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# New Thought on Matsumura–Nishida Theory in the $L_p$–$L_q$ Maximal Regularity Framework

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**Abstract.** This paper is devoted to proving the global well-posedness of initial-boundary value problem for Navier–Stokes equations describing the motion of viscous, compressible, barotropic fluid flows in a three dimensional exterior domain with non-slip boundary conditions. This was first proved by an excellent paper due to Matsumura and Nishida (Commun Math Phys 89:445–464, 1983). In [10], they used energy method and their requirement was that space derivatives of the mass density up to third order and space derivatives of the velocity fields up to fourth order belong to $L_2$ in space-time, detailed statement of Matsumura and Nishida theorem is given in Theorem 1 of Sect. 1 of context. This requirement is essentially used to estimate the $L_\infty$ norm of necessary order of derivatives in order to enclose the iteration scheme with the help of Sobolev inequalities and also to treat the material derivatives of the mass density. On the other hand, this paper gives the global wellposedness of the same problem as in [10] in $L_p$ $(1 < p \leq 2)$ in time and $L_2 \cap L_6$ in space maximal regularity class, which is an improvement of the Matsumura and Nishida theory in [10] from the point of view of the minimal requirement of the regularity of solutions. In fact, after changing the material derivatives to time derivatives by Lagrange transformation, enough estimates obtained by combination of the maximal $L_p$ $(1 < p \leq 2)$ in time and $L_2 \cap L_6$ in space regularity and $L_p$–$L_q$ decay estimate of the Stokes equations with non-slip conditions in the compressible viscous fluid flow case enable us to use the standard Banach's fixed point argument. Moreover, one of the purposes of this paper is to present a framework to prove the $L_p$–$L_q$ maximal regularity for parabolic-hyperbolic type equations with non-homogeneous boundary conditions and how to combine the maximal $L_p$–$L_q$ regularity and $L_p$–$L_q$ decay estimates of linearized equations to prove the global well-posedness of quasilinear problems in unbounded domains, which gives a new thought of proving the global well-posedness of initial-boundary value problems for systems of parabolic or parabolic-hyperbolic equations appearing in mathematical physics.

**Mathematics Subject Classification.** 35Q30, 76N10.

**Keywords.** Navier–Stokes equations, Compressible viscous barotropic fluid, Global well-posedness, The maximal $L_p$ space.

## 1. Introduction

Matsumura and Nishida [10] proved the existence of unique solutions of equations governing the flow of viscous, compressible, and heat conduction fluids in an exterior domain of 3 dimensional Euclidean space $\mathbb{R}^3$ for all times, provided the initial data are sufficiently small. Although Matsumura and Nishida [10] considered the viscous, barotropic, and heat conductive fluid, in this paper we only consider the viscous, compressible, barotropic fluid for simplicity and reprove the Matsumura and Nishida theory in view of the $L_p$ in time $(1 < p \leq 2)$ and $L_2 \cap L_6$ in space maximal regularity theorem.

To describe in more detail, we start with description of equations considered in this paper. Let $\Omega$ be a three dimensional exterior domain, that is the complement, $\Omega^c$, of $\Omega$ is a bounded domain in the three dimensional Euclidean space $\mathbb{R}^3$. Let $\Gamma$ be the boundary of $\Omega$, which is a compact $C^2$ hypersurface. Let $\rho = \rho(x,t)$ and $\mathbf{v} = (v_1(x,t), v_2(x,t), v_3(x,t))^\top$ be respective the mass density and the velocity field, where $M^\top$ denotes the transposed $M$, $t$ is a time variable and $x = (x_1, x_2, x_3) \in \Omega$. Let $\mathfrak{p} = \mathfrak{p}(\rho)$ be the

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fluid pressure, which is a smooth function defined on $(0,\infty)$ such that $\mathfrak{p}'(\rho)>0$ for $\rho>0$. We consider the following equations:

$$\begin{aligned} \partial_t\rho+\mathrm{div}\,(\rho\mathbf{v})=0 &\qquad \text{in } \Omega\times(0,T),\\ \rho(\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v})-\mathrm{Div}\,(\mu\mathbf{D}(\mathbf{v})+\nu\mathrm{div}\,\mathbf{v}\mathbf{I}-\mathfrak{p}(\rho)\mathbf{I})=0 &\qquad \text{in } \Omega\times(0,T),\\ \mathbf{v}|_\Gamma=0,\quad (\rho,\mathbf{v})|_{t=0}=(\rho_*+\theta_0,\mathbf{v}_0) &\qquad \text{in } \Omega. \end{aligned} \tag{1}$$

Here, $\partial_t=\partial/\partial t$, $\mathbf{D}(\mathbf{v})=\nabla\mathbf{v}+(\nabla\mathbf{v})^\top$ is the deformation tensor, $\mathrm{div}\,\mathbf{v}=\sum_{j=1}^3\partial v_j/\partial x_j$, for a $3\times3$ matrix $K$ with $(i,j)$ th component $K_{ij}$, $\mathrm{Div}\,K=(\sum_{j=1}^3\partial K_{1j}/\partial x_j,\sum_{j=1}^3\partial K_{2j}/\partial x_j,\sum_{j=1}^3\partial K_{3j}/\partial x_j)^\top$, $\mu$ and $\nu$ are two viscous constants such that $\mu>0$ and $\mu+\nu>0$, and $\rho_*$ is a positive constant describing the mass density of a reference body.

According to Matsumura and Nishida [10], we have the global well-posedness of Eq. (1) in the $L_2$ framework stated as follows:

**Theorem 1** ([10]). *Let $\Omega$ be a three dimensional exterior domain, the boundary of which is a smooth 2 dimensional compact hypersurface. Then, there exsits a small number $\epsilon>0$ such that for any initial data $(\theta_0,\boldsymbol{v}_0)\in H^3(\Omega)^4$ satisfying smallness condition: $\|(\theta_0,\boldsymbol{v}_0)\|_{H^3(\Omega)}\le\epsilon$ and compatibility conditions of order 1, that is $\boldsymbol{v}_0$ and $\partial_t\boldsymbol{v}|_{t=0}$ vanish at $\Gamma$, Problem (1) admits unique solutions $\rho=\rho_*+\theta$ and $\boldsymbol{v}$ with*

$$\begin{aligned} &\theta\in C^0((0,\infty),H^3(\Omega))\cap C^1((0,\infty),H^2(\Omega)),\quad \nabla\rho\in L_2((0,\infty),H^2(\Omega)^3),\\ &\boldsymbol{v}\in C^0((0,\infty),H^3(\Omega)^3)\cap C^1((0,\infty),H^1(\Omega)^3),\quad \nabla\boldsymbol{v}\in L_2((0,\infty),,H^3(\Omega)^9). \end{aligned}$$

Matsumura and Nishida [10] proved Theorem 1 essentially by energy method. One of key issues in [10] is to estimate $\sup_{t\in(0,\infty)}\|\mathbf{v}(\cdot,t)\|_{H^1_\infty(\Omega)}$ by Sobolev's inequality, namely

$$\sup_{t\in((0,\infty)}\|\mathbf{v}(\cdot,t)\|_{H^1_\infty(\Omega)}\le C\sup_{t\in(0,\infty)}\|\mathbf{v}(\cdot,t))\|_{H^3(\Omega)}. \tag{2}$$

Recently, Enomoto and Shibata [8] proved the global wellposedness of Eq. (1) for $(\theta_0,\mathbf{v}_0)\in H^2(\Omega)^4$ with small norms. Namely, they proved the following theorem.

**Theorem 2** ([8]). *Let $\Omega$ be a three dimensional exterior domain, the boundary of which is a smooth 2 dimensional compact hypersurface. Then, there exsits a small number $\epsilon>0$ such that for any initial data $(\theta_0,\boldsymbol{v}_0)\in H^2(\Omega)^4$ satisfying $\|(\theta_0,\boldsymbol{v}_0)\|_{H^2(\Omega)}\le\epsilon$ and compatibility condition: $\boldsymbol{v}_0|_\Gamma=0$, problem (1) admits unique solutions $\rho=\rho_*+\theta$ and $\boldsymbol{v}$ with*

$$\begin{aligned} &\theta\in C^0((0,\infty),H^2(\Omega))\cap C^1((0,\infty),H^1(\Omega)),\quad \nabla\rho\in L_2((0,\infty),H^1(\Omega)^3),\\ &\boldsymbol{v}\in C^0((0,\infty),H^2(\Omega)^3)\cap C^1((0,\infty),L_2(\Omega)^3),\quad \nabla\boldsymbol{v}\in L_2((0,\infty),H^2(\Omega)^9). \end{aligned}$$

The method used in the proof of Enomoto and Shibata [8] is essentially the same as that in Matsumura and Nishida [10]. Only the difference is that (2) is replaced by $\int_0^\infty\|\nabla\mathbf{v}\|^2_{L_\infty(\Omega)}\,dt\le C\int_0^\infty\|\nabla\mathbf{v}\|^2_{H^2(\Omega)}\,dt$ in [8]. As a conclusion, in the $L_2$ framework the least regularity we need is that $\nabla\rho\in L_2((0,\infty),H^1(\Omega)^3)$ and $\nabla\mathbf{v}\in L_2((0,\infty),H^2(\Omega)^9)$. In this paper, we improve this point by solving the Eq. (1) in the $L_p$-$L_q$ maximal regularity class, that is the following theorem is a main result of this paper.

**Theorem 3.** *Let $\Omega$ be an exterior domain in $\mathbb{R}^3$, whose boundary $\Gamma$ is a compact $C^2$ hypersurface and $T\in(0,\infty)$. Let $p$ be an exponent with $1<p\le2$ and set $p'=p/(p-1)$. Let $\sigma\in(0,1)$ and set $\ell=(5+\sigma)/(4+2\sigma)$ and $r=2(2+\sigma)/(4+\sigma)=(1/2+1/(2+\sigma))^{-1}$. Let $b$ be a positive constant satisfying the condition*

$$\frac{1}{p'}<b<\ell-\frac{1}{p}. \tag{3}$$

*Set*

$$\mathcal{I} = \left\{(\theta_0, \boldsymbol{v}_0) \mid \theta_0 \in \left(\bigcap_{q=2,6} H^1_q(\Omega)\right) \cap L_r(\Omega), \quad \boldsymbol{v}_0 \in \left(\bigcap_{q=2,6} B^{2(1-1/p)}_{q,p}(\Omega)^3\right) \cap L_r(\Omega)^3\right\},$$
$$\|(\theta_0, \boldsymbol{v}_0)\|_{\mathcal{I}} = \sum_{q=2,6} \|\theta_0\|_{H^1_q(\Omega)} + \sum_{q=2,6} \|\boldsymbol{v}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|(\theta_0, \boldsymbol{v}_0)\|_{L_r(\Omega)}.$$

*Then, there exists a small constant* $\epsilon \in (0,1)$ *independent of* $T$ *such that if initial data* $(\theta_0, \boldsymbol{v}_0) \in \mathcal{I}$ *satisfy the compatibility condition:* $\boldsymbol{v}_0|_\Gamma = 0$ *and the smallness condition :* $\|(\theta_0, \boldsymbol{v}_0)\|_{\mathcal{I}} \leq \epsilon^2$, *then problem* (1) *admits unique solutions* $\rho = \rho_* + \theta$ *and* $\boldsymbol{v}$ *with*

$$\begin{aligned} &\theta \in H^1_p((0,T), L_2(\Omega) \cap L_6(\Omega)) \cap L_p((0,T), H^1_2(\Omega) \cap H^1_6(\Omega)), \\ &\boldsymbol{v} \in H^1_p((0,T), L_2(\Omega)^3 \cap L_6(\Omega)^3) \cap L_p((0,T), H^2_2(\Omega)^3 \cap H^2_6(\Omega)^3). \end{aligned} \tag{4}$$

*Moreover, writing* $\|(\theta, \boldsymbol{v})\|_{H^{\ell,m}_q(\Omega)} = \|\theta\|_{H^\ell_q(\Omega)} + \|\boldsymbol{v}\|_{H^m_q(\Omega)}$ *and setting*

$$\begin{aligned} \mathcal{E}_T(\theta, \boldsymbol{v}) = &\| < t >^b (\theta, \boldsymbol{v})\|_{L_\infty((0,T), L_2(\Omega)\cap L_6(\Omega))} + \| < t >^b \nabla(\theta, \boldsymbol{v})\|_{L_p((0,T), H^{0,1}_2(\Omega))} \\ &+ \| < t >^b (\theta, \boldsymbol{v})\|_{L_p((0,T), H^{1,2}_6(\Omega))} + \| < t >^b \partial_t(\theta, \boldsymbol{v})\|_{L_p((0,T), L_2(\Omega)\cap L_6(\Omega))}, \end{aligned}$$

*we have* $\mathcal{E}_T(\theta, \boldsymbol{v}) \leq \epsilon$.

*Remark 4.* (1) $T > 0$ is taken arbitrarily and $\epsilon > 0$ is chosen independently of $T$, and so Theorem 3 tells us the global wellposedness of Eq. (1) for $(0, \infty)$ time inverval.

(2) In the $p = 2$ case, Theorem 3 gives an extension of Matsumura and Nishida theorem [10]. Roughly speaking, if we assume that $(\theta_0, \mathbf{v}_0) \in H^3_2(\Omega)^4$, then $(\theta_0, \mathbf{v}_0) \in (H^1_2(\Omega) \cap H^1_6(\Omega)) \times (H^1_2(\Omega) \cap B^1_{6,2}(\Omega))$, and so the global wellposedness holds in the class as

$$\theta \in H^1_2((0,T), H^1_2(\Omega) \cap H^1_6(\Omega)), \quad \mathbf{v} \in H^1_2((0,T), L_2(\Omega)^3 \cap L_6(\Omega)^3) \cap L_2((0,T), H^2_2(\Omega)^3 \cap H^2_6(\Omega)^3)$$

under the additional condition: $(\theta_0, \mathbf{v}_0) \in L_r(\Omega)^4$.

(3) Since we assume that $1 < p \leq 2$, it automatically follows that

$$b < \ell - \frac{1}{2} = \frac{3}{2(2+\sigma)}. \tag{5}$$

(4) Following the argument in [12, Theorem 3.8.1], we can also consider the case where $2 < p < \infty$.

As related topics, we consider the Cauchy problem, that is $\Omega = \mathbb{R}^3$ without boundary condition. Matsumura and Nishida [9] proved the global wellposedness theorem, the statement of which is essentially the same as in Theorem 1 and the proof is based on energy method. Danchin [4] proved the global wellposedness in the critical space by using the Littlewood–Paley decomposition.

