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566

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On local solutions to a free boundary problem for incompressible viscous magnetohydrodynamics in the L_p-approach

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Contents

Abstract

We consider the motion of incompressible magnetohydrodynamics (mhd) with resistivity in a domain bounded by a free surface which is coupled through the free surface with an electromagnetic field generated by a magnetic field prescribed on an exterior fixed boundary. On the free surface, transmission conditions for electromagnetic fields are imposed. We can distinguish two cases which have an essential influence on the proofs of existence: no jump of the magnetic field (Part 2) and a jump (Part 1). In the no jump case we prove local existence of solutions such that the velocity and the magnetic field belong to $W_r^{2,1}$, $r > 3$. In the case of any jump of the magnetic field we prove existence of local solutions such that the velocity belongs to $W_r^{3,3/2}$ and the magnetic field to $W_r^{2,1}$, $r > 5/2$.

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Preliminary remarks

In this paper free boundary problems for incompressible viscous magnetohydrodynamics (mhd) are considered within the L_p-approach. The paper is organized in the following way. In the Introduction (Section 1) the problem described by equations (1.1)–(1.8) contains two coupled systems:

(1) the Navier–Stokes system,
(2) the transmission problem for the magnetic field,

with corresponding initial and boundary conditions.

We consider two kinds of boundary conditions: in Part 1 we have a jump of the magnetic field on the free surface and in Part 2 there is no jump. Part 1 comprises Sections 3–7 and Part 2 Sections 8–12, and in Section 2 we introduce the notation and auxiliary results. Since different jump conditions on the free surface for the magnetic field are assumed, the considerations in Sections 3–7 and in 8–12 are totally different.

Let us describe Part 1 in more detail. Since problem (1.1)–(1.8) is a free boundary problem, we pass in Section 3 to Lagrangian coordinates to transform it to a problem in a fixed domain $\overset{1}{\Omega}_0 \cup \overset{2}{\Omega}_0$. Then the problem becomes strongly nonlinear. Next we formulate a method of successive approximations. Hence, given step n of the approximation, we obtain linear problems (1) and (2) for step $n+1$. In case (1) we derive the Stokes system with variable coefficients depending on the previous step. In case (2) we have a linearized transmission problem.

In Section 4 we examine problem (1). The existence is straightforward because we have the Stokes system. Then we can obtain an estimate for the velocity at step $n+1$ of approximation in terms of the velocity and the magnetic field at step n. The estimate is derived in anisotropic Sobolev spaces. In Section 5 we examine the transmission problem (2). To find appropriate estimates and show the existence we need the technique of regularizer. This means that problem (2) is considered locally in domains restricted to supports of functions of an appropriate partition of unity which is introduced in Section 2. The most difficult local problem is a problem in a neighborhood of a point of the free surface at the initial time. We show the existence of solutions of this last problem and derive appropriate estimates in Section 6. We mention that the solutions are constructed explicitly and estimated in Besov spaces. Finally in Section 7 we prove the existence of solutions to problem (1.1)–(1.8) using the existence results and estimates from Sections 4 and 5 and exploiting properties of the successive approximations.

We now describe Part 2. In Section 8 problem (1.1)–(1.8) is transformed into a problem defined in Lagrangian coordinates and a method of successive approximations is formulated. At any step of the method of successive approximations problems (1) and (2) are linearized. Hence, in Sections 9 and 10 the existence of solutions to the Stokes systems and to the linearized transmission problem for the magnetic fields are proved, respectively. In Section 9, compared with Section 4, we need less regularity to derive the necessary estimate for the velocity. This fact makes considerations in Sections 10 and 11 different from the corresponding ones in Sections 5 and 6. Finally, the existence of local solutions to problem (1.1)–(1.8) without any jump of the magnetic field on the free surface is proved in Section 12.

1. Introduction

We consider a free boundary problem for a magnetohydrodynamic motion in a domain $\overset{1}{\Omega}_t$ bounded by a free surface S_t. The motion interacts with an electromagnetic field located in $\overset{2}{\Omega}_t$.

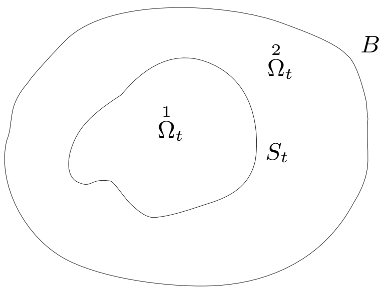

Fig. 1

In $\overset{1}{\Omega}_t$ the magnetohydrodynamic motion is described by the system of equations

$$\begin{aligned}
&v_{,t} + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) - \mu_1\overset{1}{H}\cdot\nabla\overset{1}{H} + \tfrac{1}{2}\mu_1\nabla\overset{1}{H}{}^2 = f,\\
&\operatorname{div} v = 0,\\
&\mu_1\overset{1}{H}_{,t} = -\operatorname{rot}\overset{1}{E},\\
&\operatorname{rot}\overset{1}{H} = \sigma_1(\overset{1}{E} + \mu_1 v\times\overset{1}{H}),\\
&\operatorname{div}\overset{1}{H} = 0,
\end{aligned}\tag{1.1}$$

where $v = v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t)) \in \mathbb{R}^3$ is the velocity of the fluid, $p = p(x,t) \in \mathbb{R}$ is the pressure, $\overset{1}{H}(x,t) = (\overset{1}{H}_1(x,t), \overset{1}{H}_2(x,t), \overset{1}{H}_3(x,t)) \in \mathbb{R}^3$ is the magnetic field, $\overset{1}{E} = \overset{1}{E}(x,t) = (\overset{1}{E}_1(x,t), \overset{1}{E}_2(x,t), \overset{1}{E}_3(x,t)) \in \mathbb{R}^3$ is the electric field, $f = f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}^3$ is the external force field, and $x = (x_1, x_2, x_3)$ are Cartesian coordinates. Moreover, μ_1 is the constant magnetic permeability, σ_1 is the constant electric conductivity and $\mathbb{T}(v,p)$ is the stress tensor of the form

$$\mathbb{T}(v,p)=\nu\mathbb{D}(v)-p\mathbb{I}, \tag{1.2}$$

where ν is the positive viscosity coefficient, $\mathbb{I}$ is the unit matrix and $\mathbb{D}(v)$ is the dilatation tensor of the form

$$\mathbb{D}(v)=\{v_{i,x_j}+v_{j,x_i}\}_{i,j=1,2,3}. \tag{1.3}$$

For system (1.1) the following initial and boundary conditions are prescribed:

$$\begin{aligned}&\bar n\cdot\mathbb{T}(v,p)+\mu_1\bar n\cdot\mathbb{T}(\overset{1}{H})=p_0\bar n,\\&v|_{t=0}=v(0),\quad \overset{1}{H}|_{t=0}=\overset{1}{H}(0),\quad \operatorname{div}\overset{1}{H}(0)=0,\\&\overset{1}{\Omega}_t|_{t=0}=\overset{1}{\Omega}_0,\quad S_t|_{t=0}=S_0,\end{aligned} \tag{1.4}$$

where $\bar n$ is the unit vector outward to $\overset{1}{\Omega}_t$ and normal to S_t, and

$$\mathbb{T}(\overset{1}{H})=\left\{\overset{1}{H}_i\overset{1}{H}_j-\tfrac12\overset{1}{H}{}^2\delta_{ij}\right\}_{i,j=1,2,3}. \tag{1.5}$$

The boundary condition $(1.4)_1$ implies the compatibility condition

$$\bar n(0)\cdot\mathbb{D}(v(0))\cdot\bar\tau(0)+\mu_1\bar n(0)\cdot\overset{1}{H}(0)\bar\tau(0)\cdot\overset{1}{H}(0)=0\quad\text{ on } S_0,$$

where $\bar n(0)=\bar n|_{t=0}$, $\bar\tau(0)=\bar\tau|_{t=0}$ and $\bar\tau$ is a tangent vector to S_t.

In $\overset{2}{\Omega}_t$ we have a motionless dielectric gas under a constant pressure p_0. Therefore, we only have an electromagnetic field described by the system of equations

$$\begin{aligned}&\mu_2\overset{2}{H}_{,t}=-\operatorname{rot}\overset{2}{E},\\&\sigma_2\overset{2}{E}=\operatorname{rot}\overset{2}{H},\\&\operatorname{div}\overset{2}{H}=0.\end{aligned} \tag{1.6}$$

For system (1.6) the following initial and boundary conditions are prescribed:

$$\begin{aligned}&\overset{2}{H}|_{t=0}=\overset{2}{H}(0),\quad \operatorname{div}\overset{2}{H}(0)=0,\quad \overset{2}{\Omega}_t|_{t=0}=\overset{2}{\Omega}_0,\\&\overset{2}{H}\cdot\bar\tau_\alpha|_B=H_{*\alpha},\quad \alpha=1,2,\quad \operatorname{div}\overset{2}{H}|_B=0,\end{aligned} \tag{1.7}$$

where $\bar\tau_\alpha$, $\alpha=1,2$, are tangent vectors to B.

Electromagnetic fields in $\overset{1}{\Omega}_t$ and $\overset{2}{\Omega}_t$ are coupled through S_t by the following transmission conditions:

$$\overset{1}{a}_\alpha\overset{1}{E}\cdot\bar\tau_\alpha=\overset{2}{a}_\alpha\overset{2}{E}\cdot\bar\tau_\alpha|_{S_t},\quad \overset{1}{b}_\beta\overset{1}{H}_\beta=\overset{2}{b}_\beta\overset{2}{H}_\beta|_{S_t},\quad \alpha=1,2,\quad \beta=1,2,3, \tag{1.8}$$

where $\bar\tau_1,\bar\tau_2,\bar n$ is an orthonormal system of vector fields in a neighborhood of S_t such that $\bar n|_{S_t}$ is normal and $\bar\tau_1,\bar\tau_2|_{S_t}$ are tangent to S_t.

Moreover, we assume that $\overset{i}{a}_\alpha$, $\overset{i}{b}_\beta$, $i=1,2$, $\alpha=1,2$, $\beta=1,2,3$, are constants. Nonvanishing of differences $\overset{1}{a}_\alpha-\overset{2}{a}_\alpha$, $\overset{1}{b}_\beta-\overset{2}{b}_\beta$ means the existence of jumps of the tangent component of the electric fields and the components of the magnetic field on the free surface S_t.

In Part 1 we consider the case with a jump of the magnetic field, so $\overset{1}{b}_\beta - \overset{2}{b}_\beta \neq 0$, while in Part 2 the case without jump is examined.

To prove the existence of solutions to problem (1.1)–(1.8) we transform it into two problems: the problem for the fluid motion and the problem for the electromagnetic field. Therefore, for given $\overset{1}{H}$ we have the problem for (v,p):

$$\begin{aligned} & v_{,t} + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) = f + \mu_1 \operatorname{div}\mathbb{T}(\overset{1}{H}) && \text{in } \overset{1}{\Omega}_t,\\ & \operatorname{div} v = 0 && \text{in } \overset{1}{\Omega}_t,\\ & \bar n\cdot\mathbb{T}(v,p) = -\mu_1\bar n\cdot\mathbb{T}(\overset{1}{H}) && \text{on } S_t,\\ & v|_{t=0} = v(0) && \text{in } \Omega_0, \end{aligned} \tag{1.9}$$

where we assumed that p_0 is absorbed by p.

Next for a given v, the electromagnetic field is determined by the problem

$$\begin{aligned} & \mu_1\overset{1}{H}_{,t} = -\operatorname{rot}\overset{1}{E},\quad \operatorname{rot}\overset{1}{H} = \sigma_1(\overset{1}{E} + \mu_1 v\times\overset{1}{H}),\quad \operatorname{div}\overset{1}{H} = 0 && \text{in } \overset{1}{\Omega}_t,\\ & \mu_2\overset{2}{H}_{,t} = -\operatorname{rot}\overset{2}{E},\quad \sigma_2\overset{2}{E} = \operatorname{rot}\overset{2}{H},\quad \operatorname{div}\overset{2}{H} = 0 && \text{in } \overset{2}{\Omega}_t,\\ & \overset{1}{H}|_{t=0} = \overset{1}{H}(0),\quad \operatorname{div}\overset{1}{H}(0) = 0 && \text{in } \overset{1}{\Omega}_0,\\ & \overset{2}{H}|_{t=0} = \overset{2}{H}(0),\quad \operatorname{div}\overset{2}{H}(0) = 0 && \text{in } \overset{2}{\Omega}_0,\\ & \overset{2}{H}\cdot\bar\tau'_\alpha|_B = H_{*\alpha},\quad \operatorname{div}\overset{2}{H}|_B = 0,\qquad \bar\tau'_\alpha \text{ is tangent to } B,\ \alpha = 1,2,\\ & \overset{1}{a}_\alpha\overset{1}{E}\cdot\bar\tau_\alpha = \overset{2}{a}_\alpha\overset{2}{E}\cdot\bar\tau_\alpha,\quad \overset{1}{b}_\beta\overset{1}{H}_\beta = \overset{2}{b}_\beta\overset{2}{H}_\beta \quad \text{on } S_t,\ \text{where } \overset{1}{b}_\beta \neq \overset{2}{b}_\beta \text{ in Part 1},\\ & \bar\tau_\alpha \text{ is tangent to } S_t,\quad \alpha = 1,2,\quad \beta = 1,2,3,\qquad \text{and } \overset{1}{b}_\beta = \overset{2}{b}_\beta \text{ in Part 2}. \end{aligned} \tag{1.10}$$

Since (1.9), (1.10) are free boundary problems, the natural way to treat them is passing to Lagrangian coordinates (see [Z1]). Then the domains $\overset{1}{\Omega}_t$, $\overset{2}{\Omega}_t$ and S_t are determined by the velocity of the fluid. Therefore the natural way to show the existence of solutions to (1.9), (1.10) is the method of successive approximations. Hence the paper is organized in the following way. In Section 2 we introduce the notation and some auxiliary results. In Section 3 of Part 1 and Section 8 of Part 2 the method of successive approximations is formulated. In Sections 4, 5 of Part 1 and Sections 9, 10 of Part 2 the existence of solutions to the linearized problems (1.9) and (1.10) is proved with a jump and without it, respectively.

In Section 6 of Part 1 the model problem describing the behavior of the magnetic field in a neighborhood of some point of S_t is considered. Moreover, in this section (see Lemma 6.4) we consider the Dirichlet–Neumann boundary conditions for the equation for the magnetic field

$$\begin{aligned} &H_t - \mu \operatorname{rot}^2 H = 0, \quad H \cdot \bar{\tau}_\alpha|_B = H_{*\alpha}, \quad \operatorname{div} H|_B = 0, \\ &\bar{\tau}_\alpha \text{ is tangent to } B, \quad \alpha = 1, 2. \end{aligned} \tag{1.11}$$

Finally, Section 7 is devoted to the proof of local existence of solutions to problem (1.1)–(1.10).

We point out that different transmission conditions $(1.10)_6$ imply the existence of different solutions.

Now, we formulate the main results of this paper.

Main Theorem 1.1. *Let* (A) *be problem* (1.9), (1.10) *with a jump of the magnetic field, and let* $r > 5/2$. *Let*

$$\begin{aligned} D(t) = \|v(0)\|_{W_r^{3-2/r}(\overset{1}{\Omega}_0)} + \sum_{i=1}^2 \|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)} + \|\bar{f}\|_{L_r(\overset{1}{\Omega}{}_0^t)} \\ + \sum_{\alpha=1}^2 \|H_{*\alpha}\|_{W_r^{3-1/r,3/2-1/(2r)}(B^t)}, \quad t \le T, \end{aligned} \tag{1.12}$$

be finite, where $\bar{f}$ *is equal to* f *expressed in Lagrangian coordinates. For* δ *sufficiently small, if* T *is so small that*

$$TD(T) \le \delta, \tag{1.13}$$

then there exists a local solution to problem (A) *such that*

$$\begin{aligned} &\bar{v} \in L_\infty(0, t; W_r^{3-2/r}(\overset{1}{\Omega}_0)) \cap W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t), \quad \overset{i}{\bar{H}} \in W_r^{2,1}(\overset{i}{\Omega}{}_0^t), \quad i = 1, 2, \\ &\bar{p} \in W_r^{2,1}(\overset{1}{\Omega}{}_0^t) \cap W_r^{2-1/r,1-1/(2r)}(S_0^T), \quad t \le T, \end{aligned}$$

where $\bar{v}$, $\overset{i}{\bar{H}}$, $\bar{p}$ *are equal to* v, $\overset{i}{H}$, p *expressed in Lagrangian coordinates, respectively. Moreover, there exists an increasing positive function* φ *such that*

$$\begin{aligned} \|\bar{v}\|_{L_\infty(0,t;W_r^{3-2/r}(\overset{1}{\Omega}_0))} + \|\bar{v}\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)} + \sum_{i=1}^2 \|\overset{i}{\bar{H}}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)} \\ + \|\bar{p}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|\bar{p}\|_{W_r^{2-1/r,1-1/(2r)}(S_0^t)} \le \varphi(TD(T))D(T), \quad t \le T. \end{aligned} \tag{1.14}$$

Main Theorem 1.2. *Let* (A) *be problem* (1.9), (1.10) *with no jump of the magnetic field, and let* $r > 3$. *Let*

$$\begin{aligned} D(t) = \|v(0)\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)} + \sum_{i=1}^2 \|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)} + \|\bar{f}\|_{L_r(\overset{1}{\Omega}{}_0^t)} \\ + \sum_{\alpha=1}^2 \|H_{*\alpha}\|_{W_r^{2-1/r,1-1/(2r)}(B^t)}, \quad t \le T, \end{aligned} \tag{1.15}$$

be finite, where $\bar{f}$ *is equal to* f *expressed in Lagrangian coordinates. For* δ *sufficiently small, if* T *is so small that*

$$TD(T) \le \delta, \tag{1.16}$$

then there exists a local solution to problem (A) *such that*

$$\bar v \in L_\infty(0,t;W_r^{2-2/r}(\overset{1}{\Omega}_0)) \cap W_r^{2,1}(\overset{1}{\Omega}{}_0^t), \quad \bar{\overset{i}{H}} \in W_r^{2,1}(\overset{i}{\Omega}{}_0^t), \quad i=1,2,$$

$$\bar p \in W_r^{1,0}(\overset{1}{\Omega}{}_0^t) \cap W_r^{1-1/r,1/2-1/(2r)}(S_0^T), \quad t \le T,$$

where $\bar v$, $\bar{\overset{i}{H}}$, $\bar p$ *are equal to* v, $\overset{i}{H}$, p *expressed in Lagrangian coordinates, respectively. Moreover, there exists an increasing positive function* φ *such that*

$$\|\bar v\|_{L_\infty(0,t;W_r^{2-2/r}(\overset{1}{\Omega}_0))} + \|\bar v\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \sum_{i=1}^2 \|\bar{\overset{i}{H}}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)} + \|\bar p\|_{W_r^{1,0}(\overset{1}{\Omega}{}_0^t)} + \|\bar p\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)} \le \varphi(TD(T))D(T), \quad t \le T.$$

The equations of magnetohydrodynamics (mhd) can be found in [C, LL]. Mhd is a coupling of the equations of fluid mechanics and the Maxwell equations under neglecting the displacement currents. The ideal mhd does not contain any dissipative effects. The qualitative properties of an ideal compressible mhd such as existence of k-simple waves were considered by Zajączkowski [Z2–Z6]. In this paper we are interested in the dissipative mhd which contains two dissipative mechanisms: viscosity of the fluid and finite electric conductivity. Thus

$$\int_V J^2/\sigma\, dx,$$

where V is the volume of the conductor and $J = \sigma(E + v\times H)$ is the current, describes the transfer of the kinetic energy into heat via Ohmic dissipation. In this paper we restrict our considerations to the incompressible case.

Magnetic fields influence many natural and man-made flows. They are routinely used in industry to heat, pump, stir and to levitate liquid metals. There is the terrestrial magnetic field which is maintained by fluid motion in the earth's core, the solar magnetic field which generates sunspots and solar flares, and the galactic magnetic field which is thought to influence the formation of stars from interstellar clouds. Mhd describes the flows. Formally, mhd is concerned with the mutual interaction of fluid flows and magnetic fields. The fluids must be electrically conducting and nonmagnetic. This limits us to liquid metals, hot ionised gases (plasma) and strong electrolytes. The most important example of mhd for us is the earth's dynamo. Here, the motion in the liquid-metal core of the earth twists, stretches and intensifies the terrestrial magnetic field, maintaining it against the natural processes of decay. It is the large length-scales that are important here.

The mhd became a fully fledged subject only in the late 1930s or early 1940s. The reason, probably, is that there was little incentive for nineteenth century engineers to capitalise on possibilities offered by mhd.

In the early 1960s the mhd began to be exploited in engineering. The impetus for change came largely as a result of three technological innovations:

(i) fast-breeder reactors use liquid sodium as a coolant and this needs to be pumped;
(ii) controlled thermonuclear fusion requires that the hot plasma be confined away from material surfaces by magnetic forces;
(iii) mhd power generation, in which ionised gas is propelled through a magnetic field, was thought to offer the prospect of improved power station efficiency.

The problem considered in this paper is mostly close to the controlled thermonuclear fusion, but it is a very primitive model. Here, the fusion develops in the domain $\overset{1}{\Omega}_t$ bounded by a free boundary S_t. Outside of S_t there exists an electromagnetic field which is a superposition of two fields:

1. the first field is generated by the motion of mhd fluid in $\overset{1}{\Omega}_t$, and it is transformed to the domain $\overset{2}{\Omega}_t$ by an appropriate transmission conditions on S_t;
2. the second field is generated by the magnetic field given on the fixed exterior boundary B.

The magnetic field on B is the restriction to B of a magnetic field generated by some electromagnets located at some distance from B. This kind of technical construction, although in a different geometrical framework, can be seen in CERN, where the motion of protons and antiprotons is confined to some torus thanks to a special magnetic field generated by strong electromagnets.

The problem considered is realized also in galaxies, where the motion of ionised gas is concentrated in some region and kept by a magnetic field. This phenomenon is close to the model considered in this paper up to the point where gravitation must be taken into account.

In this paper we prove the existence of solutions to problem (1.1)–(1.8) by the method of successive approximations. Therefore on each step of this method we need solvability of the linearized problem. We use the technique of regularizer to show existence of solutions of the linear problems. The magnetic field is described by the heat equation. In the regularizer technique we have four types of model problems for the heat equation:

1. near interior points of $\overset{1}{\Omega}_t$ and $\overset{2}{\Omega}_t$ we have Cauchy problems;
2. near a point of S_t we have a transmission problem;
3. near a point of B we have an initial boundary value problem.

In the third case we have problem (1.11).

The first results on existence of solutions to dissipative incompressible mhd were given by Ladyzhenskaya and Solonnikov [LS]. They considered different geometrical situations such as domains composed of two different materials separated by a fixed surface. On this surface they assumed continuity of the tangent components of the magnetic field, continuity of the normal component of the magnetic induction and continuity of the tangent components of the electric field. Moreover, the boundary of the domain was treated as an ideal conductor so $H_n = 0$ and $E_\tau = 0$. They proved the existence of weak solutions. We have to emphasize that Ladyzhenskaya ans Solonnikov [LS] considered the case that $H = \text{const}$, $E = 0$ at infinity. Padula and Solonnikov [PS] proved the existence of local regular solutions to problem (1.1)–(1.8), where $\overset{2}{H}$ is a solution to the system

$$\operatorname{rot}\overset{2}{H} = 0, \quad \operatorname{div}\overset{2}{H} = 0, \tag{1.17}$$

where on S_t the continuity of the tangent components of the magnetic field and the normal component of the induction was assumed. Additionally, S_t was governed by surface tension. Moreover $\overset{2}{H} \cdot \bar{n}|_B = 0$ was imposed.

The first result on solvability of mhd equations appeared in [LS]. Later free boundary problems to incompressible viscous mhd with resistivity were considered in [PS, SS]. The existence of solutions to the problem presented in this paper in the L_2-approach is proved in [K1, K2], where [K1] is devoted to the local existence and [K2] to the global one. In [K1,K2] the Faedo–Galerkin method is used to prove the existence of solutions. In [KZ] the existence of the fundamental basis for problem (1.10) is proved so the technique used in [K1, K2] is justified. The first results on the existence of solutions to problem (1.1)–(1.10) were given in [K3–K5]. In [S2, Z1] the existence of local and global solutions to the free boundary problem for the Navier–Stokes are proved. In this paper we base heavily on methods and techniques from [S2, Z1].

Free boundary problems for mhd equations were also considered in [FS, F]. In [FS, F] the external magnetic field satisfies the elliptic system (1.17). However in those papers the boundary condition $(1.4)_1$ contains the surface tension.

In [FSh], using the Weis theory of Fourier multipliers, two different mhd fluids interacting through a free surface are considered.

In [Le1] the local well-posedness of fluid-vacuum free boundary mhd with both kinematic viscosity and magnetic diffusivity under the gravity force is considered.

Remark A. In Part 1 we need regularity such that $\bar{v} \in W^{3,3/2}_{r_1}(\overset{1}{\Omega}{}^t_0)$, $r_1 > 5/2$, but in Part 2 the necessary regularity is such that $\bar{v} \in W^{2,1}_{r_2}(\overset{1}{\Omega}{}^t_0)$, $r_2 > 3$. Hence the proofs of local existence heavily depend on the assumed transmission conditions.

We recall that mhd is a system describing the coupling of fluid mechanics equations describing motions of fluids with the modified Maxwell equations describing electromagnetic fields. The modified Maxwell equations do not contain the current displacement.

Moreover, the coupling is such that the Lorentz force $F = j \times B$, where j is the current and B is the magnetic induction, appears in the momentum equation but the electromagnetic equations features the current described by the Ohm law

$$j = \sigma(E + v \times B),$$

where σ is the electric conductivity.

In the case of incompressible viscous fluids the equations of fluid mechanics are exactly the Navier–Stokes equations.

The physical background of mhd needs additional restrictions on the Maxwell equations:

$$\begin{aligned}
&-D_t + \frac{1}{\mu}\operatorname{rot} B = j,\\
&B_t + \operatorname{rot} D = 0,\\
&\operatorname{div} D = \varrho_e,\\
&\operatorname{div} B = 0,
\end{aligned} \tag{1.18}$$

where D is the electric induction, B is the magnetic induction, ϱ_e is the density of charges, j is the current. We assume that $D = \varepsilon E$, $B = \mu H$, where ε is the permittivity of the free space and μ is the magnetic permeability (permeability of the free space).

Applying the divergence operator to $(1.18)_1$ we derive the charge conservation equation

$$\partial_t \varrho_e + \operatorname{div} j = 0, \tag{1.19}$$

where the Gauss law $(1.18)_3$ was used.

Hence (1.19) would be violated without the displacement current D_t. However, the Maxwell correction is not needed in mhd. This means that $\frac{\partial \varrho_e}{\partial t}$ is negligible in conductors, and so we might anticipate that the contribution of $\varepsilon \frac{\partial E}{\partial t}$ in

$$\operatorname{rot} B = \mu\left[j + \varepsilon \frac{\partial E}{\partial t}\right] \tag{1.20}$$

is also small in mhd. This is readily confirmed:

$$\varepsilon \frac{\partial E}{\partial t} \sim \frac{\varepsilon}{\sigma}\frac{\partial j}{\partial t} \sim \tau_e \frac{\partial j}{\partial t} \ll j.$$

The quantity τ_e is called the charge relaxation time, and for a typical conductor has the value of around 10^{-18} sec. Then (1.20) has the form of the Ampère law

$$\operatorname{rot} B = \mu j. \tag{1.21}$$

Next (1.21) implies

$$\operatorname{div} j = 0. \tag{1.22}$$

Moreover, in mhd the charge density ϱ_e plays no significant part because the electric force qE, i.e. the Coulomb force where q is the charge, is minor in comparison with the Lorentz force

$$F_L = j \times B \tag{1.23}$$

and the contribution of $\frac{\partial \varrho_e}{\partial t}$ to the charge conservation equation is also negligible. Apparently ϱ_e is negligible and it can be dropped as the Gauss law does not influence the problem and can be ignored.

Therefore, we have

$$\begin{aligned}
&\nabla \cdot B = 0 && \text{(solenoidal nature of } B\text{)},\\
&\nabla \times E = -B_t && \text{(the Faraday law)},\\
&\nabla \times B = \mu j && \text{(the Ampère equation)},\\
&\operatorname{div} j = 0 && \text{(the charge conservation)},\\
&j = \sigma(E + v \times B) && \text{(the Ohm law)}.
\end{aligned} \tag{1.24}$$

Deriving the equations for the magnetic field

$$\operatorname{div} H = 0, \quad H_t = \operatorname{rot}(v \times H) + \frac{\mu}{\sigma}\Delta H \tag{1.25}$$

we need that the period of changes of the magnetic field is large in comparison with the so called "mean free path of electrons". Then the relation between a current and an electric field is determined by the same constant of conductivity σ as in the case of a constant field. Hence σ is a constant in the whole domain and does not depend on the magnetic field.

2. Notation and auxiliary results

First we introduce the notation employed in this paper. To prove the existence of solutions we need a partition of unity. We consider two collections of open subsets $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$, $k \in \mathfrak{M} \cup \mathfrak{N}$, such that $\bar{\omega}^{(k)} \subset \Omega^{(k)} \subset \Omega_0 = \overset{1}{\Omega}_0 \cup \overset{2}{\Omega}_0$, $\bigcup_k \omega^{(k)} = \bigcup_k \Omega^{(k)} = \Omega_0$, $\bar{\Omega}^{(k)} \cap S_0 = \emptyset$ for $k \in \mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2$, $\bar{\Omega}^{(k)} \cap S_0 \neq \emptyset$ for $k \in \mathfrak{N}_1$ and $\bar{\Omega}^{(k)} \cap B \neq \emptyset$ for $k \in \mathfrak{N}_2$, $\mathfrak{N} = \mathfrak{N}_1 \cup \mathfrak{N}_2$. Moreover, subdomains with $k \in \mathfrak{M}_i$ are contained in $\overset{i}{\Omega}_0$, $i = 1, 2$. We assume that at most N_0 of the $\Omega^{(k)}$ have nonempty intersections, and $\sup_k \operatorname{diam} \Omega^{(k)} \le 2\lambda$, $\sup_k \operatorname{diam} \omega^{(k)} \le \lambda$ for some $\lambda > 0$. Let $\zeta^{(k)}(x)$ be a smooth function such that $0 \le \zeta^{(k)}(x) \le 1$, $\zeta^{(k)}(x) = 1$ for $x \in \omega^{(k)}$, $\zeta^{(k)}(x) = 0$ for $x \in \Omega_0 \setminus \Omega^{(k)}$ and $|D_x^\nu \zeta^{(k)}(x)| \le c/\lambda^{|\nu|}$. Then $1 \le \sum_k (\zeta^{(k)}(x))^2 \le N_0$. Introducing the function

$$\eta^{(k)}(x) = \frac{\zeta^{(k)}(x)}{\sum_l (\zeta^{(l)}(x))^2}$$

we have $\eta^{(k)}(x) = 0$ for $x \in \Omega_0 \setminus \Omega^{(k)}$, $\sum_k \eta^{(k)}(x)\zeta^{(k)}(x) = 1$ and $|D_x^\nu \eta^{(k)}(x)| \le c/\lambda^{|\nu|}$. We denote by $\xi^{(k)}$ an interior point of $\omega^{(k)}$ and $\Omega^{(k)}$ for $k \in \mathfrak{M}$ and an interior point of $\bar{\omega}^{(k)} \cap S_0$ and of $\bar{\Omega}^{(k)} \cap S_0$ for $k \in \mathfrak{N}_1$ and an interior point of $\bar{\omega}^{(k)} \cap B$ and of $\bar{\Omega}^{(k)} \cap B$ for $k \in \mathfrak{N}_2$. For $k \in \mathfrak{M}_i$, $\xi^{(k)} \in \overset{i}{\Omega}_0$, $i = 1, 2$. Let $x = (x_1, x_2, x_3)$ be the Cartesian system of coordinates with the origin located in the interior of Ω_0. Then by translations and rotations we introduce a local coordinate system $y = (y_1, y_2, y_3)$ with the origin at $\xi^{(k)} \in \Omega^{(k)} \cap S_0$, $k \in \mathfrak{N}_1$, such that the part $\tilde{S}_0^{(k)} = S_0 \cap \bar{\Omega}^{(k)}$ of the boundary S_0 is described by $y_3 = F_k(y_1, y_2)$. We denote the transformation as $y = Y_k(x)$. Then we introduce new coordinates defined by

$$z_i = y_i, \quad i = 1, 2, \quad z_3 = y_3 - F_k(y_1, y_2), \quad k \in \mathfrak{N}_1.$$

We will denote this transformation by $\hat{\Omega}^{(k)} \ni \hat{\omega}^{(k)} \in z = \Phi_k(y)$ where $y \in \omega^{(k)} \subset \Omega^{(k)}$. We assume that the sets $\hat{\omega}^{(k)}$, $\hat{\Omega}^{(k)}$ are described in local coordinates at $\xi^{(k)}$ by the inequalities

$$\begin{aligned} &|y_i| < \lambda, \qquad i = 1, 2, \quad |y_3 - F_k(y_1, y_2)| < \lambda, \\ &|y_i| < 2\lambda, \qquad i = 1, 2, \quad |y_3 - F_k(y_1, y_2)| < 2\lambda, \end{aligned}$$

respectively. Moreover, $(y_1,y_2,y_3)\in\overset{1}{\Omega}_0$ if $y_3>F_k(y_1,y_2)$ and $(y_1,y_2,y_3)\in\overset{2}{\Omega}_0$ for $y_3<F_k(y_1,y_2)$. Let $\Psi_k=\Phi_k\circ Y_k$. Then $z=\Psi_k(x)$ and

$$\hat u^{(k)}(z,t)=u(\Psi_k^{-1}(z),t),\quad \tilde u^{(k)}(z,t)=\hat u^{(k)}(z,t)\hat\zeta^{(k)}(z).$$

For $k\in\mathfrak{M}$ we have

$$\tilde u^{(k)}(x,t)=u^{(k)}(x,t)\zeta^{(k)}(x).$$

For $k\in\mathfrak{N}_2$ we introduce new local coordinates with origin at $\xi^{(k)}\in B\cap\bar\Omega^{(k)}$ such that $y_3=F_k(y_1,y_2)$ describes locally $B\cap\bar\Omega^{(k)}$. We also introduce the transformation $z_i=y_i$, $i=1,2$, $z_3=y_3-F_k(y_1,y_2)$ and assume that $z=\Phi_k(y)$ belongs to $\hat\omega^{(k)}$, $\hat\Omega^{(k)}$ for $y\in\omega^{(k)}\subset\Omega^{(k)}$. Finally $\hat\omega^{(k)}$, $\hat\Omega^{(k)}$ are described by the inequalities

$$\begin{aligned}&|y_i|<\lambda,\quad i=1,2,\qquad 0<y_3-F_k(y_1,y_2)<\lambda,\\ &|y_i|<2\lambda,\quad i=1,2,\qquad 0<y_3-F_k(y_1,y_2)<2\lambda,\ \text{respectively.}\end{aligned}$$

We do not distinguish between norms of scalar and vector-valued functions. Let ω be a vector, $\omega=(\omega_1,\dots,\omega_n)$. Then

$$|\omega|=\Big(\sum_{i=1}^n|\omega_i|^2\Big)^{1/2}.$$

Let $L_p(\Omega)=\{u:\ \int_\Omega|u|^p\,dx<\infty\}$, $p\in[1,\infty]$.

We denote by $V_2^0(\Omega^T)$ the space of functions with norm

$$\|u\|_{V_2^0(\Omega^T)}=\|u\|_{L_\infty(0,T;L_2(\Omega))}+\|u\|_{L_2(0,T;H^1(\Omega))}.$$

For $l\in\mathbb{N}$, we shall use the notation $H^l(\Omega)=\{u:\sum_{|\alpha|\le l}\|D_x^\alpha u\|_{L_2(\Omega)}<\infty\}$.

We introduce the anisotropic Sobolev spaces $W_p^{3,3/2}(\Omega^T)$ with norm

$$\|u\|_{W_p^{3,3/2}(\Omega^T)}=\Big(\int\limits_{\Omega^T}(|D_x^3u|^p+|u|^p+|\partial_t^{3/2}u|^p)\,dx\,dt\Big)^{1/p},$$

where $D_x^lu=\sum_{|\alpha|=l}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}u$, $\alpha_1+\alpha_2+\alpha_3=l$, $l,\alpha_i\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $i=1,2,3$, $\Omega\subset\mathbb{R}^3$ and $\partial_t^{3/2}u$ is the fractional derivative.

We denote by c a generic constant which changes its value from formula to formula. Similarly we denote by φ a generic function which is always positive and increasing. By λ_i, $i\in\mathbb{N}$, we denote parameters that appear in imbeddings, interpolations and so on.

To examine free boundary problems in hydrodynamics we use Lagrangian coordinates which are the initial data to the Cauchy problem

$$\frac{dx}{dt}=v(x,t),\quad x|_{t=0}=\xi\in\overset{1}{\Omega}_0. \tag{2.1}$$

Therefore,

$$x=x_v(\xi,t)\equiv\xi+\int\limits_0^t\bar v(\xi,s)\,ds, \tag{2.2}$$

where $\bar v(\xi,t)=v(x_v(\xi,t),t)$. To define Lagrangian coordinates in $\overset{2}{\Omega}_t$ we need

Lemma 2.1 (see [S1]). *Let $X(\overset{1}{\Omega}_t)$ be some Sobolev space. Let $v \in X(\overset{1}{\Omega}_t)$ be a divergence free vector function. Then there exists an extension v' of v on $\overset{1}{\Omega}_t \cup \overset{2}{\Omega}_t$ such that v' is divergence free, $v'|_{\overset{1}{\Omega}_t} = v$ and there exists a constant c, independent of v, such that*

$$\|v'\|_{X(\overset{1}{\Omega}_t \cup \overset{2}{\Omega}_t)} \le c\|v\|_{X(\overset{1}{\Omega}_t)}. \tag{2.3}$$

In Definition 3.2 below the extension is defined more precisely by using Lemma 2.4 applied to problem (3.2). In view of the definition of Lagrangian coordinates we have

$$\begin{aligned}
&\overset{1}{\Omega}_t = \{x \in \mathbb{R}^3 : x = x_v(\xi,t),\ \xi \in \overset{1}{\Omega}_0\},\\
&S_t = \{x \in \mathbb{R}^3 : x = x_v(\xi,t),\ \xi \in S_0\},\\
&\overset{1}{\Omega}_t \cup \overset{2}{\Omega}_t = \{x \in \mathbb{R}^3 : x = x_{v'}(\xi,t),\ \xi \in \overset{1}{\Omega}_0 \cup \overset{2}{\Omega}_0\}.
\end{aligned}$$

To formulate our problem in Lagrangian coordinates we need the notation

$$\begin{aligned}
&\nabla_{\bar v} = \frac{\partial \xi_k}{\partial x}\frac{\partial}{\partial \xi_k}, && \mathbb{D}_{\bar v}\bar u = \nabla_{\bar v}\bar u + (\nabla_{\bar v}\bar u)^T,\\
&\mathbb{T}_{\bar v}(\bar u,\bar p) = \mathbb{D}_{\bar v}(\bar u) - \bar p\mathbb{I}, && \operatorname{div}_{\bar v}\bar v = \partial_{x_i}\xi_k\partial_{\xi_k}\bar v_i = \nabla_{\bar v}\cdot\bar v,
\end{aligned} \tag{2.4}$$

where the summation over repeated indices is assumed, and $\xi = \xi(x,t)$ is the inverse transformation to $x = x_{\bar v}(\xi,t)$. From [Z1], [S2] we have

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^3$ be a given bounded domain. Let $v \in L_2(\Omega)$ be such that*

$$E_\Omega(v) = \int_\Omega (v_{j,x_i} + v_{i,x_j})^2\,dx < \infty. \tag{2.5}$$

Then there exists a constant c, independent of v, such that

$$\|v\|^2_{H^1(\Omega)} \le c(E_\Omega(v) + \|v\|^2_{L_2(\Omega)}).$$

Consider the problem

$$\begin{aligned}
&\overset{i}{u}_t + \operatorname{rot}^2\overset{i}{u} = \overset{i}{f} && \text{in } \overset{i}{\Omega}_0, \quad i=1,2,\\
&\operatorname{div}\overset{i}{u} = 0 && \text{in } \overset{i}{\Omega}_0, \quad i=1,2,\\
&\overset{1}{u} = \overset{2}{u}, \quad \operatorname{rot}\overset{1}{u}\cdot\bar\tau_\alpha = \operatorname{rot}\overset{2}{u}\cdot\tau_\alpha && \text{on } S_0,\\
&\overset{2}{u}\cdot\bar\tau_\alpha|_B = 0, \quad \operatorname{div}\overset{2}{u} = 0 && \text{on } B,\\
&\overset{i}{u}|_{t=0} = u_0 && \text{in } \overset{i}{\Omega}_0, \quad i=1,2,
\end{aligned} \tag{2.6}$$

where $\operatorname{div}\overset{i}{f} = 0$, $i=1,2$.

Our aim is to find some inequality for weak solutions to (2.6).

Lemma 2.3. *Assume that $\overset{1}{u}$, $\overset{2}{u}$ is a solution to (2.6). Let $\overset{i}{f} \in L_1(0,t;L_2(\overset{i}{\Omega}_0))$, $\overset{i}{u}_0 \in L_2(\overset{i}{\Omega}_0)$, $i=1,2$. Then there exists a function c depending at most on B such that*

$$\sum_{i=1}^2 \|\nabla\overset{i}{u}\|^2_{L_2(\overset{i}{\Omega}_0)} \le c\sum_{i=1}^2 \big[\|\operatorname{rot}\overset{i}{u}\|^2_{L_2(\overset{i}{\Omega}_0)} + \|\overset{i}{f}\|^2_{L_1(0,t;L_2(\overset{i}{\Omega}_0))} + \|\overset{i}{u}_0\|^2_{L_2(\overset{i}{\Omega}_0)}\big]. \tag{2.7}$$

Proof. From (2.6) we have the identities

$$\overset{i}{u}_t + \operatorname{rot}^2 \overset{i}{u} - \overset{i}{f} = \overset{i}{u}_t - \Delta \overset{i}{u} - \overset{i}{f}, \quad i = 1, 2.$$

We multiply the equation by $\overset{i}{u}$ and integrate over $\overset{i}{\Omega}_0$, $i = 1, 2$. Then we derive

$$\int_{\overset{i}{\Omega}_0} \operatorname{rot}^2 \overset{i}{u} \cdot \overset{i}{u}\, dx = - \int_{\overset{i}{\Omega}_0} \Delta \overset{i}{u} \cdot \overset{i}{u}\, dx, \quad i = 1, 2.$$

Integrating by parts, subtracting and using $\overset{1}{n} = -\overset{2}{n} \equiv \bar{n}$ we obtain

$$\begin{aligned} L \equiv \sum_{i=1}^{2} \int_{\overset{i}{\Omega}_0} |\operatorname{rot} \overset{i}{u}|^2 dx &+ \int_{S_0} (\operatorname{rot} \overset{1}{u} \cdot \overset{1}{u} \times \bar{n} - \operatorname{rot} \overset{2}{u} \cdot \overset{2}{u} \times \bar{n})\, dS_0 \\ &+ \int_{B} \operatorname{rot} \overset{2}{u} \cdot \overset{2}{u} \times \bar{n}\, dB - \Big[\sum_{i=1}^{2} \int_{\overset{i}{\Omega}_0} |\nabla \overset{i}{u}|^2\, dx \\ &- \int_{S_0} (\bar{n} \cdot \nabla \overset{1}{u} \cdot \overset{1}{u} - \bar{n} \cdot \nabla \overset{2}{u} \cdot \overset{2}{u})\, dS_0 - \int_{B} \bar{n} \cdot \nabla \overset{2}{u} \cdot \overset{2}{u}\, dB \Big] = 0. \end{aligned}$$

To simplify the last but one term in L we introduce in a neighborhood of S_0 the system of orthonormal vectors $\bar{\tau}_1$, $\bar{\tau}_2$, $\bar{n}$ such that $\bar{n}|_{S_0}$ is normal to S_0 and $\bar{\tau}_\alpha|_{S_0}$, $\alpha = 1, 2$, is tangent. Next, we introduce a system of curvilinear coordinates τ_1, τ_2, n such that $x = x(\tau_1, \tau_2, n)$, where $x = (x_1, x_2, x_3)$, is a diffeomorphism in some neighborhood of S_0. Hence, locally, $x = x(\tau_1, \tau_2)$ and $n(x_1, x_2, x_3) = 0$ describe S_0. Therefore, we have the relations $x_{,\tau_\alpha} = H_\alpha \bar{\tau}_\alpha$, $\alpha = 1, 2$, $x_{,n} = H_n \bar{n}$, where H_1, H_2, H_n are the Lamé coefficients. Then, in the above mentioned neighborhood, any vector u can be expressed in the form

$$u = u_{\tau_\alpha} \bar{\tau}_\alpha + u_n \bar{n},$$

where the summation convention over repeated Greek letters is assumed.

To use the second condition of $(2.6)_3$ we express the rotation in the curvilinear coordinates,

$$\begin{aligned} (\operatorname{rot} u)_{\tau_1} &= \frac{1}{H_2 H_n} (\partial_{\tau_2}(u_n H_n) - \partial_n (u_{\tau_2} H_2)), \\ (\operatorname{rot} u)_{\tau_2} &= \frac{1}{H_1 H_n} (\partial_n (u_{\tau_1} H_1) - \partial_{\tau_1}(u_n H_n)), \\ (\operatorname{rot} u)_n &= \frac{1}{H_1 H_2} (\partial_{\tau_1}(u_{\tau_2} H_2) - \partial_{\tau_2}(u_{\tau_1} H_1)). \end{aligned}$$

Then $(2.6)_3$ implies that

$$\frac{\partial \overset{1}{u}_{\tau_\alpha}}{\partial_n} = \frac{\partial \overset{2}{u}_{\tau_\alpha}}{\partial_n}, \quad \alpha = 1, 2, \quad \text{on } S_0.$$

Hence, the last but one term in L equals

$$- \int_{S_0} (\bar{n} \cdot \nabla \overset{1}{u} \cdot \bar{n} u_n - \bar{n} \cdot \nabla \overset{2}{u} \cdot \bar{n} u_n)\, dS_0 \equiv I_1.$$

Using the above continuity results we write I_1 in the form

$$I_1 = -\int\limits_{S_0} (\bar{n}\cdot\nabla\overset{1}{u}_n - \bar{n}\cdot\nabla\overset{2}{u}_n)u_n\,dS_0 + \int\limits_{S_0} \bar{n}\cdot\nabla\bar{n}\cdot(\overset{1}{u}-\overset{2}{u})u_n\,dS_0.$$

In view of $(2.6)_3$ the second integral in I_1 vanishes.

Consider $(2.6)_2$. Expressing it in the curvilinear coordinates yields

$$0 = \operatorname{div}\overset{i}{u} = \overset{i}{u}_{\tau_\alpha,\tau_\alpha} + \overset{i}{u}_{\tau_\alpha}\operatorname{div}\bar{\tau}_\alpha + \overset{i}{u}_n\operatorname{div}\bar{n} + \overset{i}{u}_{n,n}.$$

In view of $(2.6)_3$ we see that

$$(\overset{1}{u}_{n,n} - \overset{2}{u}_{n,n})|_{S_0} = 0.$$

Hence the first integral in I_1 vanishes. Finally, $I_1 = 0$.

The third integral of L equals

$$\int\limits_B \operatorname{rot}\overset{2}{u}\,\overset{2}{u}_{\tau_\alpha}\bar{\tau}_\alpha\times\bar{n}dB = 0$$

and the last integral on the r.h.s. takes the form

$$\int\limits_B \bar{n}\cdot\nabla\overset{2}{u}\cdot\bar{n}\overset{2}{u}_n\,dB = \int\limits_B(\bar{n}\cdot\nabla\overset{2}{u}_n\overset{2}{u}_n - \bar{n}\cdot\nabla\bar{n}\cdot\overset{2}{u}\,\overset{2}{u}_n)\,dB \equiv I_2.$$

Since $u_{\tau_\alpha}|_B = 0$, $\alpha = 1,2$, the equation of continuity projected on B has the form

$$\overset{2}{u}_{n,n} = -\operatorname{div}\bar{n}\,\overset{2}{u}_n.$$

Then

$$I_2 = -\int\limits_B(\operatorname{div}\bar{n}\,\overset{2}{u}{}^2_n + \bar{n}\cdot\nabla\bar{n}\cdot\bar{n}\overset{2}{u}{}^2_n)\,dB = -\int\limits_B \operatorname{div}\bar{n}\,\overset{2}{u}{}^2_n\,dB.$$

In view of the above considerations we derive from L the inequality

$$I_3\colon\quad \sum_{i=1}^2\int\limits_{\overset{i}{\Omega}_0}|\nabla\overset{i}{u}|^2\,dx \le \sum_{i=1}^2\int\limits_{\overset{i}{\Omega}_0}|\operatorname{rot}\overset{i}{u}|^2\,dx + c|\overset{2}{u}_n|^2_{2,B}.$$

By interpolation I_3 takes the form

$$I_3\colon\quad \sum_{i=1}^2\int\limits_{\overset{i}{\Omega}_0}|\nabla\overset{i}{u}|^2\,dx \le \sum_{i=1}^2\int\limits_{\overset{i}{\Omega}_0}|\operatorname{rot}\overset{i}{u}|^2\,dx + c|\overset{2}{u}|^2_{L_2(\overset{2}{\Omega}_0)}.$$

Multiplying $(2.6)_1$ by $\overset{i}{u}$, integrating over $\overset{i}{\Omega}_0$, summing with respect to i, and using the transmission conditions yields

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^2\int\limits_{\overset{i}{\Omega}_0}|\overset{i}{u}|^2\,dx = \sum_{i=1}^2\int\limits_{\overset{i}{\Omega}_0}\overset{i}{f}\overset{i}{u}\,dx \le \Big(\sum_{i=1}^2\int\limits_{\overset{i}{\Omega}_0}|\overset{i}{f}|^2\,dx\Big)^{1/2}\Big(\sum_{i=1}^2\int\limits_{\overset{i}{\Omega}_0}|\overset{i}{u}|^2\,dx\Big)^{1/2}.$$

We integrate with respect to time and transform this to get

$$\Big(\sum_{i=1}^2\int\limits_{\overset{i}{\Omega}_0}|\overset{i}{u}|^2\,dx\Big)^{1/2} \le \int\limits_0^t\Big(\sum_{i=1}^2\int\limits_{\overset{i}{\Omega}_0}|\overset{i}{f}|^2\,dx\Big)^{1/2}dt' + \Big(\sum_{i=1}^2\int\limits_{\overset{i}{\Omega}_0}|\overset{i}{u}_0|^2\,dx\Big)^{1/2}.$$

Simplifying further, we have

$$\sum_{i=1}^{2}\|\overset{i}{u}\|^2_{L_2(\overset{i}{\Omega}_0)} \le \sum_{i=1}^{2}\|\overset{i}{f}\|_{L_1(0,t;L_2(\overset{i}{\Omega}_0))} + \sum_{i=1}^{2}\|\overset{i}{u}_0\|_{L_2(\overset{i}{\Omega}_0)}.$$

Using the estimate in I_3 we get (2.7). This concludes the proof. ■

A similar result is proved in [KZ].

