4(\Psi_{\mathbf{u}^1})-\mathcal{A}_4(\Psi_{\mathbf{u}^2})\|_{L_\infty((0,T),H^1_q(\Omega_+))}(T^{1/p}\|\partial_t\Psi_{\mathbf{u}^1}\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\nabla\mathbf{u}^1_+\|_{L_\infty((0,T),L_q(\Omega_+))}\\&\quad+\|\Psi_{\mathbf{u}^1}\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\partial_t\nabla\mathbf{u}^1_+\|_{L_p((0,T),L_q(\Omega_+))}+\|\mathbf{H}^1_+\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\partial_t\mathbf{H}^1_+\|_{L_p((0,T),L_q(\Omega_+))})\}.\end{aligned}$$

Thus, by (50), (52), (53), (66), (70), and (75), we have

$$\begin{aligned}\|\mathcal{N}_{41}\|_{L_p((0,T),H^2_q(\Omega_+))}&\leq CT^{1/p'}\{E^1_T(\mathbf{u}^1_+)+(B+E^{2,+}_T(\mathbf{H}^1_+))E^{2,+}_T(\mathbf{H}^1_+)\}E^1_T(\mathbf{u}^1_+-\mathbf{u}^2_+);\\\|\partial_t\mathcal{N}_{41}\|_{L_p((0,T),L_q(\Omega_+))}&\leq C[T^{1/p}\{(B+E^1_T(\mathbf{u}^1_+))+(B+E^{2,+}_T(\mathbf{H}^1_+))^2\}\\&\quad+T^{1/p'}\{E^1_T(\mathbf{u}^1_+)+(B+E^{2,+}_T(\mathbf{H}^1_+))E^{2,+}_T(\mathbf{H}^1_+)\}]E^1_T(\mathbf{u}^1_+-\mathbf{u}^2_+).\end{aligned}\tag{81}$$

On the other hand, by (49),

$$
\begin{aligned}
\|\mathcal{N}_{42}\|_{L_p((0,T),H^2_q(\Omega_+))} &\leq C[\mathcal{A}_4(\Psi_{\mathbf{u}^2})]_{T,2}(\|\Psi_{\mathbf{u}^1}-\Psi_{\mathbf{u}^2}\|_{L_\infty((0,T),H^2_q(\Omega_+))}\|\nabla\mathbf{u}^1_+\|_{L_p((0,T),H^2_q(\Omega_+))} \\
&\quad + \|\Psi_{\mathbf{u}^2}\|_{L_\infty((0,T),H^2_q(\Omega_+))}\|\nabla(\mathbf{u}^1_+-\mathbf{u}^2_+)\|_{L_p((0,T),H^2_q(\Omega_+))} \\
&\quad + \|\mathbf{H}^1_+-\mathbf{H}^2_+\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|(\mathbf{H}^1_+,\mathbf{H}^2_+)\|_{L_p((0,T),H^2_q(\Omega_+))} \\
&\quad + \|\mathbf{H}^1_+-\mathbf{H}^2_+\|_{L_p((0,T),H^2_q(\Omega_+))}\|(\mathbf{H}^1_+,\mathbf{H}^2_+)\|_{L_\infty((0,T),H^1_q(\Omega_+))}); \\
\|\partial_t\mathcal{N}_{42}\|_{L_p((0,T),L_q(\Omega_+))} &\leq C[\|\partial_t\mathcal{A}_4(\Psi_{\mathbf{u}^2}]\|_{L_\infty((0,T),H^1_q(\Omega_+))} \\
&\quad \times(\|\Psi_{\mathbf{u}^1}-\Psi_{\mathbf{u}^2}\|_{L_\infty((0,T),H^1_q(\Omega_+))}T^{1/p}\|\nabla\mathbf{u}^1_+\|_{L_\infty((0,T),L_q(\Omega_+))} \\
&\quad + \|\Psi_{\mathbf{u}^2}\|_{L_\infty((0,T),H^1_q(\Omega_+))}T^{1/p}\|\nabla(\mathbf{u}^1_+-\mathbf{u}^2_+)\|_{L_\infty((0,T),L_q(\Omega_+))} \\
&\quad + T^{1/p}\|\mathbf{H}^1_+-\mathbf{H}^2_+\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|(\mathbf{H}^1_+,\mathbf{H}^2_+)\|_{L_\infty((0,T),L_q(\Omega_+))}) \\
&\quad + [A_4(\Psi_{\mathbf{u}^2})]_{T,2}(T^{1/p}(\|\partial_t(\Psi_{\mathbf{u}^1}-\Psi_{\mathbf{u}^2})\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\nabla\mathbf{u}^1_+\|_{L_\infty((0,T),L_q(\Omega_+))} \\
&\quad + \|\partial_t\Psi_{\mathbf{u}^2}\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\nabla(\mathbf{u}^1_+-\mathbf{u}^2_+)\|_{L_\infty((0,T),L_q(\Omega_+))}) \\
&\quad + \|\partial_t(\mathbf{H}^1_+-\mathbf{H}^2_+)\|_{L_p((0,T),L_q(\Omega_+))}\|(\mathbf{H}^1_+,\mathbf{H}^2_+)\|_{L_\infty((0,T),H^1_q(\Omega_+))} \\
&\quad + \|\Psi_{\mathbf{u}^1}-\Psi_{\mathbf{u}^2}\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\partial_t\nabla\mathbf{u}^1_+\|_{L_p((0,T),L_q(\Omega_+))} \\
&\quad + \|\Psi_{\mathbf{u}^2}\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\partial_t\nabla(\mathbf{u}^1_+-\mathbf{u}^2_+)\|_{L_p((0,T),L_q(\Omega_+))} \\
&\quad + \|(\mathbf{H}^1_+-\mathbf{H}^2_+)\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|(\partial_t\mathbf{H}^1_+,\partial_t\mathbf{H}^2_+)\|_{L_p((0,T),L_q(\Omega_+))}].
\end{aligned}
$$

Thus, by (50), (52), (53), (66), (67),

$$
\begin{aligned}
\|\mathcal{N}_{42}\|_{L_p((0,T),H^2_q(\Omega_+))} &\leq C\{T^{1/p'}(E^1_T(\mathbf{u}^1_+)+E^1_T(\mathbf{u}^2_+))E^1_T(\mathbf{u}^1_+-\mathbf{u}^2_+) \\
&\quad + (B+T^{\frac{s}{p'(1+s)}}(E^{2,+}_T(\mathbf{H}^1_+)+E^{2,+}_T(\mathbf{H}^2_+)))E^{2,+}_T(\mathbf{H}^1_+-\mathbf{H}^2_+)\}, \\
\|\partial_t\mathcal{N}_{42}\|_{L_p((0,T),L_q(\Omega_+))} &\leq C\{T^{1/p}(B+E^1_T(\mathbf{u}^1_+)+E^1_T(\mathbf{u}^2_+))E^1_T(\mathbf{u}^1_+-\mathbf{u}^2_+) \\
&\quad + T^{1/p}(B+E^1_T(\mathbf{u}^2_+))(B+E^{2,+}_T(\mathbf{H}^1_+)+E^{2,+}_T(\mathbf{H}^2_+))E^{2,+}_T(\mathbf{H}^1_+-\mathbf{H}^2_+) \\
&\quad + T^{1/p'}(E^1_T(\mathbf{u}^1_+)+E^1_T(\mathbf{u}^2_+))E^1_T(\mathbf{u}^1_+-\mathbf{u}^2_+) \\
&\quad + (B+T^{\frac{s}{p'(1+s)}}(E^{2,+}_T(\mathbf{H}^1_+)+E^{2,+}_T(\mathbf{H}^2_+)))E^{2,+}_T(\mathbf{H}^1_+-\mathbf{H}^2_+)\},
\end{aligned}
$$

which, combined with (81), yields that

$$
\begin{aligned}
&\|\mathcal{N}_4\|_{L_p((0,T),H^2_q(\Omega_+))}+\|\partial_t\mathcal{N}_4\|_{L_p((0,T),L_q(\Omega_+))} \\
&\quad \leq C[T^{1/p'}\{E^1_T(\mathbf{u}^1_+)+E^1_T(\mathbf{u}^2_+)+(B+E^{2,+}_T(\mathbf{H}^1_+))E^{2,+}_T(\mathbf{H}^1_+)\} \\
&\qquad + T^{1/p}(B+E^1_T(\mathbf{u}^1_+)+E^1_T(\mathbf{u}^2_+)+(B+E^{2,+}_T(\mathbf{H}^1_+))^2)]E^1_+(\mathbf{u}^1_+-\mathbf{u}^2_+) \\
&\quad + [T^{1/p}(B+E^1_T(\mathbf{u}^2_+))(B+E^{2,+}_T(\mathbf{H}^1_+)+E^{2,+}_T(\mathbf{H}^2_+)) \\
&\qquad + B+T^{\frac{s}{p'(1+s)}}(E^{2,+}_T(\mathbf{H}^1_+)+E^{2,+}_T(\mathbf{H}^2_+))]E^{2,+}_T(\mathbf{H}^1_+-\mathbf{H}^2_+)\},
\end{aligned} \tag{82}
$$

where we have used $0<T<1$.

We now estimate $\mathbf{N}_5$ and $\mathcal{N}_5$. Applying (49) to the formula given in (26), we have

$$
\begin{aligned}
&\|\mathbf{N}_{5\pm}(\mathbf{u},\mathbf{H})\|_{L_p((0,T),L_q(\Omega_\pm))} \leq C[\mathcal{A}_{5\pm}(\Psi_{\mathbf{u}})]_{T,2}(T^{1/p}\|\mathbf{H}_\pm\|^2_{L_\infty((0,T),H^1_q(\Omega_\pm))} \\
&\quad + \|\Psi_{\mathbf{u}}\|_{L_\infty((0,T),H^1_q(\Omega_\pm))}\|\mathbf{H}_\pm\|_{L_p((0,T),H^2_q(\Omega_\pm))}+T^{1/p}\|\mathbf{u}_+\|_{L_\infty((0,T),H^1_q(\Omega_+))}\|\mathbf{H}_\pm\|_{L_\infty((0,T),H^1_q(\Omega_+))}) \\
&\leq C\{T^{1/p}(B+E^2_T(\mathbf{H}))(B+E^1_T(\mathbf{u}_+)+E^2_T(\mathbf{H}))+T^{1/p'}E^1_T(\mathbf{u}_+)E^2_T(\mathbf{H})\}.
\end{aligned} \tag{83}
$$