**Theorem 5** ([4]). *Let* $\Omega = \mathbb{R}^N$ $(N \geq 2)$. *Assume that* $\mu > 0$ *and* $\mu + \nu > 0$. *Let* $B^s = \dot{B}^s_{2,1}(\mathbb{R}^N)$ *and*

$$F^s = (L_2((0,\infty), B^s) \cap C((0,\infty), B^s \cap B^{s-1})) \times (L_1((0,\infty), B^{s+1}) \cap C((0,\infty), B^{s-1}))^N.$$

*Then, there exists an* $\epsilon > 0$ *such that if initial data* $\theta_0 \in B^{N/2}(\mathbb{R}^N) \cap B^{N/2-1}(\mathbb{R}^N)$ *and* $\boldsymbol{v}_0 \in B^{N/2-1}(\mathbb{R}^N)^N$ *satisfy the condition:*

$$\|\theta_0\|_{B^{N/2}(\mathbb{R}^N)\cap B^{N/2-1}(\mathbb{R}^N)} + \|\boldsymbol{v}_0\|_{B^{N/2-1}(\mathbb{R}^N)} \leq \epsilon,$$

*then problem* (1) *with* $\Omega = \mathbb{R}^N$ *and* $T = \infty$ *admits a unique solution* $\rho = \rho_* + \theta$ *and* $\boldsymbol{v}$ *with* $(\theta, \boldsymbol{v}) \in F^{N/2}$.

In the case where $\Omega = \mathbb{R}^3$ or $\mathbb{R}^N$, there are a lot of works concerning (1), but we do not mention them any more, because we are interested only in the global wellposedness in exterior domains. For more information on references, refer to Enomoto and Shibata [7].

Concerning the $L_1$ in time maximal regularity in exterior domains, the incompressible viscous fluid flows has been treated by Danchin and Mucha [5]. To obtain $L_1$ maximal regularity in time, we have to use $\dot{B}^s_{q,1}$ in space, which is slightly regular space than $H^s_q$, and the decay estimates for semigroup on

$\dot{B}^s_{q,1}$ must be needed to controle terms arising from the cut-off procedure near the boundary. Detailed arguments related with these facts can be found in [5]. To treat (1) in an exterior domain in the $L_1$ in time maximal regularity framework, we have to prepare not only $L_1$ maximal regularity for model problems in the whole space and the half space but also decay properties of semigroup in $\dot{B}^s_{q,1}$, and so this will be a future work. From Theorem 3, we may say that problem (1) can be solved in $L_p$ in time and $L_2 \cap L_6$ in space maximal regularity class for any exponet $p \in (1, 2]$.

The paper is organized as follows. In Sect. 2, Eq. (1) are rewriten in Lagrange coordinates to eliminate $\mathbf{v} \cdot \nabla\rho$ and a main result for equations with Lagrangian description is stated. In Sect. 3, we give an $L_p$–$L_q$ maximal regularity theorem in some abstract setting. In Sect. 4, we give estimates of nonlinear terms. In Sect. 5, we prove main results stated in Sect. 2. In Sect. 6, Theorem 3 is proved by using a main result in Sect. 2. In Sect. 7, we discuss the $N$ dimensonal case.

The main point of our proof is to obtain maximal regularity estimates with decay properties of solutions to linearized equations, the Stokes equations with non-slip conditions. To explain the idea, we write linearized equations as $\partial_t u - Au = f$ and $u|_{t=0} = u_0$ symbolically, where $f$ is a function corresponding to nonlinear terms and $A$ is a closed linear operator with domain $D(A)$. We write $u = u_1 + u_2$, where $u_1$ is a solution to time shifted equations: $\partial_t u_1 + \lambda_1 u_1 - Au_1 = f$ with some large positive number $\lambda_1$ and $u_2$ is a solution to compensating equations: $\partial_t u_2 - Au_2 = \lambda_1 u_1$ and $u_2|_{t=0} = u_0 - u_1|_{t=0}$. Since the fundamental solutions to time shifted equations have exponential decay properties, $u_1$ has the same decay properties as these of nonlinear terms $f$. Moreover $u_1$ belongs to the domain of $A$ for all positive time. By Duhamel principle $u_2$ is given by $u_2 = T(t)(u_0 - u_1|_{t=0}) + \lambda_1 \int_0^t T(t-s)u_1(s)\,ds$, where $\{T(t)\}_{t\geq 0}$ is a continuous analytic semigroup associated with $A$. By using $L_p$-$L_q$ decay properties of $\{T(t)\}_{t\geq 0}$ in the interval $0 < s < t-1$ and standard estimates of continuous analytic semigroup: $\|T(t-s)u_0\|_{D(A)} \leq C\|u_0\|_{D(A)}$ for $t-1 < s < t$, where $\|\cdot\|_{D(A)}$ denotes a domain norm, we obtain maximal $L_p$-$L_q$ regularity of $u_2$ with decay properties. This method seems to be a new thought to prove the global wellposedness and to be applicable to many quasilinear problems of parabolic type or parabolic-hyperbolic mixture type appearing in mathematical physics.

To end this section, symbols of functional spaces used in this paper are given. Let $L_p(\Omega)$, $H^m_p(\Omega)$ and $B^s_{q,p}(\Omega)$ denote the standard Lebesgue spaces, Sobolev spaces and Besov spaces, while their norms are written as $\|\cdot\|_{L_p(\Omega)}$, $\|\cdot\|_{H^m_p(\Omega)}$ and $\|\cdot\|_{B^s_{q,p}(\Omega)}$. We write $H^m(\Omega) = H^m_2(\Omega)$, $H^0_q(\Omega) = L_q(\Omega)$ and $W^s_q(\Omega) = B^s_{q,q}(\Omega)$. For any Banach space $X$ with norm $\|\cdot\|_X$, $L_p((a,b), X)$ and $H^m_p((a,b), X)$ denote respective the standard $X$-valued Lebesgue spaces and Sobolev spaces, while their time weighted norms are defined by

$$\| < t >^b f\|_{L_p((a,b),X)} = \begin{cases} \left(\int_a^b (< t >^b \|f(t)\|_X)^p\,dt\right)^{1/p} & (1 \leq p < \infty), \\ \mathrm{esssup}_{t\in(a,b)} < t >^b \|f(t)\|_X & (p = \infty), \end{cases}$$

where $< t >= (1+t^2)^{1/2}$. Let $X^n = \{\mathbf{v} = (u_1, \ldots, u_n)) \mid u_i \in X \ (i = 1, \ldots, n)\}$, but we write $\|\cdot\|_{X^n} = \|\cdot\|_X$ for simplicity. Let $H^{\ell,m}_q(\Omega) = \{(\rho, \mathbf{v}) \mid \rho \in H^\ell_q(\Omega), \mathbf{v} \in H^m_q(\Omega)^3\}$ and $\|(\rho,\mathbf{v})\|_{H^{\ell,m}_q(\Omega)} = \|\rho\|_{H^\ell_q(\Omega)} + \|\mathbf{v}\|_{H^m_q(\Omega)}$. The letter $C$ denotes generic constants and $C_{a,b,\cdots}$ denotes that constants depend on quantities $a$, $b$, .... $C$ and $C_{a,b,\cdots}$ may change from line to line.

## 2. Equations in Lagrange Coordinates and Statment of Main Results

To prove Theorem 3, we write Eq. (1) in Lagrange coordinates $\{y\}$. Let $\zeta = \zeta(y,t)$ and $\mathbf{u} = \mathbf{u}(y,t)$ be the mass density and the velocity field in Lagrange coordinates $\{y\}$, and for a while we assume that

$$\mathbf{u} \in H^1_p((0,T), L_6(\Omega)^3) \cap L_p((0,T), H^2_6(\Omega)^3), \tag{6}$$

and the quantity: $\| < t >^b \nabla \mathbf{u}\|_{L_p((0,T),H^1_6(\Omega)}$ is small enough for some $b > 0$ with $bp' > 1$, where $1/p + 1/p' = 1$. We consider the Lagrange transformation:

$$x = y + \int_0^t \mathbf{u}(y, s)\, ds \tag{7}$$

and assume that

$$\int_0^T \|\nabla \mathbf{u}(\cdot, t)\|_{L_\infty(\Omega)}\, dt < \delta \tag{8}$$

with some small number $\delta > 0$. If $0 < \delta < 1$, then for $x_i = y_i + \int_0^t \mathbf{u}(y_i, s)\, ds$ we have

$$|x_1 - x_2| \geq (1 - \int_0^T \|\nabla \mathbf{u}(\cdot, t)\|_{L_\infty(\Omega)}\, dt)|y_1 - y_2|,$$

and so the correspondence (7) is one to one. Moreover, applying a method due to Ströhmer [13], we see that the correspondence (7) is a $C^{1+\omega}$ ($\omega \in (0, 1/2)$) diffeomorphism from $\overline{\Omega}$ onto itself for any $t \in (0, T)$. In fact, let $J = \mathbf{I} + \int_0^t \nabla \mathbf{u}(y, s)\, ds$, which is the Jacobian of the map defined by (7), and then by Sobolev's imbedding theorem and Hölder's inequality for $\omega \in (0, 1/2)$ we have

$$\sup_{t\in(0,T)} \|\int_0^t \nabla \mathbf{u}(\cdot, s)\, ds\|_{C^\omega(\overline{\Omega})} \leq C_\omega\Bigl(\int_0^T < s >^{-bp'}\, ds\Bigr)^{1/p'}\Bigl(\int_0^T \| < s >^b \nabla \mathbf{u}(\cdot, s)\|^p_{H^1_6(\Omega)}\, ds\Bigr)^{1/p} < \infty \tag{9}$$

and we may assume that the right hand side of (9) is small enough and (8) holds in the process of constructing a solution. By (7), we have

$$\frac{\partial x}{\partial y} = \mathbf{I} + \int_0^t \frac{\partial \mathbf{u}}{\partial y}(y, s)\, ds,$$

and so choosing $\delta > 0$ small enough, we may assume that there exists a $3\times 3$ matrix $\mathbf{V}_0(\mathbf{k})$ of $C^\infty$ functions of variables $\mathbf{k}$ for $|\mathbf{k}| < \delta$, where $\mathbf{k}$ is a corresponding variable to $\int_0^t \nabla \mathbf{u}\, ds$, such that $\frac{\partial y}{\partial x} = \mathbf{I} + \mathbf{V}_0(\mathbf{k})$ and $\mathbf{V}_0(0) = 0$. Let $V_{0ij}(\mathbf{k})$ be the $(i, j)$ th component of $3 \times 3$ matrix $V_0(\mathbf{k})$, and then we have

$$\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j} + \sum_{j=1}^3 V_{0ij}(\mathbf{k})\frac{\partial}{\partial y_j}. \tag{10}$$

Let $X_t(x) = y$ be the inverse map of Lagrange transform (7) and set $\rho(x, t) = \zeta(X_t(x), t)$ and $\mathbf{v}(x, t) = \mathbf{u}(X_t(x), t)$. Setting

$$\mathcal{D}_{\rm div}\,(\mathbf{k})\nabla \mathbf{u} = \sum_{i,j=1}^3 V_{0ij}(\mathbf{k})\frac{\partial u_i}{\partial y_j},$$

we have $\mathrm{div}\, \mathbf{v} = \mathrm{div}\, \mathbf{u} + \mathcal{D}_{\rm div}\,(\mathbf{k})\mathbf{u}$. Let $\zeta = \rho_* + \eta$, and then

$$\frac{\partial}{\partial t}\rho + \mathrm{div}\,(\rho\mathbf{u}) = \frac{\partial \eta}{\partial t} + (\rho_* + \eta)(\mathrm{div}\,\mathbf{u} + \mathcal{D}_{\rm div}\,(\mathbf{k})\nabla \mathbf{u}).$$

Setting

$$\mathcal{D}_{\mathbf{D}}(\mathbf{k})\nabla \mathbf{u} = \mathbf{V}_0(\mathbf{k})\nabla \mathbf{u} + (\mathbf{V}_0(\mathbf{k})\nabla \mathbf{u})^\top, \tag{11}$$

we have $\mathbf{D}(\mathbf{v}) = \nabla \mathbf{v} + (\nabla \mathbf{v})^\top = (\mathbf{I} + \mathbf{V}_0(\mathbf{k}))\nabla \mathbf{u} + ((\mathbf{I} + \mathbf{V}_0(\mathbf{k}))\nabla \mathbf{u})^\top = \mathbf{D}(\mathbf{u}) + \mathcal{D}_{\mathbf{D}}(\mathbf{k})\nabla \mathbf{u}$. Moreover,

$$\begin{aligned}\mathrm{Div}\,(\mu\mathbf{D}(\mathbf{v}) + \nu\mathrm{div}\,\mathbf{v}\mathbf{I}) &= (\mathbf{I} + \mathbf{V}_0(\mathbf{k}))\nabla(\mu(\mathbf{D}(\mathbf{u}) + \mathcal{D}_{\mathbf{D}}(\mathbf{k})\nabla \mathbf{u}) + \nu(\mathrm{div}\,\mathbf{u} + \mathcal{D}_{\rm div}\,(\mathbf{k})\nabla \mathbf{u}) \\ &= \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}) + \nu\mathrm{div}\,\mathbf{u}\mathbf{I}) + \mathbf{V}_1(\mathbf{k})\nabla^2\mathbf{u} + (\mathbf{V}_2(\mathbf{k})\int_0^t \nabla^2\mathbf{u}\, ds)\nabla \mathbf{u}\end{aligned}$$

with

$$\begin{aligned}\mathbf{V}_1(\mathbf{k})\nabla^2\mathbf{u} &= \mu\mathcal{D}_{\mathbf{D}}(\mathbf{k})\nabla^2\mathbf{u} + \nu\mathcal{D}_{\rm div}\,(\mathbf{k})\nabla^2\mathbf{u}\mathbf{I}\\ &\quad + \mathbf{V}_0(\mathbf{k})(\mu\nabla\mathbf{D}(\mathbf{u}) + \nu\nabla{\rm div}\,\mathbf{u}\mathbf{I} + \mu\mathcal{D}_{\mathbf{D}}(\mathbf{k})\nabla^2\mathbf{u} + \nu\mathcal{D}_{\rm div}\,(\mathbf{k})\nabla^2\mathbf{u}\mathbf{I}),\\ (\mathbf{V}_2(\mathbf{k})\int_0^t\nabla\mathbf{u}\,ds)\nabla\mathbf{u} &= (\mathbf{I}+\mathbf{V}_0(\mathbf{k}))(\mu(d_{\mathbf{k}}\mathcal{D}_{\mathbf{D}}(\mathbf{k})\int_0^t\nabla^2\mathbf{u}\,ds)\nabla\mathbf{u} + \nu(d_{\mathbf{k}}\mathcal{D}_{\rm div}\,(\mathbf{k})\int_0^t\nabla^2\mathbf{u}\,ds\nabla\mathbf{u})\mathbf{I}.\end{aligned} \tag{12}$$

Here, $d_{\mathbf{k}}F(\mathbf{k})$ denotes the derivative of $F$ with respect to $\mathbf{k}$. Note that $\mathbf{V}_1(0) = 0$. Moreover, we write

$$\nabla\mathfrak{p}(\rho) = \mathfrak{p}'(\rho_*)\nabla\eta + (\mathfrak{p}'(\rho_*+\eta) - \mathfrak{p}'(\rho_*))\nabla\eta + \mathfrak{p}'(\rho_*+\eta)\mathbf{V}_0(\mathbf{k})\nabla\theta. \tag{13}$$

The material derivative $\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}$ is changed to $\partial_t\mathbf{u}$.