Let us consider the Stokes problem in a bounded domain $\Omega\subset\mathbb{R}^3$ with boundary S,

$$\begin{aligned} &\omega_{,t}-\operatorname{div}\mathbb{T}(\omega,q)=f && \text{in } \Omega^T=\Omega\times(0,T),\\ &\operatorname{div}\omega=0 && \text{in } \Omega^T,\\ &\omega=b && \text{on } S^T=S\times(0,T),\\ &\omega|_{t=0}=\omega_0 && \text{in } \Omega. \end{aligned} \tag{2.8}$$

LEMMA 2.4 (see [S1]). *Let $p\in(1,\infty)$.*

(a) *Assume $f\in W_p^{1,1/2}(\Omega^T)$, $b\in W_p^{3-1/p,3/2-1/(2p)}(S^T)$, $\omega_0\in W_p^{3-2/p}(\Omega)$. Then there exists a solution to problem (2.8) such that $\omega\in W_p^{3,3/2}(\Omega^T)$, $q\in W_p^{2,1}(\Omega^T)$ and there exists a function $c(T,S)$ such that*

$$\begin{aligned} &\|\omega\|_{W_p^{3,3/2}(\Omega^T)}+\|q\|_{W_p^{2,1}(\Omega^T)}\\ &\quad\le c(T,S)(\|f\|_{W_p^{1,1/2}(\Omega^T)}+\|b\|_{W_p^{3-1/p,3/2-1/2p}(S^T)}+\|\omega_0\|_{W_p^{3-2/p}(\Omega)}). \end{aligned} \tag{2.9}$$

(b) *Assume $f\in L_p(\Omega^T)$, $b\in W_p^{2-1/p,1-1/(2p)}(S^T)$, $\omega_0\in W_p^{2-2/p}(\Omega)$. Then there exists a solution to problem (2.8) such that $\omega\in W_p^{2,1}(\Omega^T)$, $q\in W_p^{1,0}(\Omega^T)$ and there exists a function $c(T,S)$ such that*

$$\begin{aligned} &\|\omega\|_{W_p^{2,1}(\Omega^T)}+\|q\|_{W_p^{1,0}(\Omega^T)}\\ &\quad\le c(T,S)(\|f\|_{L_p(\Omega^T)}+\|b\|_{W_p^{2-1/p,1-1/2p}(S^T)}+\|\omega_0\|_{W_p^{2-2/p}(\Omega)}). \end{aligned}$$

From [BIN, Ch. 3, Sect. 15] we recall the following multiplicative inequality.

LEMMA 2.5. *Let $\Omega\subset\mathbb{R}^n$. Assume that $u\in W_{p_2}^l(\Omega)\cap L_{p_1}(\Omega)$ with $p_1,p_2\in[1,\infty]$. Then*

$$\sum_{|\alpha|=r}\|D^\alpha u\|_{L_p(\Omega)}\le c\|u\|_{L_{p_1}(\Omega)}^{1-\theta}\|u\|_{W_{p_2}^l(\Omega)}^{\theta} \tag{2.10}$$

for $0\le r<l$ and θ satisfying

$$\frac{n}{p}-r=(1-\theta)\frac{n}{p_1}+\theta\left(\frac{n}{p_2}-l\right),\quad \frac{r}{l}\le\theta\le1. \tag{2.11}$$

Consider the problem

$$\begin{aligned} &v_t-\nu\nabla^2v+\nabla p=f,\quad \operatorname{div}v=\varrho\quad \text{in } \Omega^T,\\ &\mathbb{T}n|_S=d,\quad v|_{t=0}=v_0. \end{aligned} \tag{2.12}$$

LEMMA 2.6. *Let $r\in(1,\infty)$.*

(a) *Let $S\in W_r^{3-1/r}$, $f\in W_r^{1,1/2}(\Omega^T)$, $\varrho\in W_r^{2,1}(\Omega^T)$, $\varrho=\nabla\cdot R$, $R,R_t\in W_r^{1,0}(\Omega^T)$, $d\in W_r^{2-1/r,1-1/(2r)}(S^T)$, $v_0\in W_r^{3-2/r}(\Omega)$. Let $\varrho(x,0)=0$. Assume the compatibility* *conditions*

$$\operatorname{div} v_0 = 0, \quad \mathbb{D}(v_0)\cdot\bar n_0 - \bar n_0(\bar n_0\cdot D(v_0)\bar n_0)|_S = d|_{t=0}, \tag{2.13}$$

where $\bar n_0$ *is the unit outward vector normal to* S. *Then there exists a unique solution to problem* (2.12) *such that* $v\in W_r^{3,3/2}(\Omega^T)$, $p\in W_r^{2,1}(\Omega^T)\cap W_r^{2-1/r,1-1/(2r)}(S^T)$ *and*

$$\begin{aligned}\|v\|_{W_r^{3,3/2}(\Omega^T)} + \sup_{t\le T}\|v\|_{W_r^{3-2/r}(\Omega)} + \|p\|_{W_r^{2,1}(\Omega^T)} + \|p\|_{W_r^{2-1/r,1-1/(2r)}(S^T)}\\ \le c(T)(\|f\|_{W_r^{1,1/2}(\Omega^T)} + \|\varrho\|_{W_r^{2,1}(\Omega^T)} + \|R_t\|_{W_r^{1,0}(\Omega^T)}\\ + \|v_0\|_{W_r^{3-2/r}(\Omega)} + \|d\|_{W_r^{2-1/r,1-1/(2r)}(S^T)}),\end{aligned} \tag{2.14}$$

where $c(T)$ *is a nondecreasing function of* T.

(b) *Let* $S\in W_r^{2-1/r}$, $f\in L_r(\Omega^T)$, $\varrho\in W_r^{1,0}(\Omega^T)$, $\varrho=\nabla\cdot R$, $R,R_t\in L_r(\Omega^T)$, $d\in W_r^{1-1/r,1/2-1/(2r)}(S^T)$, $v_0\in W_r^{2-2/r}(\Omega)$. *Let* $\varrho(x,0)=0$. *Assume the compatibility conditions*

$$\operatorname{div} v_0 = 0, \quad \mathbb{D}(v_0)\cdot\bar n_0 - \bar n_0(\bar n_0\cdot D(v_0)\bar n_0)|_S = d|_{t=0},$$

where $\bar n_0$ *is the unit outward vector normal to* S. *Then there exists a unique solution to problem* (2.12) *such that* $v\in W_r^{2,1}(\Omega^T)$, $p\in W_r^{1,0}(\Omega^T)\cap W_r^{1-1/r,1/2-1/(2r)}(S^T)$ *and*

$$\begin{aligned}\|v\|_{W_r^{2,1}(\Omega^T)} + \sup_{t\le T}\|v\|_{W_r^{2-2/r}(\Omega)} + \|p\|_{W_r^{1,0}(\Omega^T)} + \|p\|_{W_r^{1-1/r,1/2-1/(2r)}(S^T)}\\ \le c(T)(\|f\|_{L_r(\Omega^T)} + \|\varrho\|_{W_r^{1,0}(\Omega^T)} + \|R_t\|_{L_r(\Omega^T)}\\ + \|v_0\|_{W_r^{2-2/r}(\Omega)} + \|d\|_{W_r^{1-1/r,1/2-1/(2r)}(S^T)}),\end{aligned}$$

where $c(T)$ *is a nondecreasing function of* T.

The above lemma is formulated in [S2] and it follows from proofs in [S4, S5].

REMARK 2.7. We recall some properties of the transformation between Eulerian and Lagrangian coordinates. Let $u=u(\xi,t)$ be given. Then the transformation is described by the relation

$$x = \xi + \int_0^t u(\xi,t')\,dt' \equiv x_u(\xi,t). \tag{2.15}$$

The Jacobian of this transformation is the matrix

$$A = \{x_{i,\xi_j}\} = \{a_{ij}\} = \Big\{\delta_{ij} + \int_0^t u_{i,\xi_j}(\xi,t')\,dt'\Big\}.$$

We have $A^{-1}=\{\xi_{j,x_i}\}=\{a^{ji}\}$, $\det A=1$ and $\mathcal{A}=(A^T)^{-1}$ is the matrix of cofactors. Denoting $\mathcal{A}=\{A_{ij}\}$ we have $a^{mj}=A_{jm}$. Since incompressible motions are considered, we have $\sum_k A_{ik,\xi_k}(\xi,t)=0$ and $\nabla_u=\mathcal{A}\cdot\nabla_\xi=\nabla_\xi\cdot\mathcal{A}^T$. Assume that S_t is given, at least locally, by the equation $F(x)=0$ and S_0 by $F(\xi)=0$. Then the normal vectors to S_t and S_0 are given, respectively, by

$$\bar n_t = \frac{\nabla_x F(x)}{|\nabla_x F(x)|}, \quad \bar n_0 = \frac{\nabla_\xi F(\xi)}{|\nabla_\xi F(\xi)|}.$$

Then

$$\bar{n}_u = \frac{\nabla_x F(x_u(\xi,t))}{|\nabla_x F(x_u(\xi,t))|}.$$

Since v is divergence free, we have the estimates

$$\|\nabla_\xi \xi_{,x}\|_{L_r(\Omega)} \le 2\int_0^t \|\bar{v}_{,\xi\xi}\|_{L_r(\Omega)}\,dt' \Big(1 + 2\int_0^t \|\bar{v}_{,\xi}\|_{L_\infty(\Omega)}\,dt'\Big),$$

$$|\mathbb{I} - \xi_x| \le 2\int_0^t |\bar{v}_{,\xi}|\,dt' \Big(1 + \int_0^t |\bar{v}_{,\xi}|\,dt'\Big),$$

where $\mathbb{I}$ is the unit matrix.

Remark 2.8. We describe spaces used in this paper. We define them by assuming finiteness of appropriate norms. Let $\Omega \subset \mathbb{R}^3$. Let $W_r^{k,l}(\Omega^T)$, $k,l \in \mathbb{N}_0$, $r \in [1,\infty]$, be the Sobolev space with norm

$$\|u\|_{W_r^{k,l}(\Omega^T)} = \Big(\int_{\Omega^T} \Big(\sum_{|\alpha|\le k} |D_x^\alpha u|^r + \sum_{i\le l} |\partial_t^i u|^r\Big)\,dx\,dt\Big)^{1/r},$$

where $D_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$, $\alpha_i \in \mathbb{N}_0$, $|\alpha| = \alpha_1+\alpha_2+\alpha_3$. Let $l \in \mathbb{R}_+$. Then $W_r^{l,l/2}(\Omega^T)$ is the space of functions with norm

$$\|u\|_{W_r^{l,l/2}(\Omega^T)} = \sum_{|\alpha|+2i\le[l]} [\|\partial_t^i D_x^\alpha u\|_{L_r(\Omega^T)} + \langle \partial_t^i D_x^\alpha u\rangle_{r,\Omega^T}^{(l)}],$$

where $[l]$ is the integer part of l,

$$\langle \partial_t^i D_x^\alpha u\rangle_{r,\Omega^T}^{(l)} = \sum_{|\alpha|+2i=[l]} \langle\!\langle \partial_t^i D_x^\alpha u\rangle\!\rangle_{r,x,\Omega^T}^{(l-[l])} + \sum_{l-|\alpha|-2i<2} \langle\!\langle \partial_t^i D_x^\alpha u\rangle\!\rangle_{r,t,\Omega^T}^{\left(\frac{l-2i-|\alpha|}{2}\right)}$$

and finally

$$\langle\!\langle u\rangle\!\rangle_{r,x,\Omega^T}^{(\alpha)} = \Big(\int_0^T dt \int_\Omega dx' \int_\Omega dx''\, \frac{|u(x',t)-u(x'',t)|^r}{|x'-x''|^{2+r\alpha}}\Big)^{1/r},$$

$$\langle\!\langle u\rangle\!\rangle_{r,t,\Omega^T}^{(\alpha)} = \Big(\int_\Omega dx \int_0^T dt' \int_0^T dt''\, \frac{|u(x,t')-u(x,t'')|^r}{|t'-t''|^{1+r\alpha}}\Big)^{1/r}.$$

Now, we recall the direct and inverse trace theorems.

Lemma 2.9 (Direct boundary trace theorem, see [S6, S7]).

(1) *Let $\Omega \subset \mathbb{R}^n$ be a domain and S be either the boundary of Ω or a subset of Ω with $\dim S = n-1$.*
(2) *Let $u \in W_r^{l,l/2}(\Omega^T)$, $l \in \mathbb{R}_+$, $r \in [1,\infty)$, $l > 1/r$, $S \in C^l$.*

Then there exists a function $\tilde{u} = u|_{S^T}$ such that

$$\tilde{u} \in W_r^{l-1/r,l/2-1/(2r)}(S^T)$$

and there exists c independent of u such that

$$\|\tilde{u}\|_{W_r^{l-1/r,l/2-1/(2r)}(S^T)} \le c\|u\|_{W_r^{l,l/2}(\Omega^t)}. \tag{2.16}$$

Lemma 2.10 (Inverse boundary trace theorem, see [S6, S7]). *Let the assumption* (1) *of Lemma* 2.9 *be satisfied. Let* $\tilde{u} \in W_r^{1-1/r,l/2-1/(2r)}(S^T)$, $l \in \mathbb{R}_+$, $l > 1/r$, $r \in [1,\infty)$, $S \in C^l$. *Then there exists a function* u *such that* $u|_{S^T} = \tilde{u}$, $u \in W_r^{l,l/2}(\Omega^T)$ *and*

$$\|u\|_{W_r^{l,l/2}(\Omega^T)} \le c\|\tilde{u}\|_{W_r^{l-1/r,l/2-1/(2r)}(S^T)}, \tag{2.17}$$

where c *does not depend on* u.

Lemma 2.11 (Direct initial trace theorem, see [S6, S7]). *Let* $u \in W_r^{l,l/2}(\Omega^T)$, $l \in \mathbb{R}_+$, $r \in (1,\infty)$, $l > 2/r$. *Then* $\tilde{u} = u|_{t=t_0}$, *where* $t_0 \in [0,T]$, *belongs to* $W_r^{l-2/r}(\Omega)$ *and*

$$\|\tilde{u}\|_{W_r^{l-2/r}(\Omega)} \le c\|u\|_{W_r^{l,l/2}(\Omega^t)}, \tag{2.18}$$

where c *does not depend on* u.

Lemma 2.12 (Inverse initial trace theorem, see [S6, S7]). *Let* $\tilde{u} \in W_r^{l-2/r}(\Omega)$, $l \in \mathbb{R}_+$, $r \in (1,\infty)$, $l > 2/r$. *Then there exists* $u \in W_r^{l,l/2}(\Omega^T)$ *such that* $u|_{t=t_0} = \tilde{u}$, $t_0 \in [0,T]$, *and*

$$\|u\|_{W_r^{l,l/2}(\Omega^t)} \le c\|\tilde{u}\|_{W_r^{l-2/r}(\Omega)}, \tag{2.19}$$

where c *does not depend on* u.

Lemma 2.13. *Let* $\Omega \subset \mathbb{R}^3$, $u \in W_r^{l,l/2}(\Omega^T)$, $l \in \mathbb{R}_+$, $r \in [1,\infty]$. *Let* $q \in [1,\infty]$, $\alpha_i \in \mathbb{N}_0$, $i = 1,2,3$, *and suppose that*

$$\frac{5}{r} - \frac{|\alpha| + 2i}{q} \le l. \tag{2.20}$$

Then $D_x^\alpha \partial_t^i u \in L_q(\Omega^T)$ *and there exists* c *independent of* u *such that*

$$\|D_x^\alpha \partial_t^i u\|_{L_q(\Omega^T)} \le c\|u\|_{W_r^{l,l/2}(\Omega^T)}. \tag{2.21}$$

Let $\mathbb{R}_+^3 = \{x \in \mathbb{R}^3 : x_3 > 0\}$. We introduce the Fourier–Laplace transform by

$$(Fu)(\xi, x_3, s) = \int_0^\infty e^{-st} \int_{\mathbb{R}^2} u(x,t) e^{-ix'\cdot\xi}\, dx'\, ds, \tag{2.22}$$

where $\operatorname{Re} s > 0$, $x' = (x_1,x_2)$, $\xi = (\xi_1,\xi_2)$, $x'\cdot\xi = x_1\xi_1 + x_2\xi_2$. Let $\tau = \sqrt{s+\xi^2}$. Then we introduce the anisotropic Bessel potential space $H_r^{l,l/2}(\mathbb{R}_+^3 \times \mathbb{R}_+)$, $l \in \mathbb{R}_+$, $r \in [1,\infty]$ (see [T1, Ch. 1, Sect. 1.5.2; Ch. 4, Sect. 4.2]) by defining

$$\|u\|_{H_r^{l,l/2}(\mathbb{R}_+^3\times\mathbb{R}_+)} = \Big[\int_{\mathbb{R}_+} dt \int_{\mathbb{R}_+^3} (|F^{-1}(1+|\tau|^2)^{l/2} Fu|^r + |\partial_{x_3}^l u|^r)\, dx\Big]^{1/r}. \tag{2.23}$$

We also introduce the Fourier–Laplace transform

$$(F_1u)(\xi, s) = \int_0^\infty e^{-st} \int_{\mathbb{R}^3} f(x,t) e^{-ix\cdot\xi}\, dx, \tag{2.24}$$

where $\operatorname{Re} s > 0$, $x = (x_1,x_2,x_3)$, $\xi = (\xi_1,\xi_2,\xi_3)$, $x\cdot\xi = x_1\xi_1 + x_2\xi_2 + x_3\xi_3$. Let $\tau = \sqrt{s+\xi^2}$. Then we introduce the anisotropic Bessel potential space $H_r^{l,l/2}(\mathbb{R}^3 \times \mathbb{R}_+)$, $l \in \mathbb{R}_+$, $r \in [1,\infty]$, by defining

$$\|u\|_{H_r^{l,l/2}(\mathbb{R}^3\times\mathbb{R}_+)} = \Big[\int_{\mathbb{R}_+} dt \int_{\mathbb{R}^3} |F_1^{-1}(1+\tau^2)^{l/2} F_1 u|^r\, dx\Big]^{1/r}. \tag{2.25}$$

For l even we have (see [T1, Sect. 2.2.2, Remark 3])

$$H_r^{l,l/2}(\Omega^T) = W_r^{l,l/2}(\Omega^T).$$

Consider the Cauchy problem

$$\begin{aligned} &\mu u_{,t} + \frac{1}{\sigma}\operatorname{rot}^2 u = f \quad && \text{in } \mathbb{R}^3 \times (0,T), \\ &\operatorname{div} u = g && \text{in } \mathbb{R}^3 \times (0,T). \end{aligned} \tag{2.26}$$

LEMMA 2.14. *Assume that* $f \in L_r(\mathbb{R}^3 \times (0,T))$, $g \in L_r(0,T; W_r^1(\mathbb{R}^3))$, $r \in (1,\infty)$. *Then there exists a solution to problem* (2.19) *such that* $u \in W_r^{2,1}(\mathbb{R}^3 \times (0,T))$ *and*

$$\|u\|_{W_r^{2,1}(\mathbb{R}^3\times(0,T))} \le c(\|f\|_{L_r(\mathbb{R}^3\times(0,T))} + \|g\|_{L_r(0,T;W_r^1(\mathbb{R}^3))}), \tag{2.27}$$

where c *does not depend on* u.

PART I (with jump of the magnetic field)

3. Method of successive approximations

Let $v_n = v_n(x,t)$ be given, $x \in \overset{1}{\Omega}_t$.

Definition 3.1. The Lagrangian coordinates in $\overset{1}{\Omega}_0$ are initial data to the Cauchy problem

$$\frac{dx}{dt} = v_n(x,t), \quad x|_{t=0} = \xi \in \overset{1}{\Omega}_0. \tag{3.1}$$

Hence the domain $\overset{1}{\Omega}_{nt}$ is defined by

$$\overset{1}{\Omega}_{nt} = \Big\{ x \in \mathbb{R}^3 : x = x^{(n)}(\xi,t) = \xi + \int_0^t \bar{v}_n(\xi,t')\,dt',\ \xi \in \overset{1}{\Omega}_0 \Big\},$$

where $\bar{v}_n(\xi,t) = v_n(x^{(n)}(\xi,t),t)$.

In free boundary problems in hydrodynamics the free boundary is built from the same fluid particles as at time $t=0$ because $v_n|_{S_{nt}}$ is tangent to S_{nt}. Then

$$S_{nt} = \{x \in \mathbb{R}^3 : x = x^{(n)}(\xi,t),\ \xi \in S_0\}.$$

To formulate problem (1.10) in Lagrangian coordinates we have to introduce them in $\overset{2}{\Omega}_0$. Since there is no velocity in $\overset{2}{\Omega}_t$, we have to introduce it artificially. We shall do it now more explicitly than in Lemma 2.1.

Definition 3.2. Let $\overset{1}{v}_n = v$ in $\overset{1}{\Omega}_t$ and construct $\overset{2}{v}_n$ in $\overset{2}{\Omega}_t$ as a solution to the nonstationary Stokes problem

$$\begin{aligned}
&\overset{2}{v}_{n,t} - \operatorname{div}\mathbb{T}(\overset{2}{v}_n, q) = 0 && \text{in } \overset{2}{\Omega}_t,\\
&\operatorname{div}\overset{2}{v}_n = 0 && \text{in } \overset{2}{\Omega}_t,\\
&\overset{2}{v}_n|_{S_t} = \overset{1}{v}_n|_{S_t}, \quad \overset{2}{v}_n|_B = 0, &&\\
&\overset{2}{v}_n|_{t=0} = \overset{2}{v}(0) && \text{in } \overset{2}{\Omega}_0,
\end{aligned} \tag{3.2}$$

where q plays the role of pressure but it is not important for any estimate for $\overset{2}{v}_n$. It is introduced to have $\overset{2}{v}_n$ divergence free. The initial data $\overset{2}{v}(0)$ is an extension of $\overset{1}{v}(0)$

through the fixed given boundary S_0, because $\overset{2}{v}(0)|_{S_0} = \overset{1}{v}(0)|_{S_0}$. The extension can be made by Lemma 2.1. The existence of solutions to (3.2) follows from Lemma 2.4(a).

Now, we can introduce Lagrangian coordinates $\overset{1}{\xi}$, $\overset{2}{\xi}$ by the Cauchy data to the problems

$$\frac{d\overset{i}{x}}{dt} = \overset{i}{v}_n(x,t), \quad x^i|_{t=0} = \overset{i}{\xi} \in \overset{i}{\Omega}_0, \quad i=1,2. \tag{3.3}$$

Then

$$\overset{i}{\Omega}_{nt} = \Big\{ \overset{i}{x} \in \mathbb{R}^3 : \overset{i}{x} = \overset{i}{x}{}^{(n)}(\overset{i}{\xi},t) = \overset{i}{\xi} + \int_0^t \overset{i}{v}_n(\overset{i}{x},t')\,dt' = \overset{i}{\xi} + \int_0^t \overset{i}{\bar{v}}_n(\xi,t')\,dt',\ \overset{i}{\xi} \in \overset{i}{\Omega}_0 \Big\}, \tag{3.4}$$

where $\overset{i}{\bar{v}}_n(\overset{i}{\xi},t) = \overset{i}{v}_n(\overset{i}{x}{}^{(n)}(\overset{i}{\xi},t),t)$, $\overset{i}{\xi} \in \overset{i}{\Omega}_0$, $i=1,2$.

Assume that $\bar{v}_n$, $\overset{1}{\bar{H}}_n$ are given. Assume that $\overset{1}{\Omega}_{nt}$ is described by (3.4). Then we linearize problem (1.9) to the form

$$\begin{aligned}
&\bar{v}_{n+1,t} - \operatorname{div}_{\bar{v}_n} \mathbb{T}_{\bar{v}_n}(\bar{v}_{n+1},\bar{p}_{n+1}) = \bar{f} + \mu_1 \operatorname{div}_{\bar{v}_n} \mathbb{T}(\overset{1}{\bar{H}}_n) && \text{in } \overset{1}{\Omega}_0 \times (0,t),\\
&\operatorname{div}_{\bar{v}_n} \bar{v}_{n+1} = 0 && \text{in } \overset{1}{\Omega}_0 \times (0,t),\\
&\bar{n}_{\bar{v}_n} \cdot \mathbb{T}_{\bar{v}_n}(\bar{v}_{n+1},\bar{p}_{n+1}) = -\mu_1 \bar{n}_{\bar{v}_n} \cdot \mathbb{T}(\overset{1}{\bar{H}}_n) && \text{on } S_0 \times (0,t),\\
&\bar{v}_{n+1}|_{t=0} = v(0) && \text{in } \overset{1}{\Omega}_0,
\end{aligned} \tag{3.5}$$

where $\nabla_{\bar{v}_n} = \frac{\partial \xi_k}{\partial x^{(n)}}\big|_{x^{(n)}=x^{(n)}(\xi,t)} \partial_{\xi_k}$. Moreover, any operator with subscript $\bar{v}_n$ means that it contains the transformed gradient $\nabla_{\bar{v}_n}$.

Assume that $\bar{v}_n$ is given and $\overset{i}{\Omega}_{nt}$ described by (3.4), $i=1,2$. Then the linearized problem (1.10) takes the form

$$\begin{aligned}
&\left.\begin{aligned}
&\mu_1 \overset{1}{\bar{H}}_{n,t} - \mu_1 \overset{1}{\bar{v}}_n \cdot \nabla_{\overset{1}{\bar{v}}_n} \overset{1}{\bar{H}}_n = -\operatorname{rot}_{\overset{1}{\bar{v}}_n} \overset{1}{\bar{E}}_n, \quad \operatorname{div}_{\overset{1}{\bar{v}}_n} \overset{1}{\bar{H}}_n = 0,\\
&\operatorname{rot}_{\overset{1}{\bar{v}}_n} \overset{1}{\bar{H}}_n = \sigma_1(\overset{1}{\bar{E}}_n + \mu_1 \overset{1}{\bar{v}}_n \times \overset{1}{\bar{H}}_n),
\end{aligned}\right\} \quad \text{in } \overset{1}{\Omega}_0 \times (0,t),\\
&\left.\begin{aligned}
&\mu_2 \overset{2}{\bar{H}}_{n,t} - \mu_2 \overset{2}{\bar{v}}_n \cdot \nabla_{\overset{2}{\bar{v}}_n} \overset{2}{\bar{H}}_n = -\operatorname{rot}_{\overset{2}{\bar{v}}_n} \overset{2}{\bar{E}}_n, \quad \operatorname{div}_{\overset{2}{\bar{v}}_n} \overset{2}{\bar{H}}_n = 0,\\
&\sigma_2 \overset{2}{\bar{E}}_n = \operatorname{rot}_{\overset{2}{\bar{v}}_n} \overset{2}{\bar{H}}_n,
\end{aligned}\right\} \quad \text{in } \overset{2}{\Omega}_0 \times (0,t),\\
&\overset{i}{\bar{H}}_n|_{t=0} = \overset{i}{H}(0), \quad \operatorname{div} \overset{i}{H}(0) = 0 \qquad \text{in } \overset{i}{\Omega}_0,\ i=1,2,\\
&\overset{2}{\bar{H}}_n \cdot \bar{\tau}_\alpha|_B = H_{*\alpha}, \quad \operatorname{div}_{\overset{2}{\bar{v}}_n} \overset{2}{\bar{H}}_n|_B = 0, \quad \alpha=1,2,\\
&\overset{1}{\bar{E}}_n \cdot \bar{\tau}_{\bar{v}_n,\alpha} = \overset{2}{\bar{E}}_n \cdot \bar{\tau}_{\bar{v}_n,\alpha}, \quad \bar{n}_{\bar{v}_n} \times \bar{\tau}_{\bar{v}_n,\alpha} \cdot (\overset{1}{b}\overset{1}{\bar{H}}_n - \overset{2}{b}\overset{2}{\bar{H}}_n) = 0, \quad \alpha=1,2, \qquad \text{on } S_0,
\end{aligned} \tag{3.6}$$

where we have used $\overset{1}{\bar{v}}_n = \overset{2}{\bar{v}}_n = \bar{v}_n$ on S_0.

We emphasize that the formulation of problem (1.10) in Lagrangian coordinates requires that the terms $\mu_i \overset{i}{v}_n \cdot \nabla \overset{i}{H}_n|_{x=x^{(n)}(\xi,t)} = \mu_i \overset{i}{\bar{v}}_n \nabla_{\overset{i}{\bar{v}}_n} \overset{i}{\bar{H}}_n$, $i=1,2$, appear in the l.h.s. of $(3.6)_{1,3}$, respectively.

4. Existence of solutions to problem (3.5) for given $\bar{v}_n$ and $\overset{1}{\bar{H}}_n$

We introduce the notation

$$v = \bar{v}_{n+1}, \quad p = \bar{p}_{n+1}, \quad u = \bar{v}_n, \quad \overset{1}{H} = \overset{1}{\bar{H}}_n. \tag{4.1}$$

Then problem (3.5) takes the form

$$\begin{aligned} & v_{,t} - \operatorname{div}_u \mathbb{T}_u(v,p) = f + \mu_1 \operatorname{div}_u \mathbb{T}(\overset{1}{H}) && \text{in } \overset{1}{\Omega}{}_0^T, \\ & \operatorname{div}_u v = 0 && \text{in } \overset{1}{\Omega}{}_0^T, \\ & \bar{n}_u \cdot \mathbb{T}_u(v,p) = -\mu_1 \bar{n}_u \cdot \mathbb{T}(\overset{1}{H}) && \text{on } S_0^T, \\ & v|_{t=0} = v(0) && \text{in } \overset{1}{\Omega}_0. \end{aligned} \tag{4.2}$$

LEMMA 4.1. *Let $r>5/2$. Assume that $f \in W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)$, $v(0)\in W_r^{3-2/r}(\overset{1}{\Omega}_0)$, $\overset{1}{H} \in W_r^{2,1}(\overset{1}{\Omega}{}_0^t)$, $u \in W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)$, $u(0)\in W_r^{3-2/r}(\overset{1}{\Omega}_0)$. Set $\overset{1}{\alpha}_u(t) = t^{1/r'}\|u\|_{L_r(0,t;W_r^3(\overset{1}{\Omega}_0))}$, $1/r+1/r'=1$. Then there exists $a>0$ and an increasing positive function φ with the following property. If t, u, $u(0)$ are so small that*

$$\varphi(t^a \|u\|_{L_r(0,t;W_r^3(\overset{1}{\Omega}_0))}) t^a [\|u\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)} + \|u(0)\|_{W_r^1(\overset{1}{\Omega}_0)}] \le 1/2,$$

then there exists a solution to (4.2) with $v \in W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)$, $p\in W_r^{2,1}(\overset{1}{\Omega}{}_0^t) \cap W_r^{2-1/r,1-1/(2r)}(S_0^t)$, and

$$\begin{aligned} & \|v\|_{W_r^{3,3/2}(\Omega_0^t)} + \sup_{t'\le t}\|v(t')\|_{W_r^{3-2/r}(\Omega_0)} + \|p\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|p\|_{W_r^{2-1/r,1-1/(2r)}(S_0^t)} \\ & \quad \le \varphi(\overset{1}{\alpha}_u(t)) t^a (\|u\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)} + \|u(0)\|_{W_r^{3-2/r}(\Omega_0)}) \|v(0)\|_{W_r^{3-2/r}(\Omega_0)} \\ & \qquad + c\|f\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)} + \varphi(\overset{1}{\alpha}_u(t)) \sup_{t'\le t}\|\overset{1}{H}(t')\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)} \cdot \|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} \\ & \qquad + c\|v(0)\|_{W_r^{3-2/r}(\overset{1}{\Omega}_0)}, \end{aligned} \tag{4.3}$$

where c does not depend on v, p, u, $\overset{1}{H}$.

Proof. We prove the existence of solutions to problem (4.2) by the following method of successive approximations:

$$
\begin{aligned}
&v_{n+1,t}-\operatorname{div}_\xi \mathbb{T}_\xi(v_{n+1},p_{n+1})=-(\operatorname{div}_\xi \mathbb{T}_\xi(v_n,p_n)-\operatorname{div}_u \mathbb{T}_u(v_n,p_n))\\
&\qquad\qquad +f+\mu_1\operatorname{div}_u\mathbb{T}(\overset{1}{H})\equiv F_1+F_2,\\
&\operatorname{div}_\xi v_{n+1}=\operatorname{div}_\xi v_n-\operatorname{div}_u v_n\equiv G=\nabla\cdot R,\\
&\bar n_\xi\cdot\mathbb{T}_\xi(v_{n+1},p_{n+1})=(\bar n_\xi\cdot\mathbb{T}_\xi(v_n,p_n)-\bar n_u\cdot\mathbb{T}_u(v_n,p_n))\\
&\qquad\qquad -\mu\bar n_u\cdot\mathbb{T}(\overset{1}{H})\equiv K_1+K_2,\\
&v_{n+1}|_{t=0}=v(0),
\end{aligned}
\tag{4.4}
$$

and $v_0=0$, $p_0=0$. The index ξ means we take the derivative with respect to ξ, and $\bar n_\xi$ is the exterior vector normal to S_0. In view of Lemma 2.6(a) we have

$$
\begin{aligned}
&\|v_{n+1}\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}+\sup_{t'\le t}\|v_{n+1}(t')\|_{W_r^{3-2/r}(\overset{1}{\Omega}_0)}+\|p_{n+1}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}+\|p_{n+1}\|_{W_r^{2-1/r,1-1/(2r)}(S_0^t)}\\
&\quad\le c(\|F_1\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)}+\|F_2\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)}+\|G\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}\\
&\qquad+\|R_t\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)}+\|K_1\|_{W_r^{2-1/r,1-1/(2r)}(S_0^t)}+\|K_2\|_{W_r^{2-1/r,1-1/(2r)}(S_0^t)}\\
&\qquad+\|v(0)\|_{W_r^{3-2/r}(\overset{1}{\Omega}_0)}).
\end{aligned}
\tag{4.5}
$$

By the properties of the transformation (2.15) we derive

$$
\begin{aligned}
\|F_1\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)}&\le c\|(\mathbb{I}-\xi_x^2)v_{n,\xi\xi}+\xi_x\xi_{xx}x_\xi v_{n,\xi}\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)}+c\|(\mathbb{I}-\xi_x)p_{n,\xi}\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)}\\
&=c\|(\mathbb{I}-\xi_x^2)v_{n,\xi\xi}+\xi_x\xi_{xx}x_\xi v_{n,\xi}\|_{L_r(\overset{1}{\Omega}{}_0^t)}\\
&\quad+c\|((\mathbb{I}-\xi_x^2)v_{n,\xi\xi}+\xi_x\xi_{xx}x_\xi v_{n,\xi})_{,\xi}\|_{L_r(\overset{1}{\Omega}{}_0^t)}\\
&\quad+c\|(\mathbb{I}-\xi_x^2)v_{n,\xi\xi}+\xi_x\xi_{xx}x_\xi v_{n,\xi}\|_{L_r(\overset{1}{\Omega}_0;L_r^{1/2}(0,t))}\\
&\quad+\|(\mathbb{I}-\xi_x)p_{n,\xi}\|_{L_r(\overset{1}{\Omega}{}_0^t)}+\|((\delta-\xi_x)p_{n,\xi})_{,\xi}\|_{L_r(\overset{1}{\Omega}{}_0^t)}\\
&\quad+\|(\mathbb{I}-\xi_x)p_{n,\xi}\|_{L_r(\overset{1}{\Omega}_0;L_r^{1/2}(0,t))}\equiv\sum_{\alpha=1}^6F_{1\alpha},
\end{aligned}
$$

where $L_r^k(\Omega)$, $L_r^k(0,t)$ denote the spaces of functions with norms

$$
\|u\|_{L_r^k(\Omega)}=\sum_{|\alpha|=k}\|\mathbb{D}_x^\alpha u\|_{L_r(\Omega)},\qquad \|u\|_{L_r^k(0,t)}=\|\partial_t^k u\|_{L_r(\Omega)},
\tag{4.6}
$$

where for k noninteger we have fractional derivatives. For F_{12}, we have

$$
F_{12}\le\|(\mathbb{I}-\xi_x^2)v_{n,\xi\xi\xi}+\xi_x\xi_{xx}x_\xi v_{n,\xi\xi}+\xi_{xx}^2x_\xi^2v_{n,\xi}+\xi_x\xi_{xxx}x_\xi^2v_{n,\xi}+\xi_x\xi_{xx}x_\xi v_{n,\xi\xi}\|_{L_r(\overset{1}{\Omega}{}_0^t)}\equiv F_{12}^1.
$$

There exists an increasing positive function φ such that the following qualitative estimates hold:

$$|\xi_x| \le \varphi(\overset{1}{\alpha}_u(t)), \quad |\xi_{xx}| \le \varphi(\overset{1}{\alpha}_u(t))\Big|\int_0^t u_{,\xi\xi}\,d\tau\Big|,$$

$$|\xi_{xxx}| \le \varphi(\overset{1}{\alpha}_u(t))\Big(\Big|\int_0^t u_{,\xi\xi}\,d\tau\Big|^2 + \Big|\int_0^t u_{,\xi\xi\xi}\,d\tau\Big|\Big).$$

Employing the above estimates in F_{12}^1 yields

$$\begin{aligned} F_{12}^1 &\le \varphi(\overset{1}{\alpha}_u(t))\Big(\overset{1}{\alpha}_u(t)\|v_{n,\xi\xi\xi}\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \Big\|\int_0^t u_{\xi\xi}\,d\tau\, v_{n,\xi\xi}\Big\|_{L_r(\overset{1}{\Omega}{}_0^t)} \\ &\quad + \Big\|\Big(\int_0^t u_{\xi\xi}\,d\tau\Big)^2 v_{n,\xi}\Big\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \Big\|\int_0^t u_{,\xi\xi\xi}\,d\tau\, v_{n,\xi}\Big\|_{L_r(\overset{1}{\Omega}{}_0^t)}\Big) \equiv F_{12}^2. \end{aligned}$$

By the Minkowski inequality the second term in F_{12}^2 is bounded by

$$\Big(\int_0^t dt'\Big|\int_0^{t'}\|u_{\xi\xi}(\tau)\|_{L_{2r}(\overset{1}{\Omega}_0)}\,d\tau\Big|^r\|v_{n,\xi\xi}\|^r_{L_{2r}(\overset{1}{\Omega}_0)}\Big)^{1/r} \equiv I_1.$$

Hence for $r > 3/2$ (which holds by our assumptions) we have

$$\begin{aligned} I_1 &\le \int_0^t \|u_{\xi\xi}(\tau)\|_{W_r^1(\overset{1}{\Omega}_0)}\,d\tau\,\Big(\int_0^t \|v_{n,\xi\xi}(t')\|^r_{W_r^1(\overset{1}{\Omega}_0)}\,dt'\Big)^{1/r} \\ &\le \overset{1}{\alpha}_u(t)\|v_n\|_{L_r(0,t;W_r^3(\overset{1}{\Omega}_0))}. \end{aligned}$$

Similarly, for $r > 3/2$ the third term in F_{12}^2 is bounded by

$$\Big(\int_0^t \|v_{n,\xi}\|^r_{L_\infty(\overset{1}{\Omega}_0)}\,dt'\Big)^{1/r}\Big(\int_0^t\|u_{\xi\xi}(\tau)\|_{L_{2r}(\overset{1}{\Omega}_0)}\,d\tau\Big)^2 \le \overset{1}{\alpha}{}_u^2(t)\|v_n\|_{L_r(0,t;W_r^3(\overset{1}{\Omega}_0))}.$$

Finally, for $r > 3/2$ the last term in F_{12}^2 is bounded by

$$\begin{aligned} &\Big(\int_0^t dt'\int_{\overset{1}{\Omega}_0}\Big|\int_0^t u_{\xi\xi\xi}\,d\tau\Big|^r|v_{n,\xi}|^r\,d\xi\Big)^{1/r} \\ &\qquad \le \Big(\int_0^t dt'\,\|v_{n,\xi}\|^r_{L_\infty(\overset{1}{\Omega}_0)}\Big)^{1/r}\int_0^t\|u_{\xi\xi\xi}\|_{L_r(\overset{1}{\Omega}_0)}\,d\tau \le \overset{1}{\alpha}_u(t)\|v_n\|_{L_r(0,t;W_r^3(\overset{1}{\Omega}_0))}. \end{aligned}$$

For F_{13}, employing the definition of the fractional derivative, we have

$$\begin{aligned} F_{13} &\le \Big(\int_{\overset{1}{\Omega}_0}\int_0^t\int_0^t \frac{|(\mathbb{I}-\xi_x^2)v_{n,\xi\xi}(t') - (\mathbb{I}-\xi_x^2)v_{n,\xi\xi}(t'')|^r}{|t'-t''|^{1+r/2}}\,d\xi\,dt'\,dt''\Big)^{1/r} \\ &\quad + \Big(\int_{\overset{1}{\Omega}_0}\int_0^t\int_0^t \frac{|\xi_x\xi_{xx}x_\xi v_{n,\xi}(t') - \xi_x\xi_{xx}x_\xi v_{n,\xi}(t'')|^r}{|t'-t''|^{1+r/2}}\,d\xi\,dt'\,dt''\Big)^{1/r} \\ &\equiv F_{13}^1 + F_{13}^2. \end{aligned}$$

First, we examine

$$F_{13}^1 \le \varphi(\overset{1}{\alpha}_u(t))\bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t \frac{\left|\int_{t''}^{t'} u_\xi\,d\tau\right|^r |v_{n,\xi\xi}|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\,d\xi\bigg)^{1/r}$$

$$+\varphi(\overset{1}{\alpha}_u(t))\bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t \frac{\left|\int_0^t u_\xi\,d\tau\right|^r |v_{n,\xi\xi}(t')-v_{n,\xi\xi}(t'')|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\,d\xi\bigg)^{1/r} \equiv F_{13}^{11}+F_{13}^{12}.$$

By the Minkowski and Hölder inequalities,

$$F_{13}^{11} \le \varphi(\overset{1}{\alpha}_u(t))\bigg(\int\limits_0^t\int\limits_0^t \frac{\left|\int_{t''}^{t'} \|u_\xi\|_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}\,d\tau\right|^r \|v_{n,\xi\xi}\|^r_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r} \equiv I_1,$$

where $1/\lambda_1+1/\lambda_2=1$. Using the imbeddings

$$\|u_\xi\|_{L_{r\lambda_1}(\overset{1}{\Omega}_0)} \le c\|u\|_{W_r^{3-2/r}(\overset{1}{\Omega}_0)},\quad \|v_{n,\xi\xi}\|_{L_{r\lambda_2}(\overset{1}{\Omega}_0)} \le c\|v_n\|_{W_r^3(\overset{1}{\Omega}_0)},$$

which hold for $r\ge 5/3$, we obtain, with some $a>0$,

$$I_1 \le \varphi(\overset{1}{\alpha}_u(t))t^a\|u\|_{L_\infty(0,t;W_r^{3-2/r}(\overset{1}{\Omega}_0))}\|v_n\|_{L_r(0,t;W_r^3(\overset{1}{\Omega}_0))}.$$

Consequently,

$$F_{13}^{12} \le \varphi(\overset{1}{\alpha}_u(t))\int\limits_0^t \|u_\xi\|_{L_\infty(\overset{1}{\Omega}_0)}\,d\tau\,\|v_n\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)} \le \varphi(\overset{1}{\alpha}_u(t))\overset{1}{\alpha}_u(t)\|v_n\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}$$

for $r\ge 3/2$. Next we examine F_{13}^2:

$$F_{13}^2 \le \varphi(\overset{1}{\alpha}_u(t))\bigg[\bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t \frac{\left|\int_{t''}^{t'} u_\xi\,d\tau\right|^r\left|\int_0^{t'} u_{\xi\xi}\,d\tau\right|^r |v_{n,\xi}(t')|^r}{|t'-t''|^{1+r/2}}\,d\xi\,dt'\,dt''\bigg)^{1/r}$$

$$+\overset{1}{\alpha}_u(t)\bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t \frac{\left|\int_{t''}^{t'} u_{\xi\xi}\,d\tau\right|^r |v_{n,\xi}(t')|^r}{|t'-t''|^{1+r/2}}\,d\xi\,dt'\,dt''\bigg)^{1/r}$$

$$+\overset{1}{\alpha}_u(t)\bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t \frac{\left|\int_0^{t''} u_{\xi\xi}\,d\tau\right|^r |v_{n,\xi}(t')-v_{n,\xi}(t'')|^r}{|t'-t''|^{1+r/2}}\,d\xi\,dt'\,dt''\bigg)^{1/r}\bigg] \equiv F_{13}^{21}.$$

By the Hölder and Minkowski inequalities the first integral in F_{13}^{21} is bounded by

$$\varphi(\overset{1}{\alpha}_u(t))t^a\|u\|_{L_\infty(0,t;W_r^{3-2/r}(\overset{1}{\Omega}_0))}\|u\|_{L_r(0,t;W_r^3(\overset{1}{\Omega}_0))}\|v_n\|_{L_r(0,t;W_r^3(\overset{1}{\Omega}_0))}$$

for $r\ge 8/5$.

Similarly, for $r\ge 5/3$, we estimate the second integral in F_{13}^{21} by

$$\varphi(\overset{1}{\alpha}_u(t))\overset{1}{\alpha}_u(t)t^a\|u\|_{L_\infty(0,t;W_r^{3-2/r}(\overset{1}{\Omega}_0))}\|v_n\|_{L_r(0,t;W_r^3(\overset{1}{\Omega}_0))}.$$

Finally, the last integral in F_{13}^{21} is bounded by

$$\varphi(\overset{1}{\alpha}_u(t))\overset{1}{\alpha}_u(t)\int_0^t \|u_{\xi\xi}\|_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}\, d\tau \left(\int_0^t\int_0^t \frac{\|v_{n,\xi}(t')-v_{n,\xi}(t'')\|^r_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}}{|t'-t''|^{1+r/2}}\, dt'\, dt''\right)^{1/r} \equiv I.$$

Using the imbeddings

$$\begin{aligned}
&\|u_{\xi\xi}\|_{L_{r\lambda_1}(\overset{1}{\Omega}_0)} \le c\|u\|_{W_r^3(\overset{1}{\Omega}_0)}, \qquad \frac{3}{r}-\frac{3}{r\lambda_1}\le 1,\\
&\left(\int_0^t\int_0^t \frac{\|v_{n,\xi}(t')-v_{n,\xi}(t'')\|^r_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}}{|t'-t''|^{1+r/2}}\, dt'\, dt''\right)^{1/r} \le c\|v_n\|_{W_r^{3,3/2}(\overset{1}{\Omega}_0)}
\end{aligned}$$

which hold under the restriction $5/r-3/(r\lambda_2)-2/r+2\le 3$, $1/\lambda_1+1/\lambda_2=1$, we obtain the condition $r\ge 3/2$ and

$$I\le \varphi(\overset{1}{\alpha}_u(t))(\overset{1}{\alpha}_u(t))^2\|v_n\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}.$$

Now F_{15} and F_{16} must be estimated. We have

$$\begin{aligned}
F_{15} &\le \varphi(\overset{1}{\alpha}_u(t))\Big(\Big\|\int_0^t u_{\xi\xi}\, d\tau\, p_{n,\xi}\Big\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \Big\|\int_0^t u_\xi\, d\tau\, p_{n,\xi\xi}\Big\|_{L_r(\overset{1}{\Omega}{}_0^t)}\Big)\\
&\le \varphi(\overset{1}{\alpha}_u(t))\overset{1}{\alpha}_u(t)\|p_n\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))} \qquad \text{for } r\ge 3/2.
\end{aligned}$$

Continuing,

$$\begin{aligned}
F_{16} &\le \left(\int_{\overset{1}{\Omega}_0}\int_0^t\int_0^t \frac{|\xi_x(t')-\xi_x(t'')|^r|p_{n,\xi}|^r}{|t'-t''|^{1+r/2}}\, d\xi\, dt'\, dt''\right)^{1/r}\\
&\quad + \varphi(\overset{1}{\alpha}_u(t))\left(\int_{\overset{1}{\Omega}_0}\int_0^t\int_0^t \frac{|\int_0^{t''} u_\xi\, d\tau|^r|p_{n,\xi}(t')-p_{n,\xi}(t'')|^r}{|t'-t''|^{1+r/2}}\, d\xi\, dt'\, dt''\right)^{1/r} \equiv F_{16}^1+F_{16}^2,
\end{aligned}$$

where

$$\begin{aligned}
F_{16}^1 &\le \varphi(\overset{1}{\alpha}_u(t))\left(\int_0^t\int_0^t \frac{\|\int_{t''}^{t'} u_\xi\, d\tau\|^r_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}\|p_{n,\xi}\|^r_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}}{|t'-t''|^{1+r/2}}\, dt'\, dt''\right)^{1/r}\\
&\le \varphi(\overset{1}{\alpha}_u(t))t^a\|u\|_{L_\infty(0,t;W_r^{3-2/r}(\overset{1}{\Omega}_0))}\|p_n\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))},
\end{aligned}$$

because $1/\lambda_1+1/\lambda_2=1$, $\|u_\xi\|_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}\le c\|u\|_{W_r^{3-2/r}(\overset{1}{\Omega}_0)}$ for $3/r-3/(r\lambda_1)\le 2-2/r$ and $\|p_{n\xi}\|_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}\le c\|p_n\|_{W_r^2(\overset{1}{\Omega}_0)}$ for $3/r-3/(r\lambda_2)\le 1$. Hence the above estimate holds for

$r \geq 5/3$. Finally,

$$F_{16}^2 \leq \varphi(\overset{1}{\alpha}_u(t)) \int_0^t \|u_\xi(\tau)\|_{L_\infty(\overset{1}{\Omega}_0)} d\tau \bigg(\int_{\overset{1}{\Omega}_0} \int_0^t \int_0^t \frac{|p_{n,\xi}(t') - p_{n,\xi}(t'')|^r}{|t'-t''|^{1+r/2}} d\xi\, dt'\, dt'' \bigg)^{1/r}$$
$$\leq \varphi(\overset{1}{\alpha}_u(t)) \overset{1}{\alpha}_u(t) \|p_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}.$$

Summarizing, the above estimates yield, for $r \geq 5/3$,

$$\|F_1\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)} \leq \varphi(\overset{1}{\alpha}_u(t))[\overset{1}{\alpha}_u(t) + t^a \|u\|_{L_\infty(0,t;W_r^{3-2/r}(\overset{1}{\Omega}_0))}] \|v_n\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}$$
$$+ \varphi(\overset{1}{\alpha}_u(t)) \overset{1}{\alpha}_u(t) \|p_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}. \tag{4.7}$$

For F_2, we have

$$\|F_2\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)} \leq \|f\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)} + \|\mathrm{div}_u\, \mathbb{T}(\overset{1}{H})\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)} \equiv F_2^1 + F_2^2,$$

where

$$F_2^2 = \|\mathrm{div}_u\, \mathbb{T}(\overset{1}{H})\|_{L_r(0,t;W_r^1(\overset{1}{\Omega}_0))} + \|\mathrm{div}_u\, \mathbb{T}(\overset{1}{H})\|_{L_r(\overset{1}{\Omega}_0;W_r^{1/2}(0,t))} \equiv F_2^{21} + F_2^{22}.$$

First we examine

$$F_2^{21} = \|\xi_x \overset{1}{H}_\xi \overset{1}{H}\|_{L_r(0,t;W_r^1(\overset{1}{\Omega}_0))}$$
$$\leq \varphi(\overset{1}{\alpha}_u(t)) \Big(\|\overset{1}{H}_\xi \overset{1}{H}\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \Big\| \int_0^t u_{\xi\xi}\, d\tau\, \overset{1}{H}_\xi \overset{1}{H} \Big\|_{L_r(\overset{1}{\Omega}{}_0^t)}$$
$$+ \|\overset{1}{H}_{\xi\xi} \overset{1}{H}\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \|\overset{1}{H}{}_\xi^2\|_{L_r(\overset{1}{\Omega}{}_0^t)} \Big) \equiv F_2^{211}.$$

We shall only examine the second, third and fourth terms in F_2^{211}. By the Minkowski and Hölder inequalities we bound the second term by

$$\varphi(\overset{1}{\alpha}_u(t)) \bigg(\int_0^t \bigg| \int_0^{t'} \|u_{\xi\xi}\|_{L_{r\lambda_1}(\overset{1}{\Omega}_0)} d\tau \bigg|^r \|\overset{1}{H}_\xi\|^r_{L_{r\lambda_2}(\overset{1}{\Omega}_0)} \|\overset{1}{H}\|^r_{L_{r\lambda_3}(\overset{1}{\Omega}_0)} dt' \bigg)^{1/r} \equiv I,$$

where $1/\lambda_1 + 1/\lambda_2 + 1/\lambda_3 = 1$. We employ the imbeddings $\|u_{\xi\xi}\|_{L_{r\lambda_1}(\overset{1}{\Omega}_0)} \leq c\|u\|_{W_r^3(\overset{1}{\Omega}_0)}$ for $3/r - 3/(r\lambda_1) \leq 1$, $\|\overset{1}{H}_\xi\|_{L_{r\lambda_2}(\overset{1}{\Omega}_0)} \leq c\|\overset{1}{H}\|_{W_r^2(\overset{1}{\Omega}_0)}$ for $3/r - 3/(r\lambda_2) \leq 1$ and $\|\overset{1}{H}\|_{L_{r\lambda_3}(\overset{1}{\Omega}_0)} \leq c\|\overset{1}{H}\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}$ for $3/r - 3/(r\lambda_3) \leq 2 - 2/r$, so the above estimate holds for $r \geq 2$. Then

$$I \leq \varphi(\overset{1}{\alpha}_u) \overset{1}{\alpha}_u \|\overset{1}{H}\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))} \sup_t \|\overset{1}{H}\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}.$$

We estimate the third term in F_2^{211} by

$$\varphi(\overset{1}{\alpha}_u)\Big(\int_0^t\int_{\overset{1}{\Omega}_0}|\overset{1}{H}_{\xi\xi}\overset{1}{H}|^r\,d\xi\,dt'\Big)^{1/r}\le\varphi(\overset{1}{\alpha}_u)\Big(\int_0^t\|\overset{1}{H}\|^r_{W_r^2(\overset{1}{\Omega}_0)}\|\overset{1}{H}\|^r_{L_\infty(\overset{1}{\Omega}_0)}\,dt'\Big)^{1/r}$$
$$\le\varphi(\overset{1}{\alpha}_u)\sup_t\|\overset{1}{H}\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}\|\overset{1}{H}\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))},$$

where we use the imbedding $\|\overset{1}{H}\|_{L_\infty(\overset{1}{\Omega}_0)}\le c\|\overset{1}{H}\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}$ which holds for $3/r\le 2-2/r$, so for $r\ge 5/2$. Finally, we bound the last term in F_2^{211} by

$$\varphi(\overset{1}{\alpha}_u(t))\Big(\int_0^t\int_{\overset{1}{\Omega}_0}|\overset{1}{H}_\xi|^{2r}\,d\xi\,dt'\Big)^{1/r}\le\varphi(\overset{1}{\alpha}_u(t))\Big(\int_0^t\|\overset{1}{H}_\xi\|^r_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}\|\overset{1}{H}_\xi\|^r_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}\,dt'\Big)^{1/r}$$
$$\le\varphi(\overset{1}{\alpha}_u(t))\sup_t|\overset{1}{H}\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}\|\overset{1}{H}\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}))},$$

where we have used $1/\lambda_1+1/\lambda_2=1$, $\|\overset{1}{H}_\xi\|_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}\le c\|\overset{1}{H}\|_{W_r^2(\overset{1}{\Omega}_0)}$ for $3/r-3/(r\lambda_1)\le 1$, $\|\overset{1}{H}_\xi\|_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}\le\|\overset{1}{H}\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}$ for $3/r-3/(r\lambda_2)\le 2-2/r$. Hence we need $r\ge 5/3$.