To estimate $\mathcal{N}_5$, we write $\mathcal{N}_{5\pm}=\mathcal{N}_{51\pm}+\mathcal{N}_{52\pm}$ with

$$
\begin{aligned}
\mathcal{N}_{51\pm} &= (\mathcal{A}_{5\pm}(\Psi_{\mathbf{u}^1})-\mathcal{A}_{5\pm}(\Psi_{\mathbf{u}^2}))(K^5_{11},K^5_{21},K^5_{31},K^5_{41},K^5_{51}), \\
\mathcal{N}_{52\pm} &= \mathcal{A}_{5\pm}(\Psi_{\mathbf{u}^2})(K^5_{11}-K^5_{12},K^5_{21}-K^5_{22},K^5_{31}-K^5_{32},K^5_{41}-K^5_{42},K^5_{51}-K^5_{52}),
\end{aligned}
$$

where $K^5_{1i} = \mathbf{H}^i_\pm \otimes \nabla \mathbf{H}^i_\pm$, $K^5_{2i} = \Psi_{\mathbf{u}^i} \otimes \nabla^2 \mathbf{H}^i_\pm$, $K^5_{3i} = \nabla \Psi_{\mathbf{u}^i} \otimes \nabla \mathbf{H}^i_\pm$, $K^5_{4i} = \delta_\pm \nabla \mathbf{u}^i_\pm \otimes \mathbf{H}^i_\pm$, and $K^5_{5i} = \mathbf{u}^i_\pm \otimes \mathbf{H}^i_\pm$. By (70) and (83), we have

$$\|\mathcal{N}_{51\pm}\|_{L_p((0,T),L_q(\Omega_\pm))} \leq CT^{1/p'}(B + E^2_T(\mathbf{H}^1))(B + E^1_T(\mathbf{u}^1_+) + E^2_T(\mathbf{H}^1))E^1_T(\mathbf{u}^1_+ - \mathbf{u}^2_+), \quad (84)$$

where we have used $0 < T < 1$. Furthermore, by (49), (50), and (66)

$$\begin{aligned}
&\|K^5_{11} - K^5_{12}\|_{L_p((0,T),L_q(\Omega_\pm))} \leq CT^{1/p}(B + E^2_T(\mathbf{H}^1) + E^2_T(\mathbf{H}^2))E^2_T(\mathbf{H}^1 - \mathbf{H}^2);\\
&\|K^5_{21} - K^5_{22}\|_{L_p((0,T),L_q(\Omega_\pm))} \leq CT^{1/p'}\{E^2_T(\mathbf{H}^1)E^1_T(\mathbf{u}^1_+ - \mathbf{u}^2_+) + E^1_T(\mathbf{u}^2_+)E^2_T(\mathbf{H}^1 - \mathbf{H}^2)\};\\
&\|K^5_{31} - K^5_{32}\|_{L_p((0,T),L_q(\Omega_\pm))} \leq CT^{1/p'}\{E^2_T(\mathbf{H}^1)E^1_T(\mathbf{u}^1_+ - \mathbf{u}^2_+) + E^1_T(\mathbf{u}^2_+)E^2_T(\mathbf{H}^1 - \mathbf{H}^2)\};\\
&\|K^5_{41} - K^5_{42}\|_{L_p((0,T),L_q(\Omega_\pm))} \leq CT^{1/p}\{(B + E^2_T(\mathbf{H}^1))E^1_T(\mathbf{u}^1_+ - \mathbf{u}^2_+) + (B + E^1_T(\mathbf{u}^2_+))E^2_T(\mathbf{H}^1 - \mathbf{H}^2)\};\\
&\|K^5_{51} - K^5_{52}\|_{L_p((0,T),L_q(\Omega_\pm))} \leq CT^{1/p}\{(B + E^2_T(\mathbf{H}^1))E^1_T(\mathbf{u}^1_+ - \mathbf{u}^2_+) + (B + E^1_T(\mathbf{u}^2_+))E^2_T(\mathbf{H}^1 - \mathbf{H}^2)\},
\end{aligned}$$

which, combined with (53), yields that

$$\begin{aligned}
\|\mathcal{N}_{52\pm}\|_{L_p((0,T),L_q(\Omega_\pm))} \leq\ & C\{T^{1/p}(B + E^2_T(\mathbf{H}^1) + E^2_T(\mathbf{H}^2) + E^1_T(\mathbf{u}^2_+)) + T^{1/p'}E^1_T(\mathbf{u}^2_+)\}E^2_T(\mathbf{H}^1 - \mathbf{H}^2)\\
&+ C\{T^{1/p'}E^2_T(\mathbf{H}^1) + T^{1/p}(B + E^2_T(\mathbf{H}^1))\}E^1_T(\mathbf{u}^1_+ - \mathbf{u}^2_+).
\end{aligned} \quad (85)$$

Combining (84) and (85) gives that

$$\begin{aligned}
\|\mathcal{N}_5\|_{L_p((0,T),L_q(\Omega))} \leq\ & C\{T^{1/p'}((B + E^2_T(\mathbf{H}^1))(B + E^1_T(\mathbf{u}^1_+) + E^2_T(\mathbf{H}^1)) + E^2_T(\mathbf{H}^1))\\
&+ T^{1/p}(B + E^2_T(\mathbf{H}^1))\}E^1_T(\mathbf{u}^1_+ - \mathbf{u}^2_+)\\
&+ C\{T^{1/p'}E^1_T(\mathbf{u}^2_+) + T^{1/p}(B + E^2_T(\mathbf{H}^1) + E^2_T(\mathbf{H}^2) + E^1_T(\mathbf{u}^2_+))\}E^2_T(\mathbf{H}^1 - \mathbf{H}^2).
\end{aligned} \quad (86)$$

We now consider $\mathbf{N}_6$ and $\mathcal{N}_6$. We have to extend them to $t < 0$. For this purpose, Let $E_\mp$ be an extension operator satisfying (59). In view of (29), we have

$$\begin{aligned}
\mathbf{N}_6(\mathbf{u}, \mathbf{H}) =\ & \mathcal{A}_{61}(\Psi_\mathbf{u})(\alpha_+^{-1}\nabla E_-[\mathbf{H}_+] - \alpha_-^{-1}\nabla E_+[\mathbf{H}_-]) + (\mathcal{B} + \mathcal{A}_{62}(\Psi_\mathbf{u}))\nabla E_-[\mathbf{u}_+] \otimes \nabla E_-[\mathbf{H}_+];\\
\mathcal{N}_6 =\ & (\mathcal{A}_{61}(\Psi_{\mathbf{u}^1}) - \mathcal{A}_{61}(\Psi_{\mathbf{u}^2}))(\alpha_+^{-1}E_-[\mathbf{H}^1_+] - \alpha_-^{-1}E_+[\mathbf{H}^1_-])\\
&+ (\mathcal{A}_{62}(\Psi_{\mathbf{u}^1}) - \mathcal{A}_{62}(\Psi_{\mathbf{u}^2}))E_-[\mathbf{u}^1_+] \otimes E_-[\mathbf{H}^1_+]\\
&+ \mathcal{A}_{61}(\Psi_{\mathbf{u}^2})(\alpha_+^{-1}\nabla(E_-[\mathbf{H}^1_+] - E_-[\mathbf{H}^2_+]) - \alpha_-^{-1}\nabla(E_+[\mathbf{H}^1_-] - E_+[\mathbf{H}^2_-]))\\
&+ (\mathcal{B} + \mathcal{A}_{62}(\Psi_{\mathbf{u}^2}))((E_-[\mathbf{u}^1_+] - E_-[\mathbf{u}^2_+]) \otimes E_-[\mathbf{H}^1_+] + E_-[\mathbf{u}^2_+] \otimes (E_-[\mathbf{H}^1_+] - E_-[\mathbf{H}^2_+])).
\end{aligned} \quad (87)$$

on $\Gamma \times (0, T)$. Define the extension operator e_T by (56) and let $E_\pm$ and $E_{\mathbb{R}^N}$ be extension operators satisfying (59) and (60), respectively. Let γ_1 and γ_2 be the positive constants appearing in respective Theorem 2 and Theorem 3 and set $\gamma_0 = \max(\gamma_1, \gamma_2)$ below. Instead of (61), we set

$$T(t)\mathbf{v}_0 = e^{(-\gamma_0 - 1 + \Delta)t}\mathbf{v}_0.$$

Then let $\mathcal{T}$ and $\mathcal{E}$ be extension operators satisfying (63) and (65), respectively. For $\ell = 0, 1$, we obtain

$$\|e^{\gamma_0 t}T(\cdot)\mathbf{v}_0\|_{L_p((0,\infty),H^{\ell+2}_q(\mathbb{R}^N))} + \|e^{\gamma_0 t}\partial_t T(\cdot)\mathbf{v}_0\|_{L_p((0,\infty),H^\ell_q(\mathbb{R}^N))} \leq C\|\mathbf{v}_0\|_{B^{\ell+2(1-1/p)}_{q,p}(\mathbb{R}^N)}$$

and

$$\begin{aligned}
&\mathcal{T}[E_\mp[\mathbf{w}_{0\pm}]](0) = E_\mp[\mathbf{w}_{0\pm}] \quad \text{in } \Omega,\\
&\|e^{\gamma_0 t}\mathcal{T}[E_\mp[\mathbf{w}_{0\pm}]]\|_{L_p(\mathbb{R},H^{2+\ell}_q(\Omega))} + \|e^{\gamma_0 t}\partial_t\mathcal{T}[E_\mp[\mathbf{w}_{0\pm}]]\|_{L_p(\mathbb{R},H^\ell_q(\Omega))} \leq C\|\mathbf{w}_0\|_{B^{2(1-1/p)+\ell}_{q,p}(\dot\Omega)}.
\end{aligned} \quad (88)$$

We now define an extension $\tilde{\mathbf{N}}_6(\mathbf{u}, \mathbf{H})$. Let $\tilde{\mathbf{N}}_6(\mathbf{u}, \mathbf{H}) = \tilde{\mathbf{N}}_{61}(\mathbf{u}, \mathbf{H}) + \tilde{\mathbf{N}}_{62}(\mathbf{u}, \mathbf{H})$ with

$$\tilde{\mathbf{N}}_{61}(\mathbf{u}, \mathbf{H}) = \mathcal{A}_{61}(e_T[\Psi_\mathbf{u}])(\alpha_+^{-1}\nabla\mathcal{E}[E_-[\mathbf{H}_+]] - \alpha_-^{-1}\nabla\mathcal{E}[E_+[\mathbf{H}_-]]);$$

$$\tilde{\mathbf{N}}_{62}(\mathbf{u},\mathbf{H}) = (\mathcal{B}+\mathcal{A}_{62}(e_T[\Psi_{\mathbf{u}}]))\mathcal{E}[E_-[\mathbf{u}_+]]\otimes\mathcal{E}[E_-[\mathbf{H}_+]].$$

To estimate $H^{1/2}_p(\mathbb{R}, L_q(\dot\Omega))$ norm, we use the following lemma,

Lemma 1. *Let $1 < p < \infty$ and $N < q < \infty$. Let*

$$f \in L_\infty(\mathbb{R}, H^1_q(\dot\Omega)) \cap H^1_\infty(\mathbb{R}, L_q(\dot\Omega)), \quad g \in H^{1/2}_p(\mathbb{R}, H^1_q(\dot\Omega)) \cap L_p(\mathbb{R}, H^1_q(\dot\Omega)).$$

Then, we have

$$\begin{aligned}&\|fg\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|fg\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))}\\&\leq C(\|\partial_t f\|_{L_\infty(\mathbb{R},L_q(\dot\Omega))} + \|f\|_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))})^{1/2}\|f\|^{1/2}_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))}(\|g\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|g\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))}).\end{aligned}$$

Proof. To prove Lemma 1, we use the fact that

$$H^{1/2}_p(\mathbb{R}, L_q(\dot\Omega)) \cap L_p(\mathbb{R}, H^{1/2}_q(\dot\Omega)) = (L_p(\mathbb{R}, L_q(\dot\Omega)), H^1_p(\mathbb{R}, L_q(\dot\Omega)) \cap L_p(\mathbb{R}, H^1_q(\dot\Omega)))_{[1/2]},$$

where $(\cdot,\cdot)_{[1/2]}$ denotes a complex interpolation functor of order $1/2$. We have

$$\begin{aligned}&\|fg\|_{H^1_p(\mathbb{R},L_q(\dot\Omega))} + \|fg\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))}\\&\quad\leq C(\|\partial_t f\|_{L_\infty(\mathbb{R},L_q(\dot\Omega))}\|g\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} + \|f\|_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))}\|g\|_{H^1_p(\mathbb{R},L_q(\dot\Omega))})\\&\quad\leq C(\|\partial_t f\|_{L_\infty(\mathbb{R},L_q(\dot\Omega))} + \|f\|_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))})(\|g\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} + \|g\|_{H^1_p(\mathbb{R},L_q(\dot\Omega))}).\end{aligned}$$

Moreover,

$$\|fg\|_{L_p(\mathbb{R},L_q(\dot\Omega))} \leq C\|f\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))}\|g\|_{L_p(\mathbb{R},L_q(\dot\Omega))}.$$

Thus, by complex interpolation, we have

$$\begin{aligned}&\|fg\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|fg\|_{L_p(\mathbb{R},H^{1/2}_q(\dot\Omega))}\\&\leq C(\|\partial_t f\|_{L_\infty(\mathbb{R},L_q(\dot\Omega))} + \|f\|_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))})^{1/2}\|f\|^{1/2}_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))}(\|g\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|g\|_{L_p(\mathbb{R},H^{1/2}_q(\dot\Omega))}).\end{aligned}$$

Moreover, we have

$$\|fg\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} \leq C\|f\|_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))}\|g\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))}.$$