Summing up, we have obtained

$$\begin{aligned}\partial_t\eta + \rho_*{\rm div}\,\mathbf{u} &= F(\eta,\mathbf{u}) && \text{in } \Omega\times(0,T),\\ \rho_*\partial_t\mathbf{u} - {\rm Div}\,(\mu\mathbf{D}(\mathbf{u}) + \nu{\rm div}\,\mathbf{u}\mathbf{I} - \mathfrak{p}'(\rho_*)\eta) &= \mathbf{G}(\eta,\mathbf{u}) && \text{in } \Omega\times(0,T),\\ \mathbf{u}|_\Gamma = 0, \quad (\eta,\mathbf{u})|_{t=0} &= (\theta_0,\mathbf{v}_0) && \text{in } \Omega.\end{aligned} \tag{14}$$

Here, we have set

$$\begin{aligned}\mathbf{k} &= \int_0^t\nabla\mathbf{u}(\cdot,s)\,ds,\\ F(\eta,\mathbf{u}) &= \rho_*\mathcal{D}_{\rm div}\,(\mathbf{k})\nabla\mathbf{u} + \eta({\rm div}\,\mathbf{u} + \mathcal{D}_{\rm div}\,(\mathbf{k})\nabla\mathbf{u}),\\ \mathbf{G}(\eta,\mathbf{u}) &= \eta\partial_t\mathbf{u} + \mathbf{V}_1(\mathbf{k})\nabla^2\mathbf{u} + (\mathbf{V}_2(\mathbf{k})\int_0^t\nabla^2\mathbf{u}\,ds)\nabla\mathbf{u}\\ &\quad - (\mathfrak{p}'(\rho_*+\eta) - \mathfrak{p}'(\rho_*))\nabla\eta - \mathfrak{p}'(\rho_*+\eta)\mathbf{V}_0(\mathbf{k})\nabla\eta\end{aligned} \tag{15}$$

and $\mathcal{D}_{\rm div}\,(\mathbf{k})\nabla\mathbf{u}$, $\mathbf{V}_1(\mathbf{k})$ and $\mathbf{V}_2(\mathbf{k})$ have been defined in (11), (12) and (13). Note that $\mathcal{D}_{\rm div}\,(0) = 0$, $\mathbf{V}_0(0) = 0$, and $\mathbf{V}_1(0) = 0$. The following theorem is a main result in this paper.

**Theorem 6.** *Let $\Omega$ be an exterior domain in $\mathbb{R}^3$, whose boundary $\Gamma$ is a compact $C^2$ hypersurface and $T\in(0,\infty)$. Let $p$ be an exponent with $1 < p \leq 2$ and set $p' = p/(p-1)$. Let $\sigma\in(0,1)$ and set $\ell = (5+\sigma)/(4+2\sigma)$ and $r = 2(2+\sigma)/(4+\sigma) = (1/2+1/(2+\sigma))^{-1}$. Let $b$ be a positive constant satisfying the condition*

$$\frac{1}{p'} < b < \ell - \frac{1}{p}. \tag{16}$$

*Set*

$$\begin{aligned}\mathcal{I} &= \left\{(\theta_0,\boldsymbol{v}_0) \mid \theta_0\in\left(\bigcap_{q=2,6}H^1_q(\Omega)\right)\cap L_r(\Omega), \quad \boldsymbol{v}_0\in\left(\bigcap_{q=2,6}B^{2(1-1/p)}_{q,p}(\Omega)^3\right)\cap L_r(\Omega)^3\right\},\\ \|(\theta_0,\boldsymbol{v}_0)\|_{\mathcal{I}} &= \sum_{q=2,6}\|\theta_0\|_{H^1_q(\Omega)} + \sum_{q=2,6}\|\boldsymbol{v}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|(\theta_0,\boldsymbol{v}_0)\|_{L_r(\Omega)}.\end{aligned}$$

*Then, there exists a small constant $\epsilon\in(0,1)$ independent of $T$ such that if initial data $(\theta_0,\boldsymbol{v}_0)\in X$ satisfy the compatibility condition: $\boldsymbol{v}_0|_\Gamma = 0$ and the smallness condition : $\|(\theta_0,\boldsymbol{v}_0)\|_{\mathcal{I}} \leq \epsilon^2$, then Problem (14) admits unique solutions $\zeta = \rho_*+\eta$ and $\boldsymbol{u}$ with*

$$\begin{aligned}&\eta\in H^1_p((0,T),H^1_2(\Omega))\cap H^1_6(\Omega)),\\ &\boldsymbol{u}\in H^1_p((0,T),L_2(\Omega)^3\cap L_6(\Omega)^3)\cap L_p((0,T),H^2_2(\Omega)^3\cap H^2_6(\Omega)^3)\end{aligned} \tag{17}$$

*possessing the estimate $E_T(\eta,\boldsymbol{u}) \leq \epsilon$. Here, we have set*

$$E_T(\eta,\boldsymbol{u}) = \mathcal{E}_T(\eta,\boldsymbol{u}) + \| < t >^b \partial_t\nabla\eta\|_{L_p((0,T),L_2(\Omega)\cap L_6(\Omega))}$$

*and $\mathcal{E}_T(\eta,\boldsymbol{u})$ is the quantity defined in Theorem 3.*

*Remark 7.* (1) The choice of $\epsilon$ is independent of $T > 0$, and so solutions of Eq. (14) exist for any time $t \in (0, \infty)$.
(2) For any natural number $m$, $B^m_{q,2}(\Omega) \subset H^m_q(\Omega)$ for $2 < q < \infty$ and $B^m_{2,2} = H^m$.
(3) Letting $\sigma > 0$ be taken a small number such that $H^2_6 \subset C^{1+\sigma}$, we see that Theorem 6 implies

$$\int_0^T \|\mathbf{u}(\cdot, s)\|_{C^{1+\sigma}(\Omega)}\, ds < \delta$$

with some small number $\delta > 0$, which guarantees that Lagrange transform given in (7) is a $C^{1+\sigma}$ diffeomorphism on $\Omega$. Moreover, Theorem 3 follows from Theorem 6, the proof of which will be given in Sect. 6 below.

## 3. $\mathcal{R}$-Bounded Solution Operators

This section gives a general framework of proving the maximal $L_p$ regularity ($1 < p < \infty$), and so problem is formulated in an abstract setting. Let $X$, $Y$, and $Z$ be three UMD Banach spaces such that $X \subset Z \subset Y$ and $X$ is dense in $Y$, where the inclusions are continuous. Let $A$ be a closed linear operator from $X$ into $Y$ and let $B$ be a linear operator from $X$ into $Z$ and also from $Z$ into $Y$. Moreover, we assume that

$$\|Ax\|_Y \leq C\|x\|_X, \quad \|Bx\|_Z \leq C\|x\|_X, \quad \|Bz\|_Y \leq C\|z\|_Z$$

with some constant $C$ for any $x \in X$ and $z \in Z$. Let $\omega \in (0, \pi/2)$ be a fixed number and set

$$\Sigma_\omega = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| < \pi - \omega\}, \quad \Sigma_{\omega,\lambda_0} = \{\lambda \in \Sigma_\omega \mid |\lambda| \geq \lambda_0\}.$$

We consider an abstract boundary value problem with parameter $\lambda \in \Sigma_{\omega,\lambda_0}$:

$$\lambda u - Au = f, \quad Bu = g. \tag{18}$$

Here, $Bu = g$ represents boundary conditions, restrictions like divergence condition for Stokes equations in the incompressible viscous fluid flows case, or both of them. The simplest example is the following:

$$\lambda u - \Delta u = f \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = g \quad \text{on } \Gamma,$$

where $\Omega$ is a uniform $C^2$ domain in $\mathbb{R}^N$, $\Gamma$ its boundary, $\nu$ the unit outer normal to $\Gamma$, and $\partial/\partial\nu = \nu\cdot\nabla$ with $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_N)$ for $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$. In this case, it is standard to choose $X = H^2_q(\Omega)$, $Y = L_q(\Omega)$, $Z = H^1_q(\Omega)$ with $1 < q < \infty$, $A = \Delta$, and $B = \partial/\partial\nu$.

Problem formulated in (18) is corresponding to parameter elliptic problems which have been studied by Agmon [1], Agmon et al. [2], Agranovich and Visik [3], Denk and Volevich [6] and references there in, and their arrival point is to prove the unique existence of solutions possessing the estimate:

$$|\lambda|\|u\|_Y + \|u\|_X \leq C(\|f\|_Y + |\lambda|^\alpha\|g\|_Y + \|g\|_Z)$$

for some $\alpha \in \mathbb{R}$. From this estimate, we can derive the generation of a continuous analytic semigroup associated with $A$ when $Bu = 0$. But to prove the maximal $L_p$ regularity with $1 < p < \infty$ for the corresponding nonstationary problem:

$$\partial_t v - Av = f, \quad Bv = g \quad \text{for } t > 0, \quad v|_{t=0} = v_0, \tag{19}$$

especially in the cases where $Bv = g \neq 0$, further consideration is needed. Below, we introduce a framework based on the Weis operator valued Fourier multiplier theorem. To state this theorem, we make a preparation.

**Definition 8.** Let $E$ and $F$ be two Banach spaces and let $\mathcal{L}(E,F)$ be the set of all bounded linear operators from $E$ into $F$. We say that an operator family $\mathcal{T} \subset \mathcal{L}(E,F)$ is $\mathcal{R}$ bounded if there exist a constant $C$ and an exponent $q \in [1,\infty)$ such that for any integer $n$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$ and $\{f_j\}_{j=1}^n \subset E$, the inequality:

$$\int_0^1 \left\|\sum_{j=1}^n r_j(u)T_jf_j\right\|_F^q du \leq C\int_0^1 \left\|\sum_{j=1}^n r_j(u)f_j\right\|_E^q du$$

is valid, where the Rademacher functions $r_k$, $k \in \mathbb{N}$, are given by $r_k : [0,1] \to \{-1,1\}$; $t \mapsto \mathrm{sign}(\sin 2^k\pi t)$. The smallest such $C$ is called $\mathcal{R}$ bound of $\mathcal{T}$ on $\mathcal{L}(E,F)$, which is denoted by $\mathcal{R}_{\mathcal{L}(E,F)}\mathcal{T}$.

For $m(\xi) \in L_\infty(\mathbb{R}\setminus\{0\}, \mathcal{L}(E,F))$, we set

$$T_mf = \mathcal{F}_\xi^{-1}[m(\xi)\mathcal{F}[f](\xi)] \quad f \in \mathcal{S}(\mathbb{R},E),$$

where $\mathcal{F}$ and $\mathcal{F}_\xi^{-1}$ denote respective Fourier transformation and inverse Fourier transformation.

**Theorem 9** (Weis's operator valued Fourier multiplier theorem). *Let $E$ and $F$ be two UMD Banach spaces. Let $m(\xi) \in C^1(\mathbb{R}\setminus\{0\}, \mathcal{L}(E,F))$ and assume that*

$$\mathcal{R}_{\mathcal{L}(E,F)}(\{m(\xi) \mid \xi \in \mathbb{R}\setminus\{0\}\}) \leq r_b$$
$$\mathcal{R}_{\mathcal{L}(E,F)}(\{\xi m'(\xi) \mid \xi \in \mathbb{R}\setminus\{0\}\}) \leq r_b$$

*with some constant $r_b > 0$. Then, for any $p \in (1,\infty)$, $T_m \in \mathcal{L}(L_p(\mathbb{R},E), L_p(\mathbb{R},F))$ and*

$$\|T_mf\|_{L_p(\mathbb{R},F)} \leq C_pr_b\|f\|_{L_p(\mathbb{R},E)}$$

*with some constant $C_p$ depending solely on $p$.*

*Remark 10.* For a proof, refer to Weis [14].

We introduce the following assumption. Recall that $\omega$ is a fixed number such that $0 < \omega < \pi/2$.

**Assumption 11.** Let $X$, $Y$ and $Z$ be UMD Banach spaces. There exist a constant $\lambda_0$, $\alpha \in \mathbb{R}$, and an operator family $\mathcal{S}(\lambda)$ with

$$\mathcal{S}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\omega,\lambda_0}, \mathcal{L}(Y\times Y\times Z, X))$$

such that for any $f \in Y$ and $g \in Z$, $u = \mathcal{S}(\lambda)(f, \lambda^\alpha g, g)$ is a solution of Eq. (18), and the estimates:

$$\mathcal{R}_{\mathcal{L}(Y\times Y\times Z,X)}(\{(\tau\partial_\tau)^\ell\mathcal{S}(\lambda) \mid \lambda \in \Sigma_{\omega,\lambda_0}\}) \leq r_b$$
$$\mathcal{R}_{\mathcal{L}(Y\times Y\times Z,Y)}(\{(\tau\partial_\tau)^\ell(\lambda\mathcal{S}(\lambda)) \mid \lambda \in \Sigma_{\omega,\lambda_0}\}) \leq r_b$$

for $\ell = 0, 1$ are valid, where $\lambda = \gamma + i\tau \in \Sigma_{\omega,\lambda_0}$. $\mathcal{S}(\lambda)$ is called an $\mathcal{R}$-bounded solution operator or an $\mathcal{R}$ solver of Eq. (18).

We now consider an initial-boundary value problem:

$$\partial_tu - Au = f \quad Bu = g \quad (t > 0), \quad u|_{t=0} = u_0. \tag{20}$$

This problem is divided into the following two equations:

$$\partial_tu - Au = f \qquad Bu = g \qquad (t \in \mathbb{R}); \tag{21}$$
$$\partial_tu - Au = 0 \qquad Bu = 0 \qquad (t > 0), \quad u|_{t=0} = u_0. \tag{22}$$

From the definition of $\mathcal{R}$-boundedness with $n = 1$ we see that $u = \mathcal{S}(\lambda)(f, 0, 0)$ satisifes equations:

$$\lambda u - Au = f, \quad Bu = 0,$$

and the estimate:

$$|\lambda|\|u\|_Y + \|u\|_X \leq C\|f\|_Y.$$

Let $\mathcal{D}(A)$ be the domain of the operator $A$ defined by

$$\mathcal{D}(A) = \{u_0 \in X \mid Bu_0 = 0\}.$$

Then, the operator $A$ generates continuous analytic semigroup $\{T_A(t)\}_{t\geq 0}$ such that $u = T_A(t)u_0$ solves Eq. (22) uniquely and the following estimates hold:

$$\|u(t)\|_Y \leq r_b e^{\lambda_0 t}\|u_0\|_Y, \quad \|\partial_t u(t)\|_Y \leq r_b\, t^{-1}\, e^{\lambda_0 t}\|u_0\|_Y, \quad \|\partial_t u(t)\|_Y \leq r_b e^{\lambda_0 t}\|u_0\|_X. \tag{23}$$

These estimates and trace method of real-interpolation theory yield the following theorem.