Next we consider

$$F_2^{22}=\|\xi_x\overset{1}{H}_\xi\overset{1}{H}\|_{L_r(\overset{1}{\Omega}_0;W_r^{1/2}(0,t))}$$
$$\le\Big(\int_{\overset{1}{\Omega}_0}\int_0^t\int_0^t\frac{|\xi_x(t')-\xi_x(t'')|^r|\overset{1}{H}_\xi(t')|^r|\overset{1}{H}(t')|^r}{|t'-t''|^{1+r/2}}\,d\xi\,dt'\,dt''\Big)^{1/r}$$
$$+\Big(\int_{\overset{1}{\Omega}_0}\int_0^t\int_0^t\frac{|\xi_x(t'')|^r|\overset{1}{H}_\xi(t')-\overset{1}{H}_\xi(t'')|^r|\overset{1}{H}(t')|^r}{|t'-t''|^{1+r/2}}\,d\xi\,dt'\,dt''\Big)^{1/r}$$
$$+\Big(\int_{\overset{1}{\Omega}_0}\int_0^t\int_0^t\frac{|\xi_x(t'')|^r|\overset{1}{H}_\xi(t'')|^r|\overset{1}{H}(t')-\overset{1}{H}(t'')|^r}{|t'-t''|^{1+r/2}}\,d\xi\,dt'\,dt''\Big)^{1/r}$$
$$\equiv I_1+I_2+I_3.$$

In view of the Minkowski and Hölder inequalities we have

$$I_1\le\Big(\int_0^t\int_0^t\frac{\big|\int_{t''}^{t'}\|u_\xi(\tau)\|_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}\,d\tau\big|^r\|\overset{1}{H}_\xi(t')\|^r_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}\|\overset{1}{H}(t')\|^r_{L_{r\lambda_3}(\overset{1}{\Omega}_0)}}{|t'-t''|^{1+r/2}}\,dt'\,dt''\Big)^{1/r}$$
$$\le t^a\overset{1}{\alpha}_u(t)\sup_{t'\le t}\|\overset{1}{H}(t')\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}\|\overset{1}{H}\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))},$$

where $1/\lambda_1+1/\lambda_2+1/\lambda_3=1$, $\|u_\xi\|_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}\le c\|u\|_{W_r^{3-2/r}(\overset{1}{\Omega}_0)}$ for $3/r-3/(r\lambda_1)\le 3-2/r$, $\|\overset{1}{H}_\xi\|_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}\le c\|\overset{1}{H}\|_{W_r^2(\overset{1}{\Omega}_0)}$ for $3/r-3/(r\lambda_2)\le 1$, and $\|\overset{1}{H}\|_{L_{r\lambda_3}(\overset{1}{\Omega}_0)}\le c\|\overset{1}{H}\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}$ for $3/r-3/(r\lambda_3)\le 2-2/r$.

Summarizing, the above estimate for I_1 holds for $r \ge 5/3$. Next

$$
\begin{aligned}
I_2 &\le \varphi(\overset{1}{\alpha}_u(t))\bigg(\int_0^t\int_0^t \frac{\|\overset{1}{H}_\xi(t')-\overset{1}{H}_\xi(t'')\|^r_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}}{|t'-t''|^{1+r/2}}\|\overset{1}{H}(t')\|_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}\,dt'\,dt''\bigg)^{1/r}\\
&\le \varphi(\overset{1}{\alpha}_u(t))\sup_{t'\le t}\|\overset{1}{H}(t')\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}\|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)},
\end{aligned}
$$

where we have used $1/\lambda_1+1/\lambda_2=1$, $\|\overset{1}{H}\|_{L_{r\lambda_2}(\overset{1}{\Omega}_0)} \le c\|\overset{1}{H}\|_{E_r^{2-2/r}(\overset{1}{\Omega}_0)}$ for $3/r-3/(r\lambda_2)\le 2-2/r$ and

$$
\bigg(\int_0^t\int_0^t \frac{\|\overset{1}{H}_\xi(t')-\overset{1}{H}_\xi(t'')\|^r_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r} \le c\|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)},
$$

which holds for $5/r-3/(r\lambda_1)-2/r+2\le 2$, so $\lambda_1=1$, $\lambda_2=\infty$. Therefore, it holds for $r>5/2$.

Finally, I_3 must be estimated. We have

$$
\begin{aligned}
I_3 &\le \varphi(\overset{1}{\alpha}_u(t))\bigg(\int_0^t\int_0^t \|\overset{1}{H}_\xi(t'')\|^r_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}\frac{\|\overset{1}{H}(t')-\overset{1}{H}(t'')\|^r_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r}\\
&\le \varphi(\overset{1}{\alpha}_u(t))\sup_{t'\le t}\|\overset{1}{H}(t')\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}\|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)},
\end{aligned}
$$

where the conditions $1/\lambda_1+1/\lambda_2=1$, $3/r-3/(r\lambda_1)+1\le 2-2/r$, $5/r-3/(r\lambda_2)-2/r+1\le 2$ must be satisfied. Hence the estimates hold for $r\ge 5/2$.

Summarizing the above estimates we get

$$
\|F_2\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)} \le c\|f\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)} + \varphi(\overset{1}{\alpha}_u(t))\sup_{t'\le t}\|\overset{1}{H}(t')\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}\cdot\|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} \tag{4.8}
$$

for $r>5/2$. Next we estimate

$$
\|G\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} \le \|G\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))} + \|G\|_{L_r(\overset{1}{\Omega}_0;W_r^1(0,t))} \equiv J_1+J_2. \tag{4.9}
$$

Qualitatively, we have

$$
\operatorname{div}_\xi v_n - \operatorname{div}_u v_n = (\mathbb{I}-\xi_x)v_{n,\xi}.
$$

The first term J_1 is bounded by the same expression as F^1_{12}. Therefore,

$$
J_1 = \|G\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))} \le \varphi(\overset{1}{\alpha}_u(t))\overset{1}{\alpha}_u(t)\|v_n\|_{L_r(0,t;W_r^3(\overset{1}{\Omega}_0))}
$$

for $r\ge 3/2$. To estimate J_2 we consider

$$
J_2 \le \varphi(\overset{1}{\alpha}_u(t))(\|u_\xi v_{n,\xi}\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \|(\mathbb{I}-\xi_x)v_{n,\xi t}\|_{L_r(\overset{1}{\Omega}{}_0^t)}) \equiv J_3+J_4.
$$

First, we examine

$$J_3=\varphi(\overset{1}{\alpha}_u(t))\Big(\int_0^t\|u_\xi v_{n,\xi}\|^r_{L_r(\overset{1}{\Omega}_0)}\,dt'\Big)^{1/r}$$
$$\le\varphi(\overset{1}{\alpha}_u(t))\Big(\int_0^t\|u_\xi\|^r_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}\|v_{n,\xi}\|^r_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}\,dt'\Big)^{1/r}\equiv J_3^1,$$

where $1/\lambda_1+1/\lambda_2=1$. Employing the interpolations

$$\|u_\xi\|_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}\le c\|u_{\xi\xi\xi}\|^{\theta_1}_{L_r(\overset{1}{\Omega}_0)}\|u_\xi\|^{1-\theta_1}_{L_r(\overset{1}{\Omega}_0)}$$

where θ_1 satisfies $3/(r\lambda_1)=(1-\theta_1)3/r+\theta_1(3/r-2)$ so $\theta_1=3/(2r\lambda_2)$, and

$$\|v_{n,\xi}\|_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}\le c\|v_{n,\xi\xi}\|^{\theta_2}_{L_r(\overset{1}{\Omega}_0)}\|v_{n,\xi}\|^{1-\theta_2}_{L_r(\overset{1}{\Omega}_0)},$$

where θ_2 satisfies $3/(r\lambda_2)=(1-\theta_2)3/r+\theta_2(3/r-2)$ so $\theta_2=3/(2r\lambda_1)$, and also the relations

$$u_\xi(t)=\int_0^t u_{\xi t'}\,dt'+u_\xi(0),\qquad v_{n,\xi}(t)=\int_0^t v_{n,\xi t'}\,dt'+v_\xi(0),$$

in J_3^1, we derive

$$J_3^1\le\varphi(\overset{1}{\alpha}_u(t))\Big(\int_0^t\|u_{\xi\xi\xi}\|^{\theta_1 r}_{L_r(\overset{1}{\Omega}_0)}\|v_{n,\xi\xi\xi}\|^{\theta_2 r}_{L_r(\overset{1}{\Omega}_0)}\,dt'\Big)^{1/r}$$
$$\cdot\Big|\int_0^t\|u_{\xi t'}\|_{L_r(\overset{1}{\Omega}_0)}\,dt'+\|u_\xi(0)\|_{L_r(\overset{1}{\Omega}_0)}\Big|^{1-\theta_1}$$
$$\cdot\Big|\int_0^t\|v_{n,\xi t'}\|_{L_r(\overset{1}{\Omega}_0)}\,dt'+\|v_\xi(0)\|_{L_r(\overset{1}{\Omega}_0)}\Big|^{1-\theta_2}\equiv J_3^2.$$

Since $\theta_1+\theta_2=3/(2r)<1$ for $r>3/2$ we obtain

$$J_3^2\le\varphi(\overset{1}{\alpha}_u(t))t^a\|u_{\xi\xi\xi}\|^{\theta_1}_{L_r(\overset{1}{\Omega}{}_0^t)}\|v_{n,\xi\xi\xi}\|^{\theta_2}_{L_r(\overset{1}{\Omega}{}_0^t)}$$
$$\cdot(t^{1/r'}\|u\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}+\|u_\xi(0)\|_{L_r(\overset{1}{\Omega}_0)})^{1-\theta_1}$$
$$\cdot(t^{1/r'}\|v_n\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}+\|v_\xi(0)\|_{L_r(\overset{1}{\Omega}_0)})^{1-\theta_2}$$
$$\le\varphi(\overset{1}{\alpha}_u(t))t^a\|u\|^{\theta_1}_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}\|v_n\|^{\theta_2}_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}$$
$$\cdot(t^{1/r'}\|u\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}+\|u(0)\|_{W_r^1(\overset{1}{\Omega}_0)})^{1-\theta_1}$$
$$\cdot(t^{1/r'}\|v_n\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}+\|v(0)\|_{W_r^1(\overset{1}{\Omega}_0)})^{1-\theta_2}$$

for some $a>0$. Finally,

$$J_4\le\varphi(\overset{1}{\alpha}_u(t))\overset{1}{\alpha}_u(t)\|v_n\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}.$$

Using the above estimates in (4.9) yields

$$
\begin{aligned}
\|G\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} &\le \varphi(\overset{1}{\alpha}_u(t))\overset{1}{\alpha}_u(t)\|v_n\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)} \\
&\quad + \varphi(\overset{1}{\alpha}_u(t))t^a\|u\|^{\theta_1}_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}\|v_n\|^{\theta_2}_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)} \\
&\quad \cdot (t^{1/r'}\|u\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)} + \|u(0)\|_{W_r^1(\overset{1}{\Omega}_0)})^{1-\theta_1} \\
&\quad \cdot (t^{1/r'}\|v_n\|_{Wr^{3,3/2}(\overset{1}{\Omega}{}_0^t)} + \|v(0)\|_{W_r^1(\overset{1}{\Omega}_0)})^{1-\theta_2}
\end{aligned}
\tag{4.10}
$$

for $r > 3/2$.

Now, we consider the expression

$$
\|R_t\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)} = \|R_t\|_{L_r(0,t;W_r^1(\overset{1}{\Omega}_0))} + \|R_t\|_{L_r(\overset{1}{\Omega}_0;W_r^{1/2}(0,t))} \equiv J_1 + J_2. \tag{4.11}
$$

Let

$$
\overset{1}{\alpha}(u) = \int_0^t u_\xi(\tau)\,d\tau. \tag{4.12}
$$

Then we examine

$$
\begin{aligned}
J_1 &= \|((\mathbb{I} - \xi_x)v_n)_{,t}\|_{L_r(0,t;W_r^1(\overset{1}{\Omega}_0))} \\
&\le \|\varphi(\overset{1}{\alpha}(u))u_\xi v_n\|_{L_r(0,t;W_r^1(\overset{1}{\Omega}_0))} + \Big\|\varphi(\overset{1}{\alpha}(u))\int_0^t u_\xi\,d\tau\, v_{nt}\Big\|_{L_r(0,t;W_r^1(\overset{1}{\Omega}_0))} \\
&\le \Big\|\varphi(\overset{1}{\alpha}(u))\int_0^t u_{\xi\xi}\,d\tau\, u_\xi v_n\Big\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \|\varphi(\overset{1}{\alpha}(u))u_{\xi\xi}v_n\|_{L_r(\overset{1}{\Omega}{}_0^t)} \\
&\quad + \|\varphi(\overset{1}{\alpha}(u))u_\xi v_{n\xi}\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \Big\|\varphi(\overset{1}{\alpha}(u))\int_0^t u_{\xi\xi}\,d\tau\int_0^t u_\xi\,d\tau\, v_{nt}\Big\|_{L_r(\overset{1}{\Omega}{}_0^t)} \\
&\quad + \Big\|\varphi(\overset{1}{\alpha}(u))\int_0^t u_{\xi\xi}\,d\tau\, v_{nt}\Big\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \Big\|\varphi(\overset{1}{\alpha}(u))\int_0^t u_\xi\,d\tau\, v_{n\xi t}\Big\|_{L_r(\overset{1}{\Omega}{}_0^t)} \equiv \sum_{\alpha=1}^6 K_\alpha.
\end{aligned}
$$

Now we estimate consecutive terms on the r.h.s. above:

$$
\begin{aligned}
K_1 &\le \varphi(\overset{1}{\alpha}_u(t))\Big\|\int_0^t u_{\xi\xi}\,d\tau\, u_\xi v_n\Big\|_{L_r(\overset{1}{\Omega}{}_0^t)} \\
&= \varphi(\overset{1}{\alpha}_u(t))\Big(\int_0^t\Big\|\int_0^{t'} u_{\xi\xi}\,d\tau\, u_\xi v_n\Big\|^r_{L_r(\overset{1}{\Omega}_0)}\,dt'\Big)^{1/r} \\
&\le \varphi(\overset{1}{\alpha}_u(t))\Big(\int_0^t\Big|\int_0^{t'}\|u_{\xi\xi}\|_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}\,d\tau\Big|^r\|u_\xi\|^r_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}\|v_n\|^r_{L_{r\lambda_3}(\overset{1}{\Omega}_0)}\,dt'\Big)^{1/r} \equiv K_1^1,
\end{aligned}
$$

where $1/\lambda_1+1/\lambda_2+1/\lambda_3=1$. Using the imbeddings

$$\|u_{\xi\xi}\|_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}\le c\|u\|_{W_r^3(\overset{1}{\Omega}_0)}\qquad \text{for } 3/r-3/(r\lambda_1)\le 1,$$

$$\|u_{\xi}\|_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}\le c\|u\|_{W_r^3(\overset{1}{\Omega}_0)}\qquad \text{for } 3/r-3/(r\lambda_2)\le 2,$$

$$\|v_n\|_{L_{r\lambda_3}(\overset{1}{\Omega}_0)}\le c\|v_n\|_{W_r^{3-3/r}(\overset{1}{\Omega}_0)}\qquad \text{for } 3/r-3/(r\lambda_3)\le 3-2/r,$$

which hold for $r\ge 4/3$, we obtain

$$K_1^1\le\varphi(\overset{1}{\alpha}_u(t))\overset{1}{\alpha}_u(t)\|u\|_{L_r(0,t;W_r^3(\overset{1}{\Omega}_0))}\|v_n\|_{L_\infty(0,t;W_r^{3-2/r}(\overset{1}{\Omega}_0))}.$$

Next,

$$\begin{aligned}K_2&\le\varphi(\overset{1}{\alpha}_u(t))\|u_{\xi\xi}v_n\|_{L_r(\overset{1}{\Omega}{}_0^t)}\\&\le\varphi(\overset{1}{\alpha}_u(t))\bigg(\int_0^t\|u_{\xi\xi}\|^r_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}\|v_n\|^r_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}\,dt'\bigg)^{1/r}\\&\le\varphi(\overset{1}{\alpha}_u(t))t^{1/r}\|u\|_{L_\infty(0,t;W_r^{3-2/r}(\overset{1}{\Omega}_0))}\|v_n\|_{L_\infty(0,t;W_r^{3-2/r}(\overset{1}{\Omega}_0))},\end{aligned}$$

where we have used $1/\lambda_1+1/\lambda_2=1$ and the imbeddings

$$\|u_{\xi\xi}\|_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}\le c\|u\|_{W_r^{3-2/r}(\overset{1}{\Omega}_0)},\qquad 3/r-3/(r\lambda_1)\le 1-2/r,$$

$$\|v_n\|_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}\le c\|v_n\|_{W_r^{3-2/r}(\overset{1}{\Omega}_0)},\qquad 3/r-3/(r\lambda_2)\le 3-2/r,$$

which hold for $r\ge 7/4$. Similarly,

$$K_3\le\varphi(\overset{1}{\alpha}_u(t))t^{1/r}\|u\|_{L_\infty(0,t;W_r^{3-2/r}(\overset{1}{\Omega}_0))}\|v_n\|_{L_\infty(0,t;W_r^{3-2/r}(\overset{1}{\Omega}_0))}$$

for $r\ge 7/4$. Next,

$$\begin{aligned}K_4&\le\varphi(\overset{1}{\alpha}_u(t))\int_0^t\|u_\xi(\tau)\|_{L_\infty(\overset{1}{\Omega}_0)}\,d\tau\,\bigg(\int_0^t\bigg|\int_0^{t'}\|u_{\xi\xi}(\tau)\|_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}\,d\tau\bigg|^r\|v_{nt'}\|^r_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}\,dt'\bigg)^{1/r}\\&\le\varphi(\overset{1}{\alpha}_u(t))(\overset{1}{\alpha}_u(t))^2\|v_n\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}\end{aligned}$$

for $1/\lambda_1+1/\lambda_2=1$, $3/r-3/(r\lambda_1)\le 1$, $3/r-3/(r\lambda_2)\le 1$ so for $r\ge 3/2$. Finally, we have

$$K_5+K_6\le\varphi(\overset{1}{\alpha}_u(t))\overset{1}{\alpha}_u(t)\|v_n\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}.$$

Summarizing, we derive

$$J_1\le\varphi(\overset{1}{\alpha}_u(t))[\overset{1}{\alpha}_u(t)(\|u\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}+1)\|v_n\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}]\tag{4.13}$$

for $r\ge 7/4$.

Finally, we estimate J_2:

$$\begin{aligned}J_2&\le\|((\mathbb{I}-\xi_x)v_n)_{,t}\|_{L_r(\overset{1}{\Omega}_0;W_r^{1/2}(0,t))}\\&\le\|\varphi(\overset{1}{\alpha}(u))u_\xi v_n\|_{L_r(\overset{1}{\Omega}_0;W_r^{1/2}(0,t))}+\|(\mathbb{I}-\xi_x)v_{nt}\|_{L_r(\overset{1}{\Omega}_0;W_r^{1/2}(0,t))}\equiv H_1+H_2.\end{aligned}$$

Continuing,

$$H_1 \le \bigg(\int\limits_{\overset{1}{\Omega}_0} \int\limits_0^t \int\limits_0^t \frac{|(\varphi(\overset{1}{\alpha}(u)) u_\xi v_n)(t') - (\varphi(\overset{1}{\alpha}(u)) u_\xi v_n)(t'')|^r}{|t'-t''|^{1+r/2}} \, d\xi \, dt' \, dt'' \bigg)^{1/r}$$

$$\le \bigg(\int\limits_{\overset{1}{\Omega}_0} \int\limits_0^t \int\limits_0^t \frac{|\varphi(\overset{1}{\alpha}(u(t'))) - \varphi(\overset{1}{\alpha}(u(t'')))|^r |u_\xi(t')|^r |v_n(t')|^r}{|t'-t''|^{1+r/2}} \, d\xi \, dt' \, dt'' \bigg)^{1/r}$$

$$+ \bigg(\int\limits_{\overset{1}{\Omega}_0} \int\limits_0^t \int\limits_0^t \frac{|\varphi(\overset{1}{\alpha}(u(t'')))|^r |u_\xi(t') - u_\xi(t'')|^r |v_n(t')|^r}{|t'-t''|^{1+r/2}} \, d\xi \, dt' \, dt'' \bigg)^{1/r}$$

$$+ \bigg(\int\limits_{\overset{1}{\Omega}_0} \int\limits_0^t \int\limits_0^t \frac{|\varphi(\overset{1}{\alpha}(u(t'')))|^r |u_\xi(t'')|^r |v_n(t') - v_n(t'')|^r}{|t'-t''|^{1+r/2}} \, d\xi \, dt' \, dt'' \bigg)^{1/r}$$

$$\equiv H_1^1 + H_1^2 + H_1^3.$$

By the Hölder inequality

$$H_1^1 \le \varphi(\overset{1}{\alpha}_u(t))$$

$$\cdot \bigg(\int\limits_0^t \int\limits_0^t \frac{\left|\int_{t''}^{t'} \|u_{,\xi}(\tau)\|_{L_{r\lambda_1}(\overset{1}{\Omega}_0)} \, d\tau\right|^r \|u_\xi(t')\|^r_{L_{r\lambda_2}(\overset{1}{\Omega}_0)} \|v_n(t')\|^r_{L_{r\lambda_3}(\overset{1}{\Omega}_0)}}{|t'-t''|^{1+r/2}} \, dt' \, dt'' \bigg)^{1/r}$$

$$\le \varphi(\overset{1}{\alpha}_u(t)) t^a \|u\|^2_{L_\infty(0,t;W_r^{3-2/r}(\overset{1}{\Omega}_0))} \|v_n\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)},$$

where we have used $1/\lambda_1 + 1/\lambda_2 + 1/\lambda_3 = 1$, $3/r - 3/(r\lambda_1) \le 2 - 2/r$, $3/r - 3/(r\lambda_2) \le 2 - 2/r$, $5/r - 3/(r\lambda_3) - 2/r \le 3$, which are satisfied for $r \ge 10/7$.

Next,

$$H_1^2 \le \varphi(\overset{1}{\alpha}_u(t)) \bigg(\int\limits_0^t \int\limits_0^t \frac{\|u_\xi(t') - u_\xi(t'')\|^r_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}}{|t'-t''|^{1+r/2}} \|v_n(t')\|^r_{L_{r\lambda_2}(\overset{1}{\Omega}_0)} \, dt' \, dt'' \bigg)^{1/r}$$

$$\le \varphi(\overset{1}{\alpha}_u(t)) \sup_{t' \le t} \|v_n(t')\|_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}$$

$$\cdot \bigg(\int\limits_0^t \int\limits_0^t \frac{|t'-t''|^{ra} \|u_\xi(t') - u_\xi(t'')\|^r_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}}{|t'-t''|^{1+r/2+ra}} \, dt' \, dt'' \bigg)^{1/r}.$$

Employing the imbeddings corresponding to the inequalities

$$5/r - 3/(r\lambda_1) - 2/r + 1 + 2a \le 2, \quad 3/r - 3/(r\lambda_2) \le 3 - 2/r, \quad 1/\lambda_1 + 1/\lambda_2 = 1,$$

which hold for $r \ge 5/(4-2a)$, we obtain the estimate

$$H_1^2 \le \varphi(\overset{1}{\alpha}_u(t)) t^a \sup_{t' \le t} \|v_n(t')\|_{W_r^{3-2/r}(\overset{1}{\Omega}_0)} \|u\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}.$$

Similarly,

$$H_1^3 \le \varphi(\overset{1}{\alpha}_u(t)) t^a \|u\|_{L_\infty(0,t;W_r^{3-2/r}(\overset{1}{\Omega}_0))} \|v_n\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}.$$

Finally,

$$H_2 \le \varphi(\overset{1}{\alpha}_u(t))(t^a + \overset{1}{\alpha}_u(t))\|v_n\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}.$$

Summarizing,

$$J_2 \le \varphi(\overset{1}{\alpha}_u(t))(t^a\|u\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)} + \overset{1}{\alpha}_u(t))\|v_n\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}, \tag{4.14}$$

which holds for $r \ge \max\{7/4, 5/(4-2a)\}$.

Finally, the norms of K_1, K_2 in (4.5) can be estimated by the same bounds as the norms of F_1, F_2, respectively.

Employing the above estimates in (4.5) yields

$$\begin{aligned}
&\|v_{n+1}\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)} + \sup_{t'\le t}\|v_{n+1}(t')\|_{W_r^{3-2/r}(\overset{1}{\Omega}_0)} \\
&\qquad\qquad + \|p_{n+1}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|p_{n+1}\|_{W_r^{2-1/r,1-1/(2r)}(S_0^t)} \\
&\quad \le \varphi(\overset{1}{\alpha}_u(t))[\overset{1}{\alpha}_u(t) + t^a\|u\|_{L_\infty(0,t;W_r^{3-2/r}(\overset{1}{\Omega}_0))} \\
&\qquad + t^a(\|u\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)} + \|u(0)\|_{W_r^1(\overset{1}{\Omega}_0)})]\|v_n\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)} \\
&\qquad + c\|f\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)} + \varphi(\overset{1}{\alpha}_u(t))\sup_{t'\le t}\|\overset{1}{H}(t')\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}\|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} \\
&\qquad + \varphi(\overset{1}{\alpha}_u(t))t^a(\|u\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)} + \|u(0)\|_{W_r^1(\overset{1}{\Omega}_0)})\|v(0)\|_{W_r^1(\overset{1}{\Omega}_0)}
\end{aligned} \tag{4.15}$$

for $r \ge 5/2$. Introducing the notation

$$\begin{aligned}
X_n(t) &= \|v_n\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)} + \sup_{t'\le t}\|v_n(t')\|_{W_r^{3-2/r}(\overset{1}{\Omega}_0)} \\
&\quad + \|p_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|p_n\|_{W_r^{2-1/r,1-1/(2r)}(S_0^t)}
\end{aligned}$$

and

$$\begin{aligned}
D &= c\|f\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)} + \varphi(\overset{1}{\alpha}_u(t))\sup_{t'\le t}\|\overset{1}{H}(t')\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}\|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} \\
&\quad + \varphi(\overset{1}{\alpha}_u(t))t^a(\|u\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)} + \|u(0)\|_{W_r^1(\overset{1}{\Omega}_0)})\|v(0)\|_{W_r^1(\overset{1}{\Omega}_0)}
\end{aligned}$$

we obtain from (4.15) the inequality

$$X_{n+1}(t) \le \varphi(\overset{1}{\alpha}_u(t))[\overset{1}{\alpha}_u(t) + t^a\|u\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)} + t^a\|u(0)\|_{W_r^1(\overset{1}{\Omega}_0)}]X_n + D. \tag{4.16}$$

Assuming that $X_n \le 2D$ and t is so small that

$$\varphi(\overset{1}{\alpha}_u(t))[\overset{1}{\alpha}_u(t) + t^a\|u\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)} + t^a\|u(0)\|_{W_r^1(\overset{1}{\Omega}_0)}] \le 1/2 \tag{4.17}$$

we obtain

$$X_n \le 2D \quad \text{ for all } n \in \mathbb{N}$$

because we have assumed that $X_0 = 0$.

To show convergence of the constructed sequence $\{v_n, p_n\}$ we introduce the differences

$$V_n = v_n - v_{n-1}, \quad P_n = p_n - p_{n-1} \tag{4.18}$$

which are solutions to the following problem derived from (4.4):

$$\begin{aligned}
&V_{n+1,t} - \operatorname{div}_\xi \mathbb{T}_\xi(V_{n+1}, P_{n+1}) \\
&\quad = -(\operatorname{div}_\xi \mathbb{T}_\xi(V_n, P_n) - \operatorname{div}_u \mathbb{T}_u(V_n, P_n)) && \text{in } \overset{1}{\Omega}{}_0^t, \\
&\operatorname{div}_\xi V_{n+1} = \operatorname{div}_\xi V_n - \operatorname{div}_u V_n && \text{in } \overset{1}{\Omega}{}_0^t, \\
&\bar{n}_\xi \cdot \mathbb{T}_\xi(V_{n+1}, P_{n+1}) = \bar{n}_\xi \cdot \mathbb{T}_\xi(V_n, P_n) - \bar{n}_u \cdot \mathbb{T}_u(V_n, P_n) && \text{on } S_0^t, \\
&V_{n+1}|_{t=0} = 0.
\end{aligned} \tag{4.19}$$

We introduce the quantity

$$\begin{aligned}
Y_n = {}& \|V_n\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)} + \sup_{t'\le t} \|V_n(t')\|_{W_r^{3-2/r}(\overset{1}{\Omega}_0)} \\
&+ \|P_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|P_n\|_{W_r^{2-1/r,1-1/(2r)}(S_0^t)}.
\end{aligned} \tag{4.20}$$

By standard considerations we obtain from problem (4.19) the inequality

$$Y_{n+1} \le \varphi(D) t^a Y_n, \qquad n \in \mathbb{N}. \tag{4.21}$$

For t sufficiently small the sequence $\{v_n, p_n\}$ converges to a limit (v, p) which is a solution to problem (4.2). This concludes the proof of Lemma 4.1. ■

5. Existence of solutions to problem (3.6)

In this section we prove existence of solutions to problem (3.6) for given $\overset{i}{\bar{v}}$, $i = 1, 2$. Therefore, it is convenient to introduce the simplified notation

$$\overset{i}{u} = \overset{i}{\bar{v}}, \quad \overset{i}{H} = \overset{i}{\bar{H}}, \quad i = 1, 2. \tag{5.1}$$

In view of (5.1) and after elimination of the electric fields, problem (3.6) takes the form

$$\begin{aligned}
&\mu_1 \overset{1}{H}_{,t} + \frac{1}{\sigma_1} \operatorname{rot}^2_{\overset{1}{u}} \overset{1}{H} = \mu_1 \operatorname{rot}_{\overset{1}{u}} (\overset{1}{u} \times \overset{1}{H}) + \mu_1 \overset{1}{u} \cdot \nabla_{\overset{1}{u}} \overset{1}{H} \equiv \overset{1}{f}_0, \\
&\operatorname{div}_{\overset{1}{u}} \overset{1}{H} = 0 && \text{in } \overset{1}{\Omega}_0, \\
&\mu_2 \overset{2}{H}_{,t} + \frac{1}{\sigma_2} \operatorname{rot}^2_{\overset{2}{u}} \overset{2}{H} = \mu_2 \overset{2}{u} \cdot \nabla_{\overset{2}{u}} \overset{2}{H} \equiv \overset{2}{f}_0, \quad \operatorname{div}_{\overset{2}{u}} \overset{2}{H} = 0 && \text{in } \overset{2}{\Omega}_0, \\
&\left(\frac{1}{\sigma_1} \operatorname{rot}_{\overset{1}{u}} \overset{1}{H} - \frac{1}{\sigma_2} \operatorname{rot}_{\overset{2}{u}} \overset{2}{H}\right) \cdot \bar{\tau}_{u\alpha} = \mu_1 \overset{1}{u} \times \overset{1}{H} \cdot \bar{\tau}_{u\alpha} \equiv k_{0\alpha}, \qquad \alpha = 1, 2, && \text{on } S_0, \\
&(\overset{1}{b}\overset{1}{H} - \overset{2}{b}\overset{2}{H}) \cdot \bar{n}_u \times \bar{\tau}_{u\alpha} = 0, \quad \alpha = 1, 2, && \text{on } S_0, \\
&\overset{i}{H}|_{t=0} = \overset{i}{H}(0), \quad i = 1, 2, \quad \overset{2}{H} \cdot \bar{\tau}_\alpha|_B = H_{*\alpha}, \quad \operatorname{div}_{\overset{2}{u}} \overset{2}{H}|_B = 0, \quad \alpha = 1, 2,
\end{aligned} \tag{5.2}$$

where $u = \overset{1}{u} = \overset{2}{u}$ on S_0.

We prove existence of solutions to problem (5.2) by the method of successive approximations. First we show the existence of solutions to the following problem with constant coefficients:

$$
\begin{aligned}
&\mu_1\overset{1}{H}_{,t}+\frac{1}{\sigma_1}\operatorname{rot}_\xi^2\overset{1}{H}=\frac{1}{\sigma_1}(\operatorname{rot}_\xi^2\overset{1}{H}-\operatorname{rot}_{\overset{1}{u}}^2\overset{1}{H})+\overset{1}{f}_0\equiv\overset{1}{f}_*,\\
&\operatorname{div}_\xi\overset{1}{H}=\operatorname{div}_\xi\overset{1}{H}-\operatorname{div}_{\overset{1}{u}}\overset{1}{H}\equiv\overset{1}{g}_*,\\
&\mu_2\overset{2}{H}_{,t}+\frac{1}{\sigma_2}\operatorname{rot}_\xi^2\overset{2}{H}=\frac{1}{\sigma_2}(\operatorname{rot}_\xi^2\overset{2}{H}-\operatorname{rot}_{\overset{2}{u}}^2\overset{2}{H})+\overset{2}{f}_0\equiv\overset{2}{f}_*,\\
&\operatorname{div}_\xi\overset{2}{H}=\operatorname{div}_\xi\overset{2}{H}-\operatorname{div}_{\overset{2}{u}}\overset{2}{H}\equiv\overset{2}{g}_*,\\
&\left(\frac{1}{\sigma_1}\operatorname{rot}_\xi\overset{1}{H}-\frac{1}{\sigma_2}\operatorname{rot}_\xi\overset{2}{H}\right)\cdot\bar\tau_\alpha=\left[\left(\frac{1}{\sigma_1}\operatorname{rot}_\xi\overset{1}{H}-\frac{1}{\sigma_1}\operatorname{rot}_{\overset{1}{u}}\overset{1}{H}\right)\right.\\
&\qquad\left.-\left(\frac{1}{\sigma_2}\operatorname{rot}_\xi\overset{2}{H}-\frac{1}{\sigma_2}\operatorname{rot}_{\overset{2}{u}}\overset{2}{H}\right)\right]\cdot\bar\tau_\alpha\\
&\qquad+\left(\frac{1}{\sigma_1}\operatorname{rot}_{\overset{1}{u}}\overset{1}{H}-\frac{1}{\sigma_2}\operatorname{rot}_{\overset{2}{u}}\overset{2}{H}\right)\cdot(\bar\tau_\alpha-\bar\tau_{u\alpha})+k_{0\alpha}\equiv k_{*\alpha},\quad \alpha=1,2,\\
&(\overset{1}{b}\overset{1}{H}-\overset{2}{b}\overset{2}{H})\cdot(\bar n\times\bar\tau_\alpha)=(\bar n\times\bar\tau_\alpha-\bar n_u\times\bar\tau_{u\alpha})\cdot(\overset{1}{b}\overset{1}{H}-\overset{2}{b}\overset{2}{H})\equiv l_{*\alpha},\quad \alpha=1,2,\\
&\overset{i}{H}|_{t=0}=\overset{i}{H}(0),\quad i=1,2,\quad \overset{2}{H}\cdot\bar\tau'_\alpha|_B=H_{*\alpha},\quad \operatorname{div}\overset{2}{H}|_B=0,\quad \alpha=1,2,
\end{aligned}
\tag{5.3}
$$

where $\bar n$ is the unit vector normal to S_0 and $\bar\tau_\alpha$, $\alpha=1,2$, are tangent vectors to S_0. Moreover, $\bar\tau'_\alpha$, $\alpha=1,2$, are tangent to B.

To treat problem (5.3) as the problem with constant coefficients it is convenient to write it in the following short form:

$$
\begin{aligned}
&\mu_i\overset{i}{H}_{,t}+\frac{1}{\sigma_i}\operatorname{rot}_\xi^2\overset{i}{H}=\overset{i}{f}_*,\quad \operatorname{div}_\xi\overset{i}{H}=\overset{i}{g}_*,\quad i=1,2,\\
&\left(\frac{1}{\sigma_1}\operatorname{rot}_\xi\overset{1}{H}-\frac{1}{\sigma_2}\operatorname{rot}_\xi\overset{2}{H}\right)\cdot\bar\tau_\alpha=k_{*\alpha},\quad \alpha=1,2,\\
&(\overset{1}{b}\overset{1}{H}-\overset{2}{b}\overset{2}{H})\cdot\bar n\times\bar\tau_\alpha=l_{*\alpha},\quad \alpha=1,2,\\
&\overset{i}{H}|_{t=0}=\overset{i}{H}(0),\quad i=1,2,\quad \overset{2}{H}\cdot\bar\tau'_\alpha|_B=H_{*\alpha},\quad \operatorname{div}\overset{2}{H}|_B=0,\quad \alpha=1,2.
\end{aligned}
\tag{5.4}
$$

REMARK 5.1. Now we are going to prove existence of solutions to problem (5.4) applying the regularizer technique (see [LSU, Ch. 4, Sect. 7]). For this we need problem (5.4) with vanishing initial data. Looking for such solutions to (5.4) that $\overset{i}{H}\in W_r^{2,1}$, $r\ge 5/2$, $i=1,2$, we need to have initial data in $W_r^{2-2/r}$. Let $\overset{i}{\tilde H}$, $i=1,2$, be divergence free extensions of initial data such that $\overset{i}{\tilde H}\in W_r^{2,1}(\overset{i}{\Omega}{}_0^t)$, $i=1,2$, and

$$
\|\overset{i}{\tilde H}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)}\le c\|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)},\quad i=1,2. \tag{5.5}
$$

Set

$$
\overset{i}{u}=\overset{i}{H}-\overset{i}{\tilde H},\qquad i=1,2. \tag{5.6}
$$

Then problem (5.4) takes the form

$$
\begin{aligned}
&\mu_i\overset{i}{u}_t + \frac{1}{\sigma_i}\operatorname{rot}^2_\xi \overset{i}{u} = \overset{i}{f}_* - \left(\mu_i\overset{i}{\tilde H}_{,t} + \frac{1}{\sigma_i}\operatorname{rot}^2_\xi \overset{i}{\tilde H}\right) \equiv \overset{i}{f}, \\
& && \text{in } \overset{i}{\Omega}{}^t_0,\ i=1,2,\\
&\operatorname{div}_\xi \overset{i}{u} = \overset{i}{g}_* \equiv \overset{i}{g},\\
&\left(\frac{1}{\sigma_1}\operatorname{rot}_\xi \overset{1}{u} - \frac{1}{\sigma_2}\operatorname{rot}_\xi \overset{2}{u}\right)\cdot\bar\tau_\alpha\\
&\quad = k_{*\alpha} - \left(\frac{1}{\sigma_1}\operatorname{rot}_\xi \overset{1}{\tilde H} - \frac{1}{\sigma_2}\operatorname{rot}_\xi \overset{2}{\tilde H}\right)\cdot\bar\tau_\alpha \equiv k_\alpha,\quad \alpha=1,2, && \text{on } S^t_0,\\
&(\overset{1}{b}\overset{1}{u} - \overset{2}{b}\overset{2}{u})\cdot\bar n\times\bar\tau_\alpha = l_{*\alpha} - (\overset{1}{b}\overset{1}{\tilde H} - \overset{2}{b}\overset{2}{\tilde H})\cdot\bar n\times\bar\tau_\alpha \equiv l_\alpha,\quad \alpha=1,2, && \text{on } S^t_0,\\
&\overset{2}{u}\cdot\bar\tau'_\alpha|_B = H_{*\alpha} - \overset{2}{\tilde H}\cdot\bar\tau'_\alpha|_B \equiv \tilde H_{*\alpha},\quad \operatorname{div}\overset{2}{u}|_B = 0,\quad \alpha=1,2,\\
&\overset{i}{u}|_{t=0} = 0,\quad i=1,2.
\end{aligned}
\tag{5.7}
$$

To express the proof of existence in a more convenient way we replace ξ by x. Then problem (5.7) takes the form

$$
\begin{aligned}
&\mu_i\overset{i}{u}_{,t} + \frac{1}{\sigma_i}\operatorname{rot}^2_x \overset{i}{u} = \overset{i}{f},\quad \operatorname{div}_x \overset{i}{u} = \overset{i}{g},\quad i=1,2,\\
&\left(\frac{1}{\sigma_1}\operatorname{rot}_x \overset{1}{u} - \frac{1}{\sigma_2}\operatorname{rot}_x \overset{2}{u}\right)\cdot\bar\tau_\alpha = k_\alpha,\quad \alpha=1,2,\\
&(\overset{1}{b}\overset{1}{u} - \overset{2}{b}\overset{2}{u})\cdot\bar n\times\bar\tau_\alpha = l_\alpha,\quad \alpha=1,2,\\
&\overset{2}{u}\cdot\bar\tau'_\alpha|_B = H_{*\alpha} - \overset{2}{\tilde H}\cdot\bar\tau'_\alpha|_B \equiv \tilde H_{*\alpha},\quad \operatorname{div}\overset{2}{u}|_B = 0,\quad \alpha=1,2,\\
&\overset{i}{u}|_{t=0} = 0,\quad i=1,2.
\end{aligned}
\tag{5.8}
$$

To apply the regularizer technique we introduce the following notation:

$$
\begin{aligned}
&L^1_i(\partial_x,\partial_t)\overset{i}{u} = \mu_i\overset{i}{u}_{,t} + \frac{1}{\sigma_i}\operatorname{rot}^2_x \overset{i}{u},\quad L^2_i(\partial_x,\partial_t)\overset{i}{u} = \operatorname{div}_x \overset{i}{u}, \qquad i=1,2,\\
&L_i = (L^1_i, L^2_i),\\
&B_{1\alpha}(\xi,\partial_x)(\overset{1}{u},\overset{2}{u}) = \left(\frac{1}{\sigma_1}\operatorname{rot}_x \overset{1}{u} - \frac{1}{\sigma_2}\operatorname{rot}_x \overset{2}{u}\right)\cdot\bar\tau_\alpha(\xi),\quad \alpha=1,2,\\
&B_{2\alpha}(\xi)(\overset{1}{u},\overset{2}{u}) = (\overset{1}{b}\overset{1}{u} - \overset{2}{b}\overset{2}{u})\cdot\bar n(\xi)\times\bar\tau_\alpha(\xi),\quad \alpha=1,2,\quad \xi = x|_{S_0},\\
&B_{3\alpha}\overset{2}{u} = \overset{2}{u}\cdot\bar\tau'_\alpha,\quad \alpha=1,2,\quad B_{33} = \operatorname{div}\overset{2}{u}.
\end{aligned}
\tag{5.9}
$$

Let $k\in\mathfrak{M}_1$ and $\overset{1}{f}{}^{(k)}(x,t) = \zeta^{(k)}(x)\overset{1}{f}(x,t)$, $\overset{1}{g}{}^{(k)}(x,t) = \zeta^{(k)}(x)\overset{1}{g}(x,t)$. Let $\overset{1}{f}{}^{(k)}\in L_r(\mathbb{R}^3\times(0,\tau))$, $\overset{1}{g}{}^{(k)}\in L_r(0,\tau;W^1_r(\mathbb{R}^3))$. We denote by $R^{(k)}$ the operator which solves the Cauchy problem with vanishing initial data

$$
L_1(\partial_x,\partial_t)\overset{1}{u}{}^{(k)}(x,t) = (\overset{1}{f}{}^{(k)}(x,t),\overset{1}{g}{}^{(k)}(x,t)).
\tag{5.10}
$$

In view of Lemma 6.6 the operator $R^{(k)}$ exists for τ sufficiently small and $\overset{1}{u}{}^{(k)}=$

$R^{(k)}(\overset{1}{f}{}^{(k)},\overset{1}{g}{}^{(k)})$. Moreover,

$$\|R^{(k)}(\overset{1}{f}{}^{(k)},\overset{1}{g}{}^{(k)})\|_{W_r^{2,1}(\mathbb{R}^3\times(0,\tau))} \le c(\|\overset{1}{f}{}^{(k)}\|_{L_r(\mathbb{R}^3\times(0,\tau))}+\|\overset{1}{g}{}^{(k)}\|_{L_r(0,\tau;W_r^1(\mathbb{R}^3))}),\quad k\in\mathfrak{M}_1. \tag{5.11}$$

For $k\in\mathfrak{M}_2$, we have a similar result for the Cauchy problem with vanishing initial data

$$L_2(\partial_x,\partial_t)\overset{2}{u}{}^{(k)}(x,t)=(\overset{2}{f}{}^{(k)}(x,t),\overset{2}{g}{}^{(k)}(x,t)). \tag{5.12}$$

Hence $\overset{2}{u}{}^{(k)}=R^{(k)}(\overset{2}{f}{}^{(k)},\overset{2}{g}{}^{(k)})$, $k\in\mathfrak{M}_2$, and

$$\|R^{(k)}(\overset{2}{f}{}^{(k)},\overset{2}{g}{}^{(k)})\|_{W_r^{2,1}(\mathbb{R}^3\times(0,\tau))} \le c(\|\overset{2}{f}{}^{(k)}\|_{L_r(\mathbb{R}^3\times(0,\tau))}+\|\overset{2}{g}{}^{(k)}\|_{L_r(0,\tau;W_r^1(\mathbb{R}^3))}),\quad k\in\mathfrak{M}_2. \tag{5.13}$$

For $k\in\mathfrak{N}_1$, the operator $R^{(k)}$ gives a solution to the following initial boundary value problem with vanishing initial data:

$$\begin{aligned} &L_i(\partial_z,\partial_t)\overset{i}{u}{}^{(k)}(z,t)=(\overset{i}{f}{}^{(k)},\overset{i}{g}{}^{(k)})\quad \text{in } \overset{i}{\mathbb{R}}\times(0,\tau),\quad i=1,2,\\ &B_{1\alpha}(\xi^{(k)},\partial_z)(\overset{1}{u}{}^{(k)}(z,t),\overset{2}{u}{}^{(k)}(z,t))=k_\alpha^{(k)},\quad \alpha=1,2,\quad z_3=0,\\ &B_{2\alpha}(\xi^{(k)})(\overset{1}{u}{}^{(k)}(z,t))=l_\alpha^{(k)},\quad \alpha=1,2,\quad z_3=0, \end{aligned} \tag{5.14}$$

where $\xi^{(k)}\in S_0$, $S_0=\{z\in\mathbb{R}^3: z_3=0\}$ and we use the definition of the partition of unity and coordinates z introduced at the beginning of Section 2 and $\overset{1}{\mathbb{R}}=\mathbb{R}^3_+=\{z: z_3>0\}$, $\overset{2}{\mathbb{R}}=\mathbb{R}^3_-=\{z: z_3<0\}$.

In view of Lemma 6.2, the operator $R^{(k)}$ is expressed by

$$\begin{aligned} (\overset{1}{u}{}^{(k)}(z,t),\overset{2}{u}{}^{(k)}(z,t))&=R^{(k)}(\overset{1}{f}{}^{(k)}(z,t),\overset{2}{f}{}^{(k)}(z,t),\overset{1}{g}{}^{(k)}(z,t),\overset{2}{g}{}^{(k)}(z,t),\\ &\quad k_1^{(k)}(z,t),k_2^{(k)}(z',t),l_1^{(k)}(z',t),l_2^{(k)}(z',t),0),\quad z'=(z_1,z_2),\ k\in\mathfrak{N}_1. \end{aligned} \tag{5.15}$$

Moreover, we have

$$\begin{aligned} &\|R^{(k)}(\overset{1}{f}{}^{(k)},\overset{2}{f}{}^{(k)},\overset{1}{g}{}^{(k)},\overset{2}{g}{}^{(k)},k_1^{(k)},k_2^{(k)},l_1^{(k)},l_2^{(k)},0)\|_{W_r^{2,1}(\overset{1}{\mathbb{R}}\times(0,\tau))\times W_r^{2,1}(\overset{2}{\mathbb{R}}\times(0,\tau))}\\ &\quad\le c\sum_{i=1}^2(\|\overset{i}{f}{}^{(k)}\|_{L_r(\overset{i}{\mathbb{R}}\times(0,\tau))}+\|\overset{i}{g}{}^{(k)}\|_{L_r(0,\tau;W_r^1(\overset{i}{\mathbb{R}}))}+\|k_i^{(k)}\|_{W_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}^2\times(0,\tau))}\\ &\qquad+\|l_i^{(k)}\|_{W_r^{2-1/r,1-1/(2r)}(\mathbb{R}^2\times(0,\tau))}),\quad k\in\mathfrak{N}_1. \end{aligned} \tag{5.16}$$

Finally, for $k\in\mathfrak{N}_2$ and Lemma 6.4, the operator $R^{(k)}$ gives a solution to the initial boundary value problem with vanishing initial data

$$\begin{aligned} &L_2(\partial_z,\partial_t)\overset{2}{u}{}^{(k)}(z,t)=(\overset{2}{f}{}^{(k)}(z,t),\overset{2}{g}{}^{(k)}(z,t)),\\ &\overset{2}{u}{}^{(k)}_\alpha|_{z_3=0}=b_\alpha^{(k)},\quad \operatorname{div}\overset{2}{u}{}^{(k)}|_{z_3=0}=0,\quad \alpha=1,2, \end{aligned} \tag{5.17}$$

where $\overset{2}{u}{}^{(k)} = R^{(k)}(\overset{2}{f}{}^{(k)}, \overset{2}{g}{}^{(k)}, b^{(k)})$, $b^{(k)} = (b_1^{(k)}, b_2^{(k)})$ and

$$\begin{aligned}\|R^{(k)}(\overset{2}{f}{}^{(k)}, \overset{2}{g}{}^{(k)})\|_{W_r^{2,1}(\mathbb{R}^3_+\times(0,\tau))} \\ \le c(\|\overset{2}{f}{}^{(k)}\|_{L_r(\mathbb{R}^3_+\times(0,\tau))} + \|\overset{2}{g}{}^{(k)}\|_{L_r(0,\tau;W_r^1(\mathbb{R}^3_+))} \\ + \|b^{(k)}\|_{W_r^{2-1/r,1-1/(2r)}(\mathbb{R}^2\times(0,\tau))}), \quad k \in \mathfrak{N}_2. \end{aligned} \tag{5.18}$$

Set

$$h = (\overset{1}{f}, \overset{2}{f}, \overset{1}{g}, \overset{2}{g}, k_1, k_2, l_1, l_2, b). \tag{5.19}$$

Then

$$\begin{aligned} h^{(k)} &= (\overset{1}{f}{}^{(k)}, \overset{1}{g}{}^{(k)}, 0,0,0,0,0,0,0) \quad \text{for } k \in \mathfrak{M}_1,\\ h^{(k)} &= (0,0,\overset{2}{f}{}^{(k)}, \overset{2}{g}{}^{(k)}, 0,0,0,0,0) \quad \text{for } k \in \mathfrak{M}_2,\\ h^{(k)}(x,t) &= \big(Z_k(\overset{1}{f}{}^{(k)}, \overset{1}{g}{}^{(k)}), Z_k(\overset{2}{f}{}^{(k)}, \overset{2}{g}{}^{(k)}), Z_k k_1^{(k)}, Z_k k_2^{(k)},\\ &\qquad Z_k l_1^{(k)}, Z_k l_2^{(k)}, 0\big) \quad \text{for } k \in \mathfrak{N}_1, \end{aligned}$$

where Z_k is the operator which transforms locally (on the support of $\zeta^{(k)}$) variables x into z (variables z are introduced in the definition of the partition of unity presented in Section 2).