Thus, combining these two inequalities gives the required estimate, which completes the proof of Lemma 1. □

Lemma 2. *Let $1 < p, q < \infty$. Then,*

$$H^1_p(\mathbb{R}, L_q(\dot\Omega)) \cap L_p(\mathbb{R}, H^2_q(\dot\Omega)) \subset H^{1/2}_p(\mathbb{R}, H^1_q(\dot\Omega))$$

and

$$\|u\|_{H^{1/2}_p(\mathbb{R},H^1_q(\dot\Omega))} \leq C(\|u\|_{L_p(\mathbb{R},H^2_q(\dot\Omega))} + \|\partial_t u\|_{L_p(\mathbb{R},L_q(\dot\Omega))}).$$

Proof. For a proof, see Shibata ([17], Proposition 1). □

By (51), we may assume that

$$\|e_T[\Psi_{\mathbf{w}}]\|_{L_\infty(\mathbb{R},L_\infty(\Omega))} \leq \delta \quad \text{for } \mathbf{w} \in \{\mathbf{u}, \mathbf{u}^1, \mathbf{u}^2\}. \tag{89}$$

By (50), (30) and the same argument as in proving (70), we have

$$\begin{aligned} \|\mathcal{A}_{6i}(e_T[\Psi_{\mathbf{w}}])\|_{L_\infty(\mathbb{R},H^2_q(\Omega)} &\leq CT^{1/p'}E^1_T(\mathbf{w}_+) \qquad \text{for } \mathbf{w} \in \{\mathbf{u},\mathbf{u}^1,\mathbf{u}^2\}; \\ \|\mathcal{A}_{6i}(e_T[\Psi_{\mathbf{u}^1}]) - \mathcal{A}_{6i}(e_T[\Psi_{\mathbf{u}^2}])\|_{L_\infty(\mathbb{R},H^2_q(\Omega))} &\leq CT^{1/p'}E^1_T(\mathbf{u}^1_+ - \mathbf{u}^2_+). \end{aligned} \tag{90}$$

Employing the same as in proving (75) yields

$$\begin{aligned} \|\partial_t(\mathcal{A}_{6i}(e_T[\Psi_{\mathbf{u}}])\|_{L_\infty(\mathbb{R},H^1_q(\Omega))} &\leq C(B + E^1_T(\mathbf{u}_+)), \\ \|\partial_t(\mathcal{A}_{6i}(e_T[\Psi_{\mathbf{u}^1}]) - \mathcal{A}_{6i}(e_T[\Psi_{\mathbf{u}^2}))\|_{L_\infty(\mathbb{R},H^1_q(\Omega))} &\leq CE^1_T(\mathbf{u}^1_+ - \mathbf{u}^2_+), \end{aligned} \tag{91}$$

and so noting that $e_T[\Psi_{\mathbf{u}}]$ vanishes for $t \notin (0,2T)$, by Lemma 1, (90) and (91), we have

$$\begin{aligned} &\|e^{-\gamma t}\tilde{\mathbf{N}}^1_6(\mathbf{u},\mathbf{H})\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma t}\tilde{\mathbf{N}}^1_6(\mathbf{u},\mathbf{H})\|_{L_p(\mathbb{R},H^1_q(\Omega))} + \gamma^{1/2}\|e^{-\gamma t}\tilde{\mathbf{N}}^1_6(\mathbf{u},\mathbf{H})\|_{L_p(\mathbb{R},L_q(\Omega))} \\ &\quad\leq CT^{1/(2p')}(B + E^1_T(\mathbf{u}_+)) \\ &\qquad\times (\|\nabla(\mathcal{E}[E_-[\mathbf{H}_+]],\mathcal{E}[E_+[\mathbf{H}_-]])\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))} + \|\nabla(\mathcal{E}[E_-[\mathbf{H}_+]],\mathcal{E}[E_+[\mathbf{H}_-]])\|_{L_p(\mathbb{R},H^1_q(\Omega))}) \end{aligned}$$

Thus, applying Lemma 2, (65), (88), (58), and (59), we have

$$\begin{aligned} &\|e^{-\gamma t}\tilde{\mathbf{N}}_{61}(\mathbf{u},\mathbf{H})\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma t}\tilde{\mathbf{N}}_{61}(\mathbf{u},\mathbf{H})\|_{L_p(\mathbb{R},H^1_q(\Omega))} + \gamma^{1/2}\|e^{-\gamma t}\tilde{\mathbf{N}}^1_6(\mathbf{u},\mathbf{H})\|_{L_p(\mathbb{R},L_q(\Omega))} \\ &\qquad\leq CT^{1/(2p')}(B + E^1_T(\mathbf{u}_+))(B + E^2_T(\mathbf{H})). \end{aligned} \tag{92}$$

We now estimate $\mathbf{N}^2_6(\mathbf{u},\mathbf{H})$. For this purpose we use the following estimate:

$$\|f\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} \leq C\|f\|^{1/2}_{H^1_p(\mathbb{R},L_q(\dot\Omega))}\|f\|^{1/2}_{L_p(\mathbb{R},L_q(\dot\Omega))}, \tag{93}$$

which follows from complex interpolation theory. Let

$$A^2_1 = \mathcal{E}[E_-[\mathbf{u}_+]] \otimes \mathcal{E}[E_-[\mathbf{H}_+]], \quad A^2_2 = \mathcal{A}_{62}(\Psi_{\mathbf{u}})A^2_1.$$

And then, $\tilde{\mathbf{N}}_{62}(\mathbf{u},\mathbf{H}) = \mathcal{B}A^2_1 + A^2_2$. We further divide A^2_1 into $A^2_1 = A^2_{11} + A^2_{12} + A^2_{21} + A^2_{22}$ with

$$\begin{aligned} A^2_{11} &= \mathcal{T}[E_-[\mathbf{u}_{0+}]] \otimes \mathcal{T}[E_-[\tilde{\mathbf{H}}_{0+}]], \\ A^2_{12} &= \mathcal{T}[E_-[\mathbf{u}_{0+}]] \otimes e_T[E_-[\mathbf{H}_+] - \mathcal{T}[E_-[\tilde{\mathbf{H}}_{0+}]]], \\ A^2_{21} &= e_T[E_-[\mathbf{u}_+] - \mathcal{T}[E_-[\mathbf{u}_{0+}]]] \otimes \mathcal{T}[E_-[\tilde{\mathbf{H}}_{0+}]], \\ A^2_{22} &= e_T[E_-[\mathbf{u}_+] - \mathcal{T}[E_-[\mathbf{u}_{0+}]]] \otimes e_T[E_-[\mathbf{H}_+] - \mathcal{T}[E_-[\tilde{\mathbf{H}}_{0+}]]] \end{aligned}$$

By (48), (49), (66), and (88),

$$\begin{aligned} &\|e^{-\gamma t}A^2_{11}\|_{H^i_p(\mathbb{R},L_q(\Omega_+))} \\ &\quad\leq C(\|e^{-\gamma t}\mathcal{T}[E_-[\mathbf{u}_{0+}]]\|_{H^i_p(\mathbb{R},L_q(\Omega_+))}\|\mathcal{T}[E_-[\tilde{\mathbf{H}}_{0+}]]\|_{L_\infty(\mathbb{R},H^1_q(\Omega_+)} \\ &\qquad\qquad + \|\mathcal{T}[E_-[\mathbf{u}_{0+}]]\|_{L_\infty(\mathbb{R},H^1_q(\Omega_+)}\|e^{-\gamma t}\mathcal{T}[E_-[\tilde{\mathbf{H}}_{0+}]]\|_{H^i_p(\mathbb{R},L_q(\Omega_+))}) \\ &\quad\leq Ce^{2(\gamma-\gamma_0)}B^2; \\ &\|e^{-\gamma t}\partial_t A^2_{12}\|_{L_p(\mathbb{R},L_q(\Omega_+))} \\ &\quad\leq C\{\|\partial_t\mathcal{T}[E_-[\mathbf{u}_{0+}]]\|_{L_p(\mathbb{R},L_q(\Omega_+))}\|e_T[E_-[\mathbf{H}_+] - \mathcal{T}[E_-[\tilde{\mathbf{H}}_{0+}]]]\|_{L_\infty(\mathbb{R},H^1_q(\Omega_+))} \\ &\quad + \|\mathcal{T}[E_-[\mathbf{u}_{0+}]]\|_{L_\infty(\mathbb{R},H^1_q(\Omega_+))}\|\partial_t e_T[E_-[\mathbf{H}_+] - \mathcal{T}[E_-[\tilde{\mathbf{H}}_{0+}]]]\|_{L_p(\mathbb{R},L_q(\Omega_+))}\} \\ &\quad\leq CB(B + E^{2,+}_T(\mathbf{H}_+)); \\ &\|e^{-\gamma t}A^2_{12}\|_{L_p(\mathbb{R},L_q(\Omega_+))} \end{aligned}$$

$$
\begin{aligned}
&\leq T^{1/p}\|\mathcal{T}[E_-[\mathbf{u}_{0+}]]\|_{L_\infty(\mathbb{R},H^1_q(\Omega_+))}\|e_T[E_-[\mathbf{H}_+]-\mathcal{T}[E_-[\tilde{\mathbf{H}}_{0+}]]\|_{L_\infty((0,2T),L_q(\Omega_+))}\\
&\leq CBT\|\partial_t e_T[E_-[\mathbf{H}_+]-\mathcal{T}[E_-[\tilde{\mathbf{H}}_{0+}]]\|_{L_p((0,2T),L_q(\Omega_+))}\\
&\leq CTB(B+E^{2,+}_T(\mathbf{H}_+)),
\end{aligned}
$$

$$\|e^{-\gamma t}\partial_t A^2_{21}\|_{L_p(\mathbb{R},L_q(\Omega_+))}\leq CB(B+E^1_T(\mathbf{u}_+));$$

$$\|e^{-\gamma t}A^2_{21}\|_{L_p(\mathbb{R},L_q(\Omega_+))}\leq CTB(B+E^1_T(\mathbf{u}_+)),$$

$$
\begin{aligned}
&\|e^{-\gamma t}\partial_t A^2_{22}\|_{L_p(\mathbb{R},L_q(\Omega_+))}\\
&\quad\leq C\{\|\partial_t e_T[E_-[\mathbf{u}_+]-\mathcal{T}[E_-[\tilde{\mathbf{u}}_{0+}]]\|_{L_p((0,2T),L_q(\Omega_+))}\|e_T[E_-[\mathbf{H}_+]-\mathcal{T}[E_-[\tilde{\mathbf{H}}_{0+}]]\|_{L_\infty((0,2T),H^1_q(\Omega_+))}\\
&\quad+\|e_T[E_-[\mathbf{u}_+]-\mathcal{T}[E_-[\tilde{\mathbf{u}}_{0+}]]\|_{L_\infty((0,2T),H^1_q(\Omega_+))}\|\partial_t e_T[E_-[\mathbf{H}_+]-\mathcal{T}[E_-[\tilde{\mathbf{H}}_{0+}]]\|_{L_p((0,2T),L_q(\Omega_+))}\\
&\quad\leq C(B+E^1_T(\mathbf{u}_+))(B+E^{2,+}_T(\mathbf{H}_+));
\end{aligned}
$$

$$
\begin{aligned}
&\|e^{-\gamma t}A^2_{22}\|_{L_p(\mathbb{R},L_q(\Omega_+))}\\
&\quad\leq CT^{1/p}\|e_T[E_-[\mathbf{u}_+]-\mathcal{T}[E_-[\tilde{\mathbf{u}}_{0+}]]\|_{L_\infty((0,2T),H^1_q(\Omega_+))}\|e_T[E_-[\mathbf{H}_+]-\mathcal{T}[E_-[\tilde{\mathbf{H}}_{0+}]]\|_{L_\infty((0,2T),L_q(\Omega_+))}\\
&\quad\leq C(B+E^1_T(\mathbf{u}_+))T\|\partial_t e_T[E_-[\mathbf{H}_+]-\mathcal{T}[E_-[\tilde{\mathbf{H}}_{0+}]]\|_{L_p((0,2T),L_q(\Omega_+))}\\
&\quad\leq CT(B+E^1_T(\mathbf{u}_+))(B+E^{2,+}_T(\mathbf{H}_+))).
\end{aligned}
$$