**Theorem 12** (Maximal regularity for initial value problem). *Let $1 < p < \infty$ and set $\mathcal{D} = (Y, \mathcal{D}(A))_{1-1/p,p}$, where $(\cdot,\cdot)_{1-1/p,p}$ denotes a real interpolation functor. Then, for any $u_0 \in \mathcal{D}$, Problem (22) admits a unique solution $u$ with*

$$e^{-\lambda_0 t}u \in L_p(\mathbb{R}_+, X) \cap H^1_p(\mathbb{R}_+, Y) \quad (\mathbb{R}_+ = (0,\infty))$$

*possessing the estimate:*

$$\|e^{-\lambda_0 t}\partial_t u\|_{L_p(\mathbb{R}_+,Y)} + \|e^{-\lambda_0 t}u\|_{L_p(\mathbb{R}_+,X)} \leq C\|u_0\|_{(Y,\mathcal{D}(A))_{1-1/p,p}}.$$

The $\mathcal{R}$-bounded solution operator plays an essential role to prove the following theorem.

**Theorem 13** (Maximal regularity for boundary value problem). *Let $1 < p < \infty$. Then for any $f$ and $g$ with $e^{-\gamma t}f \in L_p(\mathbb{R}, Y)$ and $e^{-\gamma t}g \in L_p(\mathbb{R}, Z) \cap H^\alpha_p(\mathbb{R}, Y)$ for any $\gamma \geq \lambda_0$, Problem (21) admits a unique solution $u$ with $e^{-\gamma t}u \in L_p(\mathbb{R}, X) \cap H^1_p(\mathbb{R}, Y)$ for any $\gamma \geq \lambda_0$ possessing the estimate:*

$$\begin{aligned}\|e^{-\gamma t}\partial_t u\|_{L_p(\mathbb{R}_+,Y)} + \|e^{-\gamma t}u\|_{L_p(\mathbb{R}_+,X)} &\leq C(\|e^{-\gamma t}f\|_{L_p(\mathbb{R},Y)} \\ &+ (1+\gamma)^\alpha\|e^{-\gamma t}g\|_{H^\alpha_p(\mathbb{R},Y)} + \|e^{-\gamma t}g\|_{L_p(\mathbb{R},Z)})\end{aligned}$$

*for any $\gamma \geq \lambda_0$. Here, the constant $C$ may depend on $\lambda_0$ but independent of $\gamma$ whenever $\gamma \geq \lambda_0$, and we have set*

$$H^\alpha_p(\mathbb{R}, Y) = \{h \in \mathcal{S}'(\mathbb{R}, Y) \mid \|h\|_{H^\alpha_p(\mathbb{R},Y)} := \|\mathcal{F}^{-1}_\xi[(1+|\xi|^2)^{\alpha/2}\mathcal{F}[h](\xi)]\|_{L_p(\mathbb{R},Y)} < \infty\}.$$

*Proof.* Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote respective Laplace transformation and inverse Laplace transformation defined by setting

$$\begin{aligned}\mathcal{L}[f](\lambda) &= \int_{\mathbb{R}} e^{-\lambda t}f(t)\,dt = \int_{\mathbb{R}} e^{-i\tau t}(e^{-\gamma t}f(t))\,dt = \mathcal{F}[e^{-\gamma t}f(t)](\tau) \quad (\lambda = \gamma + i\tau), \\ \mathcal{L}^{-1}[f](t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{\lambda t}f(\tau)\,d\tau = \frac{e^{\gamma t}}{2\pi}\int_{\mathbb{R}} e^{i\tau t}f(\tau)\,d\tau = e^{\gamma t}\mathcal{F}^{-1}[f](\tau).\end{aligned}$$

We consider equations:

$$\partial_t u - Au = f, \quad Bu = g \quad \text{for } t \in \mathbb{R}.$$

Applying Laplace transformation yields that

$$\lambda\mathcal{L}[u](\lambda) - A\mathcal{L}[u](\lambda) = \mathcal{L}[f](\lambda), \quad B\mathcal{L}[u](\lambda) = \mathcal{L}[g](\lambda).$$

Applying $\mathcal{R}$-bounded solution operator $\mathcal{S}(\lambda)$ yields that

$$\mathcal{L}[u](\lambda) = \mathcal{S}(\lambda)(\mathcal{L}[f](\lambda), \lambda^\alpha\mathcal{L}[g](\lambda), \mathcal{L}[g](\lambda)),$$

and so

$$u = \mathcal{L}^{-1}[\mathcal{S}(\lambda)\mathcal{L}[(f, \Lambda^\alpha g, g)](\lambda)],$$

where $\Lambda^\alpha g = \mathcal{L}^{-1}[\lambda^\alpha\mathcal{L}[g]]$. Moreover,

$$\partial_t u = \mathcal{L}^{-1}[\lambda\mathcal{S}(\lambda)\mathcal{L}[f, \Lambda^\alpha g, g)](\lambda)].$$

Using Fourier transformation and inverse Fourier transformation, we rewrite

$$u = e^{\gamma t}\mathcal{F}^{-1}[\mathcal{S}(\lambda)\mathcal{F}[e^{-\gamma t}(f, \Lambda^\alpha g, g)](\tau)](t),$$
$$\partial_t u = e^{\gamma t}\mathcal{F}^{-1}[\lambda\mathcal{S}(\lambda)\mathcal{F}[e^{-\gamma t}(f, \Lambda^\alpha g, g)](\tau)](t).$$

Applying the assumption of $\mathcal{R}$-bounded solution operators and Weis's operator valued Fourier multiplier theorem yields that

$$\|e^{-\gamma t}\partial_t u\|_{L_p(\mathbb{R},Y)} + \|e^{-\gamma t}u\|_{L_p(\mathbb{R},X)} \leq C_p r_b(\|e^{-\gamma t}f\|_{L_p(\mathbb{R},Y)} + (1+\gamma)^\alpha\|e^{-\gamma t}g\|_{H^\alpha_p(\mathbb{R},Y)} + \|e^{-\gamma t}g\|_{L_p(\mathbb{R},Z)})$$

for any $\gamma \geq \lambda_0$. The uniqueness follows from the generation of analytic semigroup and Duhamel's principle. □

We now explain our strategy to solve initial-boundary value problem:

$$\partial_t u - Au = f, \quad Bu = g \quad \text{for } t \in (0,\infty), \quad u|_{t=0} = u_0. \tag{24}$$

The point is how to get enough decay estimates. As a first step, we consider the following time shifted equations without initial data

$$\partial_t w + \lambda_1 w - Aw = f, \quad Bw = g \quad \text{for } t \in \mathbb{R}. \tag{25}$$

Then, we have the following theorem which guarantees the polynomial decay of solutions.

**Theorem 14.** *Let $\lambda_0$ be a constant appearing in Assumption 11 and let $\lambda_1 > \lambda_0$. Let $1 < p < \infty$ and $b \geq 0$. Then, for any $f$ and $g$ with $< t >^b f \in L_p(\mathbb{R}, Y)$ and $< t >^b g \in L_p(\mathbb{R}, Z) \cap H^\alpha_p(\mathbb{R}, X)$, Problem (25) admits a unique solution $w \in H^1_p(\mathbb{R}, Y) \cap L_p(\mathbb{R}, X)$ possessing the estimate:*

$$\begin{aligned}&\| < t >^b w\|_{L_p(\mathbb{R},X)} + \| < t >^b \partial_t w\|_{L_p(\mathbb{R},Y)} \\ &\quad\leq C(\| < t >^b f\|_{L_p(\mathbb{R},Y)} + \| < t >^b g\|_{H^\alpha_p(\mathbb{R},Y)} + \| < t >^b g\|_{L_p(\mathbb{R},Z)}).\end{aligned} \tag{26}$$

*Proof.* Since $ik + \lambda_1 \in \Sigma_{\omega,\lambda_0}$, for $k \in \mathbb{R}$ we set $w = \mathcal{F}^{-1}[\mathcal{M}(ik+\lambda_1)(\mathcal{F}[f], (ik)^\alpha\mathcal{F}[g], \mathcal{F}[g])]$, and then $w$ satisfies equations:

$$\partial_t w + \lambda_1 w - Aw = f, \quad Bw = g \quad \text{for } t \in \mathbb{R},$$

and the estimate:

$$\|\partial_t w\|_{L_p(\mathbb{R},Y)} + \|w\|_{L_p(\mathbb{R},X)} \leq C(\|f\|_{L_p(\mathbb{R},Y)} + \|g\|_{H^\alpha_p(\mathbb{R},Y)} + \|g\|_{L_p(\mathbb{R},Z)}). \tag{27}$$

This prove the theorem in the case where $b = 0$. When $0 < b \leq 1$, we observe that

$$\partial_t(< t >^b w) + \lambda_1(< t >^b w) - A(< t >^b w) =< t >^b f+ < t >^{b-2} tw, \quad B(< t >^b w) =< t >^b g,$$

and so noting that $\| < t >^{b-2} tw\|_Y \leq C\|w\|_Y \leq C\|w\|_X$, we have

$$\begin{aligned}&\| < t >^b w\|_{L_p((0,\infty),X)} + \| < t >^b \partial_t w\|_{L_p((0,\infty),Y)} \\ &\quad\leq C(\| < t >^{b-2} tw\|_{L_p(\mathbb{R},Y)} + \| < t >^b f\|_{L_p(\mathbb{R},Y)} + \| < t >^b g\|_{H^\alpha_p(\mathbb{R},Y)} + \| < t >^b g\|_{L_p(\mathbb{R},Z)}) \\ &\quad\leq C(\| < t >^b f\|_{L_p(\mathbb{R},Y)} + \| < t >^b g\|_{H^\alpha_p(\mathbb{R},Y)} + \| < t >^b g\|_{L_p(\mathbb{R},Z)}).\end{aligned}$$

If $b > 1$, then repeated use of this argument yields the theorem, which completes the proof of Theorem 14. □

To compensate solutions, let $v_1$ be a solution of time shifted equations:

$$\partial_t v_1 + \lambda_1 v_1 - Av_1 = \lambda_1 w, \quad Bv_1 = 0 \quad \text{for } t \in \mathbb{R}.$$

By Theorem 14,

$$\begin{aligned}&\| < t >^b \partial_t v_1\|_{L_p(\mathbb{R},Y)} + \| < t >^b v_1\|_{L_p(\mathbb{R},X)} \leq C\| < t > w\|_{L_p(\mathbb{R},Y)} \\ &\quad\leq C(\| < t >^b f\|_{L_p(\mathbb{R},Y)} + \| < t >^b g\|_{H^\alpha_p(\mathbb{R},Y)} + \| < t >^b g\|_{L_p(\mathbb{R},Z)}).\end{aligned} \tag{28}$$

Here, we used the assumption that $X$ is continuously embedded into $Y$, that is $\|w\|_Y \le C\|w\|_X$ for some constant $C$. The role of $v_1$ is to controle the compatibility conditions, that is

$$v_1 \in \mathcal{D}(A) \quad \text{for all } t \in \mathbb{R}. \tag{29}$$

Thus, if $g = 0$ in (24) like Dirichlet zero condition case, then we need not this step.

To solve Eq. (24), we now consider a second compensation function $v_2$, which is a solution of the following initial problem with zero boundary condition:

$$\partial_t v_2 - Av_2 = \lambda_1 v_1, \quad Bv_2 = 0 \text{ for } t \in (0,\infty), \quad v_2|_{t=0} = u_0 - (w|_{t=0} + v|_{t=0}). \tag{30}$$

To solve (30) with the help of semi-group $\{T_A(t)\}_{t\ge 0}$, we need the compatibility condition:

$$B(u_0 - (w|_{t=0} + v_1|_{t=0})) = Bu_0 - g|_{t=0} = 0. \tag{31}$$

Since (29) holds, assuming the compatibility condition: $Bu_0 = g|_{t=0}$, by Duhamel's principle, $v_2$ is represented as

$$v_2 = T_A(t)(u_0 - (w|_{t=0} + v_1|_{t=0}) + \int_0^t T_A(t-s)(\lambda_1 v_1(s))\,ds. \tag{32}$$

And then, $u = w + v_1 + v_2$ is a required solution of Eq. (24). Concerning the estimate of $v_2$, for $t \in (0,2)$ we use the estimate:

$$\|T_A(t)v_0\|_{D(A)} \le C\|v_0\|_{D(A)}$$

where $\|\cdot\|_{D(A)}$ denotes the norm of domain $D(A)$. And, for $t \in [2,\infty)$ we use so called $L_p$-$L_q$ decay estimate for the semigroup $\{T_A(t)\}_{t\ge0}$. In this paper, we use the $L_p$-$L_q$ decay estimate for the Stokes equations for the compressible viscous fluid, which will be given in (68) in Sect. 5 below.