Finally, $h^{(k)}(x,t) = (0,0,Z_k(\overset{2}{f}{}^{(k)}, \overset{2}{g}{}^{(k)}), 0,0,0,0, Z_k b^{(k)})$ for $k \in \mathfrak{N}_2$.

Therefore, for $u = (\overset{1}{u}, \overset{2}{u})$, we have

$$u^{(k)}(x,t) = R^{(k)}h^{(k)}, \quad k \in \mathfrak{M}_1 \cup \mathfrak{M}_2 \cup \mathfrak{N}_1 \cup \mathfrak{N}_2 \equiv \mathcal{R}. \tag{5.20}$$

Then we introduce the operator R, called a *regularizer*, by

$$Rh = \sum_k \eta^{(k)}(x) u^{(k)}(x,t). \tag{5.21}$$

Introduce the spaces

$$\begin{aligned} \mathcal{A}(r,\tau) &= W_r^{2,1}(\overset{1}{\Omega}{}_0^\tau) \times W_r^{2,1}(\overset{2}{\Omega}{}_0^\tau),\\ \mathcal{B}(r,\tau) &= L_r(\overset{1}{\Omega}{}_0^\tau) \times L_r(\overset{2}{\Omega}{}_0^\tau) \times L_r(0,\tau;W_r^1(\overset{1}{\Omega}_0)) \times L_r(0,\tau;W_r^1(\overset{2}{\Omega}_0))\\ &\quad \times W_r^{1-1/r,1/2-1/(2r)}(S_0^\tau) \times W_r^{1-1/r,1/2-1/(2r)}(S_0^\tau) \times W_r^{2-1/r,1-1/(2r)}(S_0^\tau)\\ &\quad \times W_r^{2-1/r,1-1/(2r)}(S_0^\tau) \times W_r^{2-1/r,1-1/(2r)}(B^\tau). \end{aligned} \tag{5.22}$$

Lemma 5.2. *For $r > 5/2$ and τ small,*

$$\|Rh\|_{\mathcal{A}(r,\tau)} \le c\|h\|_{\mathcal{B}(r,\tau)}. \tag{5.23}$$

Proof. This follows from estimates (5.11), (5.13), (5.16) and (5.18) and Remark 5.3. ∎

Remark 5.3. Let $u = (\overset{1}{u}, \overset{2}{u}) \in \mathcal{A}(r,\tau)$. Define

$$\|u\|_{\tilde{\mathcal{A}}(r,\tau)} = \sum_{k\in\mathcal{R}} \|u^{(k)}\|_{\mathcal{A}(r,\tau)}.$$

Similarly, for $h \in \mathcal{B}(r,\tau)$ we set

$$\|h\|_{\tilde{\mathcal{B}}(r,\tau)} = \sum_{k\in\mathcal{R}} \|h^{(k)}\|_{\mathcal{B}(r,\tau)}.$$

In view of the properties of the partition of unity the norms $\|\cdot\|_{\tilde{\mathcal{A}}}$, $\|\cdot\|_{\tilde{\mathcal{B}}}$ and $\|\cdot\|_{\mathcal{A}}$, $\|\cdot\|_{\mathcal{B}}$, are equivalent, respectively.

Let us write problem (5.8) briefly as

$$Au = h. \tag{5.24}$$

LEMMA 5.4. *Assume that* $h \in \mathcal{B}(r,\tau)$ *and* $r \ge 5/2$. *Then for sufficiently small* τ *there exists a solution to* (5.24) *such that* $u \in \mathcal{A}(r,\tau)$ *and*

$$\|u\|_{\mathcal{A}(r,\tau)} \le c\|h\|_{\mathcal{B}(r,\tau)}. \tag{5.25}$$

Proof. Repeating the considerations from [LSU, Ch. 4, Sect. 7] we show the existence of bounded operators T and W such that

$$ARh = h + Th, \quad RAu = u + Wu. \tag{5.26}$$

For sufficiently small τ and λ we show that $\|T\| < 1$, $\|W\| < 1$, which proves the lemma.

We have

$$A = (L, B), \quad L = (L_1, L_2), \quad B = (B_1, B_2, B_3).$$

First we construct the operator T (see $(5.26)_1$). We have

$$LRh = \sum_k L\eta^{(k)}u^{(k)} = \sum_k (L\eta^{(k)}u^{(k)} - \eta^{(k)}Lu^{(k)}) + \sum_k \eta^{(k)}Lu^{(k)}.$$

For $k \in \mathfrak{M}_1$,

$$Lu^{(k)} = L_1\overset{1}{u}{}^{(k)} = (\overset{1}{f}{}^{(k)}, \overset{1}{g}{}^{(k)}).$$

For $k \in \mathfrak{M}_2$,

$$Lu^{(k)} = L_2\overset{2}{u}{}^{(k)} = (\overset{2}{f}{}^{(k)}, \overset{2}{g}{}^{(k)}).$$

For $k \in \mathfrak{N}_1$,

$$\begin{aligned} Lu^{(k)} &= L(\partial_x,\partial_t)Z_kR^{(k)}Z_k^{-1}h^{(k)} = Z_kL^{(k)}(\partial_z - \nabla F\partial_{z_3},\partial_t)R^{(k)}Z_k^{-1}h^{(k)} \\ &= Z_k[L^{(k)}(\partial_z - \nabla F\partial_{z_3},\partial_t) - L^{(k)}(\partial_z,\partial_t)]R^{(k)}Z_k^{-1}h^{(k)} \\ &\quad + h^{(k)}, \end{aligned}$$

because by definition of $R^{(k)}$, $k \in \mathfrak{N}_1$, we have

$$\begin{aligned} &Z_kL^{(k)}(\partial_z,\partial_t)R^{(k)}Z_k^{-1}h \\ &\qquad = (Z_k(\overset{1}{f}{}^{(k)}, \overset{1}{g}{}^{(k)}), Z_k(\overset{2}{f}{}^{(k)}, \overset{2}{g}{}^{(k)}), Z_kk_1^{(k)}, Z_k^{-1}k_2^{(k)}, Z_kl_1^{(k)}, Z_kl_2^{(k)}, 0). \end{aligned}$$

Similar expressions can be derived for $k \in \mathfrak{N}_2$.

The operator T can be divided into two parts: $T = (T_1, T_2)$, where

$$LRh = h + T_1h, \quad BRh = h + T_2h.$$

First we describe T_1:

$$\begin{aligned} T_1h = &\sum_{k\in\mathfrak{M}_1}(L_1\eta^{(k)}u^{(k)}-\eta^{(k)}L_1u^{(k)})\\ &+\sum_{k\in\mathfrak{M}_2}(L_2\eta^{(k)}u^{(k)}-\eta^{(k)}L_2^{(k)})\\ &+\sum_{k\in\mathfrak{N}_1}[(L_1,L_2)\eta^{(k)}u^{(k)}-\eta^{(k)}(L_1,L_2)u^{(k)}]\\ &+\sum_{k\in\mathfrak{N}_1}\eta^{(k)}Z_k\big[(L_1(\partial_z-\nabla F\partial_{z_3},\partial_t),L_2(\partial_z-\nabla F\partial_{z_3},\partial_t))\\ &-(L_1(\partial_z,\partial_t),L_2(\partial_z,\partial_t))\big]R^{(k)}Z_k^{-1}h^{(k)}\\ &+\sum_{k\in\mathfrak{N}_2}(L_2\eta^{(k)}-\eta^{(k)}L_2u^{(k)})\\ &+\sum_{k\in\mathfrak{N}_2}\eta^{(k)}Z_k[L_2(\partial_z-\nabla F\partial_{z_3},\partial_t)-L_2(\partial_z,\partial_t)]R^{(k)}Z_k^{-1}h^{(k)}. \end{aligned}$$

Next, we calculate

$$\begin{aligned} T_2h = &\sum_{k\in\mathfrak{N}_1}\sum_{\alpha=1}^{2}\sum_{i=1}^{2}(B_{i\alpha}\eta^{(k)}u^{(k)}-\eta^{(k)}B_{i\alpha}u^{(k)})|_{S_0^\tau}\\ &+\sum_{k\in\mathfrak{N}_1}\sum_{\alpha=1}^{2}\sum_{i=1}^{2}\eta^{(k)}(B_{i\alpha}(x,t,\partial_x)-B_{i\alpha}(\xi^{(k)},0,\partial_x)u^{(k)})|_{S_0^\tau}\\ &+\sum_{k\in\mathfrak{N}_1}\sum_{\alpha=1}^{2}\sum_{i=1}^{2}\eta^{(k)}Z_k[B_{i\alpha}(\xi^{(k)},0,\partial_z-\nabla F\partial_{z_3})-B_{i\alpha}(\xi^{(k)},0,\partial_z)]R^{(k)}Z_k^{-1}h^{(k)}\\ &+\sum_{k\in\mathfrak{N}_2}\sum_{\alpha=1}^{3}(B_{3\alpha}\eta^{(k)}u^{(k)}-\eta^{(k)}B_{3\alpha}u^{(k)})|_B\\ &+\sum_{k\in\mathfrak{N}_1}\sum_{\alpha=1}[B_{3\alpha}(x,t,\partial_x)-B_{3\alpha}(\xi^{(k)},0,\partial_x)]u^{(k)}|_B\\ &+\sum_{k\in\mathfrak{N}_2}\sum_{\alpha=1}^{3}\eta^{(k)}Z_k[B_{3\alpha}(\xi^{(k)},0,\partial_z-\nabla F\partial_{z_3})-B_{3\alpha}(\xi^{(k)},0,\partial_z)]R^{(k)}Z_k^{-1}h^{(k)}. \end{aligned}$$

Let $T_1^{\mathfrak{M}_1}$ be the following part of the operator T_1:

$$\begin{aligned} T_1^{\mathfrak{M}_1}h &= \sum_{k\in\mathfrak{M}_1}(L_1\eta^{(k)}\overset{1}{u}{}^{(k)}-\eta^{(k)}L_1\overset{1}{u}{}^{(k)})\\ &=\sum_{k\in\mathfrak{M}_1}(-2\nabla\eta^{(k)}\nabla\overset{1}{u}{}^{(k)}-\Delta\eta^{(k)}\overset{1}{u}{}^{(k)}\\ &\quad+\nabla\eta^{(k)}_{,x_j}\overset{1}{u}{}^{(k)}_j+\eta^{(k)}_{x_j}\nabla\overset{1}{u}{}^{(k)}_j+\nabla\eta^{(k)}\operatorname{div}\overset{1}{u}{}^{(k)}). \end{aligned}$$

Hence

$$\|T_1^{\mathfrak{M}_1}h\|_{L_r(\overset{1}{\Omega}{}_0^\tau)}\le c\sum_{k\in\mathfrak{M}_1}\bigg(\frac{1}{\lambda}\|\nabla\overset{1}{u}{}^{(k)}\|_{L_r(\overset{1}{\Omega}{}_0^\tau)}+\frac{1}{\lambda^2}\|\overset{1}{u}{}^{(k)}\|_{L_r(\overset{1}{\Omega}{}_0^\tau)}\bigg).$$

Since $\overset{1}{u}{}^{(k)}|_{t=0} = 0$, we have

$$
\begin{aligned}
I_1 &\equiv \Big(\int_0^\tau \|\nabla \overset{1}{u}{}^{(k)}\|^r_{L_r(\overset{1}{\Omega}_0)}\,dt\Big)^{1/r} \le c\Big(\int_0^\tau \|\nabla^2 \overset{1}{u}{}^{(k)}\|^{r/2}_{L_r(\overset{1}{\Omega}_0)} \|\overset{1}{u}{}^{(k)}\|^{r/2}_{L_r(\overset{1}{\Omega}_0)}\,dt\Big)^{1/r}\\
&\le c\sup_{t\le\tau} \|\overset{1}{u}{}^{(k)}\|^{1/2}_{L_r(\overset{1}{\Omega}_0)} \Big(\int_0^\tau \|\nabla^2 \overset{1}{u}{}^{(k)}\|^{r/2}_{L_r(\overset{1}{\Omega}_0)}\,dt\Big)^{1/r}\\
&\le c\Big\|\int_0^\tau \overset{1}{u}{}^{(k)}_{,t}\,dt\Big\|^{1/2}_{L_r(\overset{1}{\Omega}_0)} \tau^{1/(2r)}\Big(\int_0^\tau \|\nabla^2 \overset{1}{u}{}^{(k)}\|^r_{L_r(\overset{1}{\Omega}_0)}\,dt\Big)^{1/(2r)}\\
&\le c\tau^{1/(2r)}\tau^{1/(2r')}\Big(\int_0^\tau \|\overset{1}{u}{}^{(k)}_{,t}\|^r_{L_r(\overset{1}{\Omega}_0)}\,dt\Big)^{1/(2r)}\Big(\int_0^\tau \|\nabla^2 \overset{1}{u}{}^{(k)}\|^r_{L_r(\overset{1}{\Omega}_0)}\,dt\Big)^{1/(2r)}\\
&\le c\tau^{1/2}\|\overset{1}{u}{}^{(k)}\|_{W^{2,1}_r(\overset{1}{\Omega}{}^\tau_0)}.
\end{aligned}
$$

Similarly,

$$
I_2 \equiv \|\overset{1}{u}{}^{(k)}\|_{L_r(\overset{1}{\Omega}{}^\tau_0)} \le c\tau^{1/r}\sup_{t\le\tau}\Big\|\int_0^t \overset{1}{u}{}^{(k)}_{t'}\,dt'\Big\|_{L_r(\overset{1}{\Omega}_0)} \le c\tau\|\overset{1}{u}{}^{(k)}\|_{W^{2,1}_r(\overset{1}{\Omega}{}^\tau_0)}.
$$

Hence,

$$
I_1 + I_2 \le c\tau^{1/2}\|\overset{1}{u}{}^{(k)}\|_{W^{2,1}_r(\overset{1}{\Omega}{}^\tau_0)} \le c\tau^{1/2}\|h\|_{\mathcal{B}(r,\tau)}.
$$

Therefore,

$$
\|T_1^{\mathfrak{M}_1} h\|_{L_r(\overset{1}{\Omega}{}^\tau_0)} \le c\Big(\frac{\tau^{1/2}}{\lambda} + \frac{\tau^{1/2}}{\lambda^2}\Big)\|h\|_{\mathcal{B}(r,\tau)}. \tag{5.27}
$$

Hence for τ sufficiently small the norm of $T_1^{\mathfrak{M}_1}$ can be chosen as small as we want. Consider

$$
\begin{aligned}
T_1^{\mathfrak{N}_1} h &= \sum_{k\in\mathfrak{N}_1} (L_1, L_2)\eta^{(k)}u^{(k)} - \eta^{(k)}(L_1, L_2)u^{(k)}\\
&\quad + \sum_{k\in\mathfrak{N}_1} \eta^{(k)} Z_k\big[(L_1(\partial_z - \nabla F\partial_{z_3}, t), L_2(\partial_z - \nabla F\partial_{z_3}, t))\\
&\quad - (L_1(\partial_z, \partial_t), L_2(\partial_z, \partial_t))\big]\cdot R^{(k)} Z_k^{-1} h^{(k)}\\
&= \sum_{k\in\mathfrak{N}_1}\sum_{\sigma=1}^2 \big(-2\nabla\eta^{(k)}\nabla\overset{\sigma}{u}{}^{(k)} - \Delta\eta^{(k)}\overset{\sigma}{u}{}^{(k)}\\
&\quad + \nabla\eta^{(k)}_j\overset{\sigma}{u}{}^{(k)}_j + \eta^{(k)}_{x_j}\nabla\overset{\sigma}{u}{}^{(k)}_j + \nabla\eta^{(k)}\operatorname{div}\overset{\sigma}{u}{}^{(k)}\big)\\
&\quad + \sum_{\alpha,\beta=1}^2\sum_{\sigma=1}^2 \eta^{(k)} Z_k\big[a^{(1)}_{\alpha\beta}F_{k,z_\alpha}\overset{\sigma}{\tilde u}{}^{(k)}_{,z_\beta z_3}\\
&\quad + a^{(2)}_{\alpha\beta}F_{k,z_\alpha}F_{k,z_\beta}\overset{\sigma}{\tilde u}{}^{(k)}_{,z_3 z_3} + a^{(3)}_{\alpha\beta}F_{k,z_\alpha z_\beta}\overset{\sigma}{\tilde u}{}^{(k)}_{,z_3}\big],
\end{aligned}
$$

where $a^{(j)}_{\alpha\beta}$, $j=1,2,3$, are constant coefficients and $\overset{\sigma}{\tilde{u}}{}^{(k)}(z,t)=R^{(k)}Z_k^{-1}\overset{\sigma}{u}{}^{(k)}(x,t)$. Hence,

$$\|T_1^{\mathfrak{N}_1}h\|_{L_r(\overset{1}{\Omega}{}_0^\tau\cup\overset{2}{\Omega}{}_0^\tau)}\le c\bigg(\frac{\tau^{1/2}}{\lambda}+\frac{\tau^{1/2}}{\lambda^2}\bigg)\|h\|_{\mathcal{B}(r,\tau)}+c\lambda\sum_{\sigma=1}^2\sum_{k\in\mathfrak{N}_1}\|\overset{\sigma}{u}{}^{(k)}_{,xx}\|_{L_r(\overset{\sigma}{\Omega}{}_0^\tau)}, \tag{5.28}$$

where we have used the fact that $F_{,z'z'}$, $z'\in\{z_1,z_2\}$, are bounded. Continuing, we have

$$\|T^{\mathfrak{N}_1}h\|_{L_r(\overset{1}{\Omega}{}_0^\tau\cup\overset{2}{\Omega}{}_0^\tau)}\le c\bigg(\frac{\tau^{1/2}}{\lambda^2}+\lambda\bigg)\|h\|_{\mathcal{B}(r,\tau)}. \tag{5.29}$$

Next we find the operator W which is defined by the relation

$$RAu=u+Wu.$$

Hence

$$\begin{aligned}
Wu=&\sum_{k\in\mathfrak{M}_1}\eta^{(k)}R^{(k)}(\zeta^{(k)}L_1u-L_1\zeta^{(k)}u)\\
&+\sum_{k\in\mathfrak{M}_2}\eta^{(k)}R^{(k)}(\zeta^{(k)}L_2u-L_2\zeta^{(k)}u)\\
&+\sum_{k\in\mathfrak{N}_1}\sum_{\alpha=1}^2\eta^{(k)}Z_kR^{(k)}\Big[Z_k^{-1}(\zeta^{(k)}L_\alpha u-L_\alpha\zeta^{(k)}u),\sum_{i=1}^2Z_k^{-1}(\zeta^{(k)}B_{i\alpha}u-B_{i\alpha}\zeta^{(k)}u)|_{S_0^\tau}\Big]\\
&+\sum_{k\in\mathfrak{N}_1}\sum_{\alpha=1}^2\sum_{i=1}^2\eta^{(k)}Z_kR^{(k)}[Z_k^{-1}(B_{i\alpha}(x,t,\partial_x)-B_{ik}(\xi^{(k)},0,\partial_x))\zeta^{(k)}u|_{S_0^\tau}]\\
&+\sum_{k\in\mathfrak{N}_1}\sum_{\alpha=1}^2\eta^{(k)}Z_kR^{(k)}\Big[(L_\alpha(\xi^{(k)},0,\partial_z-\nabla F\partial_{z_3})-L_\alpha(\xi^{(k)},0,\partial_z))Z_k^{-1}\zeta^{(k)}u,\\
&\qquad\sum_{i=1}^2(B_{i\alpha}(\xi^{(k)},0,\partial_z-\nabla F\partial_{z_3})-B_{i\alpha}(\xi^{(k)},0,\partial_z))Z_k^{-1}\zeta^{(k)}u|_{z_3=0}\Big]\\
&+\sum_{k\in\mathfrak{N}_2}\eta^{(k)}Z_kR^{(k)}\Big[Z_k^{-1}(\zeta^{(k)}L_2u-L_2\zeta^{(k)}u),\sum_{\alpha=1}^3Z_k^{-1}(\zeta^{(k)}B_{3\alpha}u-B_{3\alpha}\zeta^{(k)}u)|_{B^\tau}\Big]\\
&+\sum_{k\in\mathfrak{N}_2}\sum_{\alpha=1}^3\eta^{(k)}Z_kR^{(k)}Z_k^{-1}(B_{3\alpha}(x,t,\partial_x)-B_{3\alpha}(\xi^{(k)},0,\partial_x))\zeta^{(k)}u|_{B^\tau}\\
&+\sum_{k\in\mathfrak{N}_2}\eta^{(k)}Z_kR^{(k)}\Big[(L_2(\xi^{(k)},0,\partial_z-\nabla F\partial_{z_3},\partial_t)-L_2(\xi^{(k)},0,\partial_z,\partial_t))Z_k^{-1}\zeta^{(k)}u,\\
&\qquad\sum_{\alpha=1}^3(B_{3\alpha}(\xi^{(k)},0,\partial_z-\nabla F\partial_{z_3})-B_{3\alpha}(\xi^{(k)},0,\partial_z))Z_k^{-1}\zeta^{(k)}u|_{z_3=0}\Big].
\end{aligned}$$

We obtain estimates for some terms of the operator W. First we examine the term

$$W_1^{\mathfrak{M}_1}\overset{1}{u}=\sum_{k\in\mathfrak{M}_1}\eta^{(k)}R^{(k)}(\zeta^{(k)}L_1\overset{1}{u}-L_1\zeta^{(k)}\overset{1}{u}).$$

Repeating calculations leading to the estimate for $T_1^{\mathfrak{M}_1}h$, we obtain

$$\|W_1^{\mathfrak{M}_1}\overset{1}{u}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^\tau)}\le c\frac{\tau^{1/2}}{\lambda^2}\|\overset{1}{u}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^\tau)}. \tag{5.30}$$

Finally, we consider the following part of the operator W:

$$W_1^{\mathfrak{N}_1} u = \sum_{\alpha=1}^2 \sum_{k\in\mathfrak{N}_1} \eta^{(k)} Z_k R^{(k)} Z_k^{-1} (\zeta^{(k)} L_\alpha \overset{\alpha}{u} - L_\alpha \zeta^{(k)} \overset{\alpha}{u}) + \sum_{\alpha=1}^2 \sum_{k\in\mathfrak{N}_1} \eta^{(k)} Z_k R^{(k)} [L_\alpha(\partial_z - \nabla F \partial_{z_3}, \partial_t) - L_\alpha(\partial_z, \partial_t)] Z_k^{-1} \zeta^{(k)} \overset{\alpha}{u}.$$

Applying estimates used to show (5.29), we derive

$$\|W_1^{\mathfrak{N}_1} u\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^\tau \cup \overset{2}{\Omega}{}_0^\tau)} \le c\biggl(\frac{\tau^{1/2}}{\lambda^2} + \lambda\biggr) \sum_{\alpha=1}^2 \|\overset{\alpha}{u}\|_{W_r^{2,1}(\overset{\alpha}{\Omega}{}_0^\tau)}. \tag{5.31}$$

To prove the lemma we have to estimate the other terms in T and W.

Similar estimates to the above terms imply that the norms of the operators T and W are less than 1 for sufficiently small τ and λ. This concludes the proof. ■

Lemma 5.5. *Let $r > 5/2$. Assume that $\overset{i}{f}_* \in L_r(\overset{i}{\Omega}{}_0^t)$, $\overset{i}{g}_* \in L_r(0,t; W_r^1(\overset{i}{\Omega}_0))$, $k_{*i} \in W_r^{1-1/r,1/2-1/(2r)}(S_0^t)$, $l_{*i} \in W_r^{2-1/r,1-1/(2r)}(S_0^t)$, $\overset{i}{H}(0) \in W_r^{2-2/r}(\overset{i}{\Omega}_0)$, $i = 1,2$, $H_{*\alpha} \in W_r^{2-1/r,1-1/(2r)}(B^t)$, $\alpha = 1,2$. Then there exists a solution to problem* (5.4) *such that $\overset{i}{H} \in W_r^{2,1}(\overset{i}{\Omega}{}_0^t)$, $i = 1,2$, and*

$$\begin{aligned}\sum_{i=1}^2 \|\overset{i}{H}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)} \le c \sum_{i=1}^2 \Bigl(\|\overset{i}{f}_*\|_{L_r(\overset{i}{\Omega}{}_0^t)} &+ \|\overset{i}{g}_*\|_{L_r(0,t;W_r^1(\overset{i}{\Omega}_0))} \\ &+ \|k_{*i}\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)} + \|l_{*i}\|_{W_r^{2-1/r,1-1/(2r)}(S_0^t)} \\ &+ \sum_{\alpha=1}^2 \|H_{*\alpha}\|_{W_r^{2-1/r,1-1/(2r)}(B^t)} + \|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)} \Bigr).\end{aligned} \tag{5.32}$$

Proof. In view of Lemma 5.4 we have the existence of solutions to problem (5.7) with vanishing initial data and for sufficiently small existence time. Since the problem (5.7) is linear we can extend the existence step by step in time for any time interval $(0,t)$, $t = k\tau$, $k \in \mathbb{N}$. Then, in view of transformation (5.6), estimates (5.11), (5.13), (5.16), (5.18), Remark 5.3 and the estimates

$$\begin{aligned}&\|\overset{i}{f}\|_{L_r(\overset{i}{\Omega}{}_0^t)} \le \|\overset{i}{f}_*\|_{L_r(\overset{i}{\Omega}{}_0^t)} + c\|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)}, \quad i = 1,2, \\ &\|\overset{i}{g}\|_{L_r(0,t;W_r^1(\overset{i}{\Omega}_0))} \le \|\overset{i}{g}_*\|_{L_r(0,t;W_r^1(\overset{i}{\Omega}_0))} + c\|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)}, \quad i = 1,2, \\ &\|k_i\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)} \\ &\qquad\qquad \le \|k_{*i}\|_{W_r^{1-1/2,1/2-1/(2r)}(S_0^t)} + c\|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)}, \quad i = 1,2, \\ &\|l_i\|_{W_r^{2-1/r,1-1/(2r)}(S_0^t)} \le \|l_{*i}\|_{W_r^{2-1/r,1-1/(2r)}(S_0^t)} + c\|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)}, \quad i = 1,2,\end{aligned} \tag{5.33}$$

we obtain (5.32) and conclude the proof. ■

Now we estimate the r.h.s. of (5.2).

LEMMA 5.6. *Let* $r>2$. *Assume* $\overset{i}{u}\in W_r^{3,3/2}(\overset{i}{\Omega}{}_0^t)$, $i=1,2$. *Let* $\overset{i}{\alpha}_r(t)=t^{1/r'}\|\overset{i}{u}\|_{L_r(0,t;W_r^3(\overset{i}{\Omega}_0))}$, $1/r+1/r'=1$, $i=1,2$. *Assume that* $\overset{i}{H}\in W_r^{2,1}(\overset{i}{\Omega}{}_0^t)$. *Assume also that* $\overset{i}{u}(0)\in W_r^{3-2/r}(\overset{1}{\Omega}_0)$, $\overset{i}{H}(0)\in W_r^{2-2/r}(\overset{i}{\Omega}_0)$, $i=1,2$. *Then there exists* $a>0$ *and an increasing positive function* $\varphi(k)$ *nonvanishing for* $k=0$ *such that*

$$
\begin{aligned}
\|\overset{i}{f}_0\|_{L_r(\overset{i}{\Omega}{}_0^t)} &\le \sum_{i=1}^2\varphi(\overset{i}{\alpha}_r(t))t^a(t^{1/r'}\|\overset{i}{H}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)}+\|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)})\\
&\quad\cdot(t^{1/r'}\|\overset{i}{u}\|_{W_r^{3,3/2}(\overset{i}{\Omega}{}_0^t)}+\|\overset{i}{u}(0)\|_{W_r^{3-2/r}(\overset{i}{\Omega}_0)})\equiv A,
\end{aligned}
\tag{5.34}
$$

$$\|k_{0i}\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)}\le cA.$$

Proof. First we examine the r.h.s. of $(5.2)_1$. The properties of the Lagrange transformation yield

$$
\begin{aligned}
\|\overset{1}{f}_0\|_{L_r(\overset{1}{\Omega}{}_0^t)} &\le c\|\mathrm{rot}_{\overset{1}{u}}(\overset{1}{u}\times\overset{1}{H})\|_{L_r(\overset{1}{\Omega}{}_0^t)}+c\|\overset{1}{u}\cdot\nabla_{\overset{1}{u}}\overset{1}{H}\|_{L_r(\overset{1}{\Omega}{}_0^t)}\\
&\le\varphi(\overset{i}{\alpha}_r(t))(\|\overset{1}{u}_\xi\overset{1}{H}\|_{L_r(\overset{1}{\Omega}{}_0^t)}+\|\overset{1}{u}\overset{1}{H}_{,\xi}\|_{L_r(\overset{1}{\Omega}{}_0^t)})\equiv\varphi(\overset{1}{\alpha}_r(t))I.
\end{aligned}
$$

By the Hölder inequality we have

$$
\begin{aligned}
I &\le\Big(\int_0^t\|\overset{1}{u}_\xi\|^r_{L_\infty(\overset{1}{\Omega}_0)}\|\overset{1}{H}\|^r_{L_r(\overset{1}{\Omega}_0)}dt'\Big)^{1/r}+\Big(\int_0^t\|\overset{1}{u}\|^r_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}\|\overset{1}{H}_\xi\|^r_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}dt'\Big)^{1/r}\\
&\equiv I_1+I_2,\quad 1/\lambda_1+1/\lambda_2=1.
\end{aligned}
$$

Considering I_1 Lemma 2.5 implies

$$I_1\le c\sup_t\|\overset{1}{H}\|_{L_r(\overset{1}{\Omega}_0)}\Big(\int_0^t\|\overset{1}{u}_{,\xi\xi\xi}\|^{\theta_1 r}_{L_r(\overset{1}{\Omega}_0)}dt'\Big)^{1/r}\sup_t\|\overset{1}{u}\|^{1-\theta_1}_{L_r(\overset{1}{\Omega}_0)}\equiv I_1^1,$$

where $\theta_1=1/r+1/3$ and $\theta_1<1$ for $r>3/2$. Continuing,

$$
\begin{aligned}
I_1^1 &\le c\Big[t^{1/r'}\Big(\int_0^t\|\overset{1}{H}_{,t}\|^r_{L_r(\overset{1}{\Omega}_0)}dt'\Big)^{1/r}+\|\overset{1}{H}(0)\|_{L_r(\overset{1}{\Omega}_0)}\Big]\\
&\quad\cdot t^{\frac{1-\theta_1}{r}}\|\overset{1}{u}_{,\xi\xi\xi}\|^{\theta_1}_{L_r(\overset{1}{\Omega}{}_0^t)}\Big[t^{1/r'}\Big(\int_0^t\|\overset{1}{u}_{,t}\|^r_{L_r(\overset{1}{\Omega}_0)}dt'\Big)^{1/r}+\|\overset{1}{u}(0)\|_{L_r(\overset{1}{\Omega}_0)}\Big]^{1-\theta_1}\\
&\le ct^a(t^{1/r'}\|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}+\|\overset{1}{H}(0)\|_{L_r(\overset{1}{\Omega}_0)})(t^{1/r'}\|\overset{1}{u}\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}+\|\overset{1}{u}(0)\|_{L_r(\overset{1}{\Omega}_0)})
\end{aligned}
$$

for $r>3/2$ and some $a>0$.

Using the imbeddings $\|\overset{1}{u}\|_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}\le c\|\overset{1}{u}\|_{W_r^{3-2/r}(\overset{1}{\Omega}_0)}$ for $3/r-3/(r\lambda_1)\le 3-2/r$ and $\|\overset{1}{H}_\xi\|_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}\le c\|\overset{1}{H}\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}$ for $3/r-3/(r\lambda_2)\le 1-2/r$ we deduce that $I_2\le ct^{1/r}\|\overset{1}{u}\|_{L_\infty(0,t;W_r^{3-2/r}(\overset{1}{\Omega}_0))}\|\overset{1}{H}\|_{L_\infty(0,t;W_r^{2-2/r}(\overset{1}{\Omega}_0))}\le A$ for $r\ge 7/4$. Similarly,

$\|\overset{2}{f}_0\|_{L_r(\overset{2}{\Omega}{}_0^t)}$ is estimated by the above bound for $\|\overset{1}{f}_0\|_{L_r(\overset{1}{\Omega}{}_0^t)}$, where $(\overset{1}{u}, \overset{1}{H}, \overset{1}{\Omega}_0)$ is replaced by $(\overset{2}{u}, \overset{2}{H}, \overset{2}{\Omega}_0)$.

Next we estimate

$$\begin{aligned}
&\|k_{0i}\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)} = \|k_{0i}\|_{L_r(0,t;W_r^{1-1/r}(S_0))} + \|k_{0i}\|_{L_r(S_0;W_r^{1/2-1/(2r)}(0,t))}\\
&\equiv J_1 + J_2.
\end{aligned}$$

By extension, we have

$$\begin{aligned}
J_1 &\le \|k_{0i}\|_{L_r(0,t;W_r^1(\overset{1}{\Omega}_0))}\\
&\le c\|\overset{1}{u} \times \overset{1}{H} \cdot \bar{\tau}_{ui}\|_{L_r(\overset{1}{\Omega}{}_0^t)} + c\|(\overset{1}{u} \times \overset{1}{H})_{,\xi} \cdot \bar{\tau}_{ui}\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \varphi(\overset{1}{\alpha}_r(t))\Big\|\overset{1}{u} \cdot \overset{1}{H}\int_0^t \hat{u}_{,\xi\xi}\, dt'\Big\|_{L_r(\overset{1}{\Omega}{}_0^t)}\\
&\equiv J_1^1 + J_1^2 + J_1^3,
\end{aligned}$$

where $|J_1^1| + |J_1^2| \le A$ and

$$J_1^3 \le \overset{1}{\alpha}_r(t)\Big(\int_0^t \|\overset{1}{u}\overset{1}{H}\|^r_{L_\infty(\overset{1}{\Omega}_0)}\, dt'\Big)^{1/r} \equiv \overset{1}{\alpha}_r(t) J_1^4.$$

Using Lemma 2.6(a) we have

$$\begin{aligned}
J_1^4 &\le \Big(\int_0^t \|\overset{1}{u}\|^r_{L_\infty(\overset{1}{\Omega}_0)} \|\overset{1}{H}\|^r_{L_\infty(\overset{1}{\Omega}_0)}\, dt'\Big)^{1/r}\\
&\le c \sup_t \|\overset{1}{u}\|_{W_r^{3-2/r}(\overset{1}{\Omega}_0)} \sup_t \|\overset{1}{H}\|^{1-\theta}_{L_r(\overset{1}{\Omega}_0)} \Big(\int_0^t \|\overset{1}{H}\|^{r\theta}_{W_r^2(\overset{1}{\Omega}_0)}\, dt'\Big)^{1/r} \equiv J_1^5,
\end{aligned}$$

where $\theta = 3/(2r) < 1$ for $r > 3/2$ and we have also used $r \ge 5/3$. Applying the Hölder inequality gives

$$\begin{aligned}
J_1^5 &\le c(t^{1/r'}\|\overset{1}{H}_{,t'}\|_{L_r(\overset{1}{\Omega}_0)} + \|\overset{1}{H}(0)\|_{L_r(\overset{1}{\Omega}_0)})^{1-\theta}\\
&\quad\cdot t^a \|\overset{1}{u}\|_{L_\infty(0,t;W_r^{3-2/r}(\overset{1}{\Omega}_0))} \|\overset{1}{H}\|^\theta_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))}\\
&\le ct^a \|\overset{1}{u}\|_{L_\infty(0,t;W_r^{3-2/r}(\overset{1}{\Omega}_0))} \cdot (t^{1/r'}\|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|\overset{1}{H}(0)\|_{L_r(\overset{1}{\Omega})}) \le A.
\end{aligned}$$

Summarizing, we have

$$J_1 \le cA.$$

Finally, we consider

$$\begin{aligned}
J_2 &\le \|k_{0i}\|_{L_r(\overset{1}{\Omega}_0;W_r^{1/2}(0,t))}\\
&\le c\bigg(\int_{\overset{1}{\Omega}_0} d\xi \int_0^t\int_0^t \frac{|(\overset{1}{u} \times \overset{1}{H} \cdot \bar{\tau}_{ui})(t') - (\overset{1}{u} \times \overset{1}{H} \cdot \bar{\tau}_{ui})(t'')|^r}{|t' - t''|^{1+r/2}}\, dt'\, dt''\bigg)^{1/r}
\end{aligned}$$

$$\le \varphi(\overset{1}{\alpha}_r(t))\bigg(\int_{\overset{1}{\Omega}_0} d\xi \int_0^t\int_0^t \frac{|\overset{1}{u}(t')-\overset{1}{u}(t'')|^r|\overset{1}{H}(t')|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r}$$

$$+\varphi(\overset{1}{\alpha}_r(t))\bigg(\int_{\overset{1}{\Omega}_0} d\xi \int_0^t\int_0^t \frac{|\overset{1}{u}(t'')|^r|\overset{1}{H}(t')-\overset{1}{H}(t'')|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r}$$

$$+\varphi(\overset{1}{\alpha}_r(t'))\bigg(\int_{\overset{1}{\Omega}_0} d\xi \int_0^t\int_0^t \frac{|\overset{1}{u}(t'')|^r|\overset{1}{H}(t'')|^r\big|\int_{t''}^{t'} u_{,\xi}\,d\tau\big|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r}$$

$$\equiv J_2^1+J_2^2+J_2^3.$$

By the Hölder inequality, we have

$$J_2^1 \le \varphi(\overset{1}{\alpha}_r(t))\Big(\int_0^t \|\overset{1}{u}_{,t'}(t')\|^r_{L_r(\overset{1}{\Omega}_0)}\,dt'\Big)^{1/r}$$

$$\cdot\Big(\int_0^t\int_0^t \frac{|t'-t''|^{r/r'}}{|t'-t''|^{1+r/2}}\|\overset{1}{H}(t')\|^r_{L_\infty(\overset{1}{\Omega}_0)}\,dt'\,dt''\Big)^{1/r}$$

$$\le \varphi(\overset{1}{\alpha}_r(t))\|\overset{1}{u}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} t^{1/2-1/r}\|\overset{1}{H}\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))} \le cA$$

for $r>3/2$.

Performing the transformation $(\overset{1}{u},\overset{1}{H})\mapsto(\overset{1}{H},\overset{1}{u})$ we have $J_2^2\le cA$.

Finally, we estimate

$$J_2^3 \le t^{1/2-1/r}\varphi(\overset{1}{\alpha}_r(t))\Big(\int_0^t \|\overset{1}{u}_{,\xi}\|^r_{L_\infty(\overset{1}{\Omega}_0)}\,dt'\Big)^{1/r}\Big(\int_0^t\int_{\overset{1}{\Omega}_0} |\overset{1}{u}|^r|\overset{1}{H}|^r\,d\xi\,dt'\Big)^{1/r},$$

where the last factor, in view of Lemma 2.5, is bounded by

$$\Big(\int_0^t \|u\|^r_{L_{2r}(\overset{1}{\Omega}_0)}\|\overset{1}{H}\|^r_{L_{2r}(\overset{1}{\Omega}_0)}\,dt'\Big)^{1/r}$$

$$\le c\sup_t\|\overset{1}{u}\|^{1-\theta}_{L_r(\overset{1}{\Omega}_0)}\sup_t\|\overset{1}{H}\|^{1-\theta}_{L_r(\overset{1}{\Omega}_0)}\Big(\int_0^t \|\overset{1}{u}\|^{\theta r}_{W_r^2(\overset{1}{\Omega}_0)}\cdot\|\overset{1}{H}\|^{\theta r}_{W_r^2(\overset{1}{\Omega}_0)}\,dt'\Big)^{1/r}\equiv J_2^4,$$

where $\theta=3/(4r)<1/2$ for $r>3/2$. Continuing, we obtain $J_2^3\le A$.

Summarizing, $J_2\le cA$. This concludes the proof. ■

Let us consider problem (5.3). Set

$$\begin{aligned} \overset{i}{\bar f} &= \overset{i}{f}_* - \overset{i}{f}_0, && i=1,2,\\ \bar k_\alpha &= k_{*\alpha}-k_{0\alpha}, && \alpha=1,2,\\ \bar l_\alpha &= l_{*\alpha}, && \alpha=1,2. \end{aligned} \tag{5.35}$$

Lemma 5.7. *Let the assumptions of Lemma* 5.6 *hold. Let* $\overset{i}{u} \in W_r^{3,1}(\overset{i}{\Omega}{}_0^t)$, $i = 1, 2$. *Then*

$$
\begin{aligned}
&\|\overset{i}{\bar f}\|_{L_r(\overset{i}{\Omega}{}_0^t)} \le \varphi(\overset{i}{\alpha}_r(t))\overset{i}{\alpha}_r(t)\|\overset{i}{H}\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))}, \quad i = 1, 2,\\
&\sum_{\alpha=1}^2 \|\bar k_\alpha\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)}\\
&\qquad \le \sum_{i=1}^2 \varphi(\overset{i}{\alpha}_r(t))\overset{i}{\alpha}_r(t)\|\overset{i}{H}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)} \sum_{\alpha=1}^2 \|\bar l_\alpha\|_{W_r^{2-1/r,1-1/(2r)}(S_0^t)}\\
&\qquad \le \sum_{i=1}^2 t^a \varphi(\overset{i}{\alpha}_r(t))\|\overset{i}{u}\|_{L_r(0,t;W_r^3(\overset{i}{\Omega}_0))}\\
&\qquad\quad \cdot \|\overset{i}{H}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)}(t^{1/r'}\|\overset{i}{u}\|_{W_r^{3,1}(\overset{i}{\Omega}{}_0^t)} + \|\overset{i}{u}(0)\|_{L_r(\overset{i}{\Omega}_0)})\\
&\qquad\quad \cdot (t^{1/r'}\|\overset{i}{H}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)} + \|\overset{i}{H}(0)\|_{L_r(\overset{i}{\Omega}_0)}),\\
&\|\overset{i}{g}_*\|_{L_r(0,t;W_r^1(\overset{i}{\Omega}_0))} \le \overset{i}{\alpha}_r(t)\varphi(\overset{i}{\alpha}_r(t))\|\overset{i}{H}\|_{L_r(0,t;W_r^2(\overset{i}{\Omega}_0))}.
\end{aligned}
\tag{5.36}
$$

Proof. To prove the lemma we have to estimate expressions (5.35). We need some properties of the transformation between Eulerian and Lagrangian coordinates,

$$x = \xi + \int_0^t u(\xi, \tau)\, d\tau.$$

We have

$$\xi_x = \mathbb{I} - \int_0^t u_\xi(\xi, \tau)\xi_x\, d\tau,$$

where $\mathbb{I}$ is the unit matrix. Then $\frac{d}{dt}\xi_x = u_\xi(\xi, t)\xi_x$, so

$$|\xi_x| \le \exp\Big(\int_0^t |u_\xi(\xi, \tau)|\, d\tau\Big),$$

where ξ_x, u_ξ are matrices and $|\xi_x| = \sum_{i,j=1}^3 |\xi_{i,x_j}|$. First we estimate $\overset{i}{\bar f}$, $i = 1, 2$. Employing the above relations we arrive at the qualitative estimate

$$
\begin{aligned}
\|\overset{i}{\bar f}\|_{L_r(\overset{i}{\Omega}{}_0^t)} &\le \varphi(\overset{i}{\alpha}_r(t))\|(\xi_x - \mathbb{I})\operatorname{rot}_\xi^2 \overset{i}{H}\|_{L_r(\overset{i}{\Omega}{}_0^t)} + \varphi(\overset{i}{\alpha}_r(t))\|\xi_{xx}\overset{i}{H}_{,\xi}\|_{L_r(\overset{i}{\Omega}{}_0^t)}\\
&\equiv L_1 + L_2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
L_1 &\le \varphi(\overset{i}{\alpha}_r(t))\overset{i}{\alpha}_r(t)\|\overset{i}{H}\|_{L_r(0,t;W_r^2(\overset{i}{\Omega}{}_0^t))},\\
L_2 &\le \varphi(\overset{i}{\alpha}_r(t))\Big\|\int_0^t \overset{i}{u}_{\xi\xi}\, d\tau \overset{i}{H}_{,\xi}\Big\|_{L_r(\overset{i}{\Omega}{}_0^t)} \equiv \varphi(\overset{i}{\alpha}_r(t))L_2'.
\end{aligned}
$$

Applying the Hölder and Minkowski inequalities yields, for $r \ge 3/2$,

$$L_2' \le \Big(\int_0^t \Big| \int_0^{t'} \|\overset{i}{u}_{,\xi\xi}\|_{L_{r\lambda_1}(\overset{i}{\Omega}_0)} \, d\tau \Big|^r \|\overset{i}{H}_{,\xi}\|^r_{L_{r\lambda_2}(\overset{i}{\Omega}_0)} \, dt' \Big)^{1/r} \le \overset{i}{\alpha}_r(t) \|\overset{i}{H}\|_{L_r(0,t;W_r^2(\overset{i}{\Omega}_0))},$$

where $1/\lambda_1 + 1/\lambda_2 = 1$. Hence $(5.36)_1$ is proved. To show $(5.36)_2$ we consider

$$\begin{aligned} \|\bar{k}_\alpha\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)} &\le c\|\mathrm{rot}_\xi \overset{1}{H} - \mathrm{rot}_{\overset{1}{u}} \overset{1}{H}\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)} \\ &\quad + c\|\mathrm{rot}_\xi \overset{2}{H} - \mathrm{rot}_{\overset{2}{u}} \overset{2}{H}\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)} \\ &\quad + c\Big\| \Big(\frac{1}{\sigma_1} \mathrm{rot}_{\overset{1}{u}} \overset{1}{H} - \frac{1}{\sigma_2} \mathrm{rot}_{\overset{2}{u}} \overset{2}{H} \Big) (\bar{\tau}_\alpha - \bar{\tau}_{u\alpha}) \Big\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)} \\ &\equiv \sum_{i=1}^3 I_i. \end{aligned}$$

Now, we estimate the consecutive terms above. First,

$$I_1 \le c\|\mathrm{rot}_\xi \overset{1}{H} - \mathrm{rot}_{\overset{1}{u}} \overset{1}{H}\|_{L_r(0,t;W_r^1(\overset{1}{\Omega}_0))} + \|\mathrm{rot}_\xi \overset{1}{H} - \mathrm{rot}_{\overset{1}{u}} \overset{1}{H}\|_{L_r(\overset{1}{\Omega}_0;W_r^{1/2}(0,t))} \equiv I_1^1 + I_1^2,$$

where

$$I_1^1 \le \varphi(\overset{1}{\alpha}_r(t)) \overset{1}{\alpha}_r(t) \|\overset{1}{H}\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))}$$

and

$$\begin{aligned} I_1^2 &\le \Big(\int_{\overset{1}{\Omega}_0} d\xi \int_0^t \int_0^t \frac{|(\xi_x(t') - \mathbb{I})\overset{1}{H}_\xi(t') - (\xi_x(t'') - \mathbb{I})\overset{1}{H}_\xi(t'')|^r}{|t' - t''|^{1+r/2}} \, dt' \, dt'' \Big)^{1/r} \\ &\le \Big(\int_{\overset{1}{\Omega}_0} d\xi \int_0^t \int_0^t \frac{|\xi_x(t') - \xi_x(t'')|^r |\overset{1}{H}_\xi(t')|^r + |\xi_x(t'') - \mathbb{I}|^r |\overset{1}{H}_\xi(t') - \overset{1}{H}_\xi(t'')|^r}{|t' - t''|^{1+r/2}} \, dt' \, dt'' \Big)^{1/r} \\ &\le \Big(\int_{\overset{1}{\Omega}_0} d\xi \int_0^t \int_0^t \frac{|\xi_x(t') - \xi_x(t'')|^r |\overset{1}{H}_\xi(t')|^r}{|t' - t''|^{1+r/2}} \, dt' \, dt'' \Big)^{1/r} \\ &\quad + \Big(\int_{\overset{1}{\Omega}_0} d\xi \int_0^t \int_0^t \frac{|\xi_x(t'') - \mathbb{I}|^r |\overset{1}{H}_\xi(t') - \overset{1}{H}_\xi(t'')|^r}{|t' - t''|^{1+r/2}} \, dt' \, dt'' \Big)^{1/r} \equiv J_1 + J_2. \end{aligned}$$

Using the explicit form of ξ_x gives, for $r > 2$,

$$\begin{aligned} J_1 &\le \varphi(\overset{1}{\alpha}_r(t)) \Big(\int_{\overset{1}{\Omega}_0} d\xi \int_0^t \int_0^t \frac{\big| \int_{t''}^{t'} \overset{1}{u}_{,\xi}(\xi,\tau) \, d\tau \big|^r}{|t' - t''|^{1+r/2}} |\overset{1}{H}_\xi(\xi,t')|^r \, dt' \, dt'' \Big)^{1/r} \\ &\le \varphi(\overset{1}{\alpha}_r(t)) \Big(\int_0^t \|\overset{1}{u}(\tau)\|^r_{W_r^3(\overset{1}{\Omega}_0)} \, d\tau \Big)^{1/r} \Big(\int_{\overset{1}{\Omega}_0} d\xi \int_0^t \int_0^t \frac{|t' - t''|^{r/r'}}{|t' - t''|^{1+r/2}} |\overset{1}{H}_\xi(\xi,t')|^r \, dt' \, dt'' \Big)^{1/r} \\ &\le \varphi(\overset{1}{\alpha}_r(t)) t^{1/r' - 1/2} \|\overset{1}{u}\|_{L_r(0,t;W_r^3(\overset{1}{\Omega}_0))} \|\overset{1}{H}_{,\xi}\|_{L_r(\overset{1}{\Omega}_0^t)}. \end{aligned}$$

Considering J_2, we have

$$J_2 \le \varphi(\overset{1}{\alpha}_r(t))\overset{1}{\alpha}_r(t)\bigg(\int\limits_{\overset{1}{\Omega}_0} d\xi \int\limits_0^t\int\limits_0^t \frac{|\overset{1}{H}_\xi(t') - \overset{1}{H}_\xi(t'')|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r}$$
$$\le \varphi(\overset{1}{\alpha}_r(t))\overset{1}{\alpha}_r(t)\|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}$$

for $r > 2$. Summarizing,

$$I_1 \le \varphi(\overset{1}{\alpha}_r(t))\overset{1}{\alpha}_r(t)\|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}. \tag{5.37}$$

Similarly,

$$I_2 \le \varphi(\overset{2}{\alpha}_r(t))\overset{2}{\alpha}_r(t)\|\overset{2}{H}\|_{W_r^{2,1}(\overset{2}{\Omega}{}_0^t)}. \tag{5.38}$$

Next, we examine

$$I_3 \le \|\mathrm{rot}_{\overset{1}{u}}\overset{1}{H}(\bar\tau_\alpha - \bar\tau_{\overset{1}{u}\alpha})\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)} + \|\mathrm{rot}_{\overset{2}{u}}\overset{2}{H}(\bar\tau_\alpha - \bar\tau_{\overset{2}{u}\alpha})\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)} \equiv I_3^1 + I_3^2.$$

We consider I_3^1 only, because I_3^2 can be treated in the same way. We write

$$I_3^1 = \|\mathrm{rot}_{\overset{1}{u}}\overset{1}{H}\cdot(\bar\tau_\alpha - \bar\tau_{\overset{1}{u}\alpha})\|_{L_r(0,t;W_r^1(\overset{1}{\Omega}_0))} + \|\mathrm{rot}_{\overset{1}{u}}\overset{1}{H}\cdot(\bar\tau_\alpha - \bar\tau_{\overset{1}{u}\alpha})\|_{L_r(\overset{1}{\Omega}_0;W_r^{1/2}(0,t))} \equiv I_3^3 + I_3^4,$$

where, after omitting the lower order terms,

$$I_3^3 \le \varphi(\overset{1}{\alpha}_r(t))\bigg(\bigg\|\overset{1}{H}_{,\xi\xi}\int\limits_0^t \overset{1}{u}_{,\xi}\,d\tau\bigg\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \bigg\|\overset{1}{H}_{,\xi}\int\limits_0^t \overset{1}{u}_{,\xi\xi}\,d\tau\bigg\|_{L_r(\overset{1}{\Omega}{}_0^t)}\bigg)$$
$$\le \varphi(\overset{1}{\alpha}_r(t))\int\limits_0^t \|\overset{1}{u}\|_{W_r^3(\overset{1}{\Omega}_0)}\,d\tau\,\|\overset{1}{H}\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))}$$
$$\le \varphi(\overset{1}{\alpha}_r(t))\overset{1}{\alpha}_r(t)\|\overset{1}{H}\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))}$$

for $r > 3/2$. Finally, we estimate

$$I_3^4 \le \|\xi_x\overset{1}{H}_{,\xi}(\bar\tau_\alpha - \bar\tau_{u\alpha})\|_{L_r(\overset{1}{\Omega}_0;W_r^{1/2}(0,t))}$$
$$= \bigg(\int\limits_{\overset{1}{\Omega}_0} d\xi\int\limits_0^t\int\limits_0^t \frac{|\xi_x(t')\overset{1}{H}_\xi(t')(\bar\tau_\alpha - \bar\tau_{\overset{1}{u}\alpha})(t') - \xi_x(t'')\overset{1}{H}_\xi(t'')(\bar\tau_\alpha - \bar\tau_{\overset{1}{u}\alpha})(t'')|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r}$$
$$\le \varphi(\overset{1}{\alpha}_r(t))\bigg[\bigg(\int\limits_{\overset{1}{\Omega}_0} d\xi\int\limits_0^t\int\limits_0^t \frac{\big|\int_{t''}^{t'}\overset{1}{u}_{,\xi}\,d\tau\big|^r|\overset{1}{H}_{,\xi}(t')|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r}$$
$$+ \bigg(\int\limits_{\overset{1}{\Omega}_0} d\xi\int\limits_0^t\int\limits_0^t\bigg|\int\limits_0^{t'}\overset{1}{u}_\xi\,d\tau\bigg|^r \frac{|\overset{1}{H}_{,\xi}(t') - \overset{1}{H}_{,\xi}(t'')|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r}\bigg]$$

$$\le \varphi(\overset{1}{\alpha}_r(t))\bigg[t^{1/r'-1/2}\|\overset{1}{u}\|_{L_r(0,t;W_r^3(\overset{1}{\Omega}_0))}\|\overset{1}{H}\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))}$$

$$+\overset{1}{\alpha}_r(t)\bigg(\int\limits_{\overset{1}{\Omega}_0}d\xi\int\limits_0^t\int\limits_0^t\frac{|\overset{1}{H}_{,\xi}(t')-\overset{1}{H}_{,\xi}(t'')|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r}\bigg]$$

$$\le \varphi(\overset{1}{\alpha}_r(t))\overset{1}{\alpha}_r(t)\|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}.$$

Hence,

$$I_3\le\sum_{i=1}^2\varphi(\overset{i}{\alpha}_r(t))\overset{i}{\alpha}_r(t)\|\overset{i}{H}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)}. \tag{5.39}$$

From (5.37)–(5.39) estimate $(5.36)_2$ follows.