Using (93), we have

$$
\begin{aligned}
\|e^{-\gamma t}A^2_1\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega_+))}&\leq C\sum_{i,j=1}^{2}\|e^{-\gamma t}A_{ij}\|^{1/2}_{H^1_p(\mathbb{R},L_q(\dot\Omega))}\|e^{-\gamma t}A_{ij}\|^{1/2}_{L_p(\mathbb{R},L_q(\dot\Omega))}\\
&\leq C(e^{2(\gamma-\gamma_0)}B^2+T^{1/2}(B+E^1_T(\mathbf{u}_+))(B+E^{2,+}_T(\mathbf{H}_+))).
\end{aligned}\tag{94}
$$

And also, by (49), (66), (88), and (48),

$$
\begin{aligned}
\|e^{-\gamma t}A^2_{11}\|_{L_p(\mathbb{R},H^1_q(\Omega_+))}&\leq C\|e^{-\gamma t}\mathcal{T}[E_-[\mathbf{u}_{0+}]]\|_{L_p(\mathbb{R},H^1_q(\Omega_+))}\|\mathcal{T}[E_-[\tilde{\mathbf{H}}_{0+}]]\|_{L_\infty(\mathbb{R},H^1_q(\Omega_+))}\\
&\leq Ce^{2(\gamma-\gamma_0)}B^2;\\
\|e^{-\gamma t}A^2_{12}\|_{L_p(\mathbb{R},H^1_q(\Omega_+))}&\leq C\|\mathcal{T}[E_-[\mathbf{u}_{0+}]]\|_{L_\infty(\mathbb{R},H^1_q(\Omega_+))}\|e_T[E_-[\mathbf{H}_+]-\mathcal{T}[E_-[\tilde{\mathbf{H}}_{0+}]]\|_{L_p((0,2T),H^1_q(\Omega_+))}\\
&\leq CBT^{1/p}\|e_T[E_-[\mathbf{H}_+]-\mathcal{T}[E_-[\tilde{\mathbf{H}}_{0+}]]\|_{L_\infty((0,2T),H^1_q(\Omega_+))}\\
&\leq CB(B+E^{2,+}_T(\mathbf{H}_+))T^{1/p};\\
\|e^{-\gamma t}A^2_{21}\|_{L_p(\mathbb{R},H^1_q(\Omega_+))}&\leq CB(B+E^1_T(\mathbf{u}_+))T^{1/p};\\
\|e^{-\gamma t}A^2_{22}\|_{L_p(\mathbb{R},H^1_q(\Omega_+))}&\leq C\|e_T[E_-[\mathbf{H}_+]-\mathcal{T}[E_-[\tilde{\mathbf{H}}_{0+}]]\|_{L_p((0,2T),H^1_q(\Omega_+))}\\
&\quad\times\|e_T[E_-[[\mathbf{u}_+]-\mathcal{T}[E_-[\mathbf{u}_{0+}]]\|_{L_\infty((0,2T),H^1_q(\Omega_+))}\\
&\leq CT^{1/p}(B+E^1_T(\mathbf{u}_+))(B+E^{2,+}_T(\mathbf{H}_+)),
\end{aligned}
$$

and so we have

$$\|e^{-\gamma t}A^2_1\|_{L_p(\mathbb{R},H^1_q(\Omega_+))}\leq C(e^{2(\gamma-\gamma_0)}B^2+T^{1/p}(B+E^1_T(\mathbf{u}_+))(B+E^{2,+}_T(\mathbf{H}_+))),$$

which, combined with (94), yields that

$$
\begin{aligned}
&\|e^{-\gamma t}A^2_1\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega_+))}+\|e^{-\gamma t}A^2_1\|_{L_p(\mathbb{R},H^1_q(\Omega_+))}+\gamma^{1/2}\|e^{-\gamma t}A^2_1\|_{L_p(\mathbb{R},L_q(\Omega_+))}\\
&\quad\leq C\{e^{2(\gamma-\gamma_0)}B^2+(T^{1/2}+T^{1/p})(B+E^1_+(\mathbf{u}_+))(B+E^{2,+}_T(\mathbf{H}_+))\}.
\end{aligned}\tag{95}
$$

By (90), (91), (95), and Lemma 1,

$$
\begin{aligned}
&\|e^{-\gamma t}A^2_2\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega_+))}+\|e^{-\gamma t}A^2_2\|_{L_p(\mathbb{R},H^1_q(\Omega_+))}+\gamma^{1/2}\|e^{-\gamma t}A^2_2\|_{L_p(\mathbb{R},L_q(\Omega_+))}\\
&\quad\leq C((B+E^1_T(\mathbf{u}_+))T^{1/(2p')})(\|A^2_1\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))}+\|A^2_1\|_{L_p(\mathbb{R},H^1_q(\Omega_+))})\\
&\quad\leq C((B+E^1_+(\mathbf{u}_+))T^{1/(2p')})\{e^{2(\gamma-\gamma_0)}B^2+(T^{1/2}+T^{1/p})(B+E^1_+(\mathbf{u}_+))(B+E^{2,+}_T(\mathbf{H}_+))\}.
\end{aligned}\tag{96}
$$

Combining (92), (95), and (96) yields that

$$\|\tilde{\mathbf{N}}_6(\mathbf{u},\mathbf{H})\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))} + \|\tilde{\mathbf{N}}_6(\mathbf{u},\mathbf{H})\|_{L_p(\mathbb{R},H^1_q(\Omega))} + \gamma^{1/2}\|\tilde{\mathbf{N}}_6(\mathbf{u},\mathbf{H})\|_{L_p(\mathbb{R},L_q(\Omega))}$$
$$\leq C[T^{1/(2p')}(B+E^1_T(\mathbf{u}_+))(B+E^2_T(\mathbf{H}))$$
$$+(1+(B+E^1_T(\mathbf{u}_+))T^{1/(2p')})\{e^{2(\gamma-\gamma_0)}B^2+(T^{1/2}+T^{1/p})(B+E^1_+(\mathbf{u}_+))(B+E^{2,+}_T(\mathbf{H}_+))\}]. \tag{97}$$

Recalling the formula of $\mathcal{N}_6$ in (87), we define an extension, $\tilde{\mathcal{N}}_6$, of $\mathcal{N}_6$ by setting $\tilde{\mathcal{N}}_6 = \tilde{\mathcal{N}}_{61}+\tilde{\mathcal{N}}_{62}+\tilde{\mathcal{N}}_{63}+\tilde{\mathcal{N}}_{64}$ with

$$\tilde{\mathcal{N}}_{61} = (\mathcal{A}_{61}(e_T[\Psi_{\mathbf{u}^1}]) - \mathcal{A}_{61}(e_T[\Psi_{\mathbf{u}^2}]))(\alpha_+^{-1}\nabla\mathcal{E}[E_-[\mathbf{H}^1_+]] - \alpha_-^{-1}\nabla\mathcal{E}[E_+[\mathbf{H}^1_-]])$$
$$\tilde{\mathcal{N}}_{62} = (\mathcal{A}_{62}(e_T[\Psi_{\mathbf{u}^1}]) - \mathcal{A}_{62}(e_T[\Psi_{\mathbf{u}^2}]))\mathcal{E}[E_-[\mathbf{u}^1_+]]\otimes\mathcal{E}[E_-[\mathbf{H}^1_+]]$$
$$\tilde{\mathcal{N}}_{63} = \mathcal{A}_{61}(e_T[\Psi_{\mathbf{u}^2}])\{\alpha_+^{-1}\nabla(\mathcal{E}[E_-[\mathbf{H}^1_+]] - \mathcal{E}[E_-[\mathbf{H}^2_+]]) - \alpha_-^{-1}\nabla(\mathcal{E}[E_+[\mathbf{H}^1_-]] - \mathcal{E}[E_+[\mathbf{H}^2_-]])\}$$
$$\tilde{\mathcal{N}}_{64} = (\mathcal{B}+\mathcal{A}_{62}(e_T[\Psi_{\mathbf{u}^2}]))((\mathcal{E}[E_-[\mathbf{u}^1_+]] - \mathcal{E}[E_-[\mathbf{u}^2_+]])\otimes\mathcal{E}[E_-[\mathbf{H}^1_+]]$$
$$+\mathcal{E}[E_-[\mathbf{u}^2_+]]\otimes(\mathcal{E}[E_-[\mathbf{H}^1_+]] - \mathcal{E}[E_-[\mathbf{H}^2_+]])).$$

Setting $\mathcal{D}_{6i} = \mathcal{A}_{6i}(e_T[\Psi_{\mathbf{u}^1}]) - \mathcal{A}_{6i}(e_T[\Psi_{\mathbf{u}^2}])$ for notational simplicity, by (90) and (91) we have

$$(\|\partial_t\mathcal{D}_{6i}\|_{L_\infty(\mathbb{R},L_q(\dot\Omega))} + \|\mathcal{D}_{6i}\|_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))})^{1/2}\|\mathcal{D}_{6i}\|^{1/2}_{L_\infty(\mathbb{R},H^1_q(\dot\Omega))} \leq CT^{1/(2p')}E^1_T(\mathbf{u}^1_+-\mathbf{u}^2_+). \tag{98}$$

And, noting Lemma 2, we have

$$\|\nabla\mathcal{E}[E_\mp[\mathbf{H}^1_\pm]]\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))} + \|\nabla\mathcal{E}[E_\mp[\mathbf{H}^1_\pm]]\|_{L_p(\mathbb{R},H^1_q(\Omega))} \leq C(B+E^2_T(\mathbf{H}^1)),$$

which, combined with Lemma 1 and (98), yields that

$$\|e^{-\gamma t}\tilde{\mathcal{N}}_{61}\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\tilde{\mathcal{N}}_{61}\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} + \gamma^{1/2}\|e^{-\gamma t}\tilde{\mathcal{N}}_{61}\|_{L_p(\mathbb{R},L_q(\dot\Omega))}$$
$$\leq CT^{1/(2p')}(B+E^2_T(\mathbf{H}^1))E^1_T(\mathbf{u}^1_+-\mathbf{u}^2_+). \tag{99}$$

Analogously, by (98), Lemma 1 and (95), we have

$$\|e^{-\gamma t}\tilde{\mathcal{N}}_{62}\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\tilde{\mathcal{N}}_{62}\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} + \gamma^{1/2}\|e^{-\gamma t}\tilde{\mathcal{N}}_{62}\|_{L_p(\mathbb{R},L_q(\dot\Omega))}$$
$$\leq CT^{1/(2p')}\{e^{2(\gamma-\gamma_0)}B^2+(T^{1/2}+T^{1/p})(B+E^1_T(\mathbf{u}^1_+))(B+E^{2,+}_T(\mathbf{H}^1_+))\}E^1_T(\mathbf{u}^1_+-\mathbf{u}^2_+). \tag{100}$$

Since

$$\mathcal{E}[E_\mp[\mathbf{H}^1_\pm]] - \mathcal{E}[E_\mp[\mathbf{H}^2_\pm]] = e_T[\mathbf{H}^1_\pm-\mathbf{H}^2_\pm], \quad \mathcal{E}[E_-[\mathbf{u}^1_+]] - \mathcal{E}[E_-[\mathbf{u}^2_+]] = e_T(\mathbf{u}^1_+-\mathbf{u}^2_+) \tag{101}$$

as follows from (65), by Lemma 1, Lemma 2, (90), and (91), we have

$$\|e^{-\gamma t}\tilde{\mathcal{N}}_{63}\|_{H^{1/2}_p(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\tilde{\mathcal{N}}_{63}\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} + \gamma^{1/2}\|e^{-\gamma t}\tilde{\mathcal{N}}_{63}\|_{L_p(\mathbb{R},L_q(\dot\Omega))}$$
$$\leq CT^{1/(2p')}(B+E^1_T(\mathbf{u}^2_+))E^2_T(\mathbf{H}^1-\mathbf{H}^2). \tag{102}$$