## 4. Estimates of Nonlinear Terms

In what follows, let $T > 0$ be any positive time and let $b$ and $p$ be positive numbers and an exponents given in Theorem 3 and Theorem 6. Let $\mathcal{U}^i_\epsilon$ $(i = 1,2)$ be underlying spaces for linearized equations of equations (14), which is defined by

$$\begin{aligned}
&\mathcal{U}^1_T = \{\theta \in H^1_p((0,T), H^1_2(\Omega)\cap H^1_6(\Omega)) \mid \theta|_{t=0} = \theta_0, \quad \sup_{t\in(0,T)} \|\theta(\cdot,t)\|_{L_\infty(\Omega)} \le \rho_*/2\},\\
&\mathcal{U}^2_T = \{\mathbf{v} \in L_p((0,T), H^2_2(\Omega)^3\cap H^2_6(\Omega)^3)\cap H^1_p((0,T), L_2(\Omega)^3\cap L_6(\Omega)^3) \mid \\
&\mathbf{v}|_{t=0} = \mathbf{v}_0, \quad \int_0^T \|\nabla\mathbf{v}(\cdot,s)\|_{L_\infty(\Omega)}\,ds \le \delta\}.
\end{aligned} \tag{33}$$

Recall that our energy $E_T(\eta,\mathbf{u})$ has been defined by

$$\begin{aligned}
E_T(\eta,\mathbf{u}) &= \|<t>^b\, \nabla(\eta,\mathbf{u})\|_{L_p((0,T),H^{0,1}_2(\Omega))} + \|<t>^b\, (\eta,\mathbf{u})\|_{L_\infty((0,T),L_2(\Omega)\cap L_6(\Omega))}\\
&\quad + \|<t>^b\, \partial_t(\eta,\mathbf{u})\|_{L_p((0,T),H^{1,0}_2(\Omega)\cap H^{1,0}_6(\Omega))} + \|<t>^b\, (\eta,\mathbf{u})\|_{L_p((0,T),H^2_6(\Omega))}.
\end{aligned}$$

To estimate $L_{2+\sigma}$ norm, we use standard interpolation inequality:

$$\|f\|_{L_{2+\sigma}(\Omega)} \le \|f\|^{\frac{4-\sigma}{2(2+\sigma)}}_{L_2(\Omega)} \|f\|^{\frac{3\sigma}{2(2+\sigma)}}_{L_6(\Omega)} \le \frac{4-\sigma}{2(2+\sigma)}\|f\|_{L_2(\Omega)} + \frac{3\sigma}{2(2+\sigma)}\|f\|_{L_6(\Omega)}. \tag{34}$$

In estimating $L_r$ norm, we meet $L_{2+\sigma}$ norm in view of Hölder's inequality, but this norm is estimate by $L_2$ and $L_6$ norm with the help of (34). In particular, for $(\theta, \mathbf{v}) \in \mathcal{U}^1_T \times \mathcal{U}^2_T$, we know that

$$\begin{aligned}
\| < t >^b (\theta, \mathbf{v})\|_{L_\infty((0,T), L_{2+\sigma}(\Omega)} &\leq C_\sigma \sum_{q=2,6} \| < t >^b (\theta, \mathbf{v})\|_{L_\infty((0,T), L_q(\Omega))}, \\
\| < t >^b \nabla(\theta, \mathbf{v})\|_{L_p((0,T), H^{0,1}_{2+\sigma}(\Omega)} &\leq C_\sigma \sum_{q=2,6} \| < t >^b \nabla(\theta, \mathbf{v})\|_{L_p((0,T), H^{0,1}_q(\Omega))}, \\
\| < t >^b \partial_t(\theta, \mathbf{v})\|_{L_p((0,T), H^{1,0}_{2+\sigma}(\Omega)} &\leq C_\sigma \sum_{q=2,6} \| < t >^b \partial_t(\theta, \mathbf{v})\|_{L_p((0,T), H^{1,0}_q(\Omega))}.
\end{aligned} \tag{35}$$

Notice that for any $\theta \in \mathcal{U}^1_T$ we see that

$$\rho_*/2 \leq |\rho_* + \tau\theta(y,t)| \leq 3\rho_*/2 \quad \text{for } (y,t) \in \Omega \times (0,T) \text{ and } |\tau| \leq 1. \tag{36}$$

For $\mathbf{v} \in \mathcal{U}^2_T$ let $\mathbf{k}_\mathbf{v} = \int_0^t \nabla \mathbf{v}(\cdot, s)\,ds$, and then $|\mathbf{k}_\mathbf{v}(y,t)| \leq \delta$ for any $(y,t) \in \Omega \times (0,T)$. Moreover, for $q = 2, 2+\sigma$ and 6 by Hölder's inequality

$$\sup_{t\in(0,T)} \|\mathbf{k}_\mathbf{v}\|_{H^1_q(\Omega)} \leq \int_0^T \|\nabla\mathbf{v}(\cdot,t)\|_{H^1_q(\Omega)} dt \leq C\Big(\int_0^\infty < t >^{-p'b}\,dt\Big)^{1/p'} \| < t >^b \nabla\mathbf{v}\|_{L_p((0,T), H^1_q(\Omega))}, \tag{37}$$

where $bp' > 1$.

In what follows, for notational simplicity we use the following abbreviation: $\|f\|_{H^1_q(\Omega)} = \|f\|_{H^1_q}$, $\|f\|_{L_q(\Omega)} = \|f\|_{L_q}$, $\|f\|_{L_\infty((0,T),X)} = \|f\|_{L_\infty(X)}$, and $\| < t >^b f\|_{L_p((0,T),X)} = \|f\|_{L_{p,b}(X)}$. Let $(\theta, \mathbf{v}) \in \mathcal{U}^1_T \times \mathcal{U}^2_T$ and $(\theta_i, \mathbf{v}_i) \in \mathcal{U}^1_T \times \mathcal{U}^2_T$ $(i = 1,2)$. The purpose of this section is to give necessary estimates of $(F(\theta, \mathbf{v}), \mathbf{G}(\theta, \mathbf{v}))$ and difference: $(F(\theta_1, \mathbf{v}_1) - F(\theta_2, \mathbf{v}_2), \mathbf{G}(\theta_1, \mathbf{v}_1) - \mathbf{G}(\theta_2, \mathbf{v}_2)))$ to prove the global well-posedness of Eq. (14). Recall that

$$\begin{aligned}
F(\theta, \mathbf{v}) &= \rho_* \mathcal{D}_{\rm div}\,(\mathbf{k})\nabla\mathbf{v} + \theta{\rm div}\,\mathbf{v} + \theta\mathcal{D}_{\rm div}\,(\mathbf{k})\nabla\mathbf{v}, \\
\mathbf{G}(\theta, \mathbf{v}) &= \theta\partial_t\mathbf{v} + \mathbf{V}_1(\mathbf{k})\nabla^2\mathbf{v} + (\mathbf{V}_2(\mathbf{k})\int_0^t \nabla^2\mathbf{v}\,ds)\nabla\mathbf{v} \\
&\qquad - (\mathfrak{p}'(\rho_* + \theta) - \mathfrak{p}'(\rho_*))\nabla\theta - \mathfrak{p}'(\rho_* + \theta)\mathbf{V}_0(\mathbf{k})\nabla\theta.
\end{aligned} \tag{38}$$

We start with estimating $\|F(\theta, \mathbf{v})\|_{L_{p,b}(H^1_r)}$. Recall that $r^{-1} = 2^{-1} + (2+\sigma)^{-1}$ and we use the estimates:

$$\begin{aligned}
&\|fg\|_{L_{p,b}(H^1_r)} \leq C\|f\|_{L_\infty(H^1_{2+\sigma})}\|g\|_{L_{p,b}(H^1_2)}, \\
&\|fgh\|_{L_{p,b}(H^1_r)} \leq C(\|f\|_{L_\infty(H^1_6)}\|g\|_{L_\infty(H^1_{2+\sigma})} + \|f\|_{L_\infty(H^1_{2+\sigma})}\|g\|_{L_\infty(H^1_6)})\|h\|_{L_{p,b}(H^1_2)},
\end{aligned} \tag{39}$$

as follows from Hölder's inequality and Sobolev's inequality : $\|f\|_{L_\infty} \leq C\|f\|_{H^1_6}$. Let $dG(\mathbf{k})$ denote the derivative of $G(\mathbf{k})$ with respect to $\mathbf{k}$ and $C_{\rm div}$ be a constan such that $\sup_{|\mathbf{k}|<\delta} |\mathcal{D}_{\rm div}\,(\mathbf{k})| < C_{\rm div}$, $\sup_{|\mathbf{k}|<\delta} |d\mathcal{D}_{\rm div}\,(\mathbf{k})| < C_{\rm div}$, and $\sup_{|\mathbf{k}|<\delta} |d(d\mathcal{D}_{\rm div})(\mathbf{k})| < C_{\rm div}$. Then, noting $\mathcal{D}_{\rm div}\,(0) = 0$, by (37) we have

$$\|\mathcal{D}_{\rm div}\,(\mathbf{k}_\mathbf{v})\|_{H^1_q} \leq C_{\rm div}\,\|\mathbf{k}_\mathbf{v}\|_{H^1_q} \leq C\|\nabla\mathbf{v}\|_{L_{p,b}(H^1_q)} \quad \text{for } \mathbf{v} \in \mathcal{U}^2_T \text{ and } q = 2, 2+\sigma \text{ and } 6. \tag{40}$$

Moreover, for $\mathbf{v}_1$, $\mathbf{v}_2 \in \mathcal{U}^2_T$ writing

$$\mathcal{D}_{\rm div}\,(\mathbf{k}_{\mathbf{v}_1}) - \mathcal{D}_{\rm div}\,(\mathbf{k}_{\mathbf{v}_2}) = \int_0^t d\mathcal{D}_{\rm div}\,(\mathbf{k}_{\mathbf{v}_2} + \tau(\mathbf{k}_{\mathbf{v}_1} - \mathbf{k}_{\mathbf{v}_2}))\,d\tau\,(\mathbf{k}_{\mathbf{v}_1} - \mathbf{k}_{\mathbf{v}_2}),$$

and noting that $|\mathbf{k}_{\mathbf{v}_2} + \tau(\mathbf{k}_{\mathbf{v}_1} - \mathbf{k}_{\mathbf{v}_2})| = |(1-\tau)\mathbf{k}_{\mathbf{v}_2} + \tau\mathbf{k}_{\mathbf{v}_1}| \leq (1-\tau)\delta + \tau\delta = \delta$, we have

$$\begin{aligned}
&\|\mathcal{D}_{\rm div}\,(\mathbf{k}_{\mathbf{v}_1}) - \mathcal{D}_{\rm div}\,(\mathbf{k}_{\mathbf{v}_2})\|_{H^1_q} \\
&\quad \leq C_{\rm div}\,(\|\mathbf{k}_{\mathbf{v}_1} - \mathbf{k}_{\mathbf{v}_2}\|_{L_\infty(H^1_q)} + \sum_{i=1,2} \|\nabla\mathbf{k}_{\mathbf{v}_i}\|_{L_\infty(L_q)}\|\mathbf{k}_{\mathbf{v}_1} - \mathbf{k}_{\mathbf{v}_2}\|_{L_\infty(L_\infty)}) \\
&\quad \leq C(\|\nabla(\mathbf{v}_1 - \mathbf{v}_2)\|_{L_{p,b}(H^1_q)} + \sum_{i=1,2} \|\nabla\mathbf{v}_i\|_{L_{p,b}(H^1_q)}\|\nabla(\mathbf{v}_1 - \mathbf{v}_2)\|_{L_{p,b}(H^1_6)}.
\end{aligned} \tag{41}$$

Since $\theta = \theta|_{t=0} + \int_0^t \partial_s\theta\,ds$, for $X \in \{L_q, H^1_q\}$ with $q = 2,\ 2+\sigma$ and 6

$$\begin{aligned}\|\theta(\cdot,t)\|_X &\leq \|\theta_0\|_X + \int_0^T \|(\partial_s\theta)(\cdot,s)\|_X\,ds \\ &\leq \|\theta_0\|_X + \Big(\int_0^\infty <t>^{-p'b}\ dt\Big)^{1/p'}\|\partial_s\theta\|_{L_{p,b}(X)}.\end{aligned} \tag{42}$$

In particular, by Sobolev's inequality

$$\|\theta(\cdot,t)\|_{L_\infty} \leq C(\|\theta_0\|_{H^1_6} + \|\partial_t\theta\|_{L_{p,b}(H^1_6)}). \tag{43}$$

For $\theta \in \mathcal{U}^1_T$ and $\mathbf{v} \in \mathcal{U}^2_T$, combining (39), (40), (41), (42), and (43) yields that

$$\begin{aligned}\|F(\theta,\mathbf{v})\|_{L_{p,b}(H^1_r)} &\leq C[\|\nabla\mathbf{v}\|_{L_{p,b}(H^1_{2+\sigma})}\|\nabla\mathbf{v}\|_{L_{p,b}(H^1_2)} + (\|\theta_0\|_{H^1_{2+\sigma}} + \|\partial_t\theta\|_{L_{p,b}(H^1_{2+\sigma})})\|\nabla\mathbf{v}\|_{L_{p,b}(H^1_2)} \\ &+ \{(\|\theta_0\|_{H^1_6} + \|\partial_t\theta\|_{L_{p,b}(H^1_6)})\|\nabla\mathbf{v}\|_{L_{p,b}(H^1_{2+\sigma})} + (\|\theta_0\|_{H^1_{2+\sigma}} + \|\partial_t\theta\|_{L_{p,b}(H^1_{2+\sigma})})\|\nabla\mathbf{v}\|_{L_{p,b}(H^1_6)}\} \\ &\qquad\times\|\nabla\mathbf{v}\|_{L_{p,b}(H^1_2)}].\end{aligned} \tag{44}$$

Analogously, for $\theta_i \in \mathcal{U}^1_T$ and $\mathbf{v}_i \in \mathcal{U}^2_T$ $(i=1,2)$,

$$\begin{aligned}&\|F(\theta_1,\mathbf{v}_1) - F(\theta_2,\mathbf{v}_2)\|_{L_{p,b}(L_r)} \\ &\leq C[(\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_{2+\sigma})} + \sum_{i=1,2}\|\nabla\mathbf{v}_i\|_{L_{p,b}(H^1_{2+\sigma})}\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)})\|\nabla\mathbf{v}_1\|_{L_{p,b}(H^1_2)} \\ &+ \|\nabla\mathbf{v}_2\|_{L_{p,b}(H^1_{2+\sigma})}\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_2)} + \|\partial_t(\theta_1-\theta_2)\|_{L_{p,b}(H^1_{2+\sigma})}\|\nabla\mathbf{v}_1\|_{L_{p,b}(H^1_2)} \\ &+ (\|\theta_0\|_{H^1_{2+\sigma}} + \|\partial_t\theta_2\|_{L_{p,b}(H^1_{2+\sigma})})\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_2)} \\ &+ (\|\partial_t(\theta_1-\theta_2)\|_{L_{p,b}(H^1_6)}\|\nabla\mathbf{v}_1\|_{L_{p,b}(H^1_{2+\sigma})} + \|\partial_t(\theta_1-\theta_2)\|_{L_{p,b}(H^1_{2+\sigma})}\|\nabla\mathbf{v}_1\|_{L_{p,b}(H^1_6)})\|\nabla\mathbf{v}_1\|_{L_{p,b}(H^1_2)} \\ &+ \{(\|\theta_0\|_{H^1_6} + \|\partial_t\theta_2\|_{L_{p,b}(H^1_6)})(\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_{2+\sigma})} + \sum_{i=1,2}\|\nabla\mathbf{v}_i\|_{L_{p,b}(H^1_{2+\sigma})}\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)})) \\ &+ (\|\theta_0\|_{H^1_{2+\sigma}} + \|\partial_t\theta\|_{L_{p,b}(H^1_{2+\sigma})})(\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)} + \sum_{i=1,2}\|\nabla\mathbf{v}_i\|_{L_{p,b}(H^1_6)}\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)})\} \\ &\qquad\times\|\nabla\mathbf{v}_1\|_{L_{p,b}(H^1_2)} \\ &+ \{(\|\theta_0\|_{H^1_6} + \|\partial_t\theta_2\|_{L_{p,b}(H^1_6)})\|\nabla\mathbf{v}_2\|_{L_{p,b}(H^1_{2+\sigma})} + (\|\theta_0\|_{H^1_{2+\sigma}} + \|\partial_t\theta_2\|_{L_{p,b}(H^1_{2+\sigma})}\|\nabla\mathbf{v}_2\|_{L_{p,b}(H^1_6)}\} \\ &\qquad\times\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_2)}].\end{aligned} \tag{45}$$