Finally, we examine

$$\|\bar{l}_\alpha\|_{W_r^{2-1/r,1-1/(2r)}(S_0^t)}\le\varphi(\overset{1}{\alpha}_r(t),\overset{2}{\alpha}_r(t))\bigg\|\int\limits_0^t u_\xi\,d\tau\,(\overset{1}{H}-\overset{2}{H})\bigg\|_{W_r^{2-1/r,1-1/(2r)}(S_0^t)}$$

$$\le\varphi(\overset{1}{\alpha}_r(t),\overset{2}{\alpha}_r(t))\bigg(\bigg\|\int\limits_0^t\overset{1}{u}_\xi\,d\tau\,\overset{1}{H}\bigg\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}+\bigg\|\int\limits_0^t\overset{2}{u}_\xi\,d\tau\,\overset{2}{H}\bigg\|_{W_r^{2,1}(\overset{2}{\Omega}{}_0^t)}\bigg)$$

$$\equiv\varphi(\overset{1}{\alpha}_r(t),\overset{2}{\alpha}_r(t))(K_1+K_2).$$

We shall estimate K_1 only, because K_2 can be treated similarly. We have

$$K_1\le\bigg\|\int\limits_0^t\overset{1}{u}_{,\xi\xi\xi}\,d\tau\,\overset{1}{H}\bigg\|_{L_r(\overset{1}{\Omega}{}_0^t)}+\bigg\|\int\limits_0^t\overset{1}{u}_{,\xi}\,d\tau\,\overset{1}{H}_{,\xi\xi}\bigg\|_{L_r(\overset{1}{\Omega}{}_0^t)}$$

$$+\|\overset{1}{u}_{,\xi}\overset{1}{H}\|_{L_r(\overset{1}{\Omega}{}_0^t)}+\bigg\|\int\limits_0^t\overset{1}{u}_{,\xi}\,d\tau\,\overset{1}{H}_{,t}\bigg\|_{L_r(\overset{1}{\Omega}{}_0^t)}\equiv\sum_{i=1}^4K_1^i.$$

We only examine the highest order term. We have

$$K_1^1\le\bigg(\int\limits_0^t dt'\bigg|\int\limits_0^t\overset{1}{u}_{,\xi\xi\xi}\,d\tau\bigg|^r\|\overset{1}{H}\|^r_{L_\infty(\overset{1}{\Omega}_0)}\bigg)^{1/r}\le t^{1/r'}\|\overset{1}{u}\|_{L_r(0,t;W_r^3(\overset{1}{\Omega}_0))}\bigg(\int\limits_0^t\|\overset{1}{H}\|^r_{W_r^2(\overset{1}{\Omega}_0)}\bigg)^{1/r},$$

$$K_1^2\le\int\limits_0^t\|\overset{1}{u}_{,\xi}\|_{L_\infty(\overset{1}{\Omega}_0)}\,d\tau\bigg(\int\limits_0^t\|\overset{1}{H}_{,\xi\xi}\|^r_{L_r(\overset{1}{\Omega}_0)}\,dt'\bigg)^{1/r}$$

$$\le t^{1/r'}\|\overset{1}{u}\|_{L_r(0,t;W_r^3(\overset{1}{\Omega}_0))}\|\overset{1}{H}\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))}\qquad\text{for } r>3/2,$$

$$K_1^3\le\bigg(\int\limits_0^t\|\overset{1}{u}_{,\xi}\|^r_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}\|\overset{1}{H}\|^r_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}\,dt'\bigg)^{1/r}\equiv K_1^5,$$

where $1/\lambda_1+1/\lambda_2=1$. Using the interpolations (see Lemma 2.5)

$$\|\overset{1}{u}_{,\xi}\|_{L_{r\lambda_1}(\overset{1}{\Omega}_0)}\le c\|\overset{1}{u}\|^{\theta_1}_{W_r^3(\overset{1}{\Omega}_0)}\|\overset{1}{u}\|^{1-\theta_1}_{L_r(\overset{1}{\Omega}_0)},\qquad\theta_1=\frac{1}{3}+\frac{1}{r\lambda_2},$$

$$\|\overset{1}{H}\|_{L_{r\lambda_2}(\overset{1}{\Omega}_0)}\le c\|\overset{1}{H}\|^{\theta_2}_{W_r^2(\overset{1}{\Omega}_0)}\|\overset{1}{H}\|^{1-\theta_2}_{L_r(\overset{1}{\Omega}_0)},\qquad\theta_2=\frac{3}{2r\lambda_1},$$

we have

$$K_1^5 \le c \sup_t \|\overset{1}{u}\|^{1-\theta_1}_{L_r(\overset{1}{\Omega}_0)} \sup_t \|\overset{1}{H}\|^{1-\theta_2}_{L_r(\overset{1}{\Omega}_0)} \Big(\int_0^t \|\overset{1}{u}\|^{\theta_1 r}_{W_r^3(\overset{1}{\Omega}_0)} \|\overset{1}{H}\|^{\theta_2 r}_{W_r^2(\overset{1}{\Omega}_0)}\, dt'\Big)^{1/r}$$
$$\le c\big(t^{1/r'}\|\overset{1}{u}_{,t}\|_{L_r(\overset{1}{\Omega}_0)} + \|\overset{1}{u}(0)\|_{L_r(\overset{1}{\Omega}_0)}\big)^{1-\theta_1}\big(t^{1/r'}\|\overset{1}{H}_{,t}\|_{L_r(\overset{1}{\Omega}_0)} + \|\overset{1}{H}(0)\|_{L_r(\overset{1}{\Omega}_0)}\big)^{1-\theta_2}$$
$$\cdot \Big(\int_0^t \|\overset{1}{u}\|^{\theta_1 r\mu_1}_{W_r^3(\overset{1}{\Omega}_0)}\, dt'\Big)^{1/(r\mu_1)} \Big(\int_0^t \|\overset{1}{H}\|^{\theta_2 r\mu_2}_{W_r^2(\overset{1}{\Omega}_0)}\, dt'\Big)^{1/(r\mu_2)} \Big(\int_0^t dt'\Big)^{1/(r\mu_3)} \equiv K_1^6,$$

where $1/\mu_1 + 1/\mu_2 + 1/\mu_3 = 1$, $\theta_1\mu_1 = 1$, $\theta_2\mu_2 = 1$, so $1/\mu_3 = 2/3 - 1/r - 1/(2r\lambda_1) > 0$. Hence for $\lambda_1 = 1$, we need $r > 9/4$. Further, we have

$$K_1^6 \le t^a(t^{1/r'}\|\overset{1}{u}\|_{W_r^{3,1}(\overset{1}{\Omega}{}_0^t)} + \|\overset{1}{u}(0)\|_{L_r(\overset{1}{\Omega}_0)})(t^{1/r'}\|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|\overset{1}{H}(0)\|_{L_r(\overset{1}{\Omega}_0)}).$$

Finally, for $r > 3/2$,

$$K_1^4 \le \Big(\int_0^t dt' \Big|\int_0^{t'} \|\overset{1}{u}_{,\xi}(\tau)\|_{L_\infty(\overset{1}{\Omega}_0)}\, d\tau\Big|^r \|\overset{1}{H}_{,t}\|^r_{L_r(\overset{1}{\Omega}_0)}\Big)^{1/r}$$
$$\le t^{1/r'}\|\overset{1}{u}\|_{L_r(0,t;W_r^3(\overset{1}{\Omega}_0))}\|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}.$$

Summarizing the estimates we obtain $(5.36)_3$. Finally, we prove $(5.36)_4$:

$$\|\overset{i}{g}_*\|_{L_r(0,t;W_r^1(\overset{i}{\Omega}_0))} = \|\mathrm{div}_\xi \overset{i}{H} - \mathrm{div}_{\overset{i}{u}} \overset{i}{H}\|_{L_r(0,t;W_r^1(\overset{i}{\Omega}_0))}$$
$$= \|\mathrm{div}_\xi \overset{i}{H} - \mathrm{div}_{\overset{i}{u}} \overset{i}{H}\|_{L_r(\overset{i}{\Omega}{}_0^t)} + \|\nabla_\xi(\mathrm{div}_\xi \overset{i}{H} - \mathrm{div}_{\overset{i}{u}} \overset{i}{H})\|_{L_r(\overset{i}{\Omega}{}_0^t)}$$
$$\le \overset{i}{\alpha}_r(t)\varphi(\overset{i}{\alpha}_r(t))\|\nabla_\xi\overset{i}{H}\|_{L_r(\overset{i}{\Omega}{}_0^t)} + \overset{i}{\alpha}_r(t)\varphi(\overset{i}{\alpha}_r(t))\|\nabla_\xi^2\overset{i}{H}\|_{L_r(\overset{i}{\Omega}{}_0^t)}$$
$$+ \varphi(\overset{i}{\alpha}_r(t))\Big\|\int_0^t \overset{i}{u}_{\xi\xi}\, dt'\, \overset{i}{H}_\xi\Big\|_{L_r(\overset{i}{\Omega}{}_0^t)}$$
$$\le \overset{i}{\alpha}_r(t)\varphi(\overset{i}{\alpha}_r(t))\|\overset{i}{H}\|_{L_r(0,t;W_r^2(\overset{i}{\Omega}_0))},$$

where we have used the estimate for L_2'. This concludes the proof. ■

Finally, we prove the existence of solutions to problem (5.2) with given $\overset{1}{u}, \overset{2}{u}$ by the following method of successive approximations:

$$\begin{aligned} &\mu_1\overset{1}{H}_{n+1,t} + \frac{1}{\sigma_1}\mathrm{rot}_\xi^2\overset{1}{H}_{n+1} \\ &\quad = \frac{1}{\sigma_1}(\mathrm{rot}_\xi^2\overset{1}{H}_n - \mathrm{rot}^2_{\overset{1}{u}}\overset{1}{H}_n) + \mu_1\,\mathrm{rot}_{\overset{1}{u}}(\overset{1}{u}\times\overset{1}{H}_n) + \mu_1\overset{1}{u}\cdot\nabla_{\overset{1}{u}}\overset{1}{H}_n, \\ &\mathrm{div}_\xi\overset{1}{H}_{n+1} = \mathrm{div}_\xi\overset{1}{H}_n - \mathrm{div}_{\overset{1}{u}}\overset{1}{H}_n, \qquad \text{in } \overset{1}{\Omega}{}_0^t, \end{aligned} \tag{5.40}$$

$$
\begin{aligned}
&\mu_2 \overset{2}{H}_{n+1,t} + \frac{1}{\sigma_2} \operatorname{rot}_\xi^2 \overset{2}{H}_{n+1} \\
&\quad = \frac{1}{\sigma_2} (\operatorname{rot}_\xi^2 \overset{2}{H}_n - \operatorname{rot}_{\overset{2}{u}}^2 \overset{2}{H}_n) + \mu_2 \overset{2}{u} \cdot \nabla_{\overset{2}{u}} \overset{2}{H}_n, \\
&\operatorname{div}_\xi \overset{2}{H}_{n+1} = \operatorname{div}_\xi \overset{2}{H}_n - \operatorname{div}_{\overset{2}{u}} \overset{2}{H}_n \qquad\qquad \text{in } \overset{2}{\Omega}{}_0^t, \\
&\left(\frac{1}{\sigma_1} \operatorname{rot}_\xi \overset{1}{H}_{n+1} - \frac{1}{\sigma_2} \operatorname{rot}_\xi \overset{2}{H}_{n+1} \right) \cdot \bar\tau_\alpha \\
&\quad = \left[\frac{1}{\sigma_1} (\operatorname{rot}_\xi \overset{1}{H}_n - \operatorname{rot}_{\overset{1}{u}} \overset{1}{H}_n) - \frac{1}{\sigma_2} (\operatorname{rot}_\xi \overset{2}{H}_n - \operatorname{rot}_{\overset{2}{u}} \overset{2}{H}_n) \right] \cdot \bar\tau_\alpha \\
&\qquad + \left(\frac{1}{\sigma_1} \operatorname{rot}_{\overset{1}{u}} \overset{1}{H}_n - \frac{1}{\sigma_2} \operatorname{rot}_{\overset{2}{u}} \overset{2}{H}_n \right) \cdot (\bar\tau_\alpha - \bar\tau_{u\alpha}) \qquad\qquad (5.40)\ [\text{cont.}] \\
&\qquad + \mu_1 \overset{1}{u} \times \overset{1}{H}_n \cdot \bar\tau_{u\alpha}, \quad \alpha = 1,2, \qquad\qquad \text{on } S_0^t, \\
&(\overset{1}{b}\overset{1}{H}_{n+1} - \overset{2}{b}\overset{2}{H}_{n+1}) \cdot (\bar n \times \bar\tau_\alpha) \\
&\quad = (\bar n \times \bar\tau_\alpha - \bar n_u \times \bar\tau_{u\alpha})(\overset{1}{b}\overset{1}{H}_n - \overset{2}{b}\overset{2}{H}_n), \quad \alpha = 1,2, \qquad\qquad \text{on } S_0^t, \\
&\overset{i}{H}_{n+1}|_{t=0} = \overset{i}{H}(0), \quad i = 1,2, \quad \overset{2}{H}_{n+1} \cdot \bar\tau'_\alpha|_B = H_{*\alpha}, \\
&\operatorname{div}_\xi \overset{2}{H}_{n+1} = \operatorname{div}_\xi \overset{2}{H}_{n+1} - \operatorname{div}_{\overset{2}{u}} \overset{2}{H}_{n+1}, \quad \bar\tau'_\alpha \text{ is tangent to } B, \quad \alpha = 1,2.
\end{aligned}
$$

THEOREM 5.8. *Let $r > 5/2$. Assume that $\overset{i}{u}(0) \in L_r(\overset{i}{\Omega}_0)$, $\overset{i}{H}(0) \in W_r^{2-2/r}(\overset{i}{\Omega}_0)$, $\overset{i}{u} \in L_r(0,t; W_r^3(\overset{i}{\Omega}_0)) \cap W_r^{3,3/2}(\overset{i}{\Omega}{}^t)$, $i = 1,2$, $H_{*\alpha} \in W_r^{2-1/r,1-1/(2r)}(B^t)$, $\alpha = 1,2$. Then there exists T sufficiently small such that for $t \le T$ there exists a solution to problem* (5.2) *such that $\overset{i}{H} \in W_r^{2,1}(\overset{i}{\Omega}{}_0^t)$, $i = 1,2$, and*

$$
\begin{aligned}
\sum_{i=1}^2 \|\overset{i}{H}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)} \le \varphi\Big(& t^a \sum_{i=1}^2 (\|\overset{i}{u}\|_{W_r^{3,3/2}(\overset{i}{\Omega}{}_0^t)} + \|\overset{i}{u}\|_{L_r(0,t;W_r^3)(\overset{i}{\Omega}_0)}), \\
& \sum_{i=1}^2 (\|\overset{i}{u}(0)\|_{L_r(\overset{i}{\Omega}_0)} + \|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)}), \sum_{\alpha=1}^2 \|H_{*\alpha}\|_{W_r^{2-1/r,1-1/(2r)}(B^t)} \Big) \qquad (5.41)
\end{aligned}
$$

where φ is an increasing positive function.

Proof. We use the method of successive approximations described by (5.40). By zero approximation we define an extension of the initial data such that $\overset{i}{H}_0 \in W_r^{2,1}(\overset{i}{\Omega}{}_0^t)$, $i = 1,2$, and

$$
\|\overset{i}{H}_0\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)} \le c \|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)}, \quad i = 1,2.
$$

Remark 5.1 and Lemmas 5.2–5.7 imply existence of solutions to problem (5.40) and the estimate, for some $a > 0$ and some function φ,

$$\sum_{i=1}^{2} \|\overset{i}{H}_{n+1}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)} \le \varphi\Big(t^a \sum_{i=1}^{2} \|\overset{i}{H}_n\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)}, \quad t^a \sum_{i=1}^{2} (\|\overset{i}{u}\|_{W_r^{3,3/2}(\overset{i}{\Omega}{}_0^t)} + \|\overset{i}{u}\|_{L_r(0,t;W_r^3(\overset{i}{\Omega}_0))}), \quad \sum_{i=1}^{2} \|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)}, \sum_{i=1}^{2} \|\overset{i}{u}(0)\|_{L_r(\overset{i}{\Omega}_0)}, \sum_{\alpha=1}^{2} \|H_{*\alpha}\|_{W_r^{2-1/r,1-1/(2r)}(B^t)}\Big). \tag{5.42}$$

Then for t sufficiently small there exists a constant M such that

$$\varphi\Big(t^a M, t^a \sum_{i=1}^{2} (\|\overset{i}{u}\|_{W_r^{3,3/2}(\overset{i}{\Omega}{}_0^t)} + \|\overset{i}{u}\|_{L_r(0,t;W_r^3(\overset{i}{\Omega}_0))}), \quad \sum_{i=1}^{2} \|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)}, \sum_{i=1}^{2} \|\overset{i}{u}(0)\|_{L_r(\overset{i}{\Omega}_0)}, \sum_{\alpha=1}^{2} \|H_{*\alpha}\|_{W_r^{2-1/r,1-1/(2r)}(B^t)}\Big) \le M. \tag{5.43}$$

This means that

$$\sum_{i=1}^{2} \|\overset{i}{H}_n\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)} \le M \quad \text{for any } n \in \mathbb{N}. \tag{5.44}$$

Hence (5.44) implies (5.41). The convergence can be proved by using estimates for differences. These are standard considerations. We skip the calculations for simplicity. This ends the proof. ■

Remark 5.9. To prove Theorem 5.8 we need that $\overset{i}{u} \in W_r^{3,3/2}(\overset{i}{\Omega}{}_0^t)$, $i = 1, 2$. Applying the method of successive approximations formulated in Section 3 we need existence of solutions to problem (4.2) such that $v \in W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)$.

6. The model problem

Consider the following model problem:

$$\begin{aligned} &\mu_1 \overset{1}{u}_t - \frac{1}{\sigma_1}\Delta_z \overset{1}{u} = \overset{1}{\bar f}, && z_3 > 0,\\ &\operatorname{div}_z \overset{1}{u} = 0, && z_3 > 0,\\ &\mu_2 \overset{2}{u}_t - \frac{1}{\sigma_2}\Delta_z \overset{2}{u} = \overset{2}{\bar f}, && z_3 < 0,\\ &\operatorname{div}_z \overset{2}{u} = 0, && z_3 < 0, \end{aligned} \tag{6.1}$$

where $\overset{i}{\bar f}$, $i = 1, 2$, are divergence free, with the transmission conditions

$$\begin{aligned} &(a_1 \operatorname{rot}_z \overset{1}{u} - a_2 \operatorname{rot}_z \overset{2}{u}) \cdot \bar\tau_\alpha = \bar k_\alpha, \quad \alpha = 1, 2, && z_3 = 0,\\ &b_1 \overset{1}{u}_\alpha - b_2 \overset{2}{u}_\alpha = \bar l_\alpha, \quad \alpha = 1, 2, && z_3 = 0, \end{aligned} \tag{6.2}$$

where a_i, b_i, $i=1,2$, are some constants and $\bar\tau_1=(1,0,0)$, $\bar\tau_2=(0,1,0)$, and with the initial conditions

$$\overset{i}{u}|_{t=0}=\overset{i}{u}(0),\qquad i=1,2. \tag{6.3}$$

We assume that $\overset{i}{u}(0)$, $i=1,2$, are divergence free.

Remark 6.1. The transmission conditions $(6.2)_2$ can be generalized to

$$b_{1\alpha}\overset{1}{u}_\alpha-b_{2\alpha}\overset{2}{u}_\alpha=\bar l_\alpha,\qquad \alpha=1,2. \tag{6.4}$$

The considerations in the proof of Lemma 6.2 also hold for (6.4) but then they are much more complicated. Therefore we restrict ourselves to conditions (6.2).

To make the initial data homogeneous we construct divergence free extensions $\overset{i}{\tilde u}$, $i=1,2$, of the initial data $\overset{i}{u}(0)$, $i=1,2$, such that

$$\overset{i}{\tilde u}|_{t=0}=\overset{i}{u}(0),\qquad i=1,2. \tag{6.5}$$

Set

$$\overset{i}{v}=\overset{i}{u}-\overset{i}{\tilde u},\qquad i=1,2. \tag{6.6}$$

Then problem (6.1)–(6.3) takes the form

$$\begin{aligned}
&\sigma_i\mu_i\overset{i}{v}_t-\Delta\overset{i}{v}=\overset{i}{\sigma}\overset{i}{\bar f}-(\sigma_i\mu_i\overset{i}{\tilde u}_t-\Delta\overset{i}{\tilde u})\equiv\overset{i}{f},\quad i=1,2,\\
&\operatorname{div}\overset{i}{v}=0,\quad i=1,2,\\
&(a_1\operatorname{rot}\overset{1}{v}-a_2\operatorname{rot}\overset{2}{v})\cdot\bar\tau_\alpha\\
&\quad=\bar k_\alpha-(a_1\operatorname{rot}\overset{1}{\tilde u}-a_2\operatorname{rot}\overset{2}{\tilde u})\cdot\bar\tau_\alpha\equiv k_\alpha,\quad \alpha=1,2,\quad z_3=0,\\
&b_1\overset{1}{v}_\alpha-b_2\overset{2}{v}_\alpha=\bar l_\alpha-(b_1\overset{1}{\tilde u}-b_2\overset{2}{\tilde u})_\alpha\equiv l_\alpha,\quad \alpha=1,2,\quad z_3=0.
\end{aligned} \tag{6.7}$$

Expressing the transmission conditions explicitly we have

$$\begin{aligned}
&\sigma_1\mu_1\overset{1}{v}_t-\Delta\overset{1}{v}=\overset{1}{f},\quad \operatorname{div}\overset{1}{v}=0, && z_3>0,\\
&\sigma_2\mu_2\overset{2}{v}_t-\Delta\overset{2}{v}=\overset{2}{f},\quad \operatorname{div}\overset{2}{v}=0, && z_3<0,\\
&a_1(\overset{1}{v}_{2,z_3}-\overset{1}{v}_{3,z_2})-a_2(\overset{2}{v}_{2,z_3}-\overset{2}{v}_{3,z_2})=k_1, && z_3=0,\\
&a_1(\overset{1}{v}_{3,z_1}-\overset{1}{v}_{1,z_3})-a_2(\overset{2}{v}_{3,z_1}-\overset{2}{v}_{1,z_3})=k_2, && z_3=0,\\
&b_1\overset{1}{v}_i-b_2\overset{2}{v}_i=l_i,\quad i=1,2, && z_3=0.
\end{aligned} \tag{6.8}$$

Assume that $\overset{1}{u}$, $\overset{2}{u}$ are solutions to the problems

$$\begin{aligned}
&\sigma_1\mu_1\overset{1}{u}_t-\Delta\overset{1}{u}=\overset{1}{f},\quad \operatorname{div}\overset{1}{u}=0 \qquad \text{in } \mathbb{R}^3,\\
&\sigma_2\mu_2\overset{2}{u}_t-\Delta\overset{2}{u}=\overset{2}{f},\quad \operatorname{div}\overset{2}{u}=0 \qquad \text{in } \mathbb{R}^3.
\end{aligned} \tag{6.9}$$

Then the functions

$$\overset{i}{\omega}=\overset{i}{v}-\overset{i}{u},\qquad i=1,2, \tag{6.10}$$

are solutions to the problem with vanishing initial data

$$
\begin{aligned}
&\sigma_1\mu_1\overset{1}{\omega}_t-\Delta\overset{1}{\omega}=0,\quad \operatorname{div}\overset{1}{\omega}=0, && z_3>0,\\
&\sigma_2\mu_2\overset{2}{\omega}_t-\Delta\overset{2}{\omega}=0,\quad \operatorname{div}\overset{2}{\omega}=0, && z_3<0,\\
&a_1(\overset{1}{\omega}_{2,z_3}-\overset{1}{\omega}_{3,z_2})-a_2(\overset{2}{\omega}_{2,z_3}-\overset{2}{\omega}_{3,z_2})=k_1', && z_3=0,\\
&a_2(\overset{1}{\omega}_{3,z_1}-\overset{1}{\omega}_{1,z_3})-a_2(\overset{2}{\omega}_{3,z_1}-\overset{2}{\omega}_{1,z_3})=k_2', && z_3=0,\\
&b_1\overset{1}{\omega}_i-b_2\overset{2}{\omega}_i=l_i',\quad i=1,2, && z_3=0.
\end{aligned}
\tag{6.11}
$$

LEMMA 6.2. *Let $r\in(1,\infty)$. Assume that $(\overset{1}{\omega},\overset{2}{\omega})$ is a solution to (6.11). Assume that $k_i'\in H_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}^2\times\mathbb{R}_+)$, $i=1,2$, $l_i'\in H_r^{2-1/r,1-1/(2r)}(\mathbb{R}^2\times\mathbb{R}_+)$, $i=1,2$. Then there exists a solution to problem (6.11) such that $\overset{i}{\omega}\in H_r^{2,1}(\overset{i}{\mathbb{R}}\times\mathbb{R}_+)$, $\overset{i}{\omega}|_{t=0}=0$, $i=1,2$, $\overset{1}{\mathbb{R}}=\mathbb{R}^3_+$, $\overset{2}{\mathbb{R}}=\mathbb{R}^3_-$ and*

$$
\begin{aligned}
\sum_{i=1}^2\|\overset{i}{\omega}\|_{H_r^{2,1}(\overset{i}{\mathbb{R}}\times\mathbb{R}_+)}&\le c\sum_{i=1}^2\|k_i'\|_{H_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}^2\times\mathbb{R}_+)}\\
&\quad+c\sum_{i=1}^2\|l_i'\|_{H_r^{2-1/r,1-1/(2r)}(\mathbb{R}^2\times\mathbb{R}_+)}.
\end{aligned}
\tag{6.12}
$$

Proof. Apply the Fourier–Laplace transform

$$
\begin{aligned}
&(Ff)(\xi,z_3,s)\equiv\tilde f(\xi,z_3,s)=\int_0^\infty e^{-st}\int_{\mathbb{R}^2}f(z,t)e^{-i\xi\cdot z'}\,dz'dt,\quad \operatorname{Re}s>0,\\
&\xi=(\xi_1,\xi_2),\quad z'=(z_1,z_2),\quad z'\cdot\xi=z_1\xi_1+z_2\xi_2,\quad z=(z_1,z_2,z_3),
\end{aligned}
\tag{6.13}
$$

to problem (6.11). Then we have

$$
\begin{aligned}
&\tau_1^2\overset{1}{\tilde\omega}-\overset{1}{\tilde\omega}_{z_3z_3}=0,\quad i\xi_\alpha\overset{1}{\tilde\omega}_\alpha+\overset{1}{\tilde\omega}_{3,z_3}=0, && z_3>0,\\
&\tau_2^2\overset{2}{\tilde\omega}-\overset{2}{\tilde\omega}_{z_3z_3}=0,\quad i\xi_\alpha\overset{2}{\tilde\omega}_\alpha+\overset{2}{\tilde\omega}_{3,z_3}=0, && z_3<0,\\
&a_1(\overset{1}{\tilde\omega}_{2,z_3}-i\xi_2\overset{1}{\tilde\omega}_3)-a_2(\overset{2}{\tilde\omega}_{2,z_3}-i\xi_2\overset{2}{\tilde\omega}_3)=\tilde k_1', && z_3=0,\\
&a_1(i\xi_1\overset{1}{\tilde\omega}_3-\overset{1}{\tilde\omega}_{1,z_3})-a_2(i\xi_1\overset{2}{\tilde\omega}_3-\overset{2}{\tilde\omega}_{1,z_3})=\tilde k_2', && z_3=0,\\
&b_1\overset{1}{\tilde\omega}_i-b_2\overset{2}{\tilde\omega}_i=\tilde l_i',\quad i=1,2,3, && z_3=0,
\end{aligned}
\tag{6.14}
$$

where $\tau_1^2=\mu_1\sigma_1s+\xi^2$ and $\tau_2^2=\mu_2\sigma_2s+\xi^2$.

Solving $(6.14)_{1,2}$ and using the Shapiro–Lopatinskiĭ condition we obtain

$$
\begin{aligned}
&\overset{1}{\tilde\omega}=\overset{1}{A}e^{-\tau_1z_3},\quad \overset{2}{\tilde\omega}=\overset{2}{A}e^{\tau_2z_3},\\
&-\tau_1\overset{1}{A}_3+i\xi_\alpha\overset{1}{A}_\alpha=0,\quad \tau_2\overset{2}{A}_3+i\xi_\alpha\overset{2}{A}_\alpha=0.
\end{aligned}
\tag{6.15}
$$

Inserting (6.15) in the transmission conditions $(6.14)_{3,4,5}$ yields

$$\begin{aligned} &a_1(-\tau_1\overset{1}{A}_2 - i\xi_2\overset{1}{A}_3) - a_1(\tau_2\overset{2}{A}_2 - i\xi_2\overset{2}{A}_3) = \tilde{k}_1',\\ &a_1(i\xi_1\overset{1}{A}_3 + \tau_1\overset{1}{A}_1) - a_2(i\xi_1\overset{2}{A}_3 - \tau_2\overset{2}{A}_1) = \tilde{k}_2',\\ &b_1\overset{1}{A}_j - b_2\overset{2}{A}_j = \tilde{l}_j', \quad j = 1,2. \end{aligned} \tag{6.16}$$

Using $(6.15)_2$ gives

$$\begin{aligned} &a_1\left[-\tau_1\overset{1}{A}_2 - i\xi_2\left(\frac{i\xi_\alpha}{\tau_1}\overset{1}{A}_\alpha\right)\right] - a_2\left[\tau_2\overset{2}{A}_2 - i\xi_2\left(-\frac{i\xi_\alpha}{\tau_2}\overset{2}{A}_\alpha\right)\right] = \tilde{k}_1',\\ &a_1\left[i\xi_1\left(\frac{i\xi_\alpha}{\tau_1}\overset{1}{A}_\alpha\right) + \tau_1\overset{1}{A}_1\right] - a_2\left[i\xi_1\left(-\frac{i\xi_\alpha}{\tau_2}\overset{2}{A}_\alpha\right) - \tau_2\overset{2}{A}_1\right] = \tilde{k}_2',\\ &b_1\overset{1}{A}_j - b_2\overset{2}{A}_j = \tilde{l}_j', \quad j = 1,2. \end{aligned} \tag{6.17}$$

Simplifying, we get

$$\begin{aligned} &\frac{a_1}{\tau_1}[(\xi_2^2 - \tau_1^2)\overset{1}{A}_2 + \xi_1\xi_2\overset{1}{A}_1] - \frac{a_2}{\tau_2}[(\tau_2^2 - \xi_2^2)\overset{2}{A}_2 - \xi_1\xi_2\overset{2}{A}_1] = \tilde{k}_1',\\ &\frac{a_1}{\tau_1}[(\tau_1^2 - \xi_1^2)\overset{1}{A}_1 - \xi_1\xi_2\overset{1}{A}_2] - \frac{a_2}{\tau_2}[(\xi_1^2 - \tau_2^2)\overset{2}{A}_1 + \xi_1\xi_2\overset{2}{A}_2] = \tilde{k}_2'. \end{aligned} \tag{6.18}$$

Using $(6.16)_3$ yields

$$\begin{aligned} &\left[\frac{a_1b_2}{\tau_1}(\xi_2^2 - \tau_1^2) + \frac{a_2b_1}{\tau_2}(\xi_2^2 - \tau_2^2)\right]\overset{2}{A}_2 + \left(\frac{a_1b_2}{\tau_1} + \frac{a_2b_1}{\tau_2}\right)\xi_1\xi_2\overset{2}{A}_1\\ &\quad = b_1\tilde{k}_1' - \frac{a_1}{\tau_1}(\xi_2^2 - \tau_1^2)\tilde{l}_2' - \frac{a_1}{\tau_1}\xi_1\xi_2\tilde{l}_1' \equiv \tilde{h}_1,\\ &\left[\frac{a_1b_2}{\tau_1}(\tau_1^2 - \xi_1^2) + \frac{a_2b_1}{\tau_2}(\tau_2^2 - \xi_1^2)\right]\overset{2}{A}_1 - \left(\frac{a_1b_2}{\tau_1} + \frac{a_2b_1}{\tau_2}\right)\xi_1\xi_2\overset{2}{A}_2\\ &\quad = b_1\tilde{k}_2' - \frac{a_1}{\tau_1}(\tau_1^2 - \xi_1^2)\tilde{l}_1' + \frac{a_1}{\tau_1}\xi_1\xi_2\tilde{l}_2' \equiv \tilde{h}_2. \end{aligned} \tag{6.19}$$

Set

$$d_1 = \frac{a_1b_2}{\tau_1} + \frac{a_2b_1}{\tau_2}, \quad d_2 = a_1b_2\tau_1 + a_2b_1\tau_2. \tag{6.20}$$

Then (6.19) takes the form

$$\begin{aligned} &-(d_2 - d_1\xi_2^2)\overset{2}{A}_2 + d_1\xi_1\xi_2\overset{2}{A}_1 = \tilde{h}_1,\\ &-d_1\xi_1\xi_2\overset{2}{A}_2 + (d_2 - d_1\xi_1^2)\overset{2}{A}_1 = \tilde{h}_2. \end{aligned} \tag{6.21}$$

Solving (6.21) yields

$$\begin{aligned} A_1^2 &= \frac{\tilde{h}_1d_1\xi_1\xi_2 - \tilde{h}_2(d_2 - d_1\xi_2^2)}{-d_2(d_2 - d_1\xi^2)},\\ A_2^2 &= \frac{\tilde{h}_1(d_2 - d_1\xi_1^2) - \tilde{h}_2d_1\xi_1\xi_2}{-d_2(d_2 - d_1\xi^2)}. \end{aligned} \tag{6.22}$$

We have the qualitative relations

$$d_1 \sim \frac{c}{|\tau|}, \quad d_2 \sim c|\tau|, \quad \overset{2}{A}_1 \sim \frac{|\tilde{h}|}{|\tau|}, \quad \overset{2}{A}_1 \sim \frac{|\tilde{h}|}{|\tau|}, \tag{6.23}$$

where $\tilde{h}$, τ replace $(\tilde{h}_1, \tilde{h}_2)$, (τ_1, τ_2) respectively. In view of (6.15) we have

$$\begin{aligned} \overset{1}{\tilde{\omega}}_\alpha &= \overset{1}{A}_\alpha e^{-\tau_1 z_3} = \left(\frac{b_2}{b_1}\overset{2}{A}_\alpha + \frac{1}{b_1}\tilde{l}'_\alpha\right) e^{-\tau_1 z_3}, \quad \alpha = 1,2, \\ \overset{2}{\tilde{\omega}}_\alpha &= \overset{2}{A}_\alpha e^{\tau_2 z_3}, \quad \alpha = 1,2. \end{aligned} \tag{6.24}$$

Continuing,

$$\begin{aligned} \overset{1}{\tilde{\omega}}_3 &= \frac{i\xi_\alpha}{\tau_1}\overset{1}{A}_\alpha e^{-\tau_1 z_3} = \frac{i\xi_\alpha}{\tau_1}\left(\frac{b_2}{b_1}\overset{2}{A}_\alpha + \frac{1}{b_1}\tilde{l}'_\alpha\right) e^{-\tau_1 z_3}, \\ \overset{2}{\tilde{\omega}}_3 &= -\frac{i\xi_\alpha}{\tau_2}\overset{2}{A}_\alpha e^{\tau_2 z_3}, \end{aligned} \tag{6.25}$$

where the summation over $\alpha \in \{1,2\}$ is assumed.

To prove the lemma we need (see [BZ, Lemma 3.1] and also [ST, S2, Z2])

$$\|\partial_{z_3}^j e^{-\tau_1 z_3}\|_{L_r(\mathbb{R}_+)} \le c|\tau_1|^{j-1/r}, \qquad \|\partial_{z_3}^j e^{\tau_2 z_3}\|_{L_r(\mathbb{R}_-)} \le c|\tau_2|^{j-1/r}. \tag{6.26}$$

To derive estimate (6.12) we use the definition of the Bessel potential space introduced in (2.23). We have

$$\begin{aligned} &\|\overset{1}{\omega}\|_{H_r^{2,1}(\mathbb{R}^3_+\times\mathbb{R}_+)} + \|\overset{2}{\omega}\|_{H_r^{2,1}(\mathbb{R}^3_-\times\mathbb{R}_+)} \\ &= \bigg(\int_{\mathbb{R}_+} dt \int_{\mathbb{R}^3_+} \Big[|F^{-1}(|\tau_1|^2+1)\overset{1}{\tilde{\omega}}|^r + \sum_{i=1}^2 |F^{-1}\partial_{z_3}^i(|\tau_1|^{2-i}\overset{1}{\tilde{\omega}})|^r\Big]\, dz\bigg)^{1/r} \\ &\quad + \bigg(\int_{\mathbb{R}_+} dt \int_{\mathbb{R}^3_-} \Big[|F^{-1}(|\tau_2|^2+1)\overset{2}{\tilde{\omega}}|^r + \sum_{i=1}^2 |F^{-1}\partial_{z_3}^i(|\tau_2|^{2-i}\overset{2}{\tilde{\omega}})|^r\Big]\, dz\bigg)^{1/2} \equiv I_1 + I_2. \end{aligned} \tag{6.27}$$

For I_1, in view of (6.24) and (6.25) we obtain

$$\begin{aligned} I_1 \le \sum_{\alpha=1}^2 \Bigg(&\bigg(\int_{\mathbb{R}_+} dt \int_{\mathbb{R}^3_+} \Big[|F^{-1}(|\tau_1|^2+1)\overset{1}{A}_\alpha e^{-\tau_1 z_3}|^r + \sum_{i=1}^2 |F^{-1}\partial_{z_3}^i |\tau_1|^{2-i}\overset{1}{A}_\alpha e^{-\tau_1 z_3}|^r\Big]\, dz\bigg)^{1/r} \\ &+ \bigg(\int_{\mathbb{R}_+} dt \int_{\mathbb{R}^3_+} \bigg[\bigg|F^{-1}(|\tau_1|^2+1)\frac{\xi_\alpha}{\tau_1}\overset{1}{A}_\alpha e^{-\tau_1 z_3}\bigg|^r \\ &+ \sum_{i=1}^2 \bigg|F^{-1}\partial_{z_3}^i |\tau_1|^{2-i}\frac{\xi_\alpha}{\tau_1}\overset{1}{A}_\alpha e^{-\tau_1 z_3}\bigg|^r\bigg]\, dz\bigg)^{1/r}\Bigg) \equiv \sum_{i=1}^4 I_1^i. \end{aligned}$$

We only handle I_1^1 because the other terms can be treated similarly. By the Minkowski inequality (see [BIN, Ch. 2, Sect. 2.11]) we have

$$I_1^1 \le \bigg(\sum_{\alpha=1}^{2}\int\limits_{\mathbb{R}_+} dt \int\limits_{\mathbb{R}^2} dz' \,\bigg|F^{-1}(|\tau_1|^2+1)|\overset{1}{A}_\alpha|\Big(\int\limits_{\mathbb{R}_+} e^{-r|\tau_1|z_3}\,dz_3\Big)^{1/r}\bigg|^r\bigg)^{1/r}$$

$$= \bigg(\sum_{\alpha=1}^{2}\int\limits_{\mathbb{R}_+} dt \int\limits_{\mathbb{R}^2} dz' \,\bigg|F^{-1}(|\tau_1|^2+1)|\overset{1}{A}_\alpha|\frac{1}{|\tau_1|^{1/r}}\bigg|^r\bigg)^{1/r} \equiv I_1^{11}.$$

From (6.24) and (6.23) and using Besov spaces we have

$$I_1^{11} \le c\bigg(\sum_{\alpha=1}^{2}\int\limits_{\mathbb{R}_+} dt \int\limits_{\mathbb{R}^2} dz' \,\bigg|F^{-1}(|\tau_1|^2+1)\Big(\frac{|\tilde{h}_\alpha|}{|\tau_1|}+|\tilde{l}'_\alpha|\Big)\frac{1}{|\tau_1|^{1/r}}\bigg|^r\bigg)^{1/r}$$

$$\le c\sum_{\alpha=1}^{2}(\|k'_\alpha\|_{H_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}^2\times\mathbb{R}_+)} + \|l'_\alpha\|_{H_r^{2-1/r,1-1/(2r)}(\mathbb{R}^2\times\mathbb{R}_+)}),$$

where we have used $|\tilde{h}| \le c(|\tilde{k}'| + |\tau|\,|\tilde{l}'|)$. For more details, see [ZZ1–ZZ3].

Summarizing, we obtain

$$|I_1| + |I_2| \le c\sum_{\alpha=1}^{2}(\|k'_\alpha\|_{H_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}^2\times(0,t))} + \|l'_\alpha\|_{H_r^{2-1/r,1-1/(2r)}(\mathbb{R}^2\times(0,t))}).$$

This implies (6.12) and concludes the proof. ■

Remark 6.3. The transmission conditions $(6.1)_4$ can be generalized to

$$\overset{1}{b}_k\overset{1}{u}_k - \overset{2}{b}_k\overset{2}{u}_k = \bar{l}_k, \quad k = 1,2,3. \tag{6.28}$$

Taking into account problem (5.17) we have to consider the following problem:

$$\begin{aligned} &w_t - \Delta w + \nabla \operatorname{div} w = 0, \quad z_3 > 0,\\ &w|_{t=0} = 0, \quad w|_{z_3=0} = (a_1, a_2, b). \end{aligned} \tag{6.29}$$

Lemma 6.4. *Let $r \in (1,\infty)$. Assume that $a_\alpha \in H_r^{2-1/r,1-1/(2r)}(\mathbb{R}^2\times\mathbb{R}_+)$, $\alpha = 1,2$. Then there exists a solution to problem (6.29) such that $w \in H_r^{2,1}(\mathbb{R}^3_+\times\mathbb{R}_+)$ and w is divergence free. Moreover,*

$$\|w\|_{H_r^{2,1}(\mathbb{R}^3_+\times\mathbb{R}_+)} \le c\sum_\alpha \|a_\alpha\|_{H_r^{2-1/r,1-1/(2r)}(\mathbb{R}^2\times\mathbb{R}_+)}. \tag{6.30}$$

Finally, the Fourier–Laplace transforms (see (6.13)) of a_1, a_2, b satisfy the restriction

$$\tilde{b} = \frac{i\xi_\alpha\tilde{a}_\alpha}{\sqrt{s+\xi^2}}. \tag{6.31}$$

Proof. We use the summation convention over repeated Greek indices from 1 to 2. Applying the Fourier–Laplace transform (6.13) to (6.29) yields

$$\begin{aligned} &s\tilde{w}_\alpha - \tilde{w}_{\alpha,z_3z_3} + \xi^2\tilde{w}_\alpha + i\xi_\alpha(i\xi_\beta\tilde{w}_\beta + \tilde{w}_{3,z_3}) = 0,\\ &s\tilde{w}_3 - \tilde{w}_{3,z_3z_3} + \xi^2\tilde{w}_3 + \partial_{z_3}(i\xi_\beta\tilde{w}_\beta + \tilde{w}_{3,z_3}) = 0,\\ &\tilde{w}|_{z_3=0} = (\tilde{a}_1, \tilde{a}_2, \tilde{b}). \end{aligned} \tag{6.32}$$

Simplifying (6.32) yields

$$
\begin{aligned}
&(s+\xi^2)\tilde w_\alpha - \tilde w_{\alpha,z_3z_3} - \xi_\alpha\xi_\beta\tilde w_\beta + i\xi_\alpha\tilde w_{3,z_3} = 0,\\
&(s+\xi^2)\tilde w_3 + i\xi_\beta\tilde w_{\beta,z_3} = 0,\\
&\tilde w|_{z_3=0} = (\tilde a_1,\tilde a_2,\tilde b).
\end{aligned}
\tag{6.33}
$$

Multiplying $(6.33)_1$ by ξ_α and summing with respect to α gives

$$
(s+\xi^2)\xi_\alpha\tilde w_\alpha - \xi_\alpha\tilde w_{\alpha,z_3z_3} - \xi^2\xi_\beta\tilde w_\beta + i\xi^2\tilde w_{3,z_3} = 0. \tag{6.34}
$$

Introducing the quantity

$$
\tilde G = \xi_\alpha\tilde w_\alpha \tag{6.35}
$$

we obtain from (6.34), $(6.33)_{1,2}$ the following problem:

$$
\begin{aligned}
&(s+\xi^2)\tilde G - \tilde G_{,z_3z_3} - \xi^2\tilde G + i\xi^2\tilde w_{3,z_3} = 0,\\
&(s+\xi^2)\tilde w_3 + i\tilde G_{,z_3} = 0,\\
&\tilde G|_{z_3=0} = \xi_\alpha\tilde a_\alpha.
\end{aligned}
\tag{6.36}
$$

From $(6.36)_2$ we have

$$
\tilde w_3 = -\frac{i}{s+\xi^2}\tilde G_{,z_3}. \tag{6.37}
$$

Inserting this in (6.36) yields

$$
s(s+\xi^2)\tilde G - s\tilde G_{,z_3z_3} = 0.
$$

Since $\operatorname{Re} s > 0$, we get

$$
(s+\xi^2)\tilde G - \tilde G_{,z_3z_3} = 0, \qquad \tilde G|_{z_3=0} = \xi_\alpha\tilde a_\alpha. \tag{6.38}
$$

Solving $(6.38)_1$ gives

$$
\tilde G = c_1\exp(\sqrt{s+\xi^2}\,z_3) + c_2\exp(-\sqrt{s+\xi^2}\,z_3), \quad z_3>0. \tag{6.39}
$$

Since $\operatorname{Re}\sqrt{s+\xi^2} > 0$, we have to assume that $c_1 = 0$. From the boundary condition $(6.38)_2$ we obtain $c_2 = \xi_\alpha\tilde a_\alpha$. Hence

$$
\tilde G = \xi_\alpha\tilde a_\alpha\exp(-\sqrt{s+\xi^2}\,z_3). \tag{6.40}
$$

In view of (6.35) we have

$$
\xi_\alpha\tilde w_\alpha = \xi_\alpha\tilde a_\alpha\exp(-\sqrt{s+\xi^2}\,z_3), \tag{6.41}
$$

so

$$
\xi_\alpha(\tilde w_\alpha - \tilde a_\alpha\exp(-\sqrt{s+\xi^2}\,z_3)) = 0.
$$

Hence

$$
\tilde w_\alpha = \tilde a_\alpha\exp(-\sqrt{s+\xi^2}\,z_3), \quad \alpha=1,2. \tag{6.42}
$$

In view of (6.37) and (6.40) we have

$$
\tilde w_3 = \frac{i\xi_\alpha\tilde a_\alpha}{\sqrt{s+\xi^2}}\exp(-\sqrt{s+\xi^2}\,z_3). \tag{6.43}
$$

Projecting (6.43) on $z_3 = 0$ and using the boundary conditions $(6.29)_3$ we obtain the restriction (6.31). Set

$$
\tau^2 = s+\xi^2.
$$

Then

$$\|w\|_{H_r^{2,1}(\mathbb{R}_+^3\times\mathbb{R}_+)} = \sum_{\alpha=1}^2 \Big(\int_0^t\int_{\mathbb{R}_+^3} |F^{-1}(\tau^2+1+\partial_{z_3}^2)\tilde w_\alpha|^r\,dx\,dt'\Big)^{1/r} + \Big(\int_0^t\int_{\mathbb{R}^3} |F^{-1}(\tau^2+1+\partial_{z_3}^2)\tilde w_3|^r\,dx\,dt'\Big)^{1/r} \equiv I_1+I_2.$$

In view of (6.42) we have

$$I_1 = \sum_{\alpha=1}^2 \Big(\int_0^t\int_{\mathbb{R}_+^3} |F^{-1}(2\tau^2+1)\tilde a_\alpha \exp(-\tau z_3)|^r\,dx\,dt'\Big)^{1/r}.$$

In view of the Minkowski inequality we obtain

$$\begin{aligned} I_1 &\le \sum_{\alpha=1}^2 \Big(\int_0^t\int_{\mathbb{R}^2} |F^{-1}(2|\tau|^2+1)|\tilde a_\alpha|\,\|\exp(-\tau z_3)\|_{L_r(\mathbb{R}_+)}|^r\,dz'\,dt'\Big)^{1/r} \\ &\le \sum_{\alpha=1}^2 \Big(\int_0^t\int_{\mathbb{R}^2} \big|F^{-1}(2|\tau|^{2-1/r}+1)|\tilde a_\alpha|\big|^r\,dz'\,dt'\Big)^{1/r} \\ &\le c\sum_{\alpha=1}^2 \|a_\alpha\|_{H_r^{2-1/r,1-1/r}(\mathbb{R}^2\times\mathbb{R}_+)}. \end{aligned}$$

In view of (6.43) we have

$$I_2 = \Big(\int_0^t\int_{\mathbb{R}_+^3} \Big|F^{-1}(|\tau|^2+1)\frac{\xi_\alpha\tilde a_\alpha}{\tau}\exp(-\tau z_3)\Big|^r\,dz'\,dz_3\,dt'\Big)^{1/r}.$$

Using the Minkowski inequality and the estimate

$$\Big|\frac{\xi_\alpha\tilde a_\alpha}{\tau}\Big| \le c\sum_\alpha |\tilde a_\alpha|$$

we obtain

$$I_2 \le c\sum_{\alpha=1}^2 \Big(\int_0^t\int_{\mathbb{R}^2} \big|F^{-1}(|\tau|^{2-1/r}+1)|\tilde a_\alpha|\big|^r\,dz'\,dt'\Big)^{1/r} \le c\sum_{\alpha=1}^2 \|a_\alpha\|_{H_r^{2-1/r,1-1/(2r)}(\mathbb{R}^2\times\mathbb{R}_+)}. \quad \blacksquare$$

Remark 6.5. We describe condition (6.31). We have

$$\sqrt{s+\xi^2}\,\tilde b = -\tilde w_{3,z_3}|_{z_3=0}$$

and

$$i\xi_\alpha\tilde a_\alpha = \tilde w_{\alpha,z_\alpha}|_{z_3=0}.$$

Taking the inverse Fourier–Laplace transform F we get

$$w_{3,z_3} + w_{\alpha,z_\alpha} = 0, \tag{6.44}$$

so we have the equation of continuity projected on $z_3=0$.