In view of (101), by (57) we have

$$\|e^{-\gamma t}(\mathcal{E}[E_\mp[\mathbf{H}^1_\pm]] - \mathcal{E}[E_\mp[\mathbf{H}^2_\pm]])\|_{L_p(\mathbb{R},L_q(\dot\Omega))} \leq CT^{1/p}\|e_T[\mathbf{H}^1_\pm-\mathbf{H}^2_\pm]\|_{L_\infty((0,2T),L_q(\dot\Omega))}$$
$$\leq CT\|\partial_t(e_T[\mathbf{H}^1_\pm-\mathbf{H}^2_\pm]\|_{L_p((0,2T),L_q(\dot\Omega))} \leq CTE^{2,\pm}_T(\mathbf{H}^1_\pm-\mathbf{H}^2_\pm).$$

Analogously,

$$\|e^{-\gamma t}(\mathcal{E}[E_-[\mathbf{u}^1_+]] - \mathcal{E}[E_-[\mathbf{u}^2_+]])\|_{L_p(\mathbb{R},L_q(\dot\Omega))} \leq CTE^1_T(\mathbf{u}^1_+-\mathbf{u}^2_+);$$
$$\|e^{-\gamma t}(\mathcal{E}[E_\mp[\mathbf{H}^1_\pm]] - \mathcal{E}[E_\mp[\mathbf{H}^2_\pm]])\|_{L_p(\mathbb{R},H^1_q(\dot\Omega))} \leq CT^{1/p}\|e_T[\mathbf{H}^1_\pm-\mathbf{H}^2_\pm]\|_{L_\infty((0,2T),H^1_q(\dot\Omega))}$$

$$\leq CT^{1/p}E_T^{2,\pm}(\mathbf{H}_\pm^1 - \mathbf{H}_\pm^2);$$
$$\|e^{-\gamma t}(\mathcal{E}[E_-[\mathbf{u}_+^1]] - \mathcal{E}[E_-[\mathbf{u}_+^2]])\|_{L_p(\mathbb{R},H_q^1(\dot\Omega))} \leq CT^{1/p}E_T^1(\mathbf{u}_+^1 - \mathbf{u}_+^2).$$

Thus, by (93),

$$\|e^{-\gamma t}(\mathcal{E}[E_-[\mathbf{u}_+^1]] - \mathcal{E}[E_-[\mathbf{u}_+^2]]) \otimes \mathcal{E}[E_-[\mathbf{H}_+^1]]\|_{H_p^{1/2}(\mathbb{R},L_q(\dot\Omega))} \leq CT^{1/2}(B + E_T^{2,+}(\mathbf{H}_+^1))E_T^1(\mathbf{u}_+^1 - \mathbf{u}_+^2) :$$
$$\|e^{-\gamma t}\mathcal{E}[E_-[\mathbf{u}_+^1]] \otimes (\mathcal{E}[E_-[\mathbf{H}_+^1]] - \mathcal{E}[E_-[\mathbf{H}_+^2]])\|_{H_p^{1/2}(\mathbb{R},L_q(\dot\Omega))} \leq CT^{1/2}(B + E_T^1(\mathbf{u}_+^2))E_T^{2,+}(\mathbf{H}_+^1 - \mathbf{H}_+^2) :$$
$$\|e^{-\gamma t}(\mathcal{E}[E_-[\mathbf{u}_+^1]] - \mathcal{E}[E_-[\mathbf{u}_+^2]]) \otimes \mathcal{E}[E_-[\mathbf{H}_+^1]]\|_{L_p(\mathbb{R},H_q^1(\dot\Omega))} \leq CT^{1/p}(B + E_T^{2,+}(\mathbf{H}_+^1))E_T^1(\mathbf{u}_+^1 - \mathbf{u}_+^2) :$$
$$\|e^{-\gamma t}\mathcal{E}[E_-[\mathbf{u}_+^1]] \otimes (\mathcal{E}[E_-[\mathbf{H}_+^1]] - \mathcal{E}[E_-[\mathbf{H}_+^2]])\|_{L_p(\mathbb{R},H_q^1(\dot\Omega))} \leq CT^{1/p}(B + E_T^1(\mathbf{u}_+^2))E_T^{2,+}(\mathbf{H}_+^1 - \mathbf{H}_+^2),$$

which, combined with Lemma 1, (90) and (91), yields that

$$\begin{aligned}&\|e^{-\gamma t}\tilde{\mathcal{N}}_{64}\|_{H_p^{1/2}(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\tilde{\mathcal{N}}_{64}\|_{L_p(\mathbb{R},H_q^1(\dot\Omega))} + \gamma^{1/2}\|e^{-\gamma t}\tilde{\mathcal{N}}_{64}\|_{L_p(\mathbb{R},L_q(\dot\Omega))}\\ &\quad\leq C(1 + T^{1/(2p')}(B + E_T^1(\mathbf{u}_+^2)))\\ &\qquad\times (T^{1/2} + T^{1/p})((B + E_T^{2,+}(\mathbf{H}_+^1))E_T^1(\mathbf{u}_+^1 - \mathbf{u}_+^2) + (B + E_T^1(\mathbf{u}_+^2))E_T^{2,+}(\mathbf{H}_+^1 - \mathbf{H}_+^2)).\end{aligned} \tag{103}$$

Combining (99), (100), (102), and (103) yields that

$$\begin{aligned}&\|e^{-\gamma t}\tilde{\mathcal{N}}_6\|_{H_p^{1/2}(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\tilde{\mathcal{N}}_6\|_{L_p(\mathbb{R},H_q^1(\dot\Omega))} + \gamma^{1/2}\|e^{-\gamma t}\tilde{\mathcal{N}}_6\|_{L_p(\mathbb{R},L_q(\dot\Omega))}\\ &\quad\leq C\{T^{1/(2p')}(B + E_T^2(\mathbf{H}^1) + e^{2(\gamma-\gamma_0)}B^2)\\ &\qquad + (T^{1/2} + T^{1/p})\{1 + T^{1/(2p')}(B + E_T^1(\mathbf{u}_+^1) + E_T^1(\mathbf{u}_+^2))\}(B + E_T^{2,+}(\mathbf{H}_+^1))\}E_T^1(\mathbf{u}_+^1 - \mathbf{u}_+^2)\\ &\quad + C\{T^{1/(2p')}(B + E_T^1(\mathbf{u}_+^2))\\ &\qquad + (T^{1/2} + T^{1/p})(1 + T^{1/(2p')}(B + E_T^1(\mathbf{u}_+^2)))(B + E_T^1(\mathbf{u}_+^2))\}E_T^2(\mathbf{H}^1 - \mathbf{H}^2).\end{aligned} \tag{104}$$

We now consider $\mathbf{N}_7$ and $\mathcal{N}_7$. In view of (35), we define extensions of $\mathbf{N}_7$ and $\mathcal{N}_7$ to $\mathbb{R} \setminus (0, T)$ by setting

$$\begin{aligned}\tilde{\mathbf{N}}_7(\mathbf{u},\mathbf{H}) &= \mathcal{A}_7(e_T[\Psi_\mathbf{u}])\nabla(\mu_+\mathcal{E}[E_-[\mathbf{H}_+]] - \mu_-\mathcal{E}[E_+[\mathbf{H}_-]]),\\ \tilde{\mathcal{N}}_7 &= (\mathcal{A}_7(e_T[\Psi_{\mathbf{u}^1}]) - \mathcal{A}_7(e_T[\Psi_{\mathbf{u}^2}]))\nabla(\mu_+\mathcal{E}[E_-[\mathbf{H}_+^1]] - \mu_-\mathcal{E}[E_+[\mathbf{H}_-^1]])\\ &\quad + \mathcal{A}_7(e_T[\Psi_{\mathbf{u}^2}])\nabla\{\mu_+(\mathcal{E}[E_-[\mathbf{H}_+^1]] - \mathcal{E}[E_-[\mathbf{H}_+^2]]) - \mu_-(\mathcal{E}[E_+[\mathbf{H}_-^1]] - \mathcal{E}[E_+[\mathbf{H}_-^2]])\}.\end{aligned}$$

Employing the same argument as in proving (92), (99) and (102), we have

$$\begin{aligned}&\|e^{-\gamma t}\tilde{\mathbf{N}}_7(\mathbf{u},\mathbf{H})\|_{H_p^{1/2}(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\tilde{\mathbf{N}}_7(\mathbf{u},\mathbf{H})\|_{L_p(\mathbb{R},H_q^1(\dot\Omega))} + \gamma^{1/2}\|e^{-\gamma t}\tilde{\mathbf{N}}_7(\mathbf{u},\mathbf{H})\|_{L_p(\mathbb{R},L_q(\dot\Omega))}\\ &\quad\leq CT^{1/(2p')}(B + E_T^1(\mathbf{u}_+))(B + E_T^2(\mathbf{H})),\end{aligned} \tag{105}$$

$$\begin{aligned}&\|e^{-\gamma t}\tilde{\mathcal{N}}_7\|_{H_p^{1/2}(\mathbb{R},L_q(\dot\Omega))} + \|e^{-\gamma t}\tilde{\mathcal{N}}_7\|_{L_p(\mathbb{R},H_q^1(\dot\Omega))} + \gamma^{1/2}\|e^{-\gamma t}\tilde{\mathcal{N}}_7\|_{L_p(\mathbb{R},L_q(\dot\Omega))}\\ &\quad\leq CT^{1/(2p')}\{(B + E_T^2(\mathbf{H}^1))E_T^1(\mathbf{u}_+^1 - \mathbf{u}_+^2) + (B + E_T^1(\mathbf{u}_+^2))E_T^2(\mathbf{H}^1 - \mathbf{H}^2)\}.\end{aligned} \tag{106}$$

We finally define extensions of $(\mathbf{N}_8(\mathbf{u},\mathbf{H}), \mathbf{N}_9(\mathbf{u},\mathbf{H})) = \mathcal{A}_8(\Psi_\mathbf{u})[[\mathbf{H}]]$ and $\mathcal{N}_8$ by setting

$$\begin{aligned}&(\tilde{\mathbf{N}}_8(\mathbf{u},\mathbf{H}), \tilde{\mathbf{N}}_9(\mathbf{u},\mathbf{H})) = \mathcal{A}_8(e_T[\Psi_\mathbf{u}])(\mu_+\mathcal{E}[E_-[\mathbf{H}_+]] - \mu_-\mathcal{E}_-[E_+[\mathbf{H}_-]]);\\ &\tilde{\mathcal{N}}_8 = (\mathcal{A}_8(e_T[\Psi_{\mathbf{u}^1}]) - \mathcal{A}_8(e_T[\Psi_{\mathbf{u}^2}]))(\mu_+\mathcal{E}[E_-[\mathbf{H}_+^1]] - \mu_-\mathcal{E}_-[E_+[\mathbf{H}_-^1]])\\ &+ \mathcal{A}_8(e_T[\Psi_{\mathbf{u}^2}])\{\mu_+(\mathcal{E}[E_-[\mathbf{H}_+^1]] - \mathcal{E}[E_-[\mathbf{H}^2]]) - \mu_-(\mathcal{E}_-[E_+[\mathbf{H}_-^1]] - \mathcal{E}_-[E_+[\mathbf{H}_-^2]])\}.\end{aligned}$$