We now estimate $\|F(\theta,\mathbf{v})\|_{L_{p,b}(H^1_q)}$ and $\|F(\theta_1,\mathbf{v}_1) - F(\theta_2,\mathbf{v}_2)\|_{L_{p,b}(H^1_q)}$ with $q = 2$ and 6. For this purpose, we use the following estimates:

$$\begin{aligned}\|fg\|_{L_{p,b}(H^1_q)} &\leq C\{\|f\|_{L_\infty(H^1_q)}\|g\|_{L_{p,b}(H^1_6)} + \|f\|_{L_\infty(H^1_q)}\|g\|_{L_{p,b}(H^1_6)}\}, \\ \|fgh\|_{L_{p,b}(H^1_q)} &\leq C\{\|f\|_{L_\infty(H^1_q)}\|g\|_{L_\infty(H^1_6)}\|h\|_{L_{p,b}(H^1_6)} + \|f\|_{L_\infty(H^1_6)}\|g\|_{L_\infty(H^1_q)}\|h\|_{L_{p,b}(H^1_6)} \\ &\quad + \|f\|_{L_\infty(H^1_6)}\|g\|_{L_\infty(H^1_6)}\|h\|_{L_{p,b}(H^1_q)}\}.\end{aligned}$$

And then, using (40), (41), (42), we have

$$\begin{aligned}\|F(\theta,\mathbf{v})\|_{L_{p,b}(H^1_q)} &\leq C\{\|\nabla\mathbf{v}\|_{L_{p,b}(H^1_q)}\|\nabla\mathbf{v}\|_{L_{p,b}(H^1_6)} + (\|\theta_0\|_{H^1_q} + \|\partial_t\theta\|_{L_{p,b}(H^1_q)})\|\nabla\mathbf{v}\|_{L_{p,b}(H^1_6)} \\ &+ (\|\theta_0\|_{H^1_6} + \|\partial_t\theta\|_{L_{p,b}(H^1_6)})\|\nabla\mathbf{v}\|_{L_{p,b}(H^1_q)} + (\|\theta_0\|_{H^1_q} + \|\partial_t\theta\|_{L_{p,b}(H^1_q)})\|\nabla\mathbf{v}\|^2_{L_{p,b}(H^1_6)} \\ &+ (\|\theta_0\|_{H^1_6} + \|\partial_t\theta\|_{L_{p,b}(H^1_6)})\|\nabla\mathbf{v}\|_{L_{p,b}(H^1_q)}\|\nabla\mathbf{v}\|_{L_{p,b}(H^1_6)}\};\end{aligned} \tag{46}$$

$$\begin{aligned}&\|F(\theta_1,\mathbf{v}_1) - F(\theta_2,\mathbf{v}_2)\|_{L_{p,b}(H^1_q)} \\ &\leq C\{(\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_q)} + \sum_{i=1,2}\|\nabla\mathbf{v}_i\|_{L_{p,b}(H^1_q)}\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)})\|\nabla\mathbf{v}_1\|_{L_{p,b}(H^1_6)} \\ &+ (\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)} + \sum_{i=1,2}\|\nabla\mathbf{v}_i\|_{L_{p,b}(H^1_6)}\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)})\|\nabla\mathbf{v}_1\|_{L_{p,b}(H^1_q)}\end{aligned}$$

$$
\begin{aligned}
&+ \|\nabla \mathbf{v}_2\|_{L_{p,b}(H^1_q)}\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)} + \|\nabla \mathbf{v}_2\|_{L_{p,b}(H^1_6)}\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_q)}\\
&+ \|\partial_t(\theta_1-\theta_2)\|_{L_{p,b}(H^1_q)}\|\nabla \mathbf{v}_1\|_{L_{p,b}(H^1_6)} + \|\partial_t(\theta_1-\theta_2)\|_{L_{p,b}(H^1_6)}\|\nabla \mathbf{v}_1\|_{L_{p,b}(H^1_q)}\\
&+ (\|\theta_0\|_{H^1_q} + \|\partial_t\theta_2\|_{L_{p,b}(H^1_q)})\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)} + (\|\theta_0\|_{H^1_6} + \|\partial_t\theta_2\|_{L_{p,b}(H^1_6)})\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_q)}\\
&+ \|\partial_t(\theta_1-\theta_2)\|_{L_{p,b}(H^1_q)}\|\nabla \mathbf{v}_1\|^2_{L_{p,b}(H^1_6)} + \|\partial_t(\theta_1-\theta_2)\|_{L_{p,b}(H^1_6)}\|\nabla \mathbf{v}_1\|_{L_{p,b}(H^1_q)}\|\nabla \mathbf{v}_1\|_{L_{p,b}(H^1_6)}\\
&+ (\|\theta_0\|_{H^1_q} + \|\partial_t\theta_2\|_{L_{p,b}(H^1_q)})(\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)} + \sum_{i=1,2}\|\nabla \mathbf{v}_1\|_{L_{p,b}(H^1_6)}\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)})\\
&\qquad\qquad \times \|\nabla \mathbf{v}_1\|_{L_{p,b}(H^1_6)}\\
&+ (\|\theta_0\|_{H^1_6} + \|\partial_t\theta_2\|_{L_{p,b}(H^1_6)})(\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_q)} + \sum_{i=1,2}\|\nabla \mathbf{v}_1\|_{L_{p,b}(H^1_q)}\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)})\\
&\qquad\qquad \times \|\nabla \mathbf{v}_1\|_{L_{p,b}(H^1_6)}\\
&+ (\|\theta_0\|_{H^1_6} + \|\partial_t\theta_2\|_{L_{p,b}(H^1_6)})(\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)} + \sum_{i=1,2}\|\nabla \mathbf{v}_1\|_{L_{p,b}(H^1_6)}\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)})\\
&\qquad\qquad \times \|\nabla \mathbf{v}_1\|_{L_{p,b}(H^1_q)}\\
&+ (\|\theta_0\|_{H^1_q} + \|\partial_t\theta_2\|_{L_{p,b}(H^1_q)})\|\nabla \mathbf{v}_2\|_{L_{p,b}(H^1_6)}\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)}\\
&+ (\|\theta_0\|_{H^1_6} + \|\partial_t\theta_2\|_{L_{p,b}(H^1_6)})\|\nabla \mathbf{v}_2\|_{L_{p,b}(H^1_q)}\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)}\\
&+ (\|\theta_0\|_{H^1_6} + \|\partial_t\theta_2\|_{L_{p,b}(H^1_6)})\|\nabla \mathbf{v}_2\|_{L_{p,b}(H^1_6)}\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_q)}\}. \qquad (47)
\end{aligned}
$$

We next estimate $\|\mathbf{G}(\theta,\mathbf{v})\|_{L_{p,b}(L_r)}$ and $\|\mathbf{G}(\theta_1,\mathbf{v}_1)-\mathbf{G}(\theta_2,\mathbf{v}_2)\|_{L_{p,b}(L_r)}$. For this purpose, we use the estimates:

$$
\begin{aligned}
\|fg\|_{L_{p,b}(L_r)} &\le \|f\|_{L_\infty(L_{2+\sigma})}\|g\|_{L_{p,b}(L_2)},\\
\|fgh\|_{L_{p,b}(L_r)} &\le \|f\|_{L_\infty(L_\infty)}\|g\|_{L_\infty(L_{2+\sigma})}\|h\|_{L_{p,b}(L_2)}.
\end{aligned} \qquad (48)
$$

Employing the same argument as in (40) and (41) and using $\mathbf{V}_i(0)=0$ $(i=0,1)$, for $i=0,1$ we have

$$
\begin{aligned}
&\|\mathbf{V}_i(\mathbf{k})\|_{L_\infty(L_q)} \le \sup_{|\mathbf{k}|<\delta}|d\mathbf{V}_i(\mathbf{k})|\int_0^T \|\nabla\mathbf{v}(\cdot,s)\|_{L_q} \le C\|\nabla\mathbf{v}\|_{L_{p,b}(L_q)};\\
&\|\mathbf{V}_i(\mathbf{k}_{\mathbf{v}_1})-\mathbf{V}_i(\mathbf{k}_{\mathbf{v}_2})\|_{L_\infty(L_q)} \le C\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(L_q)},
\end{aligned} \qquad (49)
$$

where $q=2,2+\sigma$ and 6. Moreover, $\|\mathbf{V}_2(\mathbf{k})\|_{L_\infty(L_\infty)}=\sup_{|\mathbf{k}|<\delta}|\mathbf{V}_1(\mathbf{k})|$,

$$
\begin{aligned}
&\|\mathbf{V}_i(\mathbf{k})\|_{L_\infty(L_\infty)} \le \sup_{|\mathbf{k}|<\delta}|d\mathbf{V}_i(\mathbf{k})|\int_0^T \|\nabla\mathbf{v}(\cdot,s)\|_{H^1_6} \le C\|\nabla\mathbf{v}\|_{L_{p,b}(H^1_6)}; \quad (i=0,1),\\
&\|\mathbf{V}_i(\mathbf{k}_{\mathbf{v}_1})-\mathbf{V}_i(\mathbf{k}_{\mathbf{v}_2})\|_{L_\infty(L_\infty)} \le C\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)} \quad (i=0,1,2)
\end{aligned}
$$

as follows from $|\mathbf{V}_2(\mathbf{k}_{\mathbf{v}_1})-\mathbf{V}_2(\mathbf{k}_{\mathbf{v}_2})| \le \sup_{|\mathbf{k}|\le\delta}|(d\mathbf{V}_i)(\mathbf{k})||\mathbf{k}_{\mathbf{v}_1}-\mathbf{k}_{\mathbf{v}_2}|$. Writing

$$
\begin{aligned}
\mathfrak{p}'(\rho_*+\theta)-\mathfrak{p}'(\rho_*) &= \int_0^1 \mathfrak{p}''(\rho_*+\tau\theta)\,d\tau\,\theta,\\
\mathfrak{p}'(\rho_*+\theta_1)-\mathfrak{p}'(\rho_*+\theta_2) &= \int_0^1 \mathfrak{p}''(\rho_*+\theta_2+\tau(\theta_1-\theta_2))\,d\tau\,(\theta_1-\theta_2),
\end{aligned}
$$

by (36) and (42) we have

$$
\begin{aligned}
&\|(\mathfrak{p}'(\rho_*+\theta)-\mathfrak{p}'(\rho_*))\nabla\theta\|_{L_{p,b}(L_r)} \leq C(\|\theta_0\|_{L_{2+\sigma}}+\|\partial_t\theta\|_{L_{p,b}(L_{2+\sigma})})\|\nabla\theta\|_{L_{p,b}(L_2)},\\
&\|(\mathfrak{p}'(\rho_*+\theta_1)-\mathfrak{p}'(\rho_*))\nabla\theta_1-(\mathfrak{p}'(\rho_*+\theta_2)-\mathfrak{p}'(\rho_*))\nabla\theta_2\|_{L_{p,b}(L_r)}\\
&\quad\leq C\{\|\partial_t(\theta_1-\theta_2)\|_{L_{p,b}(L_{2+\sigma})}\|\nabla\theta\|_{L_{p,b}(L_2)}+(\|\theta_0\|_{L_{2+\sigma}}+\|\partial_t\theta_2\|_{L_{p,b}(L_{2+\sigma})}\|\nabla(\theta_1-\theta_2)\|_{L_{p,b}(L_2)}\},\\
&\|(\mathfrak{p}'(\rho_*+\theta)-\mathfrak{p}'(\rho_*))\nabla\theta\|_{L_{p,b}(L_q)} \leq C(\|\theta_0\|_{H^1_6}+\|\partial_t\theta\|_{L_{p,b}(H^1_6)}\|\nabla\theta\|_{L_{p,b}(L_q)}),\\
&\|(\mathfrak{p}'(\rho_*+\theta_1)-\mathfrak{p}'(\rho_*))\nabla\theta_1-(\mathfrak{p}'(\rho_*+\theta_2)-\mathfrak{p}'(\rho_*))\nabla\theta_2\|_{L_{p,b}(L_q)}\\
&\quad\leq C\{\|\partial_t(\theta_1-\theta_2)\|_{L_{p,b}(H^1_6)}\|\nabla\theta_1\|_{L_{p,b}(L_q)}+(\|\theta_0\|_{H^1_6}+\|\partial_t\theta_2\|_{L_{p,b}(H^1_6)}\|\nabla(\theta_1-\theta_2)\|_{L_{p,b}(L_q)}\},
\end{aligned}
\tag{50}
$$

for $q=2, 2+\sigma$ and 6. Combining these estimates above, we have

$$
\begin{aligned}
&\|\mathbf{G}(\theta,\mathbf{v})\|_{L_{p,b}(L_r)} \leq C\{(\|\theta_0\|_{L_{2+\sigma}}+\|\partial_t\theta\|_{L_{p,b}(L_{2+\sigma})})(\|\partial_t\mathbf{v}\|_{L_{p,b}(L_2)}+\|\nabla\theta\|_{L_{p,b}(L_2)})\\
&\quad+\|\nabla\mathbf{v}\|_{L_{p,b}(L_{2+\sigma})}(\|\nabla^2\mathbf{v}\|_{L_{p,b}(L_2)}+\|\nabla\theta\|_{L_{p,b}(L_2)})\};
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
&\|\mathbf{G}(\theta_1,\mathbf{v}_1)-\mathbf{G}(\theta_2,\mathbf{v}_2)\|_{L_{p,b}(L_r)} \leq C\{\|\partial_t(\theta_1-\theta_2)\|_{L_{p,b}(L_{2+\sigma})}\|\partial_t\mathbf{v}_1\|_{L_{p,b}(L_2)}\\
&\quad+(\|\theta_0\|_{L_{2+\sigma}}+\|\partial_t\theta_2\|_{L_{p,b}(L_{2+\sigma})})\|\partial_t(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(L_2)}+\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(L_2)}\|\nabla^2\mathbf{v}_1\|_{L_{p,b}(L_{2+\sigma})}\\
&\quad+\|\nabla\mathbf{v}_2\|_{L_{p,b}(L_{2+\sigma})}\|\nabla^2(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(L_2)}+\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(L_2)}\|\nabla^2\mathbf{v}_1\|_{L_{p,b}(L_{2+\sigma})}\|\nabla\mathbf{v}_1\|_{L_{p,b}(H^1_6)}\\
&\quad+\|\nabla^2(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(L_2)}\|\nabla\mathbf{v}_1\|_{L_{p,b}(L_{2+\sigma})}+\|\nabla^2\mathbf{v}_2\|_{L_{p,b}(L_{2+\sigma})}\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(L_2)}\\
&\quad+\|\partial_t(\theta_1-\theta_2)\|_{L_{p,b}(L_2)}\|\nabla\theta_1\|_{L_{p,b}(L_{2+\sigma})}+\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(L_2)}\|\nabla\theta_1\|_{L_{p,b}(L_{2+\sigma})}\\
&\quad+\|\nabla\mathbf{v}_2\|_{L_{p,b}(L_{2+\sigma})}\|\nabla(\theta_1-\theta_2)\|_{L_{p,b}(L_2)}+(\|\theta_0\|_{L_{2+\sigma}}+\|\partial_t\theta_2\|_{L_{p,b}(L_{2+\sigma})}\|\nabla(\theta_1-\theta_2)\|_{L_{p,b}(L_2)}.
\end{aligned}
\tag{52}
$$