Condition (6.44) holds on solutions to problem (6.29) proved in Lemma 6.4. Therefore (6.31) is the real condition on the coordinates of $w|_{z_3=0}$ expressed in the Fourier–Laplace transforms.

Consider the problem

$$
\begin{aligned}
&w_t - \Delta w = f - \nabla g \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+,\\
&w|_{t=0} = 0.
\end{aligned}
\tag{6.45}
$$

Lemma 6.6. *Let $r \in (1,\infty)$. Assume that $f \in L_r(\mathbb{R}^3 \times \mathbb{R}_+)$, $g \in L_r(\mathbb{R}_+; W_r^1(\mathbb{R}^3))$. Then there exists a solution to (6.45) such that $w \in H_r^{2,1}(\mathbb{R}^3 \times \mathbb{R}_+)$ and there exists c independent of w such that*

$$
\|w\|_{H_r^{2,1}(\mathbb{R}^3\times\mathbb{R}_+)} \le c(\|f\|_{L_r(\mathbb{R}^3\times\mathbb{R}_+)} + \|g\|_{L_r(\mathbb{R}_+;W_r^1(\mathbb{R}^3))}).
\tag{6.46}
$$

Proof. Applying the Fourier–Laplace transform (2.24) we obtain

$$
F_1 w = \frac{1}{\tau^2}(F_1 f + i\xi F_1 g).
$$

Using the definition (2.25) of the Bessel potential space and repeating the considerations from (6.27) we derive (6.46). This concludes the proof. ■

7. Existence of solutions to problem (1.1)–(1.8)

To prove the existence of solutions to problem (1.1)–(1.8) we use the method of successive approximations defined by problems (3.5), (3.6). Therefore in Lemma 4.1, v, p, u must be replaced by $\bar{v}_{n+1}$, $\bar{p}_{n+1}$, $\bar{v}_n$, and in Section 5 the quantities $\overset{i}{H}$, $\overset{i}{u}$ are replaced by $\overset{i}{\bar{H}}_n$, $\overset{i}{\bar{v}}_n$, $i = 1, 2$.

Let

$$
\begin{aligned}
D = \|v(0)\|_{W_r^{3-2/r}(\overset{1}{\Omega}_0)} &+ \sum_{i=1}^{2} \|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)}\\
&+ \sum_{\alpha=1}^{2} \|H_{*\alpha}\|_{W_r^{2-1/r,1-1/(2r)}(B^t)} + \|\bar{f}\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)}.
\end{aligned}
\tag{7.1}
$$

Theorem 7.1. *Let $r > 5/2$. Assume that $v(0) \in W_r^{3-2/r}(\overset{1}{\Omega}_0)$, $\overset{i}{H}(0) \in W_r^{2-2/r}(\overset{i}{\Omega}_0)$, $i = 1, 2$, $H_{*\alpha} \in W_r^{2-1/r,1-1/(2r)}(B^t)$, $\alpha = 1, 2$, $\operatorname{div} H|_B = 0$, $\bar{f} \in W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)$. Then there exists $a > 0$ with the following property. If*

$$
t_* = t^a D
\tag{7.2}
$$

is sufficiently small, then there exists a solution to problem (1.1)–(1.8) with $\bar{v} \in W_r^{3,3/2}(\overset{1}{\Omega}{}_0^T)$,

$\overset{i}{\bar{H}} \in W_r^{2,1}(\overset{i}{\Omega}{}_0^t)$, $i=1,2$, *and*

$$\|\bar v\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)} + \sum_{i=1}^2 \|\overset{i}{\bar H}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)} \le \varphi\Big(\|v(0)\|_{W_r^{3-2/r}(\overset{1}{\Omega}_0)}, \sum_{i=1}^2 \|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)}, \sum_{\alpha=1}^2 \|H_{*\alpha}\|_{W_r^{2-1/r,1-1/(2r)}(B^t)}, \|\bar f\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)}\Big), \tag{7.3}$$

where φ is an increasing positive function which is small for small arguments.

Proof. The theorem can be proved by the method of successive approximations described in the proof of Theorem 12.1 taking into account a different regularity of v. Now, we only give two steps of the proof. In view of Theorem 5.8 and Remark 5.9 the following estimate for solutions to problem (3.6) holds:

$$\sum_{i=1}^2 \|\overset{i}{\bar H}_n\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)} \le \varphi\Big(t^a D \sum_{i=1}^2 \|\overset{i}{\bar v}_n\|_{W_r^{3,3/2}(\overset{i}{\Omega}{}_0^t)}, \sum_{i=1}^2 (\|\overset{i}{v}(0)\|_{L_r(\overset{i}{\Omega}_0)} + \|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)}), \sum_{\alpha=1}^2 \|H_{*\alpha}\|_{W_r^{2-1/r,1-1/(2r)}(B^t)}\Big). \tag{7.4}$$

From Lemma 4.1 and (7.4) we have

$$\|\bar v_{n+1}\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)} + \|\nabla \bar p_{n+1}\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)} \le \varphi\Big(t^a D \|\bar v_n\|_{W_r^{3,3/2}(\overset{1}{\Omega}{}_0^t)}, \|v(0)\|_{W_r^{3-2/r}(\overset{1}{\Omega}_0)}, \sum_{i=1}^2 \|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)}, \|\bar f\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)}, \sum_{\alpha=1}^2 \|H_{*\alpha}\|_{W_r^{2-1/r,1-1/(2r)}(B^t)}\Big), \tag{7.5}$$

where we have used the construction of $\overset{2}{\bar v}_n$ by problem (3.2). Assuming that the zero approximation $\bar v_0$ is an extension of the initial data $v(0) \in W_r^{3-2/r}(\overset{1}{\Omega}_0)$, we can prove (7.3) for sufficiently small t_*. Next, through (7.3) and corresponding inequalities for differences, we prove the existence of local solutions. This concludes the proof. ∎

PART II (without jump of the magnetic field)

8. Method of successive approximations

For the convenience of the reader, we repeat some considerations of Section 3. Let $v_n = v_n(x,t)$ be given, $x \in \overset{1}{\Omega}_t$.

Definition 8.1. The Lagrangian coordinates in $\overset{1}{\Omega}_0$ are initial data to the Cauchy problem

$$\frac{dx}{dt} = v_n(x,t), \quad x|_{t=0} = \xi \in \overset{1}{\Omega}_0. \tag{8.1}$$

Hence the domain $\overset{1}{\Omega}_{nt}$ is defined by

$$\overset{1}{\Omega}_{nt} = \Big\{x \in \mathbb{R}^3 : x = x^{(n)}(\xi,t) = \xi + \int_0^t \bar{v}_n(\xi,t')\,dt',\ \xi \in \overset{1}{\Omega}_0\Big\},$$

where $\bar{v}_n(\xi,t) = v_n(x^{(n)}(\xi,t),t)$.

In free boundary problems in hydrodynamics the free boundary is built up from the same fluid particles because $v_n|_{S_{nt}}$ is tangent to S_{nt} and

$$S_{nt} = \{x \in \mathbb{R}^3 : x = x^{(n)}(\xi,t),\ \xi \in S_0\}.$$

To formulate problem (1.10) in Lagrangian coordinates we have to introduce them in $\overset{2}{\Omega}_0$. Since there is no velocity in $\overset{2}{\Omega}_t$, we introduce it artificially.

Definition 8.2. Let $\overset{1}{v}_n = v$ in $\overset{1}{\Omega}_t$, and construct $\overset{2}{v}_n$ in $\overset{2}{\Omega}_t$ as a solution to the nonstationary Stokes problem

$$\begin{aligned}
&\overset{2}{v}_{n,t} - \operatorname{div}\mathbb{T}(\overset{2}{v}_n, q) = 0 && \text{in } \overset{2}{\Omega}_t,\\
&\operatorname{div}\overset{2}{v}_n = 0 && \text{in } \overset{2}{\Omega}_t,\\
&\overset{2}{v}_n|_{S_t} = \overset{1}{v}_n|_{S_t}, \quad \overset{2}{v}_n|_B = 0, &&\\
&\overset{2}{v}_n|_{t=0} = \overset{2}{v}(0) && \text{in } \overset{2}{\Omega}_0,
\end{aligned} \tag{8.2}$$

where q plays the role of pressure but it is not important for any estimate for $\overset{2}{v}_n$. Hence $\overset{2}{v}_n$ is divergence free. The initial data $\overset{2}{v}(0)$ is an extension of $\overset{1}{v}(0)$ through the fixed given boundary S_0, because $\overset{2}{v}(0)|_{S_0} = \overset{1}{v}(0)|_{S_0}$. The extension uses Lemma 2.1. The existence of solutions to (8.2) follows from Lemma 2.4(b).

Now, we introduce Lagrangian coordinates $\overset{1}{\xi}, \overset{2}{\xi}$ by the Cauchy data to the problems

$$\frac{d\overset{i}{x}}{dt} = \overset{i}{v}_n(x,t), \quad x^i|_{t=0} = \overset{i}{\xi} \in \overset{i}{\Omega}_0, \quad i=1,2. \tag{8.3}$$

Then

$$\begin{aligned}\overset{i}{\Omega}_{nt} = \bigg\{ \overset{i}{x} \in \mathbb{R}^3 : \overset{i}{x} = \overset{i}{x}{}^{(n)}(\overset{i}{\xi},t) = \overset{i}{\xi} + \int_0^t \overset{i}{v}_n(\overset{i}{x},t')\,dt' \\ = \overset{i}{\xi} + \int_0^t \overset{i}{\bar{v}}_n(\xi,t')\,dt',\ \overset{i}{\xi} \in \overset{i}{\Omega}_0 \bigg\},\end{aligned} \tag{8.4}$$

where $\overset{i}{\bar{v}}_n(\overset{i}{\xi},t) = \overset{i}{v}_n(\overset{i}{x}{}^{(n)}(\overset{i}{\xi},t),t)$, $\overset{i}{\xi} \in \overset{i}{\Omega}_0$, $i=1,2$.

Assume that $\bar{v}_n$, $\overset{1}{\bar{H}}_n$ are given and $\overset{1}{\Omega}_{nt}$ is as above. We linearize problem (1.9) to

$$\begin{aligned}
&\bar{v}_{n+1,t} - \operatorname{div}_{\bar{v}_n} \mathbb{T}_{\bar{v}_n}(\bar{v}_{n+1},\bar{p}_{n+1}) = \bar{f} + \mu_1 \operatorname{div}_{\bar{v}_n} \mathbb{T}(\overset{1}{\bar{H}}_n) && \text{in } \overset{1}{\Omega}_0 \times (0,t),\\
&\operatorname{div}_{\bar{v}_n} \bar{v}_{n+1} = 0 && \text{in } \overset{1}{\Omega}_0 \times (0,t),\\
&\bar{n}_{\bar{v}_n} \cdot \mathbb{T}_{\bar{v}_n}(\bar{v}_{n+1},\bar{p}_{n+1}) = -\mu_1 \bar{n}_{\bar{v}_n} \cdot \mathbb{T}(\overset{1}{\bar{H}}_n) && \text{on } S_0 \times (0,t),\\
&\bar{v}_{n+1}|_{t=0} = v(0) && \text{in } \overset{1}{\Omega}_0,
\end{aligned} \tag{8.5}$$

where $\nabla_{\bar{v}_n} = \frac{\partial \xi_k}{\partial x^{(n)}}\big|_{x^{(n)}=x^{(n)}(\xi,t)} \partial_{\xi_k}$. Moreover, any operator with subscript $\bar{v}_n$ means that it contains the transformed gradient $\nabla_{\bar{v}_n}$.

The linearized problem (1.10) takes the form

$$\begin{aligned}
&\left.\begin{aligned}&\mu_1 \overset{1}{\bar{H}}_{n,t} - \mu_1 \overset{1}{\bar{v}}_n \cdot \nabla_{\overset{1}{\bar{v}}_n} \overset{1}{\bar{H}}_n = -\operatorname{rot}_{\overset{1}{\bar{v}}_n} \overset{1}{\bar{E}}_n,\\ &\operatorname{rot}_{\overset{1}{\bar{v}}_n} \overset{1}{\bar{H}}_n = \sigma_1(\overset{1}{\bar{E}}_n + \mu_1 \overset{1}{\bar{v}}_n \times \overset{1}{\bar{H}}_n),\ \operatorname{div}_{\overset{1}{\bar{v}}_n} \overset{1}{\bar{H}}_n = 0\end{aligned}\right\} && \text{in } \overset{1}{\Omega}_0 \times (0,t),\\
&\left.\begin{aligned}&\mu_2 \overset{2}{\bar{H}}_{n,t} - \mu_2 \overset{2}{\bar{v}}_n \cdot \nabla_{\overset{2}{\bar{v}}_n} \overset{2}{\bar{H}}_n = -\operatorname{rot}_{\overset{2}{\bar{v}}_n} \overset{2}{\bar{E}}_n,\\ &\sigma_2 \overset{2}{\bar{E}}_n = \operatorname{rot}_{\overset{2}{\bar{v}}_n} \overset{2}{\bar{H}}_n,\ \operatorname{div}_{\overset{2}{\bar{v}}_n} \overset{2}{\bar{H}}_n = 0\end{aligned}\right\} && \text{in } \overset{2}{\Omega}_0 \times (0,t),\\
&\overset{i}{\bar{H}}_n|_{t=0} = \overset{i}{H}(0), \quad \operatorname{div} \overset{i}{H}(0) = 0 && \text{in } \overset{i}{\Omega}_0,\ i=1,2,\\
&\overset{2}{\bar{H}}_n \cdot \bar{\tau}'_\alpha|_B = 0 = H_{*\alpha}, \quad \operatorname{div}_{\overset{2}{\bar{v}}_n} \overset{2}{\bar{H}} = 0, \quad \alpha=1,2, && \text{on } B,\\
&\overset{1}{\bar{E}}_n \cdot \bar{\tau}_{\bar{v}_n,\alpha} = \overset{2}{\bar{E}}_n \cdot \bar{\tau}_{\bar{v}_n,\alpha}, \quad \bar{n}_{\bar{v}_n} \times \bar{\tau}_{\bar{v}_n,\alpha} \cdot (\overset{1}{\bar{H}}_n - \overset{2}{\bar{H}}_n) = 0 && \text{on } S_0,
\end{aligned} \tag{8.6}$$

where we have used $\overset{1}{\bar{v}}_n = \overset{2}{\bar{v}}_n = \bar{v}_n$ on S_0.

We emphasize that formulation of problem (1.10) in Lagrangian coordinates needs the terms $\mu_i\overset{i}{\bar v}_n\cdot\nabla\overset{i}{H}_n|_{x=x^{(n)}(\xi,t)}=\mu_i\overset{i}{\bar v}_n\nabla_{\overset{i}{\bar v}_n}\overset{i}{\bar H}_n$, $i=1,2$, to be added to the l.h.s. of $(8.6)_{1,3}$, respectively.

9. Existence of solutions to problem (8.5) for given $\bar v_n$ and $\overset{1}{\bar H}_n$

We introduce the notation

$$v=\bar v_{n+1},\quad p=\bar p_{n+1},\quad u=\bar v_n,\quad \overset{1}{H}=\overset{1}{\bar H}_n. \tag{9.1}$$

Then problem (8.5) takes the form

$$\begin{aligned}
&v_{,t}-\operatorname{div}_u\mathbb{T}_u(v,p)=f+\mu_1\operatorname{div}_u\mathbb{T}(\overset{1}{H}) && \text{in } \overset{1}{\Omega}{}_0^T,\\
&\operatorname{div}_u v=0 && \text{in } \overset{1}{\Omega}{}_0^T,\\
&\bar n_u\cdot\mathbb{T}_u(v,p)=-\mu_1\bar n_u\cdot\mathbb{T}(\overset{1}{H}) && \text{on } S_0^T,\\
&v|_{t=0}=v(0) && \text{in } \overset{1}{\Omega}_0.
\end{aligned}\tag{9.2}$$

LEMMA 9.1. *Let $r>3$. Assume that $f\in L_r(\overset{1}{\Omega}{}_0^t)$, $v(0)\in W_r^{2-2/r}(\overset{1}{\Omega}_0)$, $\overset{1}{H}\in W_r^{2,1}(\overset{1}{\Omega}{}_0^t)$, $u\in W_r^{2,1}(\overset{1}{\Omega}{}_0^t)$, $u(0)\in L_r(\overset{1}{\Omega}_0)$. Let $\overset{1}{\alpha}_u(t)=t^{1/r'}\|u\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))}$, $1/r+1/r'=1$. Then there exists $a>0$ and a function φ with the following property. If t, u, $u(0)$ are so small that*

$$\varphi(\overset{1}{\alpha}_u(t))t^a(\|u\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}+\|u(0)\|_{L_r(\overset{1}{\Omega}_0)})\le 1/2,$$

then there is a solution to (9.2) with $v\in W_r^{2,1}(\overset{1}{\Omega}{}_0^T)$, $p\in W_r^{1,0}(\overset{1}{\Omega}{}_0^T)\cap W_r^{1-1/r,1/2-1/(2r)}(S_0^T)$, and

$$\begin{aligned}
&\|v\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}+\sup_{t'\le t}\|v(t')\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}+\|p\|_{W_r^{1,0}(\overset{1}{\Omega}{}_0^t)}+\|p\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)}\\
&\quad\le\varphi(\overset{1}{\alpha}_u(t))t^a(\|u\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}+\|u(0)\|_{L_r(\overset{1}{\Omega}_0)})\|v(0)\|_{L_r(\overset{1}{\Omega}_0)}+c\|f\|_{L_r(\overset{1}{\Omega}{}_0^t)}\\
&\quad\quad+\varphi(\overset{1}{\alpha}_u(t))\sup_{t'\le t}\|\overset{1}{H}(t')\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}\|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}+c\|v(0)\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}.
\end{aligned}\tag{9.3}$$

Proof. We prove the existence of solutions to problem (9.2) by the following method of successive approximations:

$$
\begin{aligned}
&v_{n+1,t} - \operatorname{div}_\xi \mathbb{T}_\xi(v_{n+1}, p_{n+1}) \\
&\qquad = -(\operatorname{div}_\xi \mathbb{T}_\xi(v_n, p_n) - \operatorname{div}_u \mathbb{T}_u(v_n, p_n)) + f + \mu_1 \operatorname{div}_u \mathbb{T}(\overset{1}{H}) \equiv F_1 + F_2, \\
&\operatorname{div}_\xi v_{n+1} = \operatorname{div}_\xi v_n - \operatorname{div}_u v_n \equiv G = \nabla \cdot R, \\
&\bar{n}_\xi \cdot \mathbb{T}_\xi(v_{n+1}, p_{n+1}) \\
&\qquad = (\bar{n}_\xi \cdot \mathbb{T}_\xi(v_n, p_n) - \bar{n}_u \cdot \mathbb{T}_u(v_n, p_n)) - \mu \bar{n}_u \cdot \mathbb{T}(\overset{1}{H}) \equiv K_1 + K_2, \\
&v_{n+1}|_{t=0} = v(0),
\end{aligned}
\tag{9.4}
$$

and $v_0 = 0$, $p_0 = 0$. The subscript ξ means that all space operators contain derivatives of the form $\nabla_\xi = \partial_\xi$ and $\bar{n}_\xi$ is the exterior normal to S_0. In view of Lemma 2.6(b) we have

$$
\begin{aligned}
&\|v_{n+1}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \sup_{t' \le t} \|v_{n+1}(t')\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)} + \|p_{n+1}\|_{W_r^{1,0}(\overset{1}{\Omega}{}_0^t)} + \|p_{n+1}\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)} \\
&\quad \le c(\|F_1\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \|F_2\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \|G\|_{W_r^{1,0}(\overset{1}{\Omega}{}_0^t)} + \|R_t\|_{L_r(\overset{1}{\Omega}{}_0^t)} \\
&\qquad + \|K_1\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)} + \|K_2\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)} + \|v(0)\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}).
\end{aligned}
\tag{9.5}
$$

Using the properties of the transformation (2.15) we derive, for $r > 3$,

$$
\begin{aligned}
\|F_1\|_{L_r(\overset{1}{\Omega}{}_0^t)} &\le c\|(\mathbb{I} - \xi_x^2) v_{n,\xi\xi} + \xi_x \xi_{xx} x_\xi v_{n,\xi}\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \|(\delta - \xi_x) p_{n,\xi}\|_{L_r(\overset{1}{\Omega}{}_0^t)} \\
&\le \varphi(\overset{1}{\alpha}_u(t)) \Big\| \int_0^t u_{,\xi}\, dt'\, v_{n,\xi\xi} \Big\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \varphi(\overset{1}{\alpha}_u(t)) \Big\| \int_0^t u_{,\xi\xi}\, dt'\, v_{n,\xi} \Big\|_{L_r(\overset{1}{\Omega}{}_0^t)} \\
&\quad + \varphi(\overset{1}{\alpha}_u(t)) \Big\| \int_0^t u_{,\xi}\, dt'\, p_{n,\xi} \Big\|_{L_r(\overset{1}{\Omega}{}_0^t)} \\
&\le \varphi(\overset{1}{\alpha}_u(t)) \overset{1}{\alpha}_u(t) (\|v_n\|_{W_r^{2,0}(\overset{1}{\Omega}{}_0^t)} + \|p_n\|_{W_r^{1,0}(\overset{1}{\Omega}{}_0^t)}),
\end{aligned}
$$

where $\mathbb{I}$ is the unit matrix, $\overset{1}{\alpha}_u(t) = t^{1/r'} \|u\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))}$, $1/r + 1/r' = 1$, and φ is a polynomial with respect to its argument. In view of Remark 2.7 the polynomial φ can be specified more precisely, but its explicit form is not important for our considerations. Next

$$
\|F_2\|_{L_r(\overset{1}{\Omega}{}_0^t)} \le c\|f\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \varphi(\overset{1}{\alpha}_u(t)) \|\overset{1}{H}\|_{L_\infty(0,t;W_r^{2-2/r}(\overset{1}{\Omega}_0))} \cdot \|\overset{1}{H}\|_{W_r^{2,0}(\overset{1}{\Omega}{}_0^t)}.
$$

Further,

$$
\begin{aligned}
\|G\|_{W_r^{1,0}(\overset{1}{\Omega}{}_0^t)} &\le c\|\operatorname{div}_\xi v_n - \operatorname{div}_u v_n\|_{W_r^{1,0}(\overset{1}{\Omega}{}_0^t)} \\
&\le c\|(\mathbb{I} - \xi_x) v_{n,\xi\xi}\|_{L_r(\overset{1}{\Omega}{}_0^t)} + c\|\xi_x \xi_{xx} x_\xi v_{n,\xi}\|_{L_r(\overset{1}{\Omega}{}_0^t)} \le \varphi(\overset{1}{\alpha}_n(t)) \overset{1}{\alpha}_n(t) \|v_n\|_{W_r^{2,0}(\overset{1}{\Omega}{}_0^t)}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\|R_{,t}\|_{L_r(\overset{1}{\Omega}{}_0^t)} &= \|((\mathbb{I} - \xi_x) v_n)_{,t}\|_{L_r(\overset{1}{\Omega}{}_0^t)} \\
&\le \varphi(\overset{1}{\alpha}_u(t)) (\|u_\xi v_n\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \overset{1}{\alpha}_u(t) \|v_{n,t}\|_{L_r(\overset{1}{\Omega}{}_0^t)}),
\end{aligned}
$$

where the term $\|u_\xi v_n\|_{L_r(\overset{1}{\Omega}{}_0^t)}$ is bounded by

$$\Big(\int_0^t dt' \int_{\overset{1}{\Omega}_0} |u_\xi v_n|^r\, d\xi\Big)^{1/r} \le \Big(\int_0^t dt'\, \|u_\xi\|^r_{L_r(\overset{1}{\Omega}_0)} \|v_n\|^r_{L_\infty(\overset{1}{\Omega}_0)}\Big)^{1/r} \equiv I.$$

Using the interpolations (see Lemma 2.5)

$$\|u_\xi\|_{L_r(\overset{1}{\Omega}_0)} \le c\|u\|^{1/2}_{W_r^2(\overset{1}{\Omega}_0)} \|u\|^{1/2}_{L_r(\overset{1}{\Omega}_0)},$$
$$\|v_n\|_{L_\infty(\overset{1}{\Omega}_0)} \le c\|v_n\|^{3/2r}_{W_r^2(\overset{1}{\Omega}_0)} \|v_n\|^{1-3/2r}_{L_r(\overset{1}{\Omega}_0)},$$

we obtain

$$\begin{aligned}
I &\le c\Big(\int_0^t dt'\, \|u\|^{r/2}_{W_r^2(\overset{1}{\Omega}_0)} \|u\|^{r/2}_{L_r(\overset{1}{\Omega}_0)} \|v_n\|^{3/2}_{W_r^2(\overset{1}{\Omega}_0)} \|v_n\|^{r-3/2}_{L_r(\overset{1}{\Omega}_0)}\Big)^{1/r}\\
&\le c \sup_t \|u\|^{1/2}_{L_r(\overset{1}{\Omega}_0)} \sup_t \|v_n\|^{1-3/2r}_{L_r(\overset{1}{\Omega}_0)} \Big(\int_0^t dt'\, \|u\|^{r/2}_{W_r^2(\overset{1}{\Omega}_0)} \|v_n\|^{3/2}_{W_r^2(\overset{1}{\Omega}_0)}\Big)^{1/r}\\
&\le c \sup_t \Big\|\int_0^t u_{t'}\, dt' + u(0)\Big\|^{1/2}_{L_r(\overset{1}{\Omega}_0)} \sup_t \Big\|\int_0^t v_{n,t'}\, dt' + v(0)\Big\|^{1-3/2r}_{L_r(\overset{1}{\Omega}_0)}\\
&\quad \cdot \Big(\int_0^t dt'\, \|u\|^{(r/2)\lambda_1}_{W_r^2(\overset{1}{\Omega}_0)}\Big)^{1/r\lambda_1} \Big(\int_0^t dt'\, \|v_n\|^{(3/2)\lambda_2}_{W_r^2(\overset{1}{\Omega}_0)}\Big)^{1/r\lambda_2} t^{1/r\lambda_3} \equiv I_1,
\end{aligned}$$

where $1/\lambda_1 + 1/\lambda_2 + 1/\lambda_3 = 1$, $\frac{r}{2}\lambda_1 = r$, $\frac{3}{2}\lambda_2 = r$, $1/\lambda_3 = 1/2 - 3/(2r)$. Continuing, we have

$$\begin{aligned}
I_1 &\le c(t^{1/(2r')}\|u_{,t}\|^{1/2}_{L_r(\overset{1}{\Omega}{}_0^t)} + \|u(0)\|^{1/2}_{L_r(\overset{1}{\Omega}_0)})(t^{\frac{1-3/(2r)}{r'}}\|v_{n,t}\|^{1-3/(2r)}_{L_r(\overset{1}{\Omega}{}_0^t)} + \|v(0)\|^{1-3/(2r)}_{L_r(\overset{1}{\Omega}{}_0^t)})\\
&\quad \cdot \|u\|^{1/2}_{W_r^{2,0}(\overset{1}{\Omega}{}_0^t)} \|v_n\|^{3/(2r)}_{W_r^{2,0}(\overset{1}{\Omega}{}_0^t)} t^{(1/r)(1/2-3/(2r))} \equiv I_2,
\end{aligned}$$

where $1/2 - 3/(2r) > 0$ because $r > 3$. Applying the Young inequalities we get

$$\begin{aligned}
I_2 \le t^a(&\|u\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} \|v_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|u(0)\|_{L_r(\overset{1}{\Omega}{}_0^t)} \|v_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}\\
&+ \|u\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} \|v(0)\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \|u(0)\|_{L_r(\overset{1}{\Omega}{}_0^t)} \|v(0)\|_{L_r(\overset{1}{\Omega}{}_0^t)}).
\end{aligned}$$

Next, we examine the boundary integrals

$$\begin{aligned}
\|K_1\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)} &= \|\bar n_\xi \mathbb{T}_\xi(v_n,p_n) - \bar n_u \mathbb{T}_u(v_n,p_n)\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)}\\
&\le c\|\bar n_\xi \mathbb{T}_\xi(v_n,p_n) - \bar n_u \mathbb{T}_u(v_n,p_n)\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)}\\
&= c\|n_\xi \mathbb{T}_\xi(v_n,p_n) - \bar n_u \mathbb{T}_u(v_n,p_n)\|_{L_r(0,t;W_r^1(\overset{1}{\Omega}_0))}\\
&\quad + c\|n_\xi \mathbb{T}_\xi(v_n,p_n) - \bar n_u \mathbb{T}_u(v_n,p_n)\|_{L_r(\overset{1}{\Omega}_0;W^{1/2}(0,t))} \equiv J_1 + J_2.
\end{aligned}$$

First, we estimate

$$
\begin{aligned}
J_1 &\le \varphi(\overset{1}{\alpha}_u(t))\Big(\Big\|\int_0^t u_{,\xi\xi}\,dt'\,(|v_{n,\xi}|+|p_n|)\Big\|_{L_r(\overset{1}{\Omega}{}_0^t)}\\
&\quad+\Big\|\int_0^t u_{,\xi}\,dt'\,(|v_{n,\xi\xi}|+|p_{n,\xi}|)\Big\|_{L_r(\overset{1}{\Omega}{}_0^t)}\Big)\\
&\le \varphi(\overset{1}{\alpha}_u(t))\overset{1}{\alpha}_u(t)\Big[\Big(\int_0^t(\|v_{n,\xi}\|^r_{L_\infty(\overset{1}{\Omega}_0)}+\|p_n\|^r_{L_\infty(\overset{1}{\Omega}_0)})\,dt'\Big)^{1/r}\\
&\quad+\|v_n\|_{L_r(0,t;W^2_r(\overset{1}{\Omega}_0))}+\|p_n\|_{L_r(0,t;W^1_r(\overset{1}{\Omega}_0))}\Big]\\
&\le \varphi(\overset{1}{\alpha}_u(t))\overset{1}{\alpha}_u(t)(\|v_n\|_{L_r(0,t;W^2_r(\overset{1}{\Omega}_0))}+\|p_n\|_{L_r(0,t;W^1_r(\overset{1}{\Omega}_0))}).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
J_2 &\le \Big(\int_{\overset{1}{\Omega}_0}d\xi\int_0^t\int_0^t dt'\,dt''\\
&\quad\cdot\frac{|\varphi(\overset{1}{\alpha}_u(t''))\int_0^{t''}u\,d\tau\,(v_{n,\xi}(t'')+p_n(t''))-\varphi(\overset{1}{\alpha}_u(t'))\int_0^{t'}u\,d\tau\,(v_{n,\xi}(t'))+p_n(t')|^r}{|t''-t'|^{1+r/2}}\Big)^{1/r}\\
&\le \Big(\int_{\overset{1}{\Omega}_0}d\xi\int_0^t\int_0^t dt'\,dt''\,\frac{|\varphi(\overset{1}{\alpha}_u(t''))-\varphi(\overset{1}{\alpha}_u(t'))|^r|\int_0^{t''}u\,d\tau|^r(|v_{n,\xi}(t'')|^r+|p_n(t'')|^r)}{|t''-t'|^{1+r/2}}\Big)^{1/r}\\
&\quad+\varphi(\overset{1}{\alpha}_u(t))\Big(\int_{\overset{1}{\Omega}_0}d\xi\int_0^t\int_0^t dt'\,dt''\,\frac{\big|\int_{t'}^{t''}u\,d\tau\big|^r(|v_{n,\xi}(t'')|^r+|p_n(t'')|^r)}{|t''-t'|^{1+r/2}}\Big)^{1/r}\\
&\quad+\varphi(\overset{1}{\alpha}_u(t))\overset{1}{\alpha}_u(t)\Big(\int_{\overset{1}{\Omega}_0}d\xi\int_0^t\int_0^t dt'\,dt''\,\Big(\frac{|v_{n,\xi}(t'')-v_{n,\xi}(t')|^r}{|t''-t'|^{1+r/2}}+\frac{|p_n(t'')-p_n(t')|^r}{|t''-t'|^{1+r/2}}\Big)\Big)^{1/r}\\
&\equiv J_2^1+J_2^2+J_2^3.
\end{aligned}
$$

Using

$$
\begin{aligned}
\varphi(\overset{1}{\alpha}_u(t''))-\varphi(\overset{1}{\alpha}_u(t')) &\le \varphi(\overset{1}{\alpha}_u(t))\int_{t'}^{t''}\|u(\tau)\|_{W^2_r(\overset{1}{\Omega}_0)}\,d\tau\\
&\le \varphi(\overset{1}{\alpha}_u(t))|t''-t'|^{1/r'}\|u\|_{L_r(0,t;W^2_r(\overset{1}{\Omega}_0))}
\end{aligned}
$$

and

$$
\sup_{\overset{1}{\Omega}_0}\int_0^t|u(\tau)|\,d\tau \le t^{1/r'}\|u\|_{L_r(0,t;W^1_r(\overset{1}{\Omega}_0))}
$$

we obtain

$$J_2^1 \le \varphi(\overset{1}{\alpha}_u(t))t^a\|u\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))}(\|v_n\|_{L_r(0,t;W_r^1(\overset{1}{\Omega}_0))} + \|p_n\|_{L_r(0,t;L_r(\overset{1}{\Omega}_0))}).$$

For J_2^2 we can get the same bound. Next,

$$J_2^3 \le \varphi(\overset{1}{\alpha}_u(t))\overset{1}{\alpha}_u(t)(\|v_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|p_n\|_{W_r^{1-1/r,1/2-1/(2r)}(S^t)}).$$

Finally,

$$\begin{aligned}&\|K_2\|_{W_r^{1-1/r,1/2-1/(2r)}(S^t)}\\ &\qquad \le c\|K_2\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)} \le c\varphi(\overset{1}{\alpha}_u(t))\sup_t \|\overset{1}{H}\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}\|\overset{1}{H}\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)}.\end{aligned}$$

Employing the above estimates in (9.5) yields

$$\begin{aligned}&\|v_{n+1}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \sup_{t'\le t}\|v_{n+1}(t')\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)} + \|p_{n+1}\|_{W_r^{1,0}(\overset{1}{\Omega}{}_0^t)} + \|p_{n+1}\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)}\\ &\quad \le \varphi(\overset{1}{\alpha}_u(t))\overset{1}{\alpha}_u(t)(\|v_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|p_n\|_{W_r^{1,0}(\overset{1}{\Omega}{}_0^t)} + \|p_n\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)})\\ &\qquad + \varphi(\overset{1}{\alpha}_u(t))t^a(\|u\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|u(0)\|_{L_r(\overset{1}{\Omega}{}_0^t)})\|v_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}\\ &\qquad + \varphi(\overset{1}{\alpha}_u(t))t^a(\|u\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|u(0)\|_{L_r(\overset{1}{\Omega}{}_0^t)})\|v(0)\|_{L_r(\overset{1}{\Omega}_0)}\\ &\qquad + c\|f\|_{L_r(\overset{1}{\Omega}{}_0^t)} + c\varphi(\overset{1}{\alpha}_u(t))\sup_t\|\overset{1}{H}\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}\|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}\\ &\qquad + c\|v(0)\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}.\end{aligned} \tag{9.6}$$

Introducing the notation

$$\begin{aligned}X_n(t) &= \|v_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|p_n\|_{W_r^{1,0}(\overset{1}{\Omega}{}_0^t)} + \|p_n\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)},\\ D(t) &= \varphi(\overset{1}{\alpha}_u(t))t^a(\|u\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|u(0)\|_{L_r(\overset{1}{\Omega}_0)})\|v(0)\|_{L_r(\overset{1}{\Omega}_0)}\\ &\quad + c\|f\|_{L_r(\overset{1}{\Omega}{}_0^t)} + c\varphi(\overset{1}{\alpha}_u(t))\sup_t\|\overset{1}{H}\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}\|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}\\ &\quad + c\|v(0)\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)},\end{aligned} \tag{9.7}$$

we express (9.6) in the form

$$X_{n+1}(t) \le \varphi(\overset{1}{\alpha}_u(t))t^a(\|u\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|u(0)\|_{L_r(\overset{1}{\Omega}_0)})X_n + D. \tag{9.8}$$

Let $X_n \le M$, $M = 2D$. Since we have assumed that t, u, $u(0)$ are so small that

$$\varphi(\overset{1}{\alpha}_u(t))t^a(\|u\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|u(0)\|_{L_r(\overset{1}{\Omega}_0)}) \le 1/2, \tag{9.9}$$

we obtain

$$X_n \le 2D \quad \text{for any } n \in \mathbb{N}_0. \tag{9.10}$$

Since $X_0 = 0$, condition (9.10) also holds for $n = 0$.

Condition (9.9) also means that for chosen sufficiently small u we can take t as large as we want.

To show convergence of the sequence $\{v_n, p_n\}$ we introduce the differences

$$V_n = v_n - v_{n-1}, \quad P_n = p_n - p_{n-1}. \tag{9.11}$$

Then problem (9.4) implies

$$\begin{aligned} &V_{n+1,t} - \operatorname{div}_\xi \mathbb{T}_\xi(V_{n+1}, P_{n+1}) \\ &\quad = -(\operatorname{div}_\xi \mathbb{T}_\xi(V_n, P_n) - \operatorname{div}_u \mathbb{T}_u(V_n, P_n)) \equiv F', \\ &\operatorname{div}_\xi V_{n+1} = \operatorname{div}_\xi V_n - \operatorname{div}_u V_n \equiv G' \equiv \nabla_\xi R', \\ &\bar{n}_\xi \cdot \mathbb{T}_\xi(V_{n+1}, P_{n+1}) = \bar{n}_\xi \mathbb{T}_\xi(V_n, P_n) - \bar{n}_u \mathbb{T}_u(V_n, P_n) \equiv K', \\ &V_{n+1}|_{t=0} = 0. \end{aligned} \tag{9.12}$$

Repeating the estimates for F_1, G, R and K_1 we have

$$\begin{aligned} &\|F'\|_{L_r(\overset{1}{\Omega}{}^t_0)} \le \varphi(\overset{1}{\alpha}_u(t))\overset{1}{\alpha}_u(t)(\|V_n\|_{W_r^{2,0}(\overset{1}{\Omega}{}^t_0)} + \|P_n\|_{W_r^{1,0}(\overset{1}{\Omega}{}^t_0)}), \\ &\|G'\|_{W_r^{1,0}(\overset{1}{\Omega}{}^t_0)} \le \varphi(\overset{1}{\alpha}_u(t))\overset{1}{\alpha}_u(t)\|V_n\|_{W_r^{2,0}(\overset{1}{\Omega}{}^t_0)}, \\ &\|R'_t\|_{L_r(\overset{1}{\Omega}{}^t_0)} \le \varphi(\overset{1}{\alpha}_u(t))t^a(\|u_{W_r^{2,1}(\overset{1}{\Omega}{}^t_0)} + \|u(0)\|_{L_r(\overset{1}{\Omega}{}^t_0)}) \cdot \|V_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}^t_0)}, \\ &\|K'\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)} \le \varphi(\overset{1}{\alpha}_u(t))(\overset{1}{\alpha}_u(t) + t^a\|u\|_{W_r^{2,1}(\overset{1}{\Omega}{}^t_0)}) \\ &\qquad \cdot (\|V_n\|_{W_r^{2,0}(\overset{1}{\Omega}{}^t_0)} + \|P_n\|_{L_r(0,t;W_r^1(\overset{1}{\Omega}_0))} + |P_n\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)}). \end{aligned} \tag{9.13}$$

Let

$$Y_n(t) = \|V_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}^t_0)} + \|P_n\|_{W_r^{1,0}(\overset{1}{\Omega}{}^t_0)} + \|P_n\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)}. \tag{9.14}$$

Applying Lemma 2.6(b) to problem (9.12), using estimates (9.13) and notation (9.14), we derive the inequality

$$Y_{n+1} \le \varphi(\overset{1}{\alpha}_u(t))t^a(\|u\|_{W_r^{2,1}(\overset{1}{\Omega}{}^t_0)} + \|u(0)\|_{L_r(\overset{1}{\Omega}{}^t_0)})Y_n. \tag{9.15}$$

Hence, condition (9.9) implies convergence of the sequence $\{v_n, p_n\}$ and concludes the proof of the lemma. ■

10. Existence of solutions to problem (8.6)

In this section we prove existence of solutions to problem (8.6) for given $\overset{i}{\bar{v}}$, $i = 1, 2$. Therefore, it is convenient to introduce the simplified notation

$$\overset{i}{u} = \overset{i}{\bar{v}}, \quad \overset{i}{H} = \overset{i}{\bar{H}}, \quad i = 1, 2. \tag{10.1}$$

In view of (10.1) and after elimination of the electric fields, problem (8.6) takes the form

$$
\begin{aligned}
&\mu_1 \overset{1}{H}_{,t} + \frac{1}{\sigma_1}\operatorname{rot}^2_{\overset{1}{u}} \overset{1}{H} = \mu_1 \operatorname{rot}_{\overset{1}{u}}(\overset{1}{u}\times \overset{1}{H}) + \mu_1 \overset{1}{u}\cdot\nabla_{\overset{1}{u}}\overset{1}{H} \equiv \overset{1}{f}_0, \\
&\operatorname{div}_{\overset{1}{u}} \overset{1}{H} = 0 && \text{in } \overset{1}{\Omega}_0, \\
&\mu_2 \overset{2}{H}_{,t} + \frac{1}{\sigma_2}\operatorname{rot}^2_{\overset{2}{u}} \overset{2}{H} = \mu_2 \overset{2}{u}\cdot\nabla_{\overset{2}{u}}\overset{2}{H} \equiv \overset{2}{f}_0, \ \operatorname{div}_{\overset{2}{u}} \overset{2}{H} = 0 && \text{in } \overset{2}{\Omega}_0, \\
&\left(\frac{1}{\sigma_1}\operatorname{rot}_{\overset{1}{u}} \overset{1}{H} - \frac{1}{\sigma_2}\operatorname{rot}_{\overset{2}{u}} \overset{2}{H}\right)\cdot\bar{\tau}_{u\alpha} = \mu_1 \overset{1}{u}\times\overset{1}{H}\cdot\bar{\tau}_{u\alpha} \equiv k_{0\alpha}, \quad \alpha=1,2, && \text{on } S_0, \\
&\overset{1}{H} - \overset{2}{H} = 0, \quad \alpha = 1,2, && \text{on } S_0, \\
&\overset{i}{H}|_{t=0} = \overset{i}{H}(0), \quad \operatorname{div}\overset{i}{H}(0) = 0, \quad i=1,2, \quad \overset{2}{H}\cdot\bar{\tau}'_\alpha|_B = H_{*\alpha}, \\
&\operatorname{div}_{\overset{2}{u}}\overset{2}{H}|_B = 0, \quad \bar{\tau}'_\alpha \text{ is tangent to } B, \quad \alpha=1,2,
\end{aligned}
\tag{10.2}
$$

where $u = \overset{1}{u} = \overset{2}{u}$ on S_0.

We prove existence of solutions to problem (10.2) by the method of successive approximations. First we show the existence of solutions to the following problem with constant coefficients:

$$
\begin{aligned}
&\mu_1 \overset{1}{H}_{,t} + \frac{1}{\sigma_1}\operatorname{rot}^2_\xi \overset{1}{H} = \frac{1}{\sigma_1}(\operatorname{rot}^2_\xi \overset{1}{H} - \operatorname{rot}^2_{\overset{1}{u}} \overset{1}{H}) + \overset{1}{f}_0 \equiv \overset{1}{f}_*, \\
&\operatorname{div}_\xi \overset{1}{H} = \operatorname{div}_\xi \overset{1}{H} - \operatorname{div}_{\overset{1}{u}} \overset{1}{H} \equiv \overset{1}{g}_*, \\
&\mu_2 \overset{2}{H}_{,t} + \frac{1}{\sigma_2}\operatorname{rot}^2_\xi \overset{2}{H} = \frac{1}{\sigma_2}(\operatorname{rot}^2_\xi \overset{2}{H} - \operatorname{rot}^2_{\overset{2}{u}} \overset{2}{H}) + \overset{2}{f}_0 \equiv \overset{2}{f}_*, \\
&\operatorname{div}_\xi \overset{2}{H} = \operatorname{div}_\xi \overset{2}{H} - \operatorname{div}_{\overset{2}{u}} \overset{2}{H} \equiv \overset{2}{g}_*, \\
&\left(\frac{1}{\sigma_1}\operatorname{rot}_\xi \overset{1}{H} - \frac{1}{\sigma_2}\operatorname{rot}_\xi \overset{2}{H}\right)\cdot\bar{\tau}_\alpha = \left[\left(\frac{1}{\sigma_1}\operatorname{rot}_\xi \overset{1}{H} - \frac{1}{\sigma_1}\operatorname{rot}_{\overset{1}{u}} \overset{1}{H}\right)\right. \\
&\qquad \left. - \left(\frac{1}{\sigma_2}\operatorname{rot}_\xi \overset{2}{H} - \frac{1}{\sigma_2}\operatorname{rot}_{\overset{2}{u}} \overset{2}{H}\right)\right]\cdot\bar{\tau}_\alpha \\
&\qquad + \left(\frac{1}{\sigma_1}\operatorname{rot}_{\overset{1}{u}} \overset{1}{H} - \frac{1}{\sigma_2}\operatorname{rot}_{\overset{2}{u}} \overset{2}{H}\right)\cdot(\bar{\tau}_\alpha - \bar{\tau}_{u\alpha}) + k_{0\alpha} \equiv k_{*\alpha}, \quad \alpha = 1,2, \\
&\overset{1}{H} = \overset{2}{H}, \\
&\overset{i}{H}|_{t=0} = \overset{i}{H}(0), \quad \operatorname{div}\overset{i}{H}(0) = 0, \quad i=1,2, \quad \overset{2}{H}\cdot\bar{\tau}'_\alpha|_B = H_{*\alpha}, \\
&\operatorname{div}_\xi \overset{2}{H} = \operatorname{div}_\xi \overset{2}{H} - \operatorname{div}_{\overset{2}{u}} \overset{2}{H} \quad \text{on } B, \quad \alpha = 1,2,
\end{aligned}
\tag{10.3}
$$

where $\bar{n}$ is the unit vector normal to S_0 and $\bar{\tau}_\alpha$, $\alpha = 1,2$, are tangent vectors to S_0. Moreover, $\bar{\tau}'_\alpha$, $\alpha = 1,2$, are tangent vectors to B.

To treat problem (10.3) as the problem with constant coefficients it is convenient to write it in the following short form:

$$
\begin{aligned}
&\mu_i \overset{i}{H}_{,t} + \frac{1}{\sigma_i}\operatorname{rot}_\xi^2 \overset{i}{H} = \overset{i}{f}_*, \quad \operatorname{div}_\xi \overset{i}{H} = \overset{i}{g}_*, \quad i=1,2,\\
&\left(\frac{1}{\sigma_1}\operatorname{rot}_\xi \overset{1}{H} - \frac{1}{\sigma_2}\operatorname{rot}_\xi \overset{2}{H}\right)\cdot\bar\tau_\alpha = k_{*\alpha}, \quad \alpha=1,2,\\
&\overset{1}{H} = \overset{2}{H},\\
&\overset{i}{H}|_{t=0} = \overset{i}{H}(0), \quad \operatorname{div}\overset{i}{H}(0)=0, \quad i=1,2, \quad \overset{2}{H}\cdot\bar\tau'_\alpha|_B = H_{*\alpha},\\
&\operatorname{div}_\xi \overset{2}{H} = \operatorname{div}_\xi \overset{2}{H} - \operatorname{div}_{\overset{2}{u}} \overset{2}{H}, \quad \alpha=1,2.
\end{aligned}
\tag{10.4}
$$

To show existence of solutions we need to obtain appropriate estimates. Considering operator $(10.4)_1$ in the whole space with some initial data, we see that it is not invertible. To make it invertible we need

Remark 10.1. Now we are going to prove existence of solutions to problem (10.4) applying the regularizer technique (see [LSU, Ch. 4, Sect. 7]). For this we need problem (10.4) with vanishing initial data. Looking for solutions to (10.4) such that $\overset{i}{H} \in W_r^{2,1}$, $r>3$, $i=1,2$, we need to have initial data in $W_r^{2-2/r}$. Let $\overset{i}{\tilde H}$, $i=1,2$, be divergence free extensions of initial data such that $\overset{i}{\tilde H} \in W_r^{2,1}(\overset{i}{\Omega}{}_0^t)$, $i=1,2$, and

$$
\|\overset{i}{\tilde H}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)} \le c\|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)}, \quad i=1,2. \tag{10.5}
$$

Then, introducing the new functions

$$
\overset{i}{u} = \overset{i}{H} - \overset{i}{\tilde H}, \quad i=1,2, \tag{10.6}
$$

problem (10.4) takes the form

$$
\begin{aligned}
&\mu_i \overset{i}{u}_t + \frac{1}{\sigma_i}\operatorname{rot}_\xi^2 \overset{i}{u} = \overset{i}{f}_* - \left(\mu_i \overset{i}{\tilde H}_{,t} + \frac{1}{\sigma_i}\operatorname{rot}_\xi^2 \overset{i}{\tilde H}\right) \equiv \overset{i}{f},\\
&\operatorname{div}_\xi \overset{i}{u} = \overset{i}{g}_* \equiv \overset{i}{g}, \qquad \text{in } \overset{i}{\Omega}{}_0^t, \quad i=1,2,\\
&\left(\frac{1}{\sigma_1}\operatorname{rot}_\xi \overset{1}{u} - \frac{1}{\sigma_2}\operatorname{rot}_\xi \overset{2}{u}\right)\cdot\bar\tau_\alpha\\
&\quad = k_{*\alpha} - \left(\frac{1}{\sigma_1}\operatorname{rot}_\xi \overset{1}{\tilde H} - \frac{1}{\sigma_2}\operatorname{rot}_\xi \overset{2}{\tilde H}\right)\cdot\bar\tau_\alpha \equiv k_\alpha, \quad \alpha=1,2, \qquad \text{on } S_0^t,\\
&\overset{1}{u} = \overset{2}{u}, \qquad \text{on } S_0^t,\\
&\overset{2}{u}\cdot\bar\tau'_\alpha|_B = H_{*\alpha} - \overset{2}{\tilde H}\cdot\bar\tau'_\alpha|_B \equiv b_\alpha,\\
&\operatorname{div}_\xi \overset{2}{u}|_B = (\operatorname{div}_\xi \overset{2}{H} - \operatorname{div}_{\overset{2}{u}} \overset{2}{H})|_B \equiv \delta,\\
&\bar\tau'_\alpha \ \text{ is tangent to } B, \quad \alpha=1,2.
\end{aligned}
\tag{10.7}
$$

To apply the partition of unity it is more convenient to replace ξ by x. Then problem (10.7) takes the form

$$\begin{aligned}
&\mu_i \overset{i}{u}_{,t} + \frac{1}{\sigma_i}\operatorname{rot}_x^2 \overset{i}{u} = \overset{i}{f}, \quad \operatorname{div}_x \overset{i}{u} = \overset{i}{g}, \quad i = 1,2,\\
&\left(\frac{1}{\sigma_1}\operatorname{rot}_x \overset{1}{u} - \frac{1}{\sigma_2}\operatorname{rot}_x \overset{2}{u}\right)\cdot \bar\tau_\alpha = k_\alpha, \quad \alpha = 1,2,\\
&\overset{1}{u} - \overset{2}{u} = 0,\\
&\overset{2}{u}\cdot \bar\tau'_\alpha|_B = b_\alpha, \quad \operatorname{div}_\xi \overset{2}{u}|_B = \delta, \quad \alpha = 1,2, \quad \overset{i}{u}|_{t=0} = 0, \quad i = 1,2.
\end{aligned} \tag{10.8}$$

To apply the regularizer technique we introduce the notation

$$\begin{aligned}
&L_i^1(\partial_x,\partial_t)\overset{i}{u} = \mu_i \overset{i}{u}_{,t} + \frac{1}{\sigma_i}\operatorname{rot}_x^2 \overset{i}{u}, \quad L_i^2(\partial_x,\partial_t)\overset{i}{u} = \operatorname{div}_x \overset{i}{u}, \qquad i = 1,2,\\
&L_i = (L_i^1, L_i^2),\\
&B_{1\alpha}(\xi,\partial_x)(\overset{1}{u},\overset{2}{u}) = \left(\frac{1}{\sigma_1}\operatorname{rot}_x \overset{1}{u} - \frac{1}{\sigma_2}\operatorname{rot}_x \overset{2}{u}\right)\cdot \bar\tau_\alpha(\xi), \quad \alpha = 1,2,\\
&B_{2\alpha}(\xi)(\overset{1}{u},\overset{2}{u}) = \overset{1}{u}_\alpha - \overset{2}{u}_\alpha, \quad \alpha = 1,2,\\
&B_{3\alpha}\overset{2}{u} = \overset{2}{u}\cdot \bar\tau'_\alpha, \quad \alpha = 1,2.
\end{aligned} \tag{10.9}$$

Let $k \in \mathfrak{M}_1$ and $\overset{1}{f}{}^{(k)}(x,t) = \zeta^{(k)}(x)\overset{1}{f}(x,t)$, $\overset{1}{g}{}^{(k)}(x,t) = \zeta^{(k)}(x)\overset{1}{g}(x,t)$. Let $\overset{1}{f}{}^{(k)} \in L_r(\mathbb{R}^3\times(0,\tau))$, $\overset{1}{g}{}^{(k)}(x,t) \in L_r(0,t;W_r^1(\mathbb{R}^3))$. We denote by $R^{(k)}$ the operator which solves the Cauchy problem with vanishing initial data

$$L_1(\partial_x,\partial_t)\overset{1}{u}{}^{(k)}(x,t) = (\overset{1}{f}{}^{(k)}(x,t), \overset{1}{g}{}^{(k)}(x,t)), \quad t \le \tau. \tag{10.10}$$

In view of Lemma 2.14 the operator $R^{(k)}$ exists for τ sufficiently small and $u^{(k)} = R^{(k)}(\overset{1}{f}{}^{(k)}, \overset{1}{g}{}^{(k)})$. Moreover,

$$\begin{aligned}
&\|R^{(k)}(\overset{1}{f}{}^{(k)}, \overset{1}{g}{}^{(k)})\|_{W_r^{2,1}(\mathbb{R}^3\times(0,\tau))}\\
&\qquad \le c(\|\overset{1}{f}{}^{(k)}\|_{L_r(\mathbb{R}^3\times(0,\tau))} + \|\overset{1}{g}{}^{(k)}\|_{L_r(0,\tau;W_r^1(\mathbb{R}^3))}), \quad k \in \mathfrak{M}_1.
\end{aligned} \tag{10.11}$$

For $k \in \mathfrak{M}_2$, we have a similar result for the Cauchy problem with vanishing initial data

$$L_2(\partial_x,\partial_t)\overset{2}{u}{}^{(k)}(x,t) = (\overset{2}{f}{}^{(k)}(x,t), \overset{2}{g}{}^{(k)}(x,t)). \tag{10.12}$$

Hence $u^{(k)} = R^{(k)}(\overset{2}{f}{}^{(k)}, \overset{1}{g}{}^{(k)})$, $k \in \mathfrak{M}_2$, and

$$\begin{aligned}
&\|R^{(k)}(\overset{2}{f}{}^{(k)}, \overset{2}{g}{}^{(k)})\|_{W_r^{2,1}(\mathbb{R}^3\times(0,\tau))}\\
&\qquad \le c(\|\overset{2}{f}{}^{(k)}\|_{L_r(\mathbb{R}^3\times(0,\tau))} + \|\overset{2}{g}{}^{(k)}\|_{L_r(0,\tau;W_r^1(\mathbb{R}^3))}), \quad k \in \mathfrak{M}_2.
\end{aligned} \tag{10.13}$$

For $k \in \mathfrak{N}_1$, the operator $R^{(k)}$ gives the solution to the following initial boundary value problem with vanishing initial data:

$$\begin{aligned}&L_i(\partial_z,\partial_t)\overset{i}{u}{}^{(k)}(z,t)=(\overset{i}{f}{}^{(k)},\overset{i}{g}{}^{(k)})\quad\text{in }\overset{i}{\mathbb{R}}\times(0,\tau),\quad i=1,2,\\&B_{1\alpha}(\xi^{(k)},\partial_z)(\overset{1}{u}{}^{(k)}(z,t),\overset{2}{u}{}^{(k)}(z,t))=k_\alpha^{(k)},\quad\alpha=1,2,\quad z_3=0,\\&B_{2\alpha}(\xi^{(k)})(\overset{1}{u}{}^{(k)}(z,t),\overset{2}{u}{}^{(k)}(z,t))=0,\quad\alpha=1,2,\quad z_3=0,\end{aligned}\tag{10.14}$$

where $\xi^{(k)}\in S_0$ and $S_0=\{z\in\mathbb{R}^3: z_3=0\}$.