By (90), (91), and (88), we have

$$\begin{aligned}&\|e^{-\gamma t}(\tilde{\mathbf{N}}_8(\mathbf{u},\mathbf{H}), \tilde{\mathbf{N}}_9(\mathbf{u},\mathbf{H}))\|_{L_p(\mathbb{R},H_q^2(\dot\Omega))} + \|e^{-\gamma t}\partial_t(\tilde{\mathbf{N}}_8(\mathbf{u},\mathbf{H}), \tilde{\mathbf{N}}_9(\mathbf{u},\mathbf{H}))\|_{L_p(\mathbb{R},L_q(\dot\Omega))}\\ &\quad\leq C\{T^{1/p'}E_T^1(\mathbf{u}_+)E_T^2(\mathbf{H}) + T^{1/p}(B + E_T^1(\mathbf{u}_+))(B + E_T^2(\mathbf{H}))\};\end{aligned} \tag{107}$$
$$\|e^{-\gamma t}\tilde{\mathcal{N}}_8\|_{L_p(\mathbb{R},H_q^2(\dot\Omega))} + \|e^{-\gamma t}\partial_t\tilde{\mathcal{N}}_8\|_{L_p(\mathbb{R},L_q(\dot\Omega))}$$

$$\leq C(T^{1/p'}E^2_T(\mathbf{H}^1)) + T^{1/p}(B + E^2_T(\mathbf{H}^1))E^1_T(\mathbf{u}^1_+ - \mathbf{u}^2_+)$$
$$+ C\{T^{1/p'}E^1_T(\mathbf{u}^2_+) + T^{1/p}(B + E^1_T(\mathbf{u}^2_+))\}E^2_T(\mathbf{H}^1 - \mathbf{H}^2). \tag{108}$$

5. A Proof of Theorem 1

We shall prove Theorem 1 by the contraction mapping principle. For this purpose, we define an underlying space $\mathbf{U}_{T,L}$ for a large number $L > 1$ and a small time $T \in (0,1)$ by setting

$$\begin{aligned}\mathbf{U}_{T,L} = \{(\mathbf{u}_+, \tilde{\mathbf{H}}_\pm) \mid\ & \mathbf{u}_+ \in H^1_p((0,T), H^1_q(\Omega_+)^N) \cap L_p((0,T), H^3_q(\Omega_+)^N), \\ & \tilde{\mathbf{H}}_\pm \in H^1_p((0,T), L_q(\Omega_\pm)^N) \cap L_p((0,T), H^2_q(\Omega_\pm)^N), \\ & \mathbf{u}_+|_{t=0} = \mathbf{u}_{0+} \ \text{ in } \Omega_+, \quad \tilde{\mathbf{H}}_\pm|_{t=0} = \tilde{\mathbf{H}}_{0\pm} \ \text{ in } \Omega_\pm, \\ & E^1_T(\mathbf{u}_+) \leq L, \quad E^{2,\pm}_T(\tilde{\mathbf{H}}_\pm) \leq L\}.\end{aligned} \tag{109}$$

Let B be a positive number for which initial data $\mathbf{u}_{0+}$ and $\tilde{\mathbf{H}}_{0\pm}$ for Equation (7) satisfying the condition (48).

Let $(\mathbf{u}_+, \tilde{\mathbf{H}}_\pm)$ be an element of $\mathbf{U}_{T,L}$, and let $\mathbf{N}_{5\pm}(\mathbf{u}, \tilde{\mathbf{H}})$, $\mathbf{N}_6(\mathbf{u}, \tilde{\mathbf{H}})$, $\mathbf{N}_7(\mathbf{u}, \tilde{\mathbf{H}})$, $\mathbf{N}_8(\mathbf{u}, \tilde{\mathbf{H}})$, and $\mathbf{N}_9(\mathbf{u}, \tilde{\mathbf{H}})$ be respective non-linear terms defined in (26), (29), (31), (33), and (34). Let $\mathbf{H}$ be a solution of equations:

$$\begin{aligned}\mu\partial_t\mathbf{H} - \alpha^{-1}\Delta\mathbf{H} &= \mathbf{N}_5(\mathbf{u}, \tilde{\mathbf{H}}) && \text{in } \dot\Omega \times (0,T), \\ [[\alpha^{-1}\mathrm{curl}\,\mathbf{H}]]\mathbf{n} &= \mathbf{N}_6(\mathbf{u}, \tilde{\mathbf{H}}) && \text{on } \Gamma \times (0,T), \\ [[\mu\mathrm{div}\,\mathbf{H}]] &= \mathbf{N}_7(\mathbf{u}, \tilde{\mathbf{H}}) && \text{on } \Gamma \times (0,T), \\ [[\mu\mathbf{H}\cdot\mathbf{n}]] &= \mathbf{N}_8(\mathbf{u}, \tilde{\mathbf{H}}) && \text{on } \Gamma \times (0,T), \\ [[\mathbf{H} - <\mathbf{H}, \mathbf{n}> \mathbf{n}]] &= \mathbf{N}_9(\mathbf{u}, \tilde{\mathbf{H}}) && \text{on } \Gamma \times (0,T), \\ \mathbf{n}_-\cdot\mathbf{H}_- = 0, \quad (\mathrm{curl}\,\mathbf{H}_-)\mathbf{n}_- &= 0 && \text{on } S_\pm \times (0,T), \\ \mathbf{H}|_{t=0} &= \tilde{\mathbf{H}}_0 && \text{in } \dot\Omega.\end{aligned} \tag{110}$$

Next, let $\mathbf{N}_1(\mathbf{u}_+, \mathbf{H}_+)$, $N_2(\mathbf{u}_+)$, $\mathbf{N}_3(\mathbf{u}_+)$ and $\mathbf{N}_4(\mathbf{u}_+, \mathbf{H}_+)$ be respective non-linear terms given in (24), (25), and (28) by replacing $\tilde{\mathbf{H}}_+$ with $\mathbf{H}_+$, where $\mathbf{H}_+ = \mathbf{H}|_{\Omega_+}$ and $\mathbf{H}$ is a solution of Equation (110). And then, let $\mathbf{v}$ be a solution of equations:

$$\begin{aligned}\rho\partial_t\mathbf{v}_+ - \mathrm{Div}\,\mathbf{T}(\mathbf{v}_+, \mathfrak{q}) &= \mathbf{N}_1(\mathbf{u}_+, \mathbf{H}_+) && \text{in } \Omega_+ \times (0,T), \\ \mathrm{div}\,\mathbf{v}_+ = \mathbf{N}_2(\mathbf{u}_+) &= \mathrm{div}\,\mathbf{N}_3(\mathbf{u}_+) && \text{in } \Omega_+ \times (0,T), \\ \mathbf{T}(\mathbf{v}_+, \mathfrak{q})\mathbf{n} &= \mathbf{N}_4(\mathbf{u}_+, \mathbf{H}_+) && \text{on } \Gamma \times (0,T), \\ \mathbf{v}_+|_{t=0} &= \mathbf{u}_{0+} && \text{in } \Omega_+.\end{aligned} \tag{111}$$

Recalling that $E^1_T(\mathbf{u}_+) \leq L$, in view of (49), (50), and (51) we choose $T > 0$ so small that

$$\|\Psi_\mathbf{u}\|_{L_\infty((0,T), L_\infty(\Omega))} \leq C\|\Psi_\mathbf{u}\|_{L_\infty((0,T), H^2_q(\Omega))} \leq CT^{1/p'}L \leq \delta.$$

Moreover, in view of (52), we choose $T > 0$ in such a way that

$$T^{1/p'}(E^1_T(\mathbf{u}_+) + B) \leq T^{1/p'}(L + B) \leq 1.$$

Let $\tilde{\mathbf{h}} = (\tilde{\mathbf{N}}_6(\mathbf{u}, \tilde{\mathbf{H}}), \mathbf{N}_7(\mathbf{u}, \tilde{\mathbf{H}}))$. and $\tilde{\mathbf{k}} = (\tilde{\mathbf{N}}_8(\mathbf{u}, \tilde{\mathbf{H}}), \tilde{\mathbf{N}}_9(\mathbf{u}, \tilde{\mathbf{H}}))$, and let $F_H(\mathbf{f}, \tilde{\mathbf{h}}, \tilde{\mathbf{k}})$ a symbol given in Theorem 3. By (83), (97), (105), and (107), we have

$$F_H(\mathbf{N}_5(\mathbf{u}, \tilde{\mathbf{H}}), \tilde{\mathbf{h}}, \tilde{\mathbf{k}}) \leq C[T^{1/p}(B+L)^2 + T^{1/p'}(B+L)^2 + T^{1/(2p')}(B+L)^2$$
$$+ (1 + (B+L)T^{1/(2p')})\{e^{2(\gamma-\gamma_0)}B^2 + (T^{1/2} + T^{1/p})(B+L)^2\}]$$

for any $\gamma \geq \gamma_0$. We fix γ as $\gamma = \gamma_0$. Let $\alpha = \min(1/p.1/p', 1/2p', 1/2, s/p'(1+s))$. Since $0 < t < 1$, there exsit positive constants M_1 and M_2 for which

$$F_H(\mathbf{N}_5(\mathbf{u}, \tilde{\mathbf{H}}), \tilde{\mathbf{h}}, \tilde{\mathbf{k}}) \leq M_1 B^2 + M_2((L+B)^2 + (L+B)^3)T^\alpha.$$

Applying Theorem 3 to Equation (110) yields that $E_T^2(\mathbf{H}) \leq Ce^{\gamma_0 T}(B + F_H(\mathbf{N}_5(\mathbf{u}, \tilde{\mathbf{H}}), \tilde{\mathbf{h}}, \tilde{\mathbf{k}}))$ for some constant C_1. Choosing $T > 0$ in such a way that $\gamma_0 T \leq 1$ gives that

$$E_T^2(\mathbf{H}) \leq C_1 e[B + M_1 B^2 + M_2((L+B)^2 + (L+B)^3)T^\alpha]. \tag{112}$$

In particular, we choose $T > 0$ so small that $M_2((L+B)^2 + (L+B)^3)T^\alpha \leq B + M_1 B^2$ and $L > 0$ so large that $4C_1 e(B + M_1 B^2) \leq L$, and then by (112)

$$E_T^{2,\pm}(\mathbf{H}^\pm) \leq L/2. \tag{113}$$

We next consider Equation (111). Let F_v be a symbol given in Theorem 2. By (68), (76), (80), (112), and (113),

$$\begin{aligned}
&F_v(\mathbf{N}_1(\mathbf{u}_+, \mathbf{H}_+), \mathbf{N}_2(\mathbf{u}_+), \mathbf{N}_3(\mathbf{u}_+), \mathbf{N}_4(\mathbf{u}_+, \mathbf{H})) \\
&\leq C(T^{1/p'}L^2 + T^{1/p}(L+B)^2 + T^{s/(p'(1+s))}(L^2 + (L+B)L^2) + (1+B)BE_T^{2,+}(\mathbf{H}_+)) \\
&\leq C[T^{1/p'}L^2 + T^{1/p}(L+B)^2 + T^{s/(p'(1+s))}(L^2 + (L+B)L^2) \\
&\qquad + (1+B)BCe\{M_1 B^2 + M_2((L+B)^2 + (L+B)B^2 + (L+B)^3)T^\alpha\}],
\end{aligned}$$

which yields that

$$F_v(\mathbf{N}_1(\mathbf{u}_+, \mathbf{H}_+), \mathbf{N}_2(\mathbf{u}_+), \mathbf{N}_3(\mathbf{u}_+), \mathbf{N}_4(\mathbf{u}_+, \mathbf{H})) \leq M_3(1+B)^2 B^2 + M_4(1+B)^2((L+B)^2 + (L+B)^3)T^\alpha$$

for some constants M_3 and M_4. Thus, applying Theorem 2 with $0 < T < 1$ to Equation (111), we have

$$E_T^1(\mathbf{v}) \leq C_2 e^{\gamma_1 T}\{B + M_3(1+B)^2 B^2 + M_4(1+B)^2((L+B)^2 + (L+B)^3)T^\alpha\}$$

for some constant C_2. Recalling that $\gamma_1 \leq \gamma_0$ and $\gamma_0 T \leq 1$, choosing $T > 0$ so small that $M_4(1+B)^2((L+B)^2 + (L+B)^3)T^\alpha \leq B + M_3(1+B)^2 B^2$ and choosing $L > 0$ so large that $L \geq 4C_2 e(B + M_3(1+B)^2 B^2)$, we have $E_T^1(\mathbf{v}) \leq L/2$, which implies that $(\mathbf{v}, \mathbf{H}_\pm) \in \mathbf{U}_{T,L}$. In particular, we set $L = 4e\max(C_1(B + M_1 B^2), C_2(B + M_3(1+B)^2 B^2))$. Let Φ be a map acting on $(\mathbf{u}, \tilde{\mathbf{H}}) \in \mathbf{U}_{T,L}$ by setting $\Phi(\mathbf{u}, \tilde{\mathbf{H}}) = (\mathbf{v}, \mathbf{H})$, and then Φ is a map from $\mathbf{U}_{T,L}$ into itself.