Finally, we estimate $\|G(\theta,\mathbf{v})\|_{L_{p,b}(L_q)}$ and $\|G(\theta_1,\mathbf{v}_1)-\mathbf{G}(\theta_2,\mathbf{v}_2)\|_{L_{p,b}(L_q)}$ with $q=2$ and 6. For this purpose, we use the following estimates:

$$
\begin{aligned}
\|fg\|_{L_{p,b}(L_q)} &\leq C\|f\|_{L_\infty(H^1_q)}\|g\|_{L_{p,b}(L_q)},\\
\|fgh\|_{L_{p,b}(L_q)} &\leq C\|f\|_{L_\infty(L_\infty)}\|g\|_{L_\infty(H^1_6)}\|h\|_{L_{p,b}(L_q)}.
\end{aligned}
$$

And then, using (49), (50), (42) and (43), for $q=2$ and 6 we have

$$
\begin{aligned}
&\|\mathbf{G}(\theta,\mathbf{v})\|_{L_{p,b}(L_q)} \leq C\{(\|\theta_0\|_{H^1_6}+\|\partial_t\theta\|_{L_{p,b}(H^1_6)})(\|\partial_t\mathbf{v}\|_{L_{p,b}(L_q)}+\|\nabla\theta\|_{L_{p,b}(L_q)})\\
&\quad+\|\nabla\mathbf{v}\|_{L_{p,b}(H^1_6)}(\|\nabla^2\mathbf{v}\|_{L_{p,b}(L_q)}+\|\nabla\theta\|_{L_{p,b}(L_q)})\};
\end{aligned}
\tag{53}
$$

$$
\begin{aligned}
&\|\mathbf{G}(\theta_1,\mathbf{v}_1)-\mathbf{G}(\theta_2,\mathbf{v}_2)\|_{L_{p,b}(L_q)} \leq C(\|\partial_t(\theta_1-\theta_2)\|_{L_{p,b}(H^1_6)}\|\partial_t\mathbf{v}_1\|_{L_{p,b}(L_q)}\\
&\quad+(\|\theta_0\|_{H^1_6}+\|\partial_t\theta_2\|_{L_{p,b}(H^1_6)})\|\partial_t(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(L_q)}+\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)}\|\nabla^2\mathbf{v}_1\|_{L_{p,b}(L_q)}\\
&\quad+\|\nabla\mathbf{v}_2\|_{L_{p,b}(H^1_6)}\|\nabla^2(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(L_q)}+\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)}\|\nabla\mathbf{v}_1\|_{L_{p,b}(H^1_6)}\|\nabla^2\mathbf{v}_1\|_{L_{p,b}(L_q)}\\
&\quad+\|\nabla^2(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(L_q)}\|\nabla\mathbf{v}_1\|_{L_{p,b}(H^1_6)}+\|\nabla^2\mathbf{v}_2\|_{L_{p,b}(L_q)}\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)}\\
&\quad+\|\partial_t(\theta_1-\theta_2)\|_{L_{p,b}(H^1_6)}\|\nabla\theta_1\|_{L_{p,b}(L_q)}+\|\nabla(\mathbf{v}_1-\mathbf{v}_2)\|_{L_{p,b}(H^1_6)}\|\nabla\theta_1\|_{L_{p,b}(L_q)}\\
&\quad+\|\nabla\mathbf{v}_2\|_{L_{p,b}(H^1_6)}\|\nabla(\theta_1-\theta_2)\|_{L_{p,b}(L_q)}+(\|\theta_0\|_{H^1_6}+\|\partial_t\theta_2\|_{L_{p,b}(H^1_6)}\|\nabla(\theta_1-\theta_2)\|_{L_{p,b}(L_q)}).
\end{aligned}
\tag{54}
$$

## 5. A Priori Estimates for Solutions of Linearized Equations

Let $\mathcal{V}_{T,\epsilon}=\{(\theta,\mathbf{v})\in\mathcal{U}^1_T\times\mathcal{U}^2_T \mid E_T(\theta,\mathbf{v})\leq\epsilon\}$. For $(\theta,\mathbf{v})\in\mathcal{V}_{T,\epsilon}$, we consider linearized equations:

$$
\begin{aligned}
\partial_t\eta+\rho_*\mathrm{div}\,\mathbf{u} &= F(\theta,\mathbf{v}) && \text{in } \Omega\times(0,T),\\
\rho_*\partial_t\mathbf{u}-\mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u})+\nu\mathrm{div}\,\mathbf{u}\mathbf{I}-\mathfrak{p}'(\rho_*)\eta) &= \mathbf{G}(\theta,\mathbf{v}) && \text{in } \Omega\times(0,T),\\
\mathbf{u}|_\Gamma=0,\quad (\eta,\mathbf{u})|_{t=0} &= (\theta_0,\mathbf{v}_0) && \text{in } \Omega.
\end{aligned}
\tag{55}
$$

We first show that Eq. (55) admit unique solutions $\eta$ and $\mathbf{u}$ with

$$\begin{aligned}&\eta \in H^1_p((0,T), H^1_2(\Omega)\cap H^1_6(\Omega)),\\ &\mathbf{u} \in H^1_p((0,T), L_2(\Omega)^3 \cap L_6(\Omega)^3) \cap L_p((0,T), H^2_2(\Omega)^3 \cap H^2_6(\Omega)^3)\end{aligned} \tag{56}$$

possessing the estimate:

$$E_T(\eta,\mathbf{u}) \leq C(\epsilon^2+\epsilon^3+\epsilon^4) \tag{57}$$

with some constant $C$ independent of $T$ and $\epsilon$.

To prove (57), we divide $\eta$ and $\mathbf{u}$ into two parts: $\eta=\eta_1+\eta_2$ and $\mathbf{u}=\mathbf{u}_1+\mathbf{u}_2$, where $\eta_1$ and $\mathbf{u}_1$ are solutions of time shifted equations:

$$\begin{aligned}\partial_t\eta_1+\lambda_1\eta_1+\rho_*\mathrm{div}\,\mathbf{u}_1 &= F(\theta,\mathbf{v}) && \text{in } \Omega\times(0,T),\\ \rho_*(\partial_t\mathbf{u}_1+\lambda\mathbf{u}_1)-\mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}_1)+\nu\mathrm{div}\,\mathbf{u}_1\mathbf{I}-\mathfrak{p}'(\rho_*)\eta_1) &= \mathbf{G}(\theta,\mathbf{v}) && \text{in } \Omega\times(0,T),\\ \mathbf{u}_1|_\Gamma=0, \quad (\eta_1,\mathbf{u}_1)|_{t=0} &= (0,0) && \text{in } \Omega,\end{aligned} \tag{58}$$

and $\eta_2$ and $\mathbf{u}_2$ are solutions to compensation equations:

$$\begin{aligned}\partial_t\eta_2+\rho_*\mathrm{div}\,\mathbf{u}_2 &= \lambda_1\eta_1 && \text{in } \Omega\times(0,T),\\ \rho_*\partial_t\mathbf{u}_2-\mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}_2)+\nu\mathrm{div}\,\mathbf{u}_2\mathbf{I}-\mathfrak{p}'(\rho_*)\eta_2) &= \rho_*\lambda_1\mathbf{u}_1 && \text{in } \Omega\times(0,T),\\ \mathbf{u}_2|_\Gamma=0, \quad (\eta_2,\mathbf{u}_2)|_{t=0} &= (\theta_0,\mathbf{v}_0) && \text{in } \Omega.\end{aligned} \tag{59}$$

We first treat with Eq. (58). For this purpose, we use results stated in Sect. 3. We consider resolvent problem corresponding to Eq. (55) given as follows:

$$\begin{aligned}\lambda\zeta+\rho_*\mathrm{div}\,\mathbf{w} &= f && \text{in } \Omega,\\ \rho_*\lambda\mathbf{w}-\mathrm{Div}\,(\mu\mathbf{D}(\mathbf{w})+\nu\mathrm{div}\,\mathbf{w}\mathbf{I}-\mathfrak{p}'(\rho_*)\zeta) &= \mathbf{g} && \text{in } \Omega,\\ \mathbf{w}|_\Gamma &= 0 && \text{in } \Omega.\end{aligned} \tag{60}$$

Enomoto and Shibata [7] proved the existence of $\mathcal{R}$ bounded solution operators associated with (60). Namely, we know the following theorem.

**Theorem 15.** *Let $\Omega$ be a uniform $C^2$ domain in $\mathbb{R}^N$. Let $0<\omega<\pi/2$ and $1<q<\infty$. Set $H^{1,0}_q(\Omega)=H^1_q(\Omega)\times L_q(\Omega)^3$ and $H^{1,2}_q(\Omega)=H^1_q(\Omega)\times H^2_q(\Omega)^3$. Then, there exist a large number $\lambda_0>0$ and operator families $\mathcal{P}(\lambda)$ and $\mathcal{S}(\lambda)$ with*

$$\mathcal{P}(\lambda)\in \mathrm{Hol}\,(\Sigma_{\omega,\lambda_0}, \mathcal{L}(H^{1,0}_q(\Omega), H^1_q(\Omega))), \quad \mathcal{S}(\lambda)\in \mathrm{Hol}\,(\Sigma_{\omega,\lambda_0}, \mathcal{L}(H^{1,0}_q(\Omega), H^2_q(\Omega)))$$

*such that for any $\lambda\in\Sigma_{\omega,\lambda_0}$ and $(f,\boldsymbol{g})\in H^{1,0}_q(\Omega)$, $\zeta=\mathcal{P}(\lambda)(f,\boldsymbol{g})$ and $\boldsymbol{w}=\mathcal{S}(\lambda)(f,\boldsymbol{g})$ are unique solutions of Stokes resolvent problem (60) and*

$$\mathcal{R}_{\mathcal{L}(H^{1,0}_q(\Omega),H^1_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\lambda^k\mathcal{P}(\lambda)) \mid \lambda\in\Sigma_{\omega,\lambda_0}\}) \leq r_b,$$
$$\mathcal{R}_{\mathcal{L}(H^{1,0}_q(\Omega),H^{2-j}_q(\Omega)^3)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{S}(\lambda)) \mid \lambda\in\Sigma_{\omega,\lambda_0}\}) \leq r_b$$

*for $\ell=0,1$, $k=0,1$ and $j=0,1,2$.*

From Theorem 14 we have the following theorem.

**Theorem 16.** *Let $1<p,q<\infty$. Let $b\geq 0$. Then, there exists a large constant $\lambda_1>0$ such that for any $(f,\boldsymbol{g})$ with $<t>^b (f,\boldsymbol{g})\in L_p((0,T),H^{1,0}_q(\Omega))$, problem:*

$$\begin{aligned}\partial_t\rho+\lambda_1\rho+\rho_*\mathrm{div}\,\boldsymbol{w} &= f && \textit{in } \Omega\times(0,T),\\ \rho_*(\partial_t\boldsymbol{w}+\lambda_1\boldsymbol{w})-\mathrm{Div}\,(\mu\boldsymbol{D}(\boldsymbol{w})+\nu\mathrm{div}\,\boldsymbol{w}\boldsymbol{I}-\mathfrak{p}'(\rho_*)\rho) &= \boldsymbol{g} && \textit{in } \Omega\times(0,T),\\ \boldsymbol{w}|_\Gamma=0, \quad (\rho,\boldsymbol{w})|_{t=0} &= (0,0) && \textit{in } \Omega,\end{aligned} \tag{61}$$

*admits unique solutions* $\rho \in H^1_p((0,T), H^1_q(\Omega))$ *and* $\boldsymbol{w} \in H^1_p((0,T), L_q(\Omega)^3) \cap L_p((0,T), H^2_q(\Omega)^3)$ *possessing the estimate:*

$$\| <t>^b (\rho, \partial_t\rho)\|_{L_p((0,T),H^1_q(\Omega))} + \| <t>^b \partial_t \boldsymbol{w}\|_{L_p((0,T),L_q(\Omega))} + \| <t>^b \boldsymbol{w}\|_{L_p((0,T),H^2_q(\Omega))} \\ \leq C\| <t>^b (f, \boldsymbol{g})\|_{L_p((0,T),H^{1,0}_q(\Omega))}.$$

*Here,* $C$ *is a constant independent of* $T > 0$*.*

*Proof.* Our situation is that $Bu = u$ and $g = 0$ in Sect. 3. Let $f_0$ and $\mathbf{g}_0$ be the zero extensions of $f$ and $\mathbf{g}$ outside of $(0,T)$. Applying Theorem 14 yields the unique existence of solutions $\rho$ and $\mathbf{w}$ defined on the whole time interval $\mathbb{R}$ possessing the estimate (26). But, what $f_0$ and $\mathbf{g_0}$ vanish for $t < 0$ implies that $\rho$ and $\mathbf{w}$ also vanish for $t < 0$, which can be proved by using the uniqueness argument due to Saito [11, Sect. 7]. Thus, these $\rho$ and $\mathbf{w}$ are required solutions to Eq. (61). This completes the proof of Theorem 16. □

We now consider Eq. (59). The corresponding Cauchy problem is equations:

$$\begin{aligned} \partial_t\zeta + \rho_*\mathrm{div}\,\mathbf{z} = 0 \qquad & \text{in } \Omega \times (0,T), \\ \rho_*\partial_t\mathbf{z} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{z}) + \nu\mathrm{div}\,\mathbf{z}\mathbf{I} - \mathfrak{p}'(\rho_*)\zeta) = 0 \qquad & \text{in } \Omega \times (0,T), \\ \mathbf{z}|_\Gamma = 0, \quad (\zeta, \mathbf{z})|_{t=0} = (\theta_0, \mathbf{v}_0) \qquad & \text{in } \Omega. \end{aligned} \tag{62}$$