In view of Lemma 11.1 the operator $R^{(k)}$ is expressed by

$$\begin{aligned}(\overset{1}{u}{}^{(k)}(z,t),\overset{2}{u}{}^{(k)}(z,t))=R^{(k)}(&\overset{1}{f}{}^{(k)}(z,t),\overset{2}{f}{}^{(k)}(z,t),\overset{1}{g}{}^{(k)}(z,t),\overset{2}{g}{}^{(k)}(z,t),\\&k_1^{(k)}(z',t),k_2^{(k)}(z',t)),\quad z'=(z_1,z_2),\quad k\in\mathfrak{N}_1,\quad t\le\tau.\end{aligned}\tag{10.15}$$

Moreover, we have

$$\begin{aligned}&\|R^{(k)}(\overset{1}{f}{}^{(k)},\overset{2}{f}{}^{(k)},\overset{1}{g}{}^{(k)},\overset{2}{g}{}^{(k)},k_1^{(k)},k_2^{(k)})\|_{W_r^{2,1}(\overset{1}{\mathbb{R}}\times(0,\tau))\times W_r^{2,1}(\overset{2}{\mathbb{R}}\times(0,\tau))}\\&\qquad\le c\sum_{i=1}^2(\|\overset{i}{f}{}^{(k)}\|_{L_r(\overset{i}{\mathbb{R}}\times(0,\tau))}+\|\overset{i}{g}{}^{(k)}\|_{L_r(0,\tau;W_r^1(\mathbb{R}^3))}\\&\qquad\quad+\|k_i^{(k)}\|_{W_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}^2\times(0,\tau))}),\quad k\in\mathfrak{N}_1.\end{aligned}\tag{10.16}$$

Finally, for $k\in\mathfrak{N}_2$, the operator $R^{(k)}$ gives a solution to the initial boundary value problem with vanishing initial data

$$\begin{aligned}&L_2(\partial_z,\partial_t)\overset{2}{u}{}^{(k)}(z,t)=(\overset{2}{f}{}^{(k)}(z,t),\overset{2}{g}{}^{(k)}(z,t))\quad\text{in }\mathbb{R}^2\times(0,\tau),\\&\overset{2}{u}{}^{(k)}\cdot\bar\tau'_\alpha=b_\alpha^{(k)},\quad\operatorname{div}\overset{2}{u}{}^{(k)}=0,\end{aligned}\tag{10.17}$$

where $\overset{2}{u}{}^{(k)}=R^{(k)}(\overset{2}{f}{}^{(k)},\overset{2}{g}{}^{(k)},b_1^{(k)},b_2^{(k)})$, δ, b are introduced in (10.7) and

$$\begin{aligned}&\|R^{(k)}(\overset{2}{f}{}^{(k)},\overset{2}{g}{}^{(k)},u_*^{(k)})\|_{W_r^{2,1}(\mathbb{R}_+^3\times(0,\tau))}\\&\qquad\le c\Big(\|\overset{2}{f}{}^{(k)}\|_{L_r(\mathbb{R}_+^3\times(0,\tau))}+\|\overset{2}{g}{}^{(k)}\|_{L_r(0,\tau;W_r^1(\mathbb{R}_+^3))}\\&\qquad\quad+\sum_{\alpha=1}^2\|b_\alpha^{(k)}\|_{W_r^{2-1/r,1-1/r}(\mathbb{R}^2\times(0,\tau))}\Big),\quad k\in\mathfrak{N}_2.\end{aligned}\tag{10.18}$$

The existence of solutions to problems (10.10), (10.12), (10.17) is shown in Section 6.

Set

$$h=(\overset{1}{f},\overset{1}{g},\overset{2}{f},\overset{2}{g},k_1,k_2,b),\tag{10.19}$$

where $b=(b_1,b_2)$. Then

$$\begin{aligned}&h^{(k)}=(\overset{1}{f}{}^{(k)},\overset{1}{g}{}^{(k)},0,0,0,0,0)\quad\text{for }k\in\mathfrak{M}_1,\\&h^{(k)}=(0,0,\overset{2}{f}{}^{(k)},\overset{2}{g}{}^{(k)},0,0,0)\quad\text{for }k\in\mathfrak{M}_2,\\&h^{(k)}(x,t)=(Z_k(\overset{1}{f}{}^{(k)},\overset{1}{g}{}^{(k)}),Z_k(\overset{2}{f}{}^{(k)},\overset{2}{g}{}^{(k)}),Z_kk_1^{(k)},Z_kk_2^{(k)},0)\quad\text{for }k\in\mathfrak{N}_1,\end{aligned}$$

where Z_k is the operator which transforms locally (on the support of $\zeta^{(k)}$) the variables z into x (the variables z are introduced in the definition of the partition of unity presented in Section 2).

Finally,

$$h^{(k)}(x,t) = (0,0,Z_k(\overset{2}{f}{}^{(k)},\overset{2}{g}{}^{(k)}),0,0,Z_k b^{(k)}) \quad \text{for } k \in \mathfrak{N}_2.$$

Therefore, for $u = (\overset{1}{u},\overset{2}{u})$, we have

$$u^{(k)}(x,t) = R^{(k)}h^{(k)}, \quad k \in \mathfrak{M}_1 \cup \mathfrak{M}_2 \cup \mathfrak{N}_1 \cup \mathfrak{N}_2 \equiv \mathcal{R}. \tag{10.20}$$

Then we introduce the operator R, called the *regularizer*, by

$$Rh = \sum_k \eta^{(k)}(x)u^{(k)}(x,t). \tag{10.21}$$

Define

$$\begin{aligned} \mathcal{A}(r,\tau) &= W_r^{2,1}(\overset{1}{\Omega}{}_0^\tau) \times W_r^{2,1}(\overset{2}{\Omega}{}_0^\tau), \\ \mathcal{B}(r,\tau) &= L_r(\overset{1}{\Omega}{}_0^\tau) \times L_r(\overset{2}{\Omega}{}_0^\tau) \times L_r(0,\tau;W_r^1(\overset{1}{\Omega}_0)) \times L_r(0,\tau;W_r^1(\overset{2}{\Omega}_0)) \\ &\quad \times W_r^{1-1/r,1/2-1/(2r)}(S_0^\tau) \\ &\quad \times W_r^{1-1/r,1/2-1/(2r)}(S_0^\tau) \times W_r^{2-1/r,1-1/(2r)}(B^\tau). \end{aligned} \tag{10.22}$$

LEMMA 10.2. *For $r > 3$ and τ small,*

$$\|Rh\|_{\mathcal{A}(r,\tau)} \le c\|h\|_{\mathcal{B}(r,\tau)}. \tag{10.23}$$

Proof. This follows from (10.11), (10.13), (10.16) and (10.18) and Remark 10.3. ■

REMARK 10.3. Let $u = (\overset{1}{u},\overset{2}{u}) \in \mathcal{A}(r,\tau)$. Define

$$\|u\|_{\tilde{\mathcal{A}}(r,\tau)} = \sum_{k\in\mathcal{R}} \|u^{(k)}\|_{\mathcal{A}(r,\tau)}.$$

Similarly, for $h \in \mathcal{B}(r,\tau)$ set

$$\|h\|_{\tilde{\mathcal{B}}(r,\tau)} = \sum_{k\in\mathcal{R}} \|h^{(k)}\|_{\mathcal{B}(r,\tau)}.$$

In view of the properties of the partition of unity, the norms $\|\cdot\|_{\tilde{\mathcal{A}}}$, $\|\cdot\|_{\tilde{\mathcal{B}}}$ and $\|\cdot\|_{\mathcal{A}}$, $\|\cdot\|_{\mathcal{B}}$ are equivalent, respectively.

Let us write problem (10.8) briefly as

$$Au = h. \tag{10.24}$$

LEMMA 10.4. *Let $r > 3$ and assume that $h \in \mathcal{B}(r,\tau)$. Then for sufficiently small τ there exists a solution to* (10.24) *such that $u \in \mathcal{A}(r,\tau)$ and*

$$\|u\|_{\mathcal{A}(r,\tau)} \le c\|h\|_{\mathcal{B}(r,\tau)}. \tag{10.25}$$

Proof. Repeating the considerations from [LSU, Ch. 4, Sect. 7] we will show existence of bounded operators T and W such that

$$ARh = h + Th, \quad RAu = u + Wu. \tag{10.26}$$

For sufficiently small τ and λ we show that $\|T\| < 1$, $\|W\| < 1$, which proves the lemma.

We have

$$A = (L, B), \quad L = (L_1, L_2), \quad B = (B_1, B_2, B_3).$$

First we construct the operator T (see $(10.26)_1$). We have

$$LRh = \sum_k L\eta^{(k)} u^{(k)} = \sum_k (L\eta^{(k)} u^{(k)} - \eta^{(k)} Lu^{(k)}) + \sum_k \eta^{(k)} Lu^{(k)}.$$

For $k \in \mathfrak{M}_1$,

$$Lu^{(k)} = L_1 \overset{1}{u}{}^{(k)} = (\overset{1}{f}{}^{(k)}, \overset{1}{g}{}^{(k)}).$$

For $k \in \mathfrak{M}_2$,

$$Lu^{(k)} = L_2 \overset{2}{u}{}^{(k)} = (\overset{2}{f}{}^{(k)}, \overset{2}{g}{}^{(k)}).$$

For $k \in \mathfrak{N}_1$,

$$\begin{aligned} Lu^{(k)} &= L(\partial_x, \partial_t) Z_k R^{(k)} Z_k^{-1} h^{(k)} = Z_k L^{(k)}(\partial_z - \nabla F \partial_{z_3}, \partial_t) R^{(k)} Z_k^{-1} h^{(k)} \\ &= Z_k [L^{(k)}(\partial_z - \nabla F \partial_{z_3}, \partial_t) - L^{(k)}(\partial_z, \partial_t)] R^{(k)} Z_k^{-1} h^{(k)} + Z_k^{-1} h^{(k)}, \end{aligned}$$

because by definition of $R^{(k)}$, $k \in \mathfrak{N}_1$, we have

$$L^{(k)}(\partial_z, \partial_t) R^{(k)} Z_k^{-1} h = Z_k^{-1} h^{(k)}.$$

Similar expressions can be derived for $k \in \mathfrak{N}_2$.

Now we recall the operators T and W introduced in the proof of Lemma 5.4. Next we estimate the same parts of these operators, but we use different Sobolev spaces. First we consider

$$\begin{aligned} T_1^{\mathfrak{M}_1} h &= \sum_{k \in \mathfrak{M}_1} T_1^{(k)} h \\ &= \sum_{k \in \mathfrak{M} 1} (-2\eta^{(k)} \nabla \overset{1}{u}{}^{(k)} - \Delta \eta^{(k)} \overset{1}{u}{}^{(k)} + \nabla \eta^{(k)}_{,x_j} \overset{1}{u}{}^{(k)}_j + \eta^{(k)}_{x_j} \nabla \overset{1}{u}{}^{(k)}_j + \nabla \eta^{(k)} \operatorname{div} \overset{1}{u}{}^{(k)}). \end{aligned}$$

Hence

$$\|T_1^{\mathfrak{M}_1} h\|_{L_r(\overset{1}{\Omega}{}_0^\tau)} \le c \sum_{k \in \mathfrak{M}_1} \left(\frac{1}{\lambda} \|\nabla \overset{1}{u}{}^{(k)}\|_{L_r(\overset{1}{\Omega}{}_0^\tau)} + \frac{1}{\lambda^2} \|\overset{1}{u}{}^{(k)}\|_{L_r(\overset{1}{\Omega}{}_0^\tau)} \right).$$

Since $\overset{1}{u}{}^{(k)}|_{t=0} = 0$, we have

$$\begin{aligned} I_1 &\equiv \Big(\int_0^\tau \|\nabla \overset{1}{u}{}^{(k)}\|^r_{L_r(\overset{1}{\Omega}_0)} dt \Big)^{1/r} \le c \Big(\int_0^\tau \|\nabla^2 \overset{1}{u}{}^{(k)}\|^{r/2}_{L_r(\overset{1}{\Omega}_0)} \|\overset{1}{u}{}^{(k)}\|^{r/2}_{L_r(\overset{1}{\Omega}_0)} dt \Big)^{1/r} \\ &\le c \sup_t \|\overset{1}{u}{}^{(k)}\|^{1/2}_{L_r(\overset{1}{\Omega}_0)} \Big(\int_0^\tau \|\nabla^2 \overset{1}{u}{}^{(k)}\|^{r/2}_{L_r(\overset{1}{\Omega}_0)} dt \Big)^{1/r} \\ &\le c \sup_t \Big\| \int_0^\tau \overset{1}{u}{}^{(k)}_{,t} \, dt \Big\|^{1/2}_{L_r(\overset{1}{\Omega}_0)} \tau^{1/(2r)} \Big(\int_0^\tau \|\nabla^2 \overset{1}{u}{}^{(k)}\|^r_{L_r(\overset{1}{\Omega}_0)} dt \Big)^{1/(2r)} \end{aligned}$$

$$\le c\tau^{1/(2r)}\tau^{1/(2r')}\Big(\int_0^\tau \|\overset{1}{u}{}^{(k)}_{,t}\|^r_{L_r(\overset{1}{\Omega}_0)}\,dt\Big)^{1/(2r)}\Big(\int_0^t \|\nabla^2\overset{1}{u}{}^{(k)}\|^r_{L_r(\overset{1}{\Omega}_0)}\,dt\Big)^{1/(2r)}$$
$$\le c\tau^{1/2}\|\overset{1}{u}{}^{(k)}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^\tau)},\quad 1/r+1/r'=1.$$

Similarly,

$$I_2\equiv\|\overset{1}{u}{}^{(k)}\|_{L_r(\overset{1}{\Omega}{}_0^\tau)}\le c\tau^{1/r}\sup_{t\le\tau}\Big\|\int_0^t \overset{1}{u}{}^{(k)}_{t'}\,dt'\Big\|_{L_r(\overset{1}{\Omega}_0)}\le c\tau\|\overset{1}{u}{}^{(k)}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^\tau)}.$$

Hence,

$$I_1+I_2\le c\tau^{1/2}\|\overset{1}{u}{}^{(k)}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^\tau)}\le c\tau^{1/2}\|h\|_{\mathcal{B}(r,\tau)}.$$

Therefore,

$$\|T_1^{\mathfrak{M}_1}h\|_{L_r(\overset{1}{\Omega}{}_0^\tau)}\le c\Big(\frac{\tau^{1/2}}{\lambda}+\frac{\tau^{1/2}}{\lambda^2}\Big)\|h\|_{\mathcal{B}(r,\tau)}. \tag{10.27}$$

Hence for τ sufficiently small the norm of $T_1^{\mathfrak{M}_1}$ can be as small as we want. Consider

$$\begin{aligned}T_1^{\mathfrak{N}_1}h&=\sum_{k\in\mathfrak{N}_1}T_1^{(k)}h\\&=\sum_{k\in\mathfrak{N}_1}\sum_{\sigma=1}^2(-2\nabla\eta^{(k)}\nabla\overset{\sigma}{u}{}^{(k)}-\Delta\eta^{(k)}\overset{\sigma}{u}{}^{(k)}+\nabla\eta^{(k)}_j\overset{\sigma}{u}{}^{(k)}_j+\eta^{(k)}_{x_j}\nabla\overset{\sigma}{u}{}^{(k)}_j+\nabla\eta^{(k)}\operatorname{div}\overset{\sigma}{u}{}^{(k)})\\&\quad+\sum_{\alpha,\beta=1}^2\sum_{\sigma=1}^2\eta^{(k)}Z_k[a^{(1)}_{\alpha\beta}F_{k,z_\alpha}\overset{\sigma}{\tilde u}{}^{(k)}_{,z_\beta z_3}+a^{(2)}_{\alpha\beta}F_{k,z_\alpha}F_{k,z_\beta}\overset{\sigma}{\tilde u}{}^{(k)}_{,z_3z_3}+a^{(3)}_{\alpha\beta}F_{k,z_\alpha z_\beta}\overset{\sigma}{\tilde u}{}^{(k)}_{,z_3}],\end{aligned}$$

where $a^{(j)}_{\alpha\beta}$, $j=1,2,3$, are constant coefficients and $\overset{\sigma}{\tilde u}{}^{(k)}(z,t)=R^{(k)}Z_k^{-1}\overset{\sigma}{u}{}^{(k)}(x,t)$. Hence,

$$\|T_1^{\mathfrak{N}_1}h\|_{L_r(\overset{1}{\Omega}{}_0^\tau\cup\overset{2}{\Omega}{}_0^\tau)}\le c\Big(\frac{\tau^{1/2}}{\lambda}+\frac{\tau^{1/2}}{\lambda^2}\Big)\|h\|_{\mathcal{B}(r,\tau)}+c\lambda\sum_{\sigma=1}^2\sum_{k\in\mathfrak{N}_1}\|\overset{\sigma}{u}{}^{(k)}_{,xx}\|_{L_r(\overset{\sigma}{\Omega}{}_0^\tau)}, \tag{10.28}$$

where we have used the fact that $F_{,z'z'}$, $z'\in\{z_1,z_2\}$, are bounded. Continuing, we have

$$\|T^{\mathfrak{N}_1}h\|_{L_r(\overset{1}{\Omega}{}_0^\tau\cup\overset{2}{\Omega}{}_0^\tau)}\le c\Big(\frac{\tau^{1/2}}{\lambda^2}+\lambda\Big)\|h\|_{\mathcal{B}(r,\tau)}. \tag{10.29}$$

Now, we obtain an estimate for the operator W. For $k\in\mathfrak{M}_1$ we have

$$R^{(k)}\zeta^{(k)}L_1\overset{1}{u}=R^{(k)}(\zeta^{(k)}L_1\overset{1}{u}-L_1\zeta^{(k)}\overset{1}{u})+R^{(k)}L_1\zeta^{(k)}\overset{1}{u},$$

where

$$\sum_{k\in\mathfrak{M}_1}R^{(k)}L_1\zeta^{(k)}\overset{1}{u}=\sum_{k\in\mathfrak{M}_1}\zeta^{(k)}\overset{1}{u}.$$

Hence

$$W_1^{\mathfrak{M}_1}\overset{1}{u}=\sum_{k\in\mathfrak{M}_1}\eta^{(k)}R^{(k)}(\zeta^{(k)}L_1\overset{1}{u}-L_1\zeta^{(k)}\overset{1}{u}).$$

Repeating the calculations leading to the estimate for $T_1^{\mathfrak{M}_1}h$, we obtain

$$\|W_1^{\mathfrak{M}_1}\overset{1}{u}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^\tau)}\le c\frac{\tau^{1/2}}{\lambda^2}\|\overset{1}{u}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^\tau)}. \tag{10.30}$$

Finally, we consider the following part of the operator W:

$$W_1^{\mathfrak{N}_1}u=\sum_{\alpha=1}^2\sum_{k\in\mathfrak{N}_1}\eta^{(k)}R^{(k)}L_\alpha\overset{\alpha}{u}=\sum_{\alpha=1}^2\sum_{k\in\mathfrak{N}_1}\eta^{(k)}Z_kR^{(k)}Z_k^{-1}(\zeta^{(k)}L_\alpha\overset{\alpha}{u}-L_\alpha\zeta^{(k)}\overset{\alpha}{u})$$
$$+\sum_{\alpha=1}^2\sum_{k\in\mathfrak{N}_1}\eta^{(k)}Z_kR^{(k)}[L_\alpha(\partial_z-\nabla F\partial_{z_3},\partial_t)-L_\alpha(\partial_z,\partial_t)]Z_k^{-1}\zeta^{(k)}\overset{\alpha}{u}.$$

Applying estimates used to show (10.29), we derive

$$\|W_1^{\mathfrak{N}_1}u\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^\tau\cup\overset{2}{\Omega}{}_0^\tau)}\le c\bigg(\frac{\tau^{1/2}}{\lambda^2}+\lambda\bigg)\sum_{\alpha=1}^2\|\overset{\alpha}{u}\|_{W_r^{2,1}(\overset{\alpha}{\Omega}{}_0^\tau)}. \tag{10.31}$$

To prove the lemma we have to estimate the operators $T_1^{\mathfrak{M}_1}$, $T_1^{\mathfrak{N}_2}$, $W_1^{\mathfrak{M}_2}$, $W_1^{\mathfrak{N}_2}$ and also boundary operators.

Similar estimates to those above imply that the norms of the operators T and W are less than 1 for sufficiently small τ and λ. To prove existence of solutions to problem (10.17) we use Lemma 6.4 (Part 1) so condition (1) $\operatorname{div}\overset{2}{u}{}^{(k)}|_{\mathbb{R}^2}=0$ must hold. But in problem (8.6) it is assumed that (2) $\operatorname{div}\overset{2}{H}|_B=0$. The difference of (1) and (2) is small for small τ and small support of $\zeta^{(k)}$. Thanks to the smallness of the norms of T and W, existence of solutions to problem (10.4) can be proved. This concludes the proof. ∎

Lemma 10.5. *Let $r>3$. Assume that $\overset{i}{f}_*\in L_r(\overset{i}{\Omega}{}_0^t)$, $\overset{i}{g}_*\in L_r(0,t;W_r^1(\overset{i}{\Omega}_0))$, $k_{*i}\in W_r^{1-1/r,1/2-1/(2r)}(S_0^t)$, $H_{*\alpha}\in W_r^{2-1/r,1-1/(2r)}(B^t)$, $\alpha=1,2$, $\overset{i}{H}(0)\in W_r^{2-2/r}(\overset{i}{\Omega}_0)$, $i=1,2$. Then there exists a solution to problem* (10.4) *such that $\overset{i}{H}\in W_r^{2,1}(\overset{i}{\Omega}{}_0^t)$, $i=1,2$, and*

$$\sum_{i=1}^2\|\overset{i}{H}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)}\le c\sum_{i=1}^2\Big(\|\overset{i}{f}_*\|_{L_r(\overset{i}{\Omega}{}_0^t)}+\|\overset{i}{g}_*\|_{L_r(0,t;W_r^1(\overset{i}{\Omega}_0))}$$
$$+\|k_{*i}\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)}$$
$$+\sum_{\alpha=1}^2\|H_{*\alpha}\|_{W_r^{2-1/r,1-1/(2r)}(B^t)}+\|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)}\Big). \tag{10.32}$$

Proof. In view of Lemma 10.4 we have existence of solutions to problem (10.7) with vanishing initial data and for sufficiently small existence time. Since problem (10.7) is linear, we can extend the solution step by step in time to any time interval $(0,t)$, $t=k\tau$, $k\in\mathbb{N}$. Then, in view of transformation (10.6), estimates (10.11), (10.13), (10.16), (10.18), Remark 10.3 and the estimates, for $i=1,2$,

$$\begin{aligned}
&\|\overset{i}{f}\|_{L_r(\overset{i}{\Omega}{}_0^t)}\le\|\overset{i}{f}_*\|_{L_r(\overset{i}{\Omega}{}_0^t)}+c\|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)},\\
&\|\overset{i}{g}\|_{L_r(0,t;W_r^1(\overset{i}{\Omega}_0))}\le\|\overset{i}{g}_*\|_{L_r(0,t;W_r^1(\overset{i}{\Omega}_0))}+c\|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)},\\
&\|k_i\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)}\le\|k_{*i}\|_{W_r^{1-1/2,1/2-1/(2r)}(S_0^t)}+c\|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)},
\end{aligned} \tag{10.33}$$

we obtain (10.32) and conclude the proof. ∎

Now we estimate the r.h.s. of (10.2).

Lemma 10.6. *Let $r>3$. Assume $\overset{i}{u}\in W_r^{2,1}(\overset{i}{\Omega}{}_0^t)$, $i=1,2$. Let $\overset{i}{\alpha}_r(t)=t^{1/r'}\|\overset{i}{u}\|_{L_r(0,t;W_r^2(\overset{i}{\Omega}_0))}$, $1/r+1/r'=1$, $i=1,2$. Assume that $\overset{i}{H}\in W_r^{2,1}(\overset{i}{\Omega}{}_0^t)$. Assume also that $\overset{i}{u}(0)$, $\overset{i}{H}(0)\in L_r(\overset{i}{\Omega}_0)$, $i=1,2$. Then there exists $a>0$ and an increasing positive function $\varphi=\varphi(k)$ nonvanishing for $k=0$ such that*

$$
\begin{aligned}
\|\overset{i}{f}_0\|_{L_r(\overset{i}{\Omega}{}_0^t)} &\le \sum_{i=1}^2 \varphi(\overset{i}{\alpha}_r(t))t^a(t^{1/r'}\|\overset{i}{H}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)}+\|\overset{i}{H}(0)\|_{L_r(\overset{i}{\Omega}_0)})\\
&\quad\cdot\Big(t^{1/r'}\|\overset{i}{u}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)}+\|\overset{i}{u}(0)\|_{L_r(\overset{i}{\Omega}_0)}\\
&\quad+\sum_{\alpha=1}^2\|H_{*\alpha}\|_{W_r^{2-1/r,1-1/(2r)}(B^t)}\Big)\equiv A,\\
\|k_{0i}\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)} &\le cA.
\end{aligned}
\tag{10.34}
$$

Proof. First we examine the r.h.s. of $(10.2)_1$. Using the properties of the Lagrange transformation we get

$$
\begin{aligned}
\|\overset{1}{f}_0\|_{L_r(\overset{1}{\Omega}{}_0^t)} &\le \|\mathrm{rot}_{\overset{1}{u}}(\overset{1}{u}\times\overset{1}{H})\|_{L_r(\overset{1}{\Omega}{}_0^t)}+\|\overset{1}{u}\cdot\nabla_{\overset{1}{u}}\overset{1}{H}\|_{L_r(\overset{1}{\Omega}{}_0^t)}\\
&\le \varphi(\overset{1}{\alpha}_r(t))(\|\overset{1}{u}_\xi\overset{1}{H}\|_{L_r(\overset{1}{\Omega}{}_0^t)}+\|\overset{1}{u}\overset{1}{H}_{,\xi}\|_{L_r(\overset{1}{\Omega}{}_0^t)})\equiv\varphi(\overset{1}{\alpha}_r(t))I.
\end{aligned}
$$

By the Hölder inequality we have

$$
\begin{aligned}
I &\le \Big(\int_0^t\|\overset{1}{u}_\xi\|^r_{L_\infty(\overset{1}{\Omega}_0)}\|\overset{1}{H}\|^r_{L_r(\overset{1}{\Omega}_0)}dt'\Big)^{1/r}+\Big(\int_0^t\|\overset{1}{u}\|^r_{L_r(\overset{1}{\Omega}_0)}\|\overset{1}{H}_\xi\|^r_{L_\infty(\overset{1}{\Omega}_0)}dt'\Big)^{1/r}\\
&\equiv I_1+I_2.
\end{aligned}
$$

For I_1, Lemma 2.5 implies

$$
I_1\le c\sup_t\|\overset{1}{H}\|_{L_r(\overset{1}{\Omega}_0)}\Big(\int_0^t\|\overset{1}{u}_{,\xi\xi}\|^{\theta_1 r}_{L_r(\overset{1}{\Omega}_0)}dt'\Big)^{1/r}\sup_t\|\overset{1}{u}\|^{1-\theta_1}_{L_r(\overset{1}{\Omega}_0)}\equiv I_1^1,
$$

where $\theta_1=3/(2r)+1/2$ and $\theta_1<1$ for $r>3$. Continuing,

$$
\begin{aligned}
I_1^1 &\le c\Big[t^{1/r'}\Big(\int_0^t\|\overset{1}{H}_{,t}\|^r_{L_r(\overset{1}{\Omega}_0)}dt'\Big)^{1/r}+\|\overset{1}{H}(0)\|_{L_r(\overset{1}{\Omega}_0)}\Big]\\
&\quad\cdot t^{\frac{1-\theta_1}{r}}\|\overset{1}{u}_{,\xi\xi}\|^{\theta_1}_{L_r(\overset{1}{\Omega}{}_0^t)}\Big[t^{1/r'}\Big(\int_0^t\|\overset{1}{u}_{,t}\|^r_{L_r(\overset{1}{\Omega}_0)}dt'\Big)^{1/r}+\|\overset{1}{u}(0)\|_{L_r(\overset{1}{\Omega}_0)}\Big]^{1-\theta_1}\\
&\le ct^a(t^{1/r'}\|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}+\|\overset{1}{H}(0)\|_{L_r(\overset{1}{\Omega}_0)})(t^{1/r'}\|\overset{1}{u}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}+\|\overset{1}{u}(0)\|_{L_r(\overset{1}{\Omega}_0)}).
\end{aligned}
$$

For I_2 we get the same bound. Similarly, $\|\overset{2}{f}_0\|_{L_r(\overset{2}{\Omega}{}_0^t)}$ is estimated by the above bound for $\|\overset{1}{f}_0\|_{L_r(\overset{1}{\Omega}{}_0^t)}$, where $(\overset{1}{u},\overset{1}{H},\overset{1}{\Omega}_0)$ is replaced by $(\overset{2}{u},\overset{2}{H},\overset{2}{\Omega}_0)$.

Next we estimate

$$\begin{aligned}\|k_{0i}\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)} &= \|k_{0i}\|_{L_r(0,t;W_r^{1-1/r}(S_0))} + \|k_{0i}\|_{L_r(S_0;W_r^{1/2-1/(2r)}(0,t))}\\ &\equiv J_1 + J_2.\end{aligned}$$

By extension, we have

$$\begin{aligned}J_1 \le \|k_{0i}\|_{L_r(0,t;W_r^1(\overset{1}{\Omega}_0))} &\le c\|\overset{1}{u}\times\overset{1}{H}\cdot\bar{\tau}_{ui}\|_{L_r(\overset{1}{\Omega}{}_0^t)} + c\|(\overset{1}{u}\times\overset{1}{H})_{,\xi}\cdot\bar{\tau}_{ui}\|_{L_r(\overset{1}{\Omega}{}_0^t)}\\ &\quad + \varphi(\overset{1}{\alpha}_r(t))\Big\|\overset{1}{u}\cdot\overset{1}{H}\int_0^t \hat{u}_{,\xi\xi}\,dt'\Big\|_{L_r(\overset{1}{\Omega}{}_0^t)} \equiv J_1^1 + J_1^2 + J_1^3,\end{aligned}$$

where $|J_1^1| + |J_1^2| \le A$ and

$$J_1^3 \le \overset{1}{\alpha}_r(t)\Big(\int_0^t \|\overset{1}{u}\overset{1}{H}\|^r_{L_\infty(\overset{1}{\Omega}_0)}\,dt'\Big)^{1/r} \equiv \overset{1}{\alpha}_r(t)J_1^4.$$

Using Lemma 2.5 we have

$$\begin{aligned}J_1^4 &\le \Big(\int_0^t \|\overset{1}{u}\|^r_{L_\infty(\overset{1}{\Omega}_0)}\|\overset{1}{H}\|^r_{L_\infty(\overset{1}{\Omega}_0)}\,dt'\Big)^{1/r}\\ &\le c\sup_t\|\overset{1}{u}\|^{1-\theta}_{L_r(\overset{1}{\Omega}_0)}\sup_t\|\overset{1}{H}\|^{1-\theta}_{L_r(\overset{1}{\Omega}_0)}\Big(\int_0^t\|\overset{1}{u}\|^{r\theta}_{W_r^2(\overset{1}{\Omega}_0)}\|\overset{1}{H}\|^{r\theta}_{W_r^2(\overset{1}{\Omega}_0)}\,dt'\Big)^{1/r} \equiv J_1^5,\end{aligned}$$

where $\theta = 3/(2r) < 1/2$ for $r > 3$. Applying the Hölder inequality gives

$$\begin{aligned}J_1^5 &\le c(t^{1/r'}\|\overset{1}{u}_{,t'}\|_{L_r(\overset{1}{\Omega}_0)} + \|\overset{1}{u}(0)\|_{L_r(\overset{1}{\Omega}_0)})^{1-\theta}\\ &\quad\cdot(t^{1/r'}\|\overset{1}{H}_{,t'}\|_{L_r(\overset{1}{\Omega}_0)} + \|\overset{1}{H}(0)\|_{L_r(\overset{1}{\Omega}_0)})^{1-\theta}\\ &\quad\cdot t^a\|\overset{1}{u}\|^\theta_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))}\|\overset{1}{H}\|^\theta_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))} \le A.\end{aligned}$$

Summarizing, we have

$$J_1 \le cA.$$

Finally, we consider

$$\begin{aligned}J_2 &\le \|k_{0i}\|_{L_r(\overset{1}{\Omega}_0;W_r^{1/2}(0,t))}\\ &\le c\bigg(\int_{\overset{1}{\Omega}_0} d\xi\int_0^t\int_0^t \frac{|(\overset{1}{u}\times\overset{1}{H}\cdot\bar{\tau}_{ui})(t') - (\overset{1}{u}\times\overset{1}{H}\cdot\bar{\tau}_{ui})(t'')|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r}\end{aligned}$$

$$
\begin{aligned}
&\le \varphi(\overset{1}{\alpha}_r(t))\bigg(\int\limits_{\overset{1}{\Omega}_0} d\xi \int\limits_0^t\int\limits_0^t \frac{|\overset{1}{u}(t')-\overset{1}{u}(t'')|^r|\overset{1}{H}(t')|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r}\\
&\quad+\varphi(\overset{1}{\alpha}_r(t))\bigg(\int\limits_{\overset{1}{\Omega}_0} d\xi \int\limits_0^t\int\limits_0^t \frac{|\overset{1}{u}(t'')|^r|\overset{1}{H}(t')-\overset{1}{H}(t'')|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r}\\
&\quad+\varphi(\overset{1}{\alpha}_r(t'))\bigg(\int\limits_{\overset{1}{\Omega}_0} d\xi \int\limits_0^t\int\limits_0^t \frac{|\overset{1}{u}(t'')|^r|\overset{1}{H}(t'')|^r\big|\int_{t''}^{t'} u_{,\xi}\,d\tau\big|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r}\\
&\equiv J_2^1+J_2^2+J_2^3.
\end{aligned}
$$

By the Hölder inequality, we have

$$
\begin{aligned}
J_2^1 &\le \varphi(\overset{1}{\alpha}_r(t))\Big(\int\limits_0^t \|\overset{1}{u}_{,t'}(t')\|^r_{L_r(\overset{1}{\Omega}_0)}\,dt'\Big)^{1/r}\\
&\quad\cdot\Big(\int\limits_0^t\int\limits_0^t \frac{|t'-t''|^{r/r'}}{|t'-t''|^{1+r/2}}\|\overset{1}{H}(t')\|^r_{L_\infty(\overset{1}{\Omega}_0)}\,dt'\,dt''\Big)^{1/r}\\
&\le \varphi(\overset{1}{\alpha}_r(t))\|\overset{1}{u}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} t^{1/2-1/r}\|\overset{1}{H}\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))}\le cA.
\end{aligned}
$$

Performing the transformation $(\overset{1}{u},\overset{1}{H})\mapsto(\overset{1}{H},\overset{1}{u})$ we obain $J_2^2\le cA$.

Finally, we estimate

$$
J_2^3\le t^{1/2-1/r}\varphi(\overset{1}{\alpha}_r(t))\Big(\int\limits_0^t\|\overset{1}{u}_{,\xi}\|^r_{L_\infty(\overset{1}{\Omega}_0)}\,dt'\Big)^{1/r}\Big(\int\limits_0^t\int\limits_{\overset{1}{\Omega}_0}|\overset{1}{u}|^r|\overset{1}{H}|^r\,d\xi\,dt'\Big)^{1/r},
$$

where the last factor, in view of Lemma 2.5, is bounded by

$$
\begin{aligned}
\Big(\int\limits_0^t\|\overset{1}{u}\|^r_{L_{2r}(\overset{1}{\Omega}_0)}\|\overset{1}{H}\|^r_{L_{2r}(\overset{1}{\Omega}_0)}\,dt'\Big)^{1/r} &\le c\sup_t\|\overset{1}{u}\|^\theta_{L_r(\overset{1}{\Omega}_0)}\sup_t\|\overset{1}{H}\|^\theta_{L_r(\overset{1}{\Omega}_0)}\\
&\quad\cdot\Big(\int\limits_0^t\|\overset{1}{u}\|^{\theta r}_{W_r^2(\overset{1}{\Omega}_0)}\cdot\|\overset{1}{H}\|^{\theta r}_{W_r^2(\overset{1}{\Omega}_0)}\,dt'\Big)^{1/r}\equiv J_2^4,
\end{aligned}
$$

where $\theta=3/(4r)<1/2$ for $r>3$. Continuing, we obtain $J_2^3\le A$.

Summarizing, $J_2\le cA$. This concludes the proof. ∎

Set

$$
\begin{aligned}
\overset{i}{\bar f}&=\overset{i}{f}_*-\overset{i}{f}_0, &&i=1,2,\\
\bar k_\alpha&=k_{*\alpha}-k_{0\alpha}, &&\alpha=1,2.
\end{aligned}
\tag{10.35}
$$

LEMMA 10.7. *Let the assumptions of Lemma* 10.6 *hold. Let* $\overset{i}{u}\in W_r^{2,1}(\overset{i}{\Omega}{}_0^t)$, $i=1,2,$ *Then,* *for* $i = 1, 2$,

$$
\begin{aligned}
&\|\overset{i}{\bar f}\|_{L_r(\overset{i}{\Omega}{}^t_0)} \le \varphi(\overset{i}{\alpha}_r(t))\overset{i}{\alpha}_r(t)\|\overset{i}{H}\|_{L_r(0,t;W^2_r(\overset{1}{\Omega}_0))},\\
&\sum_{\alpha=1}^2 \|\bar k_\alpha\|_{W_r^{1-1/r,1/2-1/(2r)}(S^t_0)} \le \sum_{i=1}^2 \varphi(\overset{i}{\alpha}_r(t))\overset{i}{\alpha}_r(t)\|\overset{i}{H}\|_{W_r^{2,1}(\overset{i}{\Omega}{}^t_0)},\\
&\|\overset{i}{g}_*\|_{L_r(0,t;W^1_r(\overset{i}{\Omega}_0))} \le \overset{i}{\alpha}_r(t)\varphi(\overset{i}{\alpha}_r(t))\|\overset{i}{H}\|_{L_r(0,t;W^2_r(\overset{i}{\Omega}_0))}.
\end{aligned}
\tag{10.36}
$$

Proof. To prove the lemma we have to estimate expressions (10.35). We need some properties of the transformation between Eulerian and Lagrangian coordinates

$$
x = \xi + \int_0^t u(\xi,\tau)\,d\tau.
$$

We have $\xi_x = \mathbb{I} - \int_0^t u_\xi(\xi,\tau)\xi_x\,d\tau$, where $\mathbb{I}$ is the unit matrix. Then

$$
\frac{d}{dt}\xi_x = u_\xi(\xi,t)\xi_x,
$$

so

$$
|\xi_x| \le \exp\Big(\int_0^t |u_\xi(\xi,\tau)|\,d\tau\Big),
$$

where ξ_x, u_ξ are matrices and $|\xi_x| = \sum_{i,j=1}^3 |\xi_{i,x_j}|$. First we estimate $\overset{i}{\bar f}$, $i = 1, 2$. Employing the above relations and using the form of $\overset{i}{f}_*$ from $(10.3)_{1,2}$, we arrive at the qualitative estimate

$$
\|\overset{i}{\bar f}\|_{L_r(\overset{i}{\Omega}{}^t_0)} \le \varphi(\overset{i}{\alpha}_r(t))\|(\xi_x - \mathbb{I})\operatorname{rot}^2_\xi \overset{i}{H}\|_{L_r(\overset{i}{\Omega}{}^t_0)} + \varphi(\overset{i}{\alpha}_r(t))\|\xi_{xx}\overset{i}{H}_{,\xi}\|_{L_r(\overset{i}{\Omega}{}^t_0)} \equiv L_1 + L_2.
$$

Hence,

$$
\begin{aligned}
&L_1 \le \varphi(\overset{i}{\alpha}_r(t))\overset{i}{\alpha}_r(t)\|\overset{i}{H}\|_{L_r(0,t;W^2_r(\overset{i}{\Omega}{}^t_0))},\\
&L_2 \le \varphi(\overset{i}{\alpha}_r(t))\Big\|\int_0^t \overset{i}{u}_{\xi\xi}\,d\tau\,\overset{i}{H}_{,\xi}\Big\|_{L_r(\overset{i}{\Omega}{}^t_0)} \equiv \varphi(\overset{i}{\alpha}_r(t))L_2'.
\end{aligned}
$$

Applying the Hölder and Minkowski inequalities yields

$$
L_2' \le \bigg(\int_0^t\Big|\int_0^{t'} \|\overset{i}{u}_{,\xi\xi}\|_{L_r(\overset{i}{\Omega}_0)}\,d\tau\Big|^r \|\overset{i}{H}\|^r_{L_\infty(\overset{i}{\Omega}_0)}\,dt'\bigg)^{1/r} \le \overset{i}{\alpha}_r(t)\|\overset{i}{H}\|_{L_r(0,t;W^2_r(\overset{i}{\Omega}_0))}
$$

for $r>3$. Hence $(10.36)_1$ is proved. To show $(10.36)_2$ we consider

$$\begin{aligned}\|\bar k_\alpha\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)} &\le c\|\mathrm{rot}_\xi \overset{1}{H}-\mathrm{rot}_{\overset{1}{u}}\overset{1}{H}\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)}\\ &+c\|\mathrm{rot}_\xi \overset{2}{H}-\mathrm{rot}_{\overset{2}{u}}\overset{2}{H}\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)}\\ &+c\bigg\|\bigg(\frac{1}{\sigma_1}\mathrm{rot}_{\overset{1}{u}}\overset{1}{H}-\frac{1}{\sigma_2}\mathrm{rot}_{\overset{2}{u}}\overset{2}{H}\bigg)(\bar\tau_\alpha-\bar\tau_{u\alpha})\bigg\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)}\equiv\sum_{i=1}^3 I_i.\end{aligned}$$

First, we examine

$$I_1\le c\|\mathrm{rot}_\xi \overset{1}{H}-\mathrm{rot}_{\overset{1}{u}}\overset{1}{H}\|_{L_r(0,t;W_r^1(\overset{1}{\Omega}_0))}+c\|\mathrm{rot}_\xi \overset{1}{H}-\mathrm{rot}_{\overset{1}{u}}\overset{1}{H}\|_{L_r(\overset{1}{\Omega}_0;W_r^{1/2}(0,t))}\equiv I_1^1+I_1^2,$$

where

$$I_1^1\le\varphi(\overset{1}{\alpha}_r(t))\overset{1}{\alpha}_r(t)\|\overset{1}{H}\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))}$$

and

$$\begin{aligned}I_1^2&\le\bigg(\int_{\overset{1}{\Omega}_0}d\xi\int_0^t\!\!\int_0^t\frac{|(\xi_x(t')-\mathbb{I})\overset{1}{H}_\xi(t')-(\xi_x(t'')-\mathbb{I})\overset{1}{H}_\xi(t'')|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r}\\ &\le\bigg(\int_{\overset{1}{\Omega}_0}d\xi\int_0^t\!\!\int_0^t\frac{|\xi_x(t')-\xi_x(t'')|^r|\overset{1}{H}_\xi(t')|^r+|\xi_x(t'')-\mathbb{I}|^r|\overset{1}{H}_\xi(t')-\overset{1}{H}_\xi(t'')|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r}\\ &\le\bigg(\int_{\overset{1}{\Omega}_0}d\xi\int_0^t\!\!\int_0^t\frac{|\xi_x(t')-\xi_x(t'')|^r|\overset{1}{H}_\xi(t')|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r}\\ &\quad+\bigg(\int_{\overset{1}{\Omega}_0}d\xi\int_0^t\!\!\int_0^t\frac{|\xi_x(t'')-\mathbb{I}|^r|\overset{1}{H}_\xi(t')-\overset{1}{H}_\xi(t'')|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r}\equiv J_1+J_2.\end{aligned}$$

Using the explicit form of ξ_x gives

$$\begin{aligned}J_1&\le\varphi(\overset{1}{\alpha}_r(t))\bigg(\int_{\overset{1}{\Omega}_0}d\xi\int_0^t\!\!\int_0^t\frac{\big|\int_{t''}^{t'}\overset{1}{u}_{,\xi}(\xi,\tau)\,d\tau\big|^r}{|t'-t''|^{1+r/2}}|\overset{1}{H}_\xi(\xi,t')|^r\,dt'\,dt''\bigg)^{1/r}\\ &\le\varphi(\overset{1}{\alpha}_r(t))\bigg(\int_0^t\|\overset{1}{u}(\tau)\|^r_{W_r^2(\overset{1}{\Omega}_0)}\,d\tau\bigg)^{1/r}\\ &\quad\cdot\bigg(\int_{\overset{1}{\Omega}_0}d\xi\int_0^t\!\!\int_0^t\frac{|t'-t''|^{r/r'}}{|t'-t''|^{1+r/2}}|\overset{1}{H}_\xi(\xi,t')|^r\,dt'\,dt''\bigg)^{1/r}\\ &\le\varphi(\overset{1}{\alpha}_r(t))t^{1/r'-1/2}\|\overset{1}{u}\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))}\|\overset{1}{H}_{,\xi}\|_{L_r(\overset{1}{\Omega}{}_0^t)}.\end{aligned}$$

Considering J_2, we have

$$J_2 \le \varphi(\overset{1}{\alpha}_r(t))\overset{1}{\alpha}_r(t)\bigg(\int\limits_{\overset{1}{\Omega}_0} d\xi \int\limits_0^t\int\limits_0^t \frac{|\overset{1}{H}_\xi(t') - \overset{1}{H}_\xi(t'')|^r}{|t'-t''|^{1+r/2}}\, dt'\, dt''\bigg)^{1/r}$$
$$\le \varphi(\overset{1}{\alpha}_r(t))\overset{1}{\alpha}_r(t)\|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}.$$

Summarizing, we have

$$I_1 \le \varphi(\overset{1}{\alpha}_r(t))\overset{1}{\alpha}_r(t)\|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}. \tag{10.37}$$

Similarly,

$$I_2 \le \varphi(\overset{2}{\alpha}_r(t))\overset{2}{\alpha}_r(t)\|\overset{2}{H}\|_{W_r^{2,1}(\overset{2}{\Omega}{}_0^t)}. \tag{10.38}$$

Next,

$$I_3 \le \|\mathrm{rot}_{\overset{1}{u}} \overset{1}{H}(\bar\tau_\alpha - \bar\tau_{\overset{1}{u}\alpha})\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)} + \|\mathrm{rot}_{\overset{2}{u}} \overset{2}{H}(\bar\tau_\alpha - \bar\tau_{\overset{2}{u}\alpha})\|_{W_r^{1,1/2}(\overset{1}{\Omega}{}_0^t)} \equiv I_3^1 + I_3^2.$$

We consider I_3^1 only, because I_3^2 can be treated in the same way. We write

$$I_3^1 = \|\mathrm{rot}_{\overset{1}{u}} \overset{1}{H}\cdot(\bar\tau_\alpha - \bar\tau_{\overset{1}{u}\alpha})\|_{L_r(0,t;W_r^1(\overset{1}{\Omega}_0))} + \|\mathrm{rot}_{\overset{1}{u}} \overset{1}{H}\cdot(\bar\tau_\alpha - \bar\tau_{\overset{1}{u}\alpha})\|_{L_r(\overset{1}{\Omega}_0;W_r^{1/2}(0,t))} \equiv I_3^3 + I_3^4,$$

where, after omitting the lower order terms,

$$I_3^3 \le \varphi(\overset{1}{\alpha}_r(t))\Big(\Big\|\overset{1}{H}_{,\xi\xi}\int\limits_0^t \overset{1}{u}_{,\xi}\,d\tau\Big\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \Big\|\overset{1}{H}_{,\xi}\int\limits_0^t \overset{1}{u}_{,\xi\xi}\,d\tau\Big\|_{L_r(\overset{1}{\Omega}{}_0^t)}\Big)$$
$$\le \varphi(\overset{1}{\alpha}_r(t))\int\limits_0^t \|\overset{1}{u}\|_{W_r^2(\overset{1}{\Omega}_0)}\,d\tau\,\|\overset{1}{H}\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))} \le \varphi(\overset{1}{\alpha}_r(t))\overset{1}{\alpha}_r(t)\|\overset{1}{H}\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))}.$$

Finally, we estimate

$$I_3^4 \le \|\xi_x\overset{1}{H}_{,\xi}(\bar\tau_\alpha - \bar\tau_{u\alpha})\|_{L_r(\overset{1}{\Omega}_0;W_r^{1/2}(0,t))}$$
$$= \bigg(\int\limits_{\overset{1}{\Omega}_0} d\xi\int\limits_0^t\int\limits_0^t \frac{|\xi_x(t')\overset{1}{H}_\xi(t')(\bar\tau_\alpha - \bar\tau_{\overset{1}{u}\alpha})(t') - \xi_x(t'')\overset{1}{H}_\xi(t'')(\bar\tau_\alpha - \bar\tau_{\overset{1}{u}\alpha})(t'')|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r}$$
$$\le \varphi(\overset{1}{\alpha}_r(t))\bigg[\bigg(\int\limits_{\overset{1}{\Omega}_0} d\xi\int\limits_0^t\int\limits_0^t \frac{\big|\int_{t''}^{t'}\overset{1}{u}_{,\xi}\,d\tau\big|^r|\overset{1}{H}_{,\xi}(t')|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r}$$
$$+\bigg(\int\limits_{\overset{1}{\Omega}_0} d\xi\int\limits_0^t\int\limits_0^t\bigg|\int\limits_0^{t'}\overset{1}{u}_\xi\,d\tau\bigg|^r\frac{|\overset{1}{H}_{,\xi}(t') - \overset{1}{H}_{,\xi}(t'')|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\bigg)^{1/r}\bigg]$$

$$\leq \varphi(\overset{1}{\alpha}_r(t))\Big[t^{1/r'-1/2}\|\overset{1}{u}\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))}\|\overset{1}{H}\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))} + \overset{1}{\alpha}_r(t)\Big(\int\limits_{\overset{1}{\Omega}_0} d\xi \int\limits_0^t\int\limits_0^t \frac{|\overset{1}{H}_{,\xi}(t') - \overset{1}{H}_{,\xi}(t'')|^r}{|t'-t''|^{1+r/2}}\,dt'\,dt''\Big)^{1/r}\Big]$$

$$\leq \varphi(\overset{1}{\alpha}_r(t))\overset{1}{\alpha}_r(t)\|\overset{1}{H}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}.$$

Hence,

$$I_3 \leq \sum_{i=1}^2 \varphi(\overset{i}{\alpha}_r(t))\overset{i}{\alpha}_r(t)\|\overset{i}{H}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)}. \tag{10.39}$$

From (10.37), (10.38), and (10.39), estimate $(10.36)_2$ follows. Similarly, $(10.36)_3$ can be shown. This concludes the proof. ∎

Finally, we prove existence of solutions to problem (10.1) with given $\overset{1}{u}, \overset{2}{u}$ by the following method of successive approximations:

$$\begin{aligned}
&\mu_1\overset{1}{H}_{n+1,t} + \frac{1}{\sigma_1}\operatorname{rot}_\xi^2 \overset{1}{H}_{n+1}\\
&\quad= \frac{1}{\sigma_1}(\operatorname{rot}_\xi^2 \overset{1}{H}_n - \operatorname{rot}_{\overset{1}{u}}^2 \overset{1}{H}_n) + \mu_1 \operatorname{rot}_{\overset{1}{u}}(\overset{1}{u}\times \overset{1}{H}_n) + \mu_1\overset{1}{u}\cdot\nabla_{\overset{1}{u}}\overset{1}{H}_n,\\
&\operatorname{div}_\xi \overset{1}{H}_{n+1} = \operatorname{div}_\xi \overset{1}{H}_n - \operatorname{div}_{\overset{1}{u}} \overset{1}{H}_n \qquad \text{in } \overset{1}{\Omega}{}_0^t,\\
&\mu_2\overset{2}{H}_{n+1,t} + \frac{1}{\sigma_2}\operatorname{rot}_\xi^2 \overset{2}{H}_{n+1}\\
&\quad= \frac{1}{\sigma_2}(\operatorname{rot}_\xi^2 \overset{2}{H}_n - \operatorname{rot}_{\overset{2}{u}}^2 \overset{2}{H}_n) + \mu_2\overset{2}{u}\cdot\nabla_{\overset{2}{u}}\overset{2}{H}_n,\\
&\operatorname{div}_\xi \overset{2}{H}_{n+1} = \operatorname{div}_\xi \overset{2}{H}_n - \operatorname{div}_{\overset{2}{u}} \overset{2}{H}_n \qquad \text{in } \overset{2}{\Omega}{}_0^t,\\
&\Big(\frac{1}{\sigma_1}\operatorname{rot}_\xi \overset{1}{H}_{n+1} - \frac{1}{\sigma_2}\operatorname{rot}_\xi \overset{2}{H}_{n+1}\Big)\cdot\bar{\tau}_\alpha\\
&\quad= \Big[\frac{1}{\sigma_1}(\operatorname{rot}_\xi \overset{1}{H}_n - \operatorname{rot}_{\overset{1}{u}} \overset{1}{H}_n) - \frac{1}{\sigma_2}(\operatorname{rot}_\xi \overset{2}{H}_n - \operatorname{rot}_{\overset{2}{u}} \overset{2}{H}_n)\Big]\cdot\bar{\tau}_\alpha\\
&\qquad+ \Big(\frac{1}{\sigma_1}\operatorname{rot}_{\overset{1}{u}} \overset{1}{H}_n - \frac{1}{\sigma_2}\operatorname{rot}_{\overset{2}{u}} \overset{2}{H}_n\Big)\cdot(\bar{\tau}_\alpha - \bar{\tau}_{u\alpha})\\
&\qquad+ \mu_1\overset{1}{u}\times\overset{1}{H}_n\cdot\bar{\tau}_{u\alpha}, \quad \alpha=1,2, \qquad \text{on } S_0^t,\\
&\overset{1}{H}_{n+1} - \overset{2}{H}_{n+1} = 0 \qquad \text{on } S_0^t,\\
&\overset{i}{H}_{n+1}|_{t=0} = \overset{i}{H}(0), \quad \operatorname{div}\overset{i}{H}(0) = 0, \quad i=1,2, \quad \overset{2}{H}_{n+1}\cdot\bar{\tau}'_\alpha|_B = H_{*\alpha},\\
&\operatorname{div}\overset{2}{H}_{n+1}|_B = 0, \quad \alpha = 1,2.
\end{aligned} \tag{10.40}$$

Let $\overset{i}{\tilde{H}}$ be a divergence free extension of the initial data, so $\overset{i}{\tilde{H}}|_{t=0} = \overset{i}{H}(0)$, $i=1,2$. We assume that $\overset{i}{H}_0 = \overset{i}{\tilde{H}}$, $i = 1, 2$. Recall that in this section we have replaced $\overset{i}{\bar{H}}$ by $\overset{i}{H}$, $i = 1, 2$ (see (10.1).