We now prove that Φ is a contraction map. Let $(\mathbf{u}_+^i, \tilde{\mathbf{H}}^i) \in \mathbf{U}_{T,L}$ $(i = 1, 2)$ and set $(\mathbf{v}_+^i, \mathbf{H}^i) = \Phi(\mathbf{u}_+^i, \tilde{\mathbf{H}}^i)$. In view of (51), (52), and (89), choosing $T > 0$ as smaller if necessary, we may assume that

$$\|\Psi_{\mathbf{u}^i}\|_{L_\infty((0,T), L_\infty(\Omega))} \leq \delta, \quad \|e_T[\Psi_{\mathbf{u}^i}]\|_{L_\infty(\mathbb{R}, L_\infty(\Omega))} \leq \delta, \quad T^{1/p'}(E_T^1(\mathbf{u}_i) + B) \leq 1$$

for $i = 1, 2$. Set

$$\begin{aligned}
&\mathbf{v}_+ = \mathbf{v}_+^1 - \mathbf{v}_+^2, \quad \mathbf{H} = \mathbf{H}^1 - \mathbf{H}^2, \quad \mathcal{N}_i = \mathbf{N}_i(\mathbf{u}_+^1, \mathbf{H}_+^1) - \mathbf{N}_i(\mathbf{u}_+^2, \mathbf{H}_+^2), \quad \mathcal{N}_2 = \mathbf{N}_2(\mathbf{u}_+^1) - N_2(\mathbf{u}_+^2), \\
&\mathcal{N}_3 = \mathbf{N}_3(\mathbf{u}_+^1) - \mathbf{N}_3(\mathbf{u}_+^2), \quad \mathcal{N}_j = \mathbf{N}_j(\mathbf{u}_+^1, \tilde{\mathbf{H}}_+^1) - \mathbf{N}_j(\mathbf{u}_+^2, \tilde{\mathbf{H}}_+^2), \quad \mathcal{N}_k = N_k(\mathbf{u}_+^1, \tilde{\mathbf{H}}^1) - N_k(\mathbf{u}_+^2, \tilde{\mathbf{H}}^2)
\end{aligned}$$

for $i = 1, 4$, $j = 5, 6, 9$ and $k = 7, 8$. Noticing that $\mathbf{v}_+^1|_{t=0} = \mathbf{v}_+^2|_{t=0} = \mathbf{u}_{0+}$, and $\mathbf{H}_\pm^1|_{t=0} = \mathbf{H}_\pm^2|_{t=0} = \tilde{\mathbf{H}}_{0\pm}$, by (110), (111) we see that $\mathbf{H}$ satisfies the following equations:

$$\begin{aligned} \mu\partial_t \mathbf{H} - \alpha^{-1}\Delta\mathbf{H} &= \mathcal{N}_5 && \text{in } \dot{\Omega}\times(0,T),\\ [[\alpha^{-1}\mathrm{curl}\,\mathbf{H}]]\mathbf{n} = \mathcal{N}_6, \quad [[\mu\mathrm{div}\,\mathbf{H}]] &= \mathcal{N}_7 && \text{on } \Gamma\times(0,T),\\ [[\mu\mathbf{H}\cdot\mathbf{n}]] = \mathcal{N}_8, \quad [[\mathbf{H}- <\mathbf{H},\mathbf{n}>\mathbf{n}]] &= \mathcal{N}_9 && \text{on } \Gamma\times(0,T),\\ \mathbf{n}_\pm\cdot\mathbf{H}_\pm = 0, \quad (\mathrm{curl}\,\mathbf{H}_\pm)\mathbf{n}_\pm &= 0 && \text{on } S_\pm\times(0,T),\\ \mathbf{H}|_{t=0} &= 0 && \text{in } \dot{\Omega} \end{aligned} \tag{114}$$

and that $\mathbf{v}_+$ satisfies the following equations:

$$\begin{aligned} \rho\partial_t\mathbf{v}_+ - \mathrm{Div}\,\mathbf{T}(\mathbf{v}_+,\mathfrak{q}) &= \mathcal{N}_1 && \text{in } \Omega_+\times(0,T),\\ \mathrm{div}\,\mathbf{v}_+ = \mathcal{N}_2 &= \mathrm{div}\,\mathcal{N}_3 && \text{in } \Omega_+\times(0,T),\\ \mathbf{T}(\mathbf{v}_+,\mathfrak{q})\mathbf{n} &= \mathcal{N}_4 && \text{on } \Gamma\times(0,T),\\ \mathbf{v}_+ &= 0 && \text{on } S_+\times(0,T),\\ \mathbf{v}_+|_{t=0} &= 0 && \text{in } \Omega_+. \end{aligned} \tag{115}$$

Set $\tilde{\mathcal{H}} = (\mathcal{N}_6,\mathcal{N}_7)$ and $\tilde{\mathcal{K}} = (\mathcal{N}_8,\mathcal{N}_9)$. By (86), (104), (106), and (108), we have

$$\begin{aligned} F_H(\mathcal{N}_5,\tilde{\mathcal{H}},\tilde{\mathcal{K}}) \le C[&\{T^{1/p'}((B+L)^2+(B+L)) + T^{1/p}(B+L) + T^{1/(2p')}(B+L)\}E^1_T(\mathbf{u}^1-\mathbf{u}^2)\\ &+\{T^{1/p'}L + T^{1/p}(B+L) + T^{1/(2p')}(B+L)\}E^2_T(\tilde{\mathbf{H}}^1-\tilde{\mathbf{H}}^2)\\ &+\{T^{1/(2p')}(B+L+e^{\gamma-\gamma_0}B^2) + (T^{1/2}+T^{1/p})(1+T^{1/(2p')}(B+L))(B+L)\}E^1_T(\mathbf{u}^1-\mathbf{u}^2)\\ &+\{T^{1/(2p')}(B+L) + (T^{1/2}+T^{1/p})(1+T^{1/(2p')}(B+L))(B+L)\}E^2_T(\tilde{\mathbf{H}}^1-\tilde{\mathbf{H}}^2)] \end{aligned}$$

for any $\gamma \ge \gamma_0$. Thus, choosing $\gamma = \gamma_0$ and noting $0 < T < 1$, we have

$$F_H(\mathcal{N}_5,\tilde{\mathcal{H}},\tilde{\mathcal{K}}) \le C(B+L+(B+L)^2)T^\alpha(E^1_T(\mathbf{u}^1_+-\mathbf{u}^2_+) + E^2_T(\tilde{\mathbf{H}}^1-\tilde{\mathbf{H}}^2)). \tag{116}$$

Applying Theorem 3 to Equation (114) and using (116) gives that

$$E^1_T(\mathbf{H}^1-\mathbf{H}^2) \le M_5(B+L+(B+L)^2)T^\alpha(E^1_T(\mathbf{u}^1_+-\mathbf{u}^2_+) + E^2_T(\tilde{\mathbf{H}}^1-\tilde{\mathbf{H}}^2)) \tag{117}$$

for some constant M_5, where we have used $\gamma_2 \le \gamma_0$ and $\gamma_0 T \le 1$. Moreover, by (73), (79), and (82)

$$\begin{aligned} F_v(\mathcal{N}_1,\mathcal{N}_2,\mathcal{N}_3,\mathcal{N}_4) \le\ & C\{(T^{1/p'}+T^{1/p})((L+B)+(L+B)^2)\}E^1_T(\mathbf{u}^1_+-\mathbf{u}^2_+)\\ &+C\{T^{1/p}(B+L)^2 + (1+T^{1/p}(B+L))(B+T^{s/p'(1+s)}L)\}E^2_T(\mathbf{H}^1-\mathbf{H}^2)\}, \end{aligned}$$

which, combined with (117), leads to

$$\begin{aligned} F_v(\mathcal{N}_1,\mathcal{N}_2,\mathcal{N}_3,\mathcal{N}_4) \le\ & C(B+L+(B+L)^2)T^\alpha E^1_T(\mathbf{u}^1_+-\mathbf{u}^2_+)\\ &+CM_5(B+L+(B+L)^2)^2T^\alpha(E^1_T(\mathbf{u}^1_+-\mathbf{u}^2_+) + E^2_T(\tilde{\mathbf{H}}^1-\tilde{\mathbf{H}}^2)). \end{aligned}$$

Thus, applying Theorem 2 to Equation (115) leads to

$$\begin{aligned} E^1_T(\mathbf{v}^1_+-\mathbf{v}^2_+) \le M_6\{&(B+L+(B+L)^2)\\ &+(B+L+(B+L)^2)^2\}T^\alpha(E^1_T(\mathbf{u}^1_+-\mathbf{u}^2_+) + E^2_T(\tilde{\mathbf{H}}^1-\tilde{\mathbf{H}}^2)) \end{aligned} \tag{118}$$

for some M_6, where we have used $\gamma_1 \le \gamma_0$ and $\gamma_0 T \le 1$. Choosing $T > 0$ so small that $M_5(B+L+(B+L)^2)T^\alpha \le 1/4$ in (117) and $M_6\{(B+L+(B+L)^2)+(B+L+(B+L)^2)^2\}$ $T^\alpha \le 1/4$ in (118) gives that

$$E^1_T(\mathbf{v}^1_+-\mathbf{v}^2_+) + E^2_T(\mathbf{H}_1-\mathbf{H}_2) \le (1/2)(E^1_T(\mathbf{u}^1_+-\mathbf{u}^2_+) + E^2_T(\tilde{\mathbf{H}}^1-\tilde{\mathbf{H}}^2)),$$

which shows that the Φ is a contraction map. Thus, the Banach fixed point theorem yields the unique existence of a fixed point, $(\mathbf{u}_+, \tilde{\mathbf{H}}_\pm) \in \mathbf{U}_{T,L}$, of the map Φ, which is a unique solution of Equation (7). This completes the proof of Theorem 1.

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Appendix A. A Proof of Theorem 2

Appendix A.1

To prove the maximal L_p-L_q regularity, according to Shibata [14,16,17] a main step is to prove the existence of $\mathcal{R}$-solver for the following model problem:

$$\begin{aligned} \lambda\mathbf{u} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I}) = \mathbf{f} & \quad \text{in } \mathbb{R}^N_+, \\ \mathrm{div}\,\mathbf{u} = g = \mathrm{div}\,\mathbf{g} & \quad \text{in } \mathbb{R}^N_+, \\ (\mu\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I})\mathbf{n} = \mathbf{h} & \quad \text{on } \mathbb{R}^N_0, \end{aligned} \tag{A1}$$

where $\mathbb{R}^N_+ = \{x = (x_1, \ldots, x_N) \mid x_N > 0\}$, $\mathbb{R}^N_0 = \{x = (x_1, \ldots, x_{N-1}, 0) \mid (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}\}$, and $\mathbf{n} = (0, \ldots, 0, -1)$. The λ is a complex parameter ranging in $\Sigma_{\epsilon,\lambda_0} = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \le \pi - \epsilon, |\lambda| \ge \lambda_0\}$ with $0 < \epsilon < \pi/2$ and $\lambda_0 > 0$. In [14,16], the existence of $\mathcal{R}$-solvers, $\mathcal{S}(\lambda)$, $\mathcal{P}(\lambda)$, were proved for Equation (A1), which satisfy the following properties:

(1) $\mathcal{S}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+), H^2_q(\mathbb{R}^N_+)^N))$, $\quad \mathcal{P}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+), H^1_q(\mathbb{R}^N_+) + \hat{H}^1_{q,0}(\mathbb{R}^N_+)))$.

(2) Problem (A1) admits unique solutions $\mathbf{u} = \mathcal{S}(\lambda)F_\lambda(\mathbf{f}, g, \mathbf{g}, \mathbf{h})$ and $\mathfrak{p} = \mathcal{P}(\lambda)F_\lambda(\mathbf{f}, g, \mathbf{g}, \mathbf{h})$ for any $(\mathbf{f}, g, \mathbf{g}, \mathbf{h}) \in X_q$ and $\lambda \in \Sigma_{\epsilon,\lambda_0}$, where $F_\lambda(\mathbf{f}, g, \mathbf{g}, \mathbf{h}) = (\mathbf{f}, g, \lambda^{1/2}g, \lambda\mathbf{g}, \mathbf{h}, \lambda^{1/2}\mathbf{h})$,.

(3) $\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+), H^{2-j}_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{S}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r_b \quad (j = 0, 1, 2)$,
$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+), L_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^\ell(\nabla\mathcal{P}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r_b$
for $\ell = 0, 1$ with some constant r_b depending on λ_0 and ϵ.

Here, $\lambda = \gamma + i\tau \in \mathbb{C}$, $\mathcal{R}_{\mathcal{L}(X,Y)}\mathcal{T}$ denotes the $\mathcal{R}$ norm of an operator family $\mathcal{T} \subset \mathcal{L}(X, Y)$, $\mathcal{L}(X, Y)$ being the set of all bounded linear operators from X into Y,

$$\mathcal{X}_q(\mathbb{R}^N_+) = \{(F_1, \ldots, F_6) \mid F_1, F_4, F_6 \in L_q(\mathbb{R}^N_+)^N, \quad F_2 \in H^1_q(\mathbb{R}^N_+), \quad F_3 \in L_q(\mathbb{R}^N_+), \quad F_5 \in H^1_q(\mathbb{R}^N_+)^N\};$$
$$X_q(\mathbb{R}^N_+) = \{(\mathbf{f}, g, \mathbf{g}, \mathbf{h}) \mid \mathbf{f} \in L_q(\mathbb{R}^N_+)^N, \ g \in H^1_q(\mathbb{R}^N_+), \ \mathbf{g} \in L_q(\mathbb{R}^N_+)^N, \ \mathbf{h} \in H^1_q(\mathbb{R}^N_+)^N, \ g = \mathrm{div}\,\mathbf{g}\}.$$

The F_1, F_2, F_3, F_4, F_5, and F_6 are corresponding variables to $\mathbf{f}$, g, $\lambda^{1/2}g$, $\lambda\mathbf{g}$, $\mathbf{h}$, and $\lambda^{1/2}\mathbf{h}$, respectively. The norm of $\mathcal{X}_q(\mathbb{R}^N_+)$ is defined by setting

$$\|(F_1, \ldots, F_6)\|_{\mathcal{X}_q(\mathbb{R}^N_+)} = \|(F_1, F_3, F_4, F_6)\|_{L_q(\mathbb{R}^N_+)} + \|(F_2, F_5)\|_{H^1_q(\mathbb{R}^N_+)}.$$

In particular, we know an unique existence of solutions $\mathbf{u} \in H^2_q(\mathbb{R}^N_+)^N$ and $\mathfrak{p} \in H^1_q(\mathbb{R}^N_+) + \hat{H}^1_{q,0}(\mathbb{R}^N_+)$ (Here, we just give an idea of obtaining third order regularities. An idea also is found in ([19], Appendix 6.2). To prove Theorem 2 exactly from the $\mathcal{R}$-bounded solution operators point of view, we have to start returning the non-zero $\mathbf{f}$, g and $\mathbf{g}$ situation to the situation where $\mathbf{f} = g = \mathbf{g} = 0$, which needs an idea. We will give an exact proof of Theorem 2 in a forthcoming paper.) of Equation (A1) possessing the estimate:

$$\begin{aligned}&\|\lambda\mathbf{u}\|_{L_q(\mathbb{R}^N_+)} + \|\mathbf{u}\|_{H^2_q(\mathbb{R}^N_+)} + \|\nabla\mathfrak{p}\|_{L_q(\mathbb{R}^N_+)} \\ &\quad\leq C(\|\mathbf{f}\|_{L_q(\mathbb{R}^N_+)} + \|(g,\mathbf{h})\|_{H^1_q(\mathbb{R}^N_+)} + \|\lambda^{1/2}(g,\mathbf{h})\|_{L_q(\mathbb{R}^N_+)} + \|\lambda\mathbf{g}\|_{L_q(\mathbb{R}^N_+)}).\end{aligned} \tag{A2}$$

We now prove that $\mathbf{u} \in H^3_q(\mathbb{R}^N_+)^N$ and $\nabla\mathfrak{p} \in H^1_q(\mathbb{R}^N_+)^N$ provided that $\mathbf{f} \in H^1_q(\mathbb{R}^N_+)^N$, $g \in H^2_q(\mathbb{R}^N_+)$, $\mathbf{g} \in H^1_q(\mathbb{R}^N_+)^N$, and $\mathbf{h} \in H^2_q(\mathbb{R}^N_+)^N$. Moreover, $\mathbf{u}$ and $\mathfrak{p}$ satisfy the estimate:

$$\begin{aligned}&\|\lambda\mathbf{u}\|_{H^1_q(\mathbb{R}^N_+)} + \|\mathbf{u}\|_{H^3_q(\mathbb{R}^N_+)} + \|\nabla\mathfrak{p}\|_{H^1_q(\mathbb{R}^N_+)} \\ &\quad\leq C(\|\mathbf{f}\|_{H^1_q(\mathbb{R}^N_+)} + \|(g,\mathbf{h})\|_{H^2_q(\mathbb{R}^N_+)} + \|\lambda(g,\mathbf{h})\|_{L_q(\mathbb{R}^N_+)} + \|\lambda\mathbf{g}\|_{H^1_q(\mathbb{R}^N_+)}).\end{aligned} \tag{A3}$$

In fact, differentiating Equation (A1) with respect to tangential variables x_j ($j = 1, \ldots, N-1$) and noting that $\partial_j\mathbf{u}$ and $\partial_j\mathfrak{p}$ satisfy equations replacing $\mathbf{f}$, $g = \operatorname{div}\mathbf{g}$, and $\mathbf{h}$ with $\partial_j\mathbf{f}$, $\partial_j g = \operatorname{div}\partial_j\mathbf{g}$ and $\partial_j\mathbf{h}$, by (A2) and the uniqueness of solutions we see that $\partial_j\mathbf{u} \in H^2_q(\mathbb{R}^N_+)^N$ and $\nabla\partial_j\mathfrak{p} \in L_q(\mathbb{R}^N_+)^N$, and

$$\begin{aligned}&\|\lambda\partial_j\mathbf{u}\|_{L_q(\mathbb{R}^N_+)} + \|\partial_j\mathbf{u}\|_{H^2_q(\mathbb{R}^N_+)} + \|\nabla\partial_j\mathfrak{p}\|_{L_q(\mathbb{R}^N_+)} \\ &\quad\leq C(\|\partial_j\mathbf{f}\|_{L_q(\mathbb{R}^N_+)} + \|(\partial_j g,\partial_j\mathbf{h})\|_{H^1_q(\mathbb{R}^N_+)} + \|\lambda^{1/2}(\partial_j g,\partial_j\mathbf{h})\|_{L_q(\mathbb{R}^N_+)} + \|\lambda\partial_j\mathbf{g}\|_{L_q(\mathbb{R}^N_+)})\end{aligned} \tag{A4}$$

for $j = 1, \ldots, N-1$. To estimate $\partial_N\mathbf{u}$ and $\partial_N\mathfrak{p}$, we start with estimating $\partial_N u_N$. In fact, from the divergence equations it follows that $\partial_N u_N = -\sum_{j=1}^{N-1}\partial_j u_j + g$, and so

$$\lambda\partial_N u_N = -\sum_{j=1}^{N-1}\lambda\partial_j u_j + \lambda g, \quad \partial^3_N u_N = -\sum_{j=1}^{N-1}\partial^2_N\partial_j u_j + \partial^2_N g,$$

which, combined with (A4) yields that

$$\begin{aligned}&\|\lambda\partial_N u_N\|_{L_q(\mathbb{R}^N_+)} + \|\partial^3_N u_N\|_{L_q(\mathbb{R}^N_+)} \\ &\leq C\Big\{\sum_{j-1}^{N-1}(\|\partial_j\mathbf{f}\|_{L_q(\mathbb{R}^N_+)} + \|(\partial_j g,\partial_j\mathbf{h})\|_{H^1_q(\mathbb{R}^N_+)} + \|\lambda^{1/2}(\partial_j g,\partial_j\mathbf{h})\|_{L_q(\mathbb{R}^N_+)} + \|\lambda\partial_j\mathbf{g}\|_{L_q(\mathbb{R}^N_+)}) \\ &+\|(\lambda g,\partial^2_N g)\|_{L_q(\mathbb{R}^N_+)}\Big\}.\end{aligned} \tag{A5}$$

From the the N-th component of the first equation of Equation (A1) and $\operatorname{div}\mathbf{u} = g$, we have

$$\lambda u_N - \mu\Delta u_N - \mu\partial_N g + \partial_N\mathfrak{p} = f_N,$$

and so, we see that $\partial^2_N\mathfrak{p} \in L_q(\mathbb{R}^N_+)$ and

$$\|\partial^2_N\mathfrak{p}\|_{L_q(\mathbb{R}^N_+)} \leq \|\partial_N f_N\|_{L_q(\mathbb{R}^N_+)} + \mu\|\partial^2_N g\|_{L_q(\mathbb{R}^N_+)} + \|\lambda\partial_N u_N\|_{L_q(\mathbb{R}^N_+)} + \mu\|u_N\|_{H^3_q(\mathbb{R}^N_+)}. \tag{A6}$$

From Equation (A1), we have

$$\begin{aligned}\lambda u_j - \mu\Delta u_j = f_j - \partial_j p + \mu\partial_j g &\quad \text{in } \mathbb{R}^N_+, \\ \partial_N u_j = -\partial_j u_N + \mu^{-1}h_j &\quad \text{on } \mathbb{R}^N_0.\end{aligned}$$

Differentiating the first equation of the above set of equations with respect to x_N and setting $\partial_N u_j = v$, we have

$$\begin{aligned}\lambda v - \mu\Delta v = \partial_N f_j - \partial_j\partial_N\mathfrak{p} + \mu\partial_N\partial_j g &\quad \text{in } \mathbb{R}^N_+, \\ v = -\partial_j u_N + \mu^{-1}h_j &\quad \text{on } \mathbb{R}^N_0.\end{aligned}$$

Thus, setting $w = v + \partial_j u_N - \mu^{-1} h_j$, we have

$$\begin{aligned} \lambda w - \mu \Delta w &= \partial_N f_j - \partial_j \partial_N \mathfrak{p} + \mu \partial_N \partial_j g + (\lambda - \Delta)(\partial_j u_N - \mu^{-1} h_j) && \text{in } \mathbb{R}^N_+, \\ w &= 0 && \text{on } \mathbb{R}^N_0. \end{aligned}$$

Thus, by a known estimate for the Dirichlet problem, we have

$$\begin{aligned} &\|\lambda \partial_N u_j\|_{L_q(\mathbb{R}^N_+)} + \|\partial_N u_j\|_{H^2_q(\mathbb{R}^N_+)} \\ &\leq C\{\|\partial_N f_j\|_{L_q(\mathbb{R}^N_+)} + \|\partial_j \partial_N \mathfrak{p}\|_{L_q(\mathbb{R}^N_+)} + \|\partial_j \partial_N g\|_{L_q(\mathbb{R}^N_+)} + \|\lambda \partial_j u_N\|_{L_q(\mathbb{R}^N_+)} \\ &+ \|\lambda h_j\|_{L_q(\mathbb{R}^N_+)} + \|\partial_j u_N\|_{H^2_q(\mathbb{R}^N_+)} + \|h_j\|_{H^2_q(\mathbb{R}^N_+)}\}. \end{aligned} \tag{A7}$$

Noting that $\|\lambda^{1/2}(\partial_j g, \partial_j \mathbf{h})\|_{L_q(\mathbb{R}^N_+)} \leq C(\|\lambda(g, \mathbf{h})\|_{L_q(\mathbb{R}^N_+)} + \|(g, \mathbf{h})\|_{H^2_q(\mathbb{R}^N_+)})$ and combining (A2) and (A4)–(A7), we have (A3).

Localizing the problem and using the argument above, we can show Theorem 2.

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