THEOREM 10.8. *Let $r > 3$. Assume that $\overset{i}{u}(0) \in L_r(\overset{i}{\Omega}_0)$, $\overset{i}{H}(0) \in W_r^{2-2/r}(\overset{i}{\Omega}_0)$, $\overset{i}{u} \in W_r^{2,1}(\overset{i}{\Omega}{}^t)$, $i = 1, 2$, $H_* \in W_r^{2-1/r,1-1/(2r)}(B^t)$. Then there exists T sufficiently small such that for $t \le T$ there exists a solution to problem* (10.1) *such that $\overset{i}{H} \in W_r^{2,1}(\overset{i}{\Omega}{}_0^t)$, $i = 1, 2$, and*

$$\sum_{i=1}^{2} \|\overset{i}{H}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)} \le \varphi\Big(t^a \sum_{i=1}^{2} \|\overset{i}{u}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)}, \sum_{i=1}^{2} \|\overset{i}{u}(0)\|_{L_r(\overset{i}{\Omega}_0)}, \sum_{i=1}^{2} \|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)}, \sum_{\alpha=1}^{2} \|H_{*\alpha}\|_{W_r^{2-1/r,1-1/(2r)}(B^t)}\Big), \tag{10.41}$$

where φ is an increasing positive function.

Proof. We use the method of successive approximations described by (10.40). By zero approximation we assume an extension of the initial data such that $\overset{i}{H}_0 \in W_r^{2,1}(\overset{i}{\Omega}{}_0^t)$, $i = 1, 2$, and

$$\|\overset{i}{H}_0\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)} \le c \|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)}, \quad i = 1, 2.$$

Remark 10.1 and Lemmas 10.2–10.7 imply existence of solutions to problem (10.40) and the estimate

$$\sum_{i=1}^{2} \|\overset{i}{H}_{n+1}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)} \le \varphi\Big(t^a \sum_{i=1}^{2} \overset{i}{H}_n\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)}, t^a \sum_{i=1}^{2} \|\overset{i}{u}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)}, \sum_{i=1}^{2} \|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)}, \sum_{i=1}^{2} \|\overset{i}{u}(0)\|_{L_r(\overset{i}{\Omega}_0)}, \sum_{\alpha=1}^{2} \|H_{*\alpha}\|_{W_r^{2-1/r,1-1/(2r)}(B^t)}\Big), \tag{10.42}$$

where φ is an increasing positive function. Then for t sufficiently small there exists a constant M such that

$$\varphi\Big(t^a M, t^a \sum_{i=1}^{2} \|\overset{i}{u}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)}, \sum_{i=1}^{2} \|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)}, \sum_{i=1}^{2} \|\overset{i}{u}(0)\|_{L_r(\overset{i}{\Omega}_0)}, \sum_{\alpha=1}^{2} \|H_{*\alpha}\|_{W_r^{2-1/r,1-1/(2r)}(B^t)}\Big) \le M. \tag{10.43}$$

This means that

$$\sum_{i=1}^{2} \|\overset{i}{H}_n\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)} \le M \quad \text{for any } n \in \mathbb{N}. \tag{10.44}$$

Hence (10.43) implies (10.41). To prove convergence we introduce the differences

$$\overset{i}{h}_n = \overset{i}{H}_n - \overset{i}{H}_{n-1}, \quad i = 1, 2, \tag{10.45}$$

which are solutions to the problem

$$
\begin{aligned}
&\mu_1\overset{1}{h}_{n+1,t}+\frac{1}{\sigma_1}\operatorname{rot}^2_\xi\overset{1}{h}_{n+1}\\
&\quad=\frac{1}{\sigma_1}(\operatorname{rot}^2_\xi\overset{1}{h}_n-\operatorname{rot}^2_{\overset{1}{u}}\overset{1}{h}_n)+\mu_1\operatorname{rot}_{\overset{1}{u}}(\overset{1}{u}\times\overset{1}{h}_n)+\mu_1\overset{1}{u}\cdot\nabla_{\overset{1}{u}}\overset{1}{h}_n \qquad \text{in } \overset{1}{\Omega}{}^t_0,\\
&\operatorname{div}_\xi\overset{1}{h}_{n+1}=\operatorname{div}_\xi\overset{1}{h}_n-\operatorname{div}_{\overset{1}{u}}\overset{1}{h}_n,\\
&\mu_2\overset{2}{h}_{n+1,t}+\frac{1}{\sigma_2}\operatorname{rot}^2_\xi\overset{2}{h}_{n+1}\\
&\quad=\frac{1}{\sigma_2}(\operatorname{rot}^2_\xi\overset{2}{h}_n-\operatorname{rot}^2_{\overset{2}{u}}\overset{2}{h}_n)+\mu_2\overset{2}{u}\cdot\nabla_{\overset{2}{u}}\overset{2}{h}_n \qquad \text{in } \overset{2}{\Omega}{}^t_0,\\
&\operatorname{div}_\xi\overset{2}{h}_{n+1}=\operatorname{div}_\xi\overset{2}{h}_n-\operatorname{div}_{\overset{2}{u}}\overset{2}{h}_n,\\
&\left(\frac{1}{\sigma_1}\operatorname{rot}_\xi\overset{1}{h}_{n+1}-\frac{1}{\sigma_2}\operatorname{rot}_\xi\overset{2}{h}_{n+1}\right)\cdot\bar\tau_\alpha\\
&\quad=\left[\frac{1}{\sigma_1}(\operatorname{rot}_\xi\overset{1}{h}_n-\operatorname{rot}_{\overset{1}{u}}\overset{1}{h}_n)-\frac{1}{\sigma_2}(\operatorname{rot}_\xi\overset{2}{h}_n-\operatorname{rot}_{\overset{2}{u}}\overset{2}{h}_n)\right]\cdot\bar\tau_\alpha\\
&\qquad+\left(\frac{1}{\sigma_1}\operatorname{rot}_{\overset{1}{u}}\overset{1}{h}_n-\frac{1}{\sigma_2}\operatorname{rot}_{\overset{2}{u}}\overset{2}{h}_n\right)\cdot(\bar\tau_\alpha-\bar\tau_{u\alpha})\\
&\qquad+\mu_1\overset{1}{u}\times\overset{1}{h}_n\cdot\bar\tau_{u\alpha},\quad \alpha=1,2, \qquad \text{on } S^t_0,\\
&\overset{1}{h}_{n+1}-\overset{2}{h}_{n+1}=0 \qquad \text{on } S^t_0,\\
&\overset{i}{h}_{n+1}|_{t=0}=0,\quad i=1,2,\quad \overset{2}{h}_{n+1}\cdot\bar\tau'_\alpha|_B=0,\quad \alpha=1,2,\\
&\operatorname{div}\overset{2}{h}_{n+1}|_B=0.
\end{aligned}
\tag{10.46}
$$

Using $\overset{i}{u}\in W^{2,1}_r(\overset{i}{\Omega}{}^t_0)$, $i=1,2$, and the technique of regularizer we obtain the estimate

$$
\sum_{i=1}^2\|\overset{i}{h}_{n+1}\|_{W^{2,1}_r(\overset{i}{\Omega}{}^t_0)}\le c(M)t^a\sum_{i=1}^2\|\overset{i}{h}_n\|_{W^{2,1}_r(\overset{i}{\Omega}{}^t_0)}. \tag{10.47}
$$

Hence for t so small that $C(M)t^a<1$ we get the existence of solutions to problem (10.1) by standard considerations of the method of successive approximations. The solution $(\overset{1}{H},\overset{2}{H})$ belongs to $W^{2,1}_r(\overset{1}{\Omega}{}^t_0)\times W^{2,1}_r(\overset{2}{\Omega}{}^t_0)$ and estimate (10.41) holds. This concludes the proof. ∎

11. The model problem

In this section we consider the following model problem:

$$
\begin{aligned}
&\mu_1\overset{1}{u}_{,t}+\frac{1}{\sigma_1}\operatorname{rot}^2_z\overset{1}{u}=\overset{1}{f},\ \operatorname{div}_z\overset{1}{u}=0,\qquad z_3>0,\\
&\mu_2\overset{2}{u}_{,t}+\frac{1}{\sigma_2}\operatorname{rot}^2_z\overset{2}{u}=\overset{2}{f},\ \operatorname{div}_z\overset{2}{u}=0,\qquad z_3<0,
\end{aligned}
\tag{11.1}
$$

$$
\begin{aligned}
&\left(\frac{1}{\sigma_1}\operatorname{rot}_z \overset{1}{u} - \frac{1}{\sigma_2}\operatorname{rot}_z \overset{2}{u}\right)\cdot\bar\tau_\alpha = \bar k_\alpha, \quad \alpha=1,2, && z_3=0,\\
&\overset{1}{u}_\alpha - \overset{2}{u}_\alpha = 0, \quad \alpha=1,2, && z_3=0,\\
&\overset{1}{u}\to 0 && \text{as } z_3\to+\infty,\\
&\overset{2}{u}\to 0 && \text{as } z_3\to-\infty,\\
&\overset{i}{u}|_{t=0} = \overset{i}{u}(0),\ i=1,2,
\end{aligned}
\tag{11.1 [cont.]}
$$

where $\bar\tau_1=(1,0,0)$, $\bar\tau_2=(0,1,0)$, $\bar n=(0,0,1)$, $\bar\tau_1\times\bar n=(0,-1,0)$, $\bar\tau_2\times\bar n=(1,0,0)$, $z=(z_1,z_2,z_3)$ are the Cartesian coordinates. Let $\overset{1}{\mathbb{R}}=\mathbb{R}^3_+=\{z\in\mathbb{R}^3: z_3>0\}$, $\overset{2}{\mathbb{R}}=\mathbb{R}^3_-=\{z\in\mathbb{R}^3: z_3<0\}$.

We assume that $\overset{i}{\bar f}$, $\overset{i}{u}(0)$, $i=1,2$, are divergence free. To make the initial data homogeneous we construct divergence free extensions $\overset{i}{\tilde u}$, $i=1,2$, of the initial data $\overset{i}{u}(0)$, $i=1,2$, such that

$$
\overset{i}{\tilde u}|_{t=0} = \overset{i}{u}(0), \quad i=1,2. \tag{11.2}
$$

Set

$$
\overset{i}{v} = \overset{i}{u} - \overset{i}{\tilde u}, \quad i=1,2. \tag{11.3}
$$

Then problem (11.1) takes the form

$$
\begin{aligned}
&\sigma_1\mu_1\overset{1}{v}_t - \Delta\overset{1}{v} = \sigma_1\overset{1}{\bar f} - \sigma_1\left(\mu_1\overset{1}{\tilde u}_t - \frac{1}{\sigma_1}\Delta\overset{1}{\tilde u}\right) \equiv \overset{1}{f}, && z_3>0,\\
&\operatorname{div}_z\overset{1}{v}=0, && z_3>0,\\
&\sigma_2\mu_2\overset{2}{v}_t - \Delta\overset{2}{v} = \sigma_2\overset{2}{\bar f} - \sigma_2\left(\mu_2\overset{2}{\tilde u}_t - \frac{1}{\sigma_2}\Delta\overset{2}{\tilde u}\right) \equiv \overset{2}{f} && z_3<0,\\
&\operatorname{div}_z\overset{2}{v}=0, && z_3<0,\\
&(a_1\operatorname{rot}_z\overset{1}{v} - a_2\operatorname{rot}_z\overset{2}{v})\cdot\bar\tau_\alpha\\
&\quad = \bar k_\alpha - (a_1\operatorname{rot}_z\overset{1}{\tilde u} - a_2\operatorname{rot}_z\overset{3}{\tilde u})\cdot\bar\tau_\alpha \equiv k_\alpha, \quad \alpha=1,2, && z_3=0,\\
&\overset{1}{v}_\alpha = \overset{2}{v}_\alpha, \quad \alpha=1,2, && z_3=0.
\end{aligned}
\tag{11.4}
$$

where we have used $\overset{1}{u}_\alpha(0)=\overset{2}{u}_\alpha(0)$ on $z_3=0$ so also the extensions $\overset{1}{\tilde u}$, $\overset{2}{\tilde u}$ satisfy the condition $\overset{1}{\tilde u}_\alpha=\overset{2}{\tilde u}_\alpha$, $\alpha=1,2$, on $z_3=0$.

Expressing the boundary conditions explicitly we have

$$
\begin{aligned}
&\sigma_1\mu_1\overset{1}{v}_t - \Delta\overset{1}{v} = \overset{1}{f}, \quad \operatorname{div}\overset{1}{v}=0, && z_3>0,\\
&\sigma_2\mu_2\overset{2}{v}_t - \Delta\overset{2}{v} = \overset{2}{f}, \quad \operatorname{div}\overset{2}{v}=0, && z_3<0,
\end{aligned}
\tag{11.5}
$$

$$
\begin{aligned}
&a_1(\overset{1}{v}_{2,z_3}-\overset{1}{v}_{3,z_2})-a_2(\overset{2}{v}_{2,z_3}-\overset{2}{v}_{3,z_2})=k_1, && z_3=0,\\
&a_1(\overset{1}{v}_{3,z_1}-\overset{1}{v}_{1,z_3})-a_2(\overset{2}{v}_{3,z_1}-\overset{2}{v}_{1,z_3})=k_2, && z_3=0,\\
&\overset{1}{v}_\alpha=\overset{2}{v}_\alpha,\quad \alpha=1,2, && z_3=0.
\end{aligned}
\tag{11.5 [cont.]}
$$

Assume that $\overset{1}{\delta}$, $\overset{2}{\delta}$ are solutions to the problems

$$
\begin{aligned}
&\sigma_1\mu_1\overset{1}{\delta}_t-\Delta\overset{1}{\delta}=\overset{1}{f},\quad \operatorname{div}\overset{1}{\delta}=0, && z_3>0,\\
&\sigma_{2\mu_2}\overset{2}{\delta}_t-\Delta\overset{2}{\delta}=\overset{2}{f},\quad \operatorname{div}\overset{2}{\delta}=0, && z_3<0.
\end{aligned}
\tag{11.6}
$$

To find solutions to (11.6) we have to transform it to

$$
\sigma_1\mu_1\overset{1}{\delta}_t-\Delta\overset{1}{\delta}=\overset{1}{\bar f}\quad \text{in } \mathbb{R}^3,
\tag{11.7}
$$

where

$$
\overset{1}{\bar f}=\begin{cases}\overset{1}{f}, & z_3>0,\\ 0, & z_3<0,\end{cases}
$$

and

$$
\sigma_2\mu_2\overset{2}{\delta}_t-\Delta\overset{2}{\delta}=\overset{2}{\bar f}\quad \text{in } \mathbb{R}^3
\tag{11.8}
$$

where

$$
\overset{2}{\bar f}=\begin{cases}0, & z_3>0,\\ \overset{2}{f}, & z_3<0.\end{cases}
$$

Let G be the fundamental solution to the heat equation. Then solutions to (11.7) and (11.8) have the form

$$
\overset{1}{\delta}=G*\overset{1}{\bar f},\qquad \overset{2}{\delta}=G*\overset{2}{\bar f}.
\tag{11.9}
$$

Set

$$
\overset{i}{\omega}=\overset{i}{v}-\overset{i}{\delta},\qquad i=1,2,
\tag{11.10}
$$

which are solutions to the problem

$$
\begin{aligned}
&\sigma_1\mu_1\overset{1}{\omega}_t-\Delta\overset{1}{\omega}=0,\quad \operatorname{div}\overset{1}{\omega}=0, && z_3>0,\\
&\sigma_2\mu_2\overset{2}{\omega}_t-\Delta\overset{2}{\omega}=0,\quad \operatorname{div}\overset{2}{\omega}=0, && z_3<0,\\
&a_1(\overset{1}{\omega}_{2,z_3}-\overset{1}{\omega}_{3,z_2})-a_2(\overset{2}{\omega}_{2,z_3}-\overset{2}{\omega}_{3,z_2})\\
&\quad =k_1-[a_1(\overset{1}{\delta}_{2,z_3}-\overset{1}{\delta}_{3,z_2})-a_2(\overset{2}{\delta}_{2,z_3}-\overset{2}{\delta}_{3,z_2})]\equiv k_1', && z_3=0,\\
&a_1(\overset{1}{\omega}_{3,z_1}-\overset{1}{\omega}_{1,z_3})-a_2(\overset{2}{\omega}_{3,z_1}-\overset{2}{\omega}_{1,z_3})\\
&\quad =k_2-[a_1(\overset{1}{\delta}_{3,z_1}-\overset{1}{\delta}_{1,z_3})-a_2(\overset{2}{\delta}_{3,z_1}-\overset{2}{\delta}_{1,z_3})]\equiv k_2', && z_3=0,\\
&\overset{1}{\omega}_\alpha-\overset{2}{\omega}_\alpha=-(\overset{1}{\delta}_\alpha-\overset{2}{\delta}_\alpha)\equiv l_\alpha,\quad \alpha=1,2, && z_3=0.
\end{aligned}
\tag{11.11}
$$

The problem differs from problem (6.11) only in the boundary conditions. Therefore the proof of Lemma 6.2 can be repeated to get

Lemma 11.1. *Let* $r \in (1,\infty)$. *Assume* $\overset{i}{\bar f} \in L_r(\overset{i}{\mathbb{R}} \times \mathbb{R}_+)$, $\bar k_i \in W_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}^2 \times \mathbb{R}_+)$, $\overset{i}{u}(0) \in W_r^{2-2/r}(\overset{i}{\mathbb{R}})$, $i = 1,2$. *Then there exists a solution to problem* (11.1) *such that* $\overset{i}{u} \in W_r^{2,1}(\overset{i}{\mathbb{R}} \times \mathbb{R}_+)$, $i = 1,2$, *and*

$$\sum_{i=1}^2 \|\overset{i}{u}\|_{W_r^{2,1}(\overset{i}{\mathbb{R}}\times\mathbb{R}_+)} \le c\sum_{i=1}^2 (\|\overset{i}{\bar f}\|_{L_r(\overset{i}{\mathbb{R}}\times\mathbb{R}_+)} + \|\bar k_i\|_{W_r^{1-1/r,1/2-1/(2r)}(\mathbb{R}^2\times\mathbb{R}_+)} + \|\overset{i}{u}(0)\|_{W_r^{2-2/r}(\overset{i}{\mathbb{R}})}). \tag{11.12}$$

12. Existence of solutions to problem (1.1)–(1.8)

To prove the existence of solutions to problem (1.1)–(1.8) we use the method of successive approximations defined by problems (8.5), (8.6). Therefore in Lemma 9.1, v, p, u must be replaced by $\bar v_{n+1}$, $\bar p_{n+1}$, $\bar v_n$, and in Section 10 the quantities $\overset{i}{H}$, $\overset{i}{u}$ are replaced by $\overset{i}{\bar H}_n$, $\overset{i}{\bar v}_n$, $i = 1,2$.

The transmission condition (1.11), where there is no jump for the magnetic field across S_t, simplifies the considerations in this part. Otherwise, the estimate for l_i in (11.2) and the boundary condition $(8.6)_7$ for H imply that v must belong to $W_r^{3,3/2}$ (see Lemma 4.1).

Let

$$\begin{aligned} D &= \|v(0)\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)} + \sum_{i=1}^2 \|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)} \\ &\quad + \|\bar f\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \sum_{\alpha=1}^2 \|H_{*\alpha}\|_{W_r^{2-1/r,1-1/(2r)}(B^t)}. \end{aligned} \tag{12.1}$$

Theorem 12.1. *Let* $r > 3$. *Assume that* $v(0) \in W_r^{2-2/r}(\overset{1}{\Omega}_0)$, $\overset{i}{H}(0) \in W_r^{2-2/r}(\overset{i}{\Omega}_0)$, $i = 1,2$, $\bar f \in L_r(\overset{1}{\Omega}{}_0^t)$. *Then there exists* $a > 0$ *with the following property. If*

$$t_* = t^a D \tag{12.2}$$

is sufficiently small, then there exists a solution to problem (1.1)–(1.8) *with* $\bar v \in W_r^{2,1}(\overset{1}{\Omega}{}_0^T)$, $\overset{i}{\bar H} \in W_r^{2,1}(\overset{i}{\Omega}{}_0^t)$, $i = 1,2$, *and*

$$\begin{aligned} \|\bar v\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \sum_{i=1}^2 \|\overset{i}{\bar H}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)} &\le \varphi\Big(\|v(0)\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}, \sum_{i=1}^2 \|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)}, \\ &\quad \|\bar f\|_{L_r(\overset{1}{\Omega}{}_0^t)}, \sum_{\alpha=1}^2 \|H_{*\alpha}\|_{W_r^{2-1/r,1-1/(2r)}(B^t)}\Big), \end{aligned} \tag{12.3}$$

where φ *is an increasing positive function which is small for small arguments.*

Proof. To prove existence of solutions we use the method of successive approximations described by problems (9.4) and (10.40). But in this case the problems must be coupled. Therefore, we replace u by v_n and $\overset{1}{H}$ by $\overset{1}{H}_n$ in (5.4), and $\overset{i}{u}$ by $\overset{i}{v}_n$, $i=1,2$, in (10.40), where $\overset{1}{v}_n = v_n$ and $\overset{2}{v}_n$ is determined by problem (8.2). Under the above assumptions (9.4) takes the form

$$
\begin{aligned}
&v_{n+1,t} - \operatorname{div}_\xi \mathbb{T}_\xi(v_{n+1},p_{n+1}) = -(\operatorname{div}_\xi \mathbb{T}_\xi(v_n,p_n) - \operatorname{div}_{v_n} \mathbb{T}_{v_n}(v_n,p_n)) \\
&\qquad\qquad + f + \mu_1 \operatorname{div}_{v_n} \mathbb{T}(\overset{1}{H}_n) && \text{in } \overset{1}{\Omega}{}_0^t, \\
&\operatorname{div}_\xi v_{n+1} = \operatorname{div}_\xi v_n - \operatorname{div}_{v_n} v_n && \text{in } \overset{1}{\Omega}{}_0^t, \\
&\bar n_\xi \cdot \mathbb{T}_\xi(v_{n+1},p_{n+1}) = \bar n_\xi \cdot \mathbb{T}_\xi(v_n,p_n) - \bar n_{v_n}\cdot \mathbb{T}_{v_n}(v_n,p_n) \\
&\qquad\qquad - \mu \bar n_{v_n} \mathbb{T}(\overset{1}{H}_n) && \text{on } S_0^t, \\
&v_{n+1}|_{t=0} = v(0) && \text{in } \overset{1}{\Omega}_0,
\end{aligned}
\tag{12.4}
$$

where v_0 is an extension of the initial data such that $v_0|_{t=0} = v(0)$ and $p_0 = 0$.

Problem (10.40) has the form

$$
\begin{aligned}
&\mu_1 \overset{1}{H}_{n+1,t} + \frac{1}{\sigma_1}\operatorname{rot}^2 \overset{1}{H}_{n+1} \\
&\quad = \frac{1}{\sigma_1}(\operatorname{rot}^2_\xi \overset{1}{H}_n - \operatorname{rot}^2_{\overset{1}{v}_n} \overset{1}{H}_n) + \mu_1 \operatorname{rot}_{\overset{1}{v}_n}(\overset{1}{v}_n \times \overset{1}{H}_n) + \mu_1 \overset{1}{v}_n \cdot \nabla_{\overset{1}{v}_n} \overset{1}{H}_n && \text{in } \overset{1}{\Omega}{}_0^t, \\
&\operatorname{div}_\xi \overset{1}{H}_{n+1} = \operatorname{div}_\xi \overset{1}{H}_n - \operatorname{div}_{\overset{1}{v}_n} \overset{1}{H}_n && \text{in } \overset{1}{\Omega}{}_0^t, \\
&\mu_2 \overset{2}{H}_{n+1,t} + \frac{1}{\sigma_2}\operatorname{rot}^2 \overset{2}{H}_{n+1} \\
&\quad = \frac{1}{\sigma_2}(\operatorname{rot}^2_\xi \overset{2}{H}_n - \operatorname{rot}^2_{\overset{2}{v}_n} \overset{2}{H}_n) + \mu_2 \overset{2}{v}_n \cdot \nabla_{\overset{2}{v}_n} \overset{2}{H}_n && \text{in } \overset{2}{\Omega}{}_0^t, \\
&\operatorname{div}_\xi \overset{2}{H}_{n+1} = \operatorname{div}_\xi \overset{2}{H}_n - \operatorname{div}_{\overset{2}{v}_n} \overset{2}{H}_n && \text{in } \overset{2}{\Omega}{}_0^t, \\
&\left(\frac{1}{\sigma_1}\operatorname{rot}_\xi \overset{1}{H}_{n+1} - \frac{1}{\sigma_2}\operatorname{rot}_\xi \overset{2}{H}_{n+1}\right)\cdot \bar\tau_\alpha \\
&\quad = \left[\frac{1}{\sigma_1}(\operatorname{rot}_\xi \overset{1}{H}_n - \operatorname{rot}_{\overset{1}{v}_n} \overset{1}{H}_n) - \frac{1}{\sigma_2}(\operatorname{rot}_\xi \overset{2}{H}_n - \operatorname{rot}_{\overset{2}{v}_n} \overset{2}{H}_n)\right]\cdot \bar\tau_\alpha \\
&\qquad + \left(\frac{1}{\sigma_1}\operatorname{rot}_{\overset{1}{v}_n} H_n^1 - \frac{1}{\sigma_2}\operatorname{rot}_{\overset{2}{v}_n} \overset{2}{H}_n\right)\cdot(\tau_\alpha - \bar\tau_{v_n\alpha}) \\
&\qquad + \mu_1 \overset{1}{v}_n \times \overset{1}{H}_n \cdot \bar\tau_{v_n\alpha}, \quad \alpha = 1,2, && \text{on } S_0^t, \\
&\overset{1}{H}_{n+1} = \overset{2}{H}_{n+1} && \text{on } S_0^t, \\
&\overset{i}{H}_{n+1}|_{t=0} = \overset{i}{H}(0), \quad \operatorname{div} \overset{i}{H}(0) = 0, \quad i=1,2, && \text{in } \overset{1}{\Omega}_0 \cup \overset{2}{\Omega}_0, \\
&\overset{2}{H}_{n+1}\cdot \bar\tau'_\alpha = H_{*\alpha}, \quad \alpha=1,2, \quad \operatorname{div} \overset{2}{H}_{n+1} = 0 && \text{on } B^t,
\end{aligned}
\tag{12.5}
$$

where the zero approximation $\overset{i}{H}_0$, $i=1,2$, is equal to the divergence free extension of the initial data $\overset{i}{H}(0)$, $i=1,2$.

To obtain estimates for solutions to problems (12.4) and (12.5) we recall the notation

$$\nabla_u = \frac{\partial \xi_j}{\partial x}\partial_{\xi_j}$$

where $x=\xi+\int_0^t u(\xi,t')\,dt'$ and $x_{i,\xi_j}=\delta_{ij}+\int_0^t u_{i,\xi_j}(\xi,t')\,dt'$ and ξ_{j,x_i} is the inverse matrix to x_{i,ξ_j}, $i,j=1,2,3$. Let

$$\alpha_u(t)=t^{1/r'}\|u\|_{L_r(0,t;W_r^2(\Omega))},\quad 1/r+1/r'=1.$$

Recall that $r>3$. In view of [S4–S6] there exists a unique solution to problem (12.4) and, for some $a>0$,

$$\begin{aligned}
&\|v_{n+1}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}+\|p_{n+1}\|_{W_r^{1,0}(\overset{1}{\Omega}{}_0^t)}+\|p_{n+1}\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)}\\
&\quad\le\varphi(\alpha_{v_n}(t))\alpha_{v_n}(t)[\|v_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}+\|p_n\|_{W_r^{1,0}(\overset{1}{\Omega}{}_0^t)}+\|p_n\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)}]\\
&\qquad+\varphi(\alpha_{v_n}(t))t^a\|\overset{1}{H}_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}+c(\|f\|_{L_r(\overset{1}{\Omega}{}_0^t)}+\|v(0)\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}).
\end{aligned}\tag{12.6}$$

By the technique of regularizer (see Section 10) there exists a unique solution to problem (10.5) and we have the estimate

$$\begin{aligned}
\sum_{i=1}^2\|\overset{i}{H}_{n+1}\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)}&\le\sum_{i=1}^2[\varphi(\alpha_{\overset{i}{v}_n}(t))\cdot\alpha_{\overset{i}{v}_n}(t)\|\overset{i}{H}_n\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)}\\
&\quad+\varphi(\alpha_{\overset{i}{v}_n}(t))t^a\|\overset{i}{v}_n\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)}\|\overset{i}{H}_n\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)}+\|\overset{i}{H}(0)\|_{W_r^{2-2/r}(\overset{i}{\Omega}_0)}].
\end{aligned}\tag{12.7}$$

We show estimates of the last terms on the r.h.s. of $(12.5)_{1,3}$. We examine

$$\begin{aligned}
\|\overset{i}{v}_n\nabla\overset{i}{H}_n\|_{L_r(\overset{i}{\Omega}{}_0^t)}&\le\sup_t\|\overset{i}{v}_n\|_{L_r(\overset{i}{\Omega}_0)}\|\nabla\overset{i}{H}_n\|_{L_r(0,t;L_\infty(\overset{i}{\Omega}_0))}\\
&\le\Big\|\int_0^t\overset{i}{v}_{n,t}\,dt'\Big\|_{L_r(\overset{i}{\Omega}_0)}\|\overset{i}{H}_n\|_{L_r(0,t;W_r^2(\overset{i}{\Omega}))}\\
&\le ct^{1/r'}\|\overset{i}{v}_n\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)}\|\overset{i}{H}_n\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)},\quad\frac1r+\frac1{r'}=1.
\end{aligned}$$

Introduce the quantity

$$\begin{aligned}
X_n(t)&=\|v_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}+\|p_n\|_{W_r^{1,0}(\overset{1}{\Omega}{}_0^t)}\\
&\quad+\|p_n\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)}+\sum_{i=1}^2\|\overset{i}{H}_n\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)}.
\end{aligned}\tag{12.8}$$

Recall that $v_n=\overset{1}{v}_n\in W_r^{2,1}(\overset{1}{\Omega}{}_0^t)$, $v_n|_{t=0}=v(0)\in W_r^{2-2/r}(\overset{1}{\Omega}_0)$.

Then Lemma 2.4 implies existence of solutions to problem (3.2) such that $\overset{2}{v}_n \in W_r^{2,1}(\overset{2}{\Omega}{}_0^t)$ and

$$\|\overset{2}{v}_n\|_{W_r^{2,1}(\overset{2}{\Omega}{}_0^t)} \le c(\|\overset{1}{v}_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|v(0)\|_{W_r^{2-2/r}(\overset{1}{\Omega}_0)}). \tag{12.9}$$

In view of notation (12.8) and estimate (12.9) we obtain from (12.6) and (12.7) the inequality

$$X_{n+1}(t) \le \varphi(t^a X_n(t), D)t^a X_n(t) + cD. \tag{12.10}$$

Then for t sufficiently small there exists a quantity $\varphi(D)$ such that

$$X_n(t) \le \varphi(D) \equiv M \quad \text{for any } n \in \mathbb{N}. \tag{12.11}$$

To apply the method of successive approximations we need to show convergence of the constructed sequence. For this purpose we set

$$V_{n+1} = \bar{v}_{n+1} - \bar{v}_n, \quad P_{n+1} = \bar{p}_{n+1} - \bar{p}_n, \quad \overset{i}{h}_{n+1} = \overset{i}{\bar{H}}_{n+1} - \overset{i}{\bar{H}}_n, \tag{12.12}$$

$i = 1, 2$. To formulate the problem for differences we use problems (12.4) and (12.5). Hence (12.4) implies the following problem for V_{n+1} and P_{n+1}:

$$\begin{aligned}
&V_{n+1,t} - \operatorname{div}_\xi \mathbb{T}_\xi(V_{n+1}, P_{n+1}) = -(\operatorname{div}_\xi \mathbb{T}_\xi(V_n, P_n) \\
&\quad - \operatorname{div}_{\overset{1}{\bar{v}}_n} \mathbb{T}_{\overset{1}{\bar{v}}_n}(V_n, P_n)) - (\operatorname{div}_{\overset{1}{\bar{v}}_n} \mathbb{T}_{\overset{1}{\bar{v}}_n}(\overset{1}{\bar{v}}_{n-1}, \bar{p}_{n-1}) - \operatorname{div}_{\overset{1}{\bar{v}}_{n-1}} \mathbb{T}_{\overset{1}{\bar{v}}_{n-1}}(\overset{1}{\bar{v}}_{n-1}, \bar{p}_{n-1})) \\
&\quad + \mu_1(\operatorname{div}_{\overset{1}{\bar{v}}_n} \mathbb{T}(\overset{1}{\bar{H}}_n) - \operatorname{div}_{\overset{1}{\bar{v}}_{n-1}} \mathbb{T}(\overset{1}{\bar{H}}_{n-1})) \equiv I_1 + I_2 + I_3, \\
&\operatorname{div}_\xi V_{n+1} = (\operatorname{div}_\xi V_n - \operatorname{div}_{\overset{1}{\bar{v}}_n} V_n) + (\operatorname{div}_{\overset{1}{\bar{v}}_n} \overset{1}{\bar{v}}_{n-1} - \operatorname{div}_{\overset{1}{\bar{v}}_{n-1}} \overset{1}{\bar{v}}_{n-1}) \equiv I_4 + I_5, \\
&\bar{n}_\xi \cdot \mathbb{T}_\xi(V_{n+1}, P_{n+1}) = (\bar{n}_\xi \cdot \mathbb{T}_\xi(V_n, P_n) - \bar{v}_{\overset{1}{\bar{v}}_n} \cdot \mathbb{T}_{\overset{1}{\bar{v}}_n}(V_n, P_n)) \\
&\quad + (\bar{n}_{\overset{1}{\bar{v}}_n} \mathbb{T}_{\overset{1}{\bar{v}}_n}(\overset{1}{\bar{v}}_{n-1}, \bar{p}_{n-1}) - \bar{n}_{\overset{1}{\bar{v}}_{n-1}} \mathbb{T}_{\overset{1}{\bar{v}}_{n-1}}(\overset{1}{\bar{v}}_{n-1}, \bar{p}_{n-1})) \equiv I_6 + I_7, \\
&V_{n+1}|_{t=0} = 0.
\end{aligned} \tag{12.13}$$

By the technique of regularizer (see Section 10) we have existence of solutions V_{n+1}, P_{n+1} to problem (12.13) and the estimate

$$\begin{aligned}
&\|V_{n+1}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|P_{n+1}\|_{L_r(0,t;W_r^1(\overset{1}{\Omega}_0))} + \|P_{n+1}\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)} \\
&\quad \le c\Big(\sum_{i=1}^3 \|I_i\|_{L_r(\overset{1}{\Omega}{}_0^t)} + \sum_{i=4}^5 \|I_i\|_{L_r(0,t;W_r^1(\overset{1}{\Omega}_0))} + \sum_{i=6}^7 \|I_i\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)}\Big).
\end{aligned} \tag{12.14}$$

Using the above mentioned properties of the transformation between Lagrangian and

Eulerian coordinates we have

$$
\begin{aligned}
&\|I_1\|_{L_r(\overset{1}{\Omega}{}_0^t)} \le \varphi(\alpha_{v_n}(t))\alpha_{v_n}(t)(\|V_n\|_{L_r(0,t;W_r^2(\overset{1}{\Omega}_0))} + \|P_n\|_{L_r(0,t;W_r^1(\overset{1}{\Omega}_0))}),\\
&\|I_2\|_{L_r(\overset{1}{\Omega}{}_0^t)} \le \varphi(\alpha_{v_n}(t),\alpha_{v_{n-1}}(t))t^a\|V_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)},\\
&\|I_3\|_{L_r(\overset{1}{\Omega}{}_0^t)} \le \varphi(\alpha_{v_n}(t),\alpha_{v_{n-1}}(t))[t^a\|V_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + t^a\|\overset{1}{\bar H}_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}\|\overset{1}{h}_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}],\\
&\|I_6\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)} \le \varphi(\alpha_{v_n}(t))\alpha_{v_n}(t)(\|V_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|P_n\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)}),\\
&\|I_7\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)} \le \varphi(\alpha_{v_n}(t),\alpha_{v_{n-1}}(t))(\alpha_{v_n}(t)+\alpha_{v_{n-1}}(t))\\
&\qquad\qquad \cdot\|V_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)}(\|v_{n-1}\|_{W_r^{2,1}(\Omega^t)} + \|p_{n-1}\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)}).
\end{aligned}\tag{12.15}
$$

Bounds of $\|I_4\|$ and $\|I_5\|$ are similar to the bounds of $\|I_1\|_{L_r}$ and $\|I_2\|_{L_r}$. Plugging (12.15) in (12.14) yields

$$
\begin{aligned}
&\|V_{n+1}\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|P_{n+1}\|_{L_r(0,t;W_r^1(\overset{1}{\Omega}_0))} + \|P_{n+1}\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)}\\
&\qquad\qquad \le \varphi(M)t^aY_n(t),
\end{aligned}\tag{12.16}
$$

where

$$
\begin{aligned}
Y_n(t) &= \|V_n\|_{W_r^{2,1}(\overset{1}{\Omega}{}_0^t)} + \|P_{n+1}\|_{L_r(0,t;W_r^1(\overset{1}{\Omega}_0))}\\
&\quad + \|P_n\|_{W_r^{1-1/r,1/2-1/(2r)}(S_0^t)} + \sum_{i=1}^2 \|\overset{i}{h}_n\|_{W_r^{2,1}(\overset{i}{\Omega}{}_0^t)},
\end{aligned}\tag{12.17}
$$

where

$$
\overset{i}{h}_n = \overset{i}{\bar H}_n - \overset{i}{\bar H}_{n-1}, \quad i=1,2. \tag{12.18}
$$

From (12.5) we have the following problem for $\overset{1}{h}_{n+1}$ and $\overset{2}{h}_{n+1}$:

$$
\begin{aligned}
&\mu_1\overset{1}{h}_{n+1,t} + \frac{1}{\sigma_1}\operatorname{rot}_\xi^2\overset{1}{h}_{n+1} = \frac{1}{\sigma_1}(\operatorname{rot}_\xi^2\overset{1}{h}_n - \operatorname{rot}^2_{\overset{1}{v}_n}\overset{1}{h}_n)\\
&\quad - \frac{1}{\sigma_1}(\operatorname{rot}^2_{\overset{1}{v}_n}\overset{1}{\bar H}_{n-1} - \operatorname{rot}_{\overset{1}{v}_{n-1}}\overset{1}{\bar H}_{n-1}) + \mu_1(\operatorname{rot}_{\overset{1}{v}_n}(\overset{1}{v}_n\times\overset{1}{\bar H}_n)\\
&\quad - \operatorname{rot}_{\overset{1}{v}_{n-1}}(\overset{1}{v}_{n-1}\times\overset{1}{\bar H}_{n-1})) + \mu_1(\overset{1}{v}_n\cdot\nabla_{\overset{1}{v}_n}\overset{1}{\bar H}_n - \overset{1}{v}_{n-1}\cdot\nabla_{\overset{1}{v}_{n-1}}\overset{1}{\bar H}_{n-1}) \qquad \text{in } \overset{1}{\Omega}{}_0^t,\\
&\operatorname{div}_\xi\overset{1}{h}_{n+1} = \operatorname{div}_\xi\overset{1}{h}_n - \operatorname{div}_{\overset{1}{v}_n}\overset{1}{h}_n + \operatorname{div}_{\overset{1}{v}_n}\overset{1}{\bar H}_{n-1} - \operatorname{div}_{\overset{1}{v}_{n-1}}\overset{1}{\bar H}_{n-1},\\
&\mu_2\overset{2}{h}_{n+1,t} + \frac{1}{\sigma_2}\operatorname{rot}_\xi^2\overset{2}{h}_{n+1} = \frac{1}{\sigma_2}(\operatorname{rot}_\xi^2\overset{2}{h}_n - \operatorname{rot}^2_{\overset{2}{v}_n}\overset{2}{h}_n)\\
&\quad - \frac{1}{\sigma_2}(\operatorname{rot}^2_{\overset{2}{v}_n}\overset{2}{\bar H}_{n-1} - \operatorname{rot}^2_{\overset{2}{v}_{n-1}}\overset{2}{\bar H}_{n-1})\\
&\quad + \mu_2(\overset{2}{v}_n\cdot\nabla_{\overset{2}{v}_n}\overset{2}{\bar H}_n - \overset{2}{v}_{n-1}\cdot\nabla_{\overset{2}{v}_{n-1}}\overset{2}{\bar H}_{n-1}) \qquad \text{in } \overset{2}{\Omega}{}_0^t,
\end{aligned}\tag{12.19}
$$

$$
\begin{aligned}
&\operatorname{div}_\xi \overset{2}{h}_{n+1} = \operatorname{div}_\xi \overset{2}{h}_n - \operatorname{div}_{\overset{2}{v}_n} \overset{2}{h}_n + \operatorname{div}_{\overset{2}{v}_n} \overset{2}{\bar H}_{n-1} - \operatorname{div}_{\overset{2}{v}_{n-1}} \overset{2}{\bar H}_{n-1} \qquad \text{in } \overset{2}{\Omega}{}^t_0,\\
&\left(\frac{1}{\sigma_1}\operatorname{rot}_\xi \overset{1}{h}_n - \frac{1}{\sigma_2}\operatorname{rot}_\xi \overset{2}{h}_n\right)\cdot \bar\tau_\alpha\\
&\quad = \bigg[\frac{1}{\sigma_1}(\operatorname{rot}_\xi \overset{1}{h}_n - \operatorname{rot}_{\overset{1}{v}_n} \overset{1}{h}_n) - \frac{1}{\sigma_1}(\operatorname{rot}_{\overset{1}{v}_n} \overset{1}{\bar H}_{n-1} - \operatorname{rot}_{\overset{1}{v}_{n-1}} \overset{1}{\bar H}_{n-1})\\
&\quad - \frac{1}{\sigma_2}(\operatorname{rot}_\xi \overset{2}{h}_n - \operatorname{rot}_{\overset{2}{v}_n} \overset{2}{h}_n) + \frac{1}{\sigma_2}(\operatorname{rot}_{\overset{2}{v}_n} \overset{2}{\bar H}_{n-1} - \operatorname{rot}_{\overset{2}{v}_{n-1}} \overset{2}{\bar H}_{n-1})\bigg]\cdot\bar\tau_\alpha\\
&\quad + \bigg[\frac{1}{\sigma_1}(\operatorname{rot}_\xi \overset{1}{h}_n - \operatorname{rot}_{\overset{1}{v}_n} \overset{1}{h}_n + \operatorname{rot}_{\overset{1}{v}_n} \overset{1}{\bar H}_{n-1} - \operatorname{rot}_{\overset{1}{v}_{n-1}} \overset{1}{\bar H}_{n-1})\\
&\quad - \frac{1}{\sigma_2}(\operatorname{rot}_\xi \overset{2}{h}_n - \operatorname{rot}_{\overset{2}{v}_n} \overset{2}{h}_n + \operatorname{rot}_{\overset{2}{v}_n} \overset{2}{\bar H}_{n-1} - \operatorname{rot}_{\overset{2}{v}_{n-1}} \overset{2}{\bar H}_{n-1})\bigg](\bar\tau_\alpha - \bar\tau_{v_n\alpha}) \qquad (12.19)\ \text{[cont.]}\\
&\quad + \bigg[\frac{1}{\sigma_1}\operatorname{rot}_{\overset{1}{v}_{n-1}} \overset{1}{\bar H}_{n-1} - \frac{1}{\sigma_2}\operatorname{rot}_{\overset{2}{v}_{n-1}} \overset{2}{\bar H}_{n-1}\bigg](\bar\tau_{v_n\alpha} - \bar\tau_{v_{n-1}\alpha})\\
&\quad + \mu_1(\overset{1}{v}_n \times \overset{1}{\bar H}_n \cdot \bar\tau_{\overset{1}{v}_{n\alpha}} - \overset{1}{v}_{n-1} \times \overset{1}{\bar H}_{n-1} \cdot \bar\tau_{\overset{1}{v}_{n-1}\alpha}), \quad \alpha = 1,2, \quad \text{on } S^t_0,\\
&\overset{1}{h}_{n+1} - \overset{2}{h}_{n+1} = 0 \qquad \text{on } S^t_0,\\
&\overset{i}{h}_{n+1}|_{t=0} = 0,\\
&\overset{2}{h}_{n+1}\cdot\bar\tau'_\alpha|_{B^t} = 0, \quad \operatorname{div}\overset{2}{h}_{n+1}|_{B^t} = 0, \quad \alpha = 1,2, \qquad \text{on } B^t,
\end{aligned}
$$

where $\overset{1}{v}_n = v_n$ and $\overset{2}{v}_n$ is described by problem (8.2). We emphasize that $\overset{i}{v}_n$, $\overset{i}{v}_{n-1}$ in (12.19) should be replaced by $\overset{i}{\bar v}_n$, $\overset{i}{\bar v}_{n-1}$, $i = 1,2$.

By the technique of regularizer (see Section 10) we get existence of solutions to problem (12.19) and the estimate

$$
\sum_{i=1}^{2} \|\overset{i}{h}_{n+1}\|_{W^{2,1}_r(\overset{i}{\Omega}{}^t_0)} \le \varphi(M) t^a Y_n(t) \tag{12.20}
$$

for some $a > 0$. From (12.16) and (12.20),

$$
Y_{n+1}(t) \le \varphi(M) t^a Y_n(t). \tag{12.21}
$$

Hence (12.11) and (12.21) imply existence of solutions to problem (1.1)–(1.8) for t sufficiently small. This concludes the proof. ■

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