As was seen in Sect. 3, Theorem 15 implies generation of continuous analytic semigroup $\{T(t)\}_{t\geq 0}$ associated with equations (62). Thus, by Duhamel's principle we have

$$(\eta_2, \mathbf{u}_2) = T(t)(\theta_0, \mathbf{v}_0) + \int_0^t T(t-s)(\lambda_1\eta_1(\cdot,s), \rho_*\lambda_1\mathbf{u}_1(\cdot,s))\,ds. \tag{63}$$

Now, we shall estimate $(\eta_1, \mathbf{u}_1)$ and $(\eta_2, \mathbf{u}_2)$. Applying Theorem 16 to Eq. (58) yields that

$$\begin{aligned} \| <t>^b \partial_t(\eta_1, \mathbf{u}_1)\|_{L_p((0,T),H^{1,0}_q(\Omega))} + \| <t>^b (\eta_1, \mathbf{u}_1)\|_{L_p((0,T),H^{1,2}_q(\Omega))} \\ \leq C\| <t>^b (F(\theta,\mathbf{v}), \mathbf{G}(\theta,\mathbf{v}))\|_{L_p((0,T),H^{1,0}_q(\Omega))} \end{aligned} \tag{64}$$

for $q = r$, 2 and 6. Recalling that $\|(\theta_0, \mathbf{v}_0)\|_{\mathcal{I}} \leq \epsilon^2$ and $E_T(\theta, \mathbf{v}) \leq \epsilon$, by (34), (44), (46), (51), (53), and (64), we have

$$\| <t>^b \partial_t(\eta_1, \mathbf{u}_1)\|_{L_p((0,T),H^{1,0}_q(\Omega))} + \| <t>^b (\eta_1, \mathbf{u}_1)\|_{L_p((0,T),H^{1,2}_q(\Omega))}) \leq C(\epsilon^2 + \epsilon^3 + \epsilon^4). \tag{65}$$

for $q = r$, 2, and 6. Here, $C$ is a constant independent of $T$ and $\epsilon$. By the trace method of real interpolation theorem,

$$\| <t>^b \mathbf{u}_1\|_{L_\infty((0,T),L_q(\Omega))} \leq C(\| <t>^b \partial_t\mathbf{u}_1\|_{L_p((0,T),L_q(\Omega))} + \| <t>^b \mathbf{u}_1\|_{L_p((0,T),H^2_q(\Omega))}),$$

and so by (65),

$$\| <t>^b \mathbf{u}_1\|_{L_\infty((0,T),L_q(\Omega))} \leq C(\epsilon^2 + \epsilon^3 + \epsilon^4), \tag{66}$$

for $q = 2$ and 6, which, combined with (65), yields that

$$E_T(\eta_1, \mathbf{u}_1) \leq C(\epsilon^2 + \epsilon^3 + \epsilon^4) \tag{67}$$

with some constant $C > 0$ independent of $T \in (0, \infty)$.

To estimate $\eta_2$ and $\mathbf{u}_2$, we shall use the following $L_p$-$L_q$ decay estimates due to Enomoto and Shibata [8]. Setting $(\theta, \mathbf{v}) = T(t)(f, \mathbf{g})$, we have

$$\begin{aligned} \|(\theta, \mathbf{v})(\cdot,t)\|_{L_p(\Omega)} &\leq C_{p,q}t^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}[(f,\mathbf{g})]_{p,q} \quad (t>1); \\ \|\nabla(\theta, \mathbf{v})(\cdot,t)\|_{L_p(\Omega)} &\leq C_{p,q}t^{-\sigma(p,q)}[(f,\mathbf{g})]_{p,q} \quad (t>1); \\ \|\nabla^2\mathbf{v}(\cdot,t)\|_{L_p(\Omega)} &\leq C_{p,q}t^{-\frac{3}{2q}}[(f,\mathbf{g})]_{p,q} \quad (t>1); \\ \|\partial_t(\theta, \mathbf{v})(\cdot,t)\|_{L_p(\Omega)} &\leq Ct^{-\frac{3}{2q}}[(f,\mathbf{g})]_{p,q} \quad (t>1). \end{aligned} \tag{68}$$

Here, $1 \le q \le 2 \le p < \infty$, $[(f,\mathbf{g})]_{p,q} = \|(f,\mathbf{g})\|_{H^{1,0}_p(\Omega)} + \|(f,\mathbf{g})\|_{L_q(\Omega)}$, and

$$\sigma(p,q) = \frac{3}{2}\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{1}{2} \quad (2 \le p \le 3), \quad \text{and} \quad \frac{3}{2q} \quad (p \ge 3).$$

Moreover, we use

$$\|(\theta,\mathbf{v})(\cdot,t)\|_{H^{1,2}_q(\Omega)} \le M\|(f,\mathbf{g})\|_{H^{1,2}_q(\Omega)} \quad (0 < t < 2), \tag{69}$$

for $q = 2$, $2+\sigma$, and 6, which follows from standard estimates for continuous analytic semigroup. In (63), we set

$$(\eta_2^1, \mathbf{u}_2^1) = T(t)(\theta_0, \mathbf{v}_0), \quad (\eta_2^2, \mathbf{u}_2^2) = \int_0^t T(t-s)(\lambda_1\eta_1(\cdot,s), \rho_*\lambda_1\mathbf{u}_1(\cdot,s))\,ds.$$

Recall that

$$\ell = \frac{1}{2} + \frac{3}{2(2+\sigma)} = \frac{1}{2} + \frac{3}{2}\Big(\frac{1}{2} + \frac{1}{2+\sigma} - \frac{1}{2}\Big) = \frac{3}{2}\Big(\frac{1}{2} + \frac{1}{2+\sigma} - \frac{1}{6}\Big),$$

and

$$\sigma(2,r) = \ell, \quad \sigma(2+\sigma,r) = \frac{1}{2} + \frac{3}{4} > \ell, \quad \sigma(6,r) = \ell, \quad \ell \le 3/2r.$$

In particular, we use (68) estimate with decay rate $\ell$, replacing $q$ with $r$, except for the first inequality in (68).

We first consider the case where $T > 2$. Direct use of (68) with $q = r$ for $t \in (1,T)$ and (69) for $t \in (0,2]$ yields immediately that

$$E_T(\eta_2^1, \mathbf{u}_2^1) \le C\|(\theta_0,\mathbf{v}_0)\|_{\mathcal{I}} \le C\epsilon^2. \tag{70}$$

Here, the estimate of $\sup_{1<t<T} t^b\|\eta_2^1(\cdot,t), \mathbf{u}_2^1(\cdot,t)\|_{L_q(\Omega)}$ is a little bit exceptional. In fact, since $b \le \ell - 1/2 \le (3/2)(1/r - 1/q)$ for $q = 2$ and 6 as follows from (5), we have

$$\sup_{1<t<T} t^b\|(\eta_2^1(\cdot,t), \mathbf{u}_2^1(\cdot,t))\|_{L_q(\Omega)} \le C(\|(\theta_0,\mathbf{v}_0)\|_{L_r(\Omega)} + \|(\theta_0,\mathbf{v}_0)\|_{H^{1,0}_q(\Omega)}).$$

To estimate $(\eta_2^2, \mathbf{u}_2^2)$, we set

$$[[(\eta_1,\mathbf{u}_1)(\cdot,s)]] = \|(\eta_1,\mathbf{u}_1)(\cdot,s)\|_{L_r(\Omega)} + \sum_{q=2,6}(\|(\eta_1,\mathbf{u}_1)(\cdot,s)\|_{H^{1,2}_q(\Omega)} + \|\partial_t(\eta_1,\mathbf{u}_1)(\cdot,s)\|_{H^{1,0}_q(\Omega)}).$$

We set

$$\tilde{E}_T(\eta_1,\mathbf{u}_1) := \Big(\int_0^T (< t >^b\, [[\eta_1,\mathbf{u}_1)(\cdot,t)]])^p\,dt\Big)^{1/p},$$

and then, by (65) we have

$$\tilde{E}_T(\eta_1,\mathbf{u}_1) \le C(\epsilon^2 + \epsilon^3 + \epsilon^4). \tag{71}$$

First we consider the case: $2 \le t \le T$. Let $(\eta_3,\mathbf{u}_3) = (\nabla\eta_2^2, \bar\nabla^1\nabla\mathbf{u}_2^2)$ when $q = 2$, and $(\eta_3,\mathbf{u}_3) = (\bar\nabla^1\eta_2^2, \bar\nabla^2\mathbf{u}_2^2)$ when $q = 6$. Here, $\bar\nabla^m f = (\partial_x^\alpha f \mid |\alpha| \le m)$. And then,

$$\begin{aligned}
&\|(\eta_3,\mathbf{u}_3)(\cdot,t)\|_{L_q(\Omega)}\\
&\le C\Big\{\int_0^{t/2} + \int_{t/2}^{t-1} + \int_{t-1}^t\Big\}\|(\nabla,\bar\nabla^1\nabla) \text{ or } (\bar\nabla^1,\bar\nabla^2)T(t-s)(\lambda_1\eta_1, \rho_*\lambda_1\mathbf{u}_1)(\cdot,s)\|_{L_q(\Omega)}\,ds\\
&= I_q + II_q + III_q.
\end{aligned}$$

By (68) and $bp' > 1$ (cf. (16)), we have

$$I_q(t) \le C\int_0^{t/2} (t-s)^{-\ell}[[(\eta_1,\mathbf{u}_1)]]\,ds$$

$$
\begin{aligned}
&\leq C(t/2)^{-\ell}\int_0^{t/2} <s>^{-b}<s>^b [[(\eta_1,\mathbf{u}_1)(\cdot,s)]]\,ds\\
&\leq Ct^{-\ell}\Big(\int_0^T <s>^{-bp'}\,ds\Big)^{1/p'}\Big(\int_0^T (<s>^b [[(\eta_1,\mathbf{u}_1)(\cdot,s)]])^p\,ds\Big)^{1/p}\\
&\leq Ct^{-\ell}\tilde{E}_T(\eta_1,\mathbf{u}_1).
\end{aligned}
$$

Recalling that $(\ell-b)p>1$ (cf. (16)), we have

$$\int_1^T (<t>^b I_q(t))^p\,dt \leq C\tilde{E}_T(\eta_1,\mathbf{u}_1)^p.$$

We next estimate $II_q(t)$. By (68) we have

$$II_q(t)\leq C\int_{t/2}^{t-1}(t-s)^{-\ell}[[(\eta_1,\mathbf{u}_1)(\cdot,s)]]\,ds.$$

By Hölder's inequality and $<t>^b\leq C_b<s>^b$ for $s\in(t/2,t-1)$, we have

$$
\begin{aligned}
<t>^b II_q(t) &\leq C\int_{t/2}^{t-1}(t-s)^{-\ell/p'}(t-s)^{-\ell/p}<s>^b [[(\eta_1,\mathbf{u}_1)(\cdot,s)]]\,ds\\
&\leq C\Big(\int_{t/2}^{t-1}(t-s)^{-\ell}\,ds\Big)^{1/p'}\Big(\int_{t/2}^{t-1}(t-s)^{-\ell}(<s>^b [[(\eta_1,\mathbf{u}_1)(\cdot,s)]])^p\,ds\Big)^{1/p}.
\end{aligned}
$$

Setting $\int_1^\infty s^{-\ell}\,ds=L$, by Fubini's theorem we have

$$
\begin{aligned}
\int_2^T (<t>^b II_q(t))^p\,dt &\leq CL^{p/p'}\int_1^{T-1}(<s>^b [[(\eta_1,\mathbf{u}_1)(\cdot,s)]])^p\Big(\int_{s+1}^{2s}(t-s)^{-\ell}\,dt\Big)\,ds\\
&\leq CL^p\tilde{E}_T(\eta_1,\mathbf{u}_1)^p.
\end{aligned}
$$

Using a standard estimate (69) for continuous analytic semigroup, we have

$$III_q(t)\leq C\int_{t-1}^t \|(\eta_1,\mathbf{u}_1)(\cdot,s)\|_{H^{1,2}_q(\Omega)}\,ds\leq C\int_{t-1}^t [[(\eta_1,\mathbf{u}_1)(\cdot,s)]]\,ds.$$

Thus, employing the same argument as in estimating $II_q(t)$, we have

$$\int_2^T (<t>^b III_q(t))^p\,dt\leq C\tilde{E}_T(\eta_1,\mathbf{u}_1)^p.$$

Combining these three estimates yields that

$$\int_2^T (<t>^b \|(\eta_3,\mathbf{u}_3)(\cdot,t)\|_{L_q(\Omega)})^p\,dt\leq C\tilde{E}_T(\eta_1,\mathbf{u}_1)^p, \tag{72}$$

when $T>2$.

For $0<t<\min(2,T)$, using (69) and employing the same argument as in estimating $III_q(t)$ above, we have

$$\int_0^{\min(2,T)} (<t>^b \|(\eta_3,\mathbf{u}_3)(\cdot,t)\|_{L_q(\Omega)})^p\,dt\leq C\tilde{E}_T(\eta_1,\mathbf{u}_1)^p,$$

which, combined with (72), yields that

$$\int_0^T (<t>^b \|(\eta_3,\mathbf{u}_3)(\cdot,t)\|_{L_q(\Omega)})^p\,dt\leq C\tilde{E}_T(\eta_1,\mathbf{u}_1)^p \tag{73}$$

for $q=2$ and 6.

Since

$$\partial_t(\eta_2^2,\mathbf{u}_2^2)=-\lambda_1(\eta_1,\rho_*\mathbf{u}_1)(\cdot,t)-\lambda_1\int_0^t \partial_t T(t-s)(\eta_1,\rho_*\mathbf{u}_1)(\cdot,s)\,ds,$$

employing the same argument as in proving (73), we have

$$\int_0^T (< t >^b \|\partial_t(\eta_2^2, \mathbf{u}_2^2)(\cdot,t)\|_{L_q(\Omega)})^p\,dt \le C\tilde{E}_T(\eta_1,\mathbf{u}_1)^p \tag{74}$$

for $q = 2$ and 6.

We now estimate $\sup_{2<t<T} < t >^b \|(\eta_2^2, \mathbf{u}_2^2)\|_{L_q(\Omega)}$ for $q = 2$ and 6. Let $q = 2$ and 6 in what follows. For $2 < t < T$,

$$\begin{aligned}\|(\eta_2^2, \mathbf{u}_2^2)(\cdot,t)\|_{L_q(\Omega)} &\le C\Big\{\int_0^{t/2} + \int_{t/2}^{t-1} + \int_{t-1}^t\Big\}\|T(t-s)(\lambda_1\eta_1, \lambda_1\rho_*\mathbf{u}_1)(\cdot,s)\|_{L_q(\Omega)}\,ds\\ &= I_{q,0} + II_{q,0} + III_{q,0}.\end{aligned}$$

By (68), we have

$$\begin{aligned}I_{q,0}(t) &\le C\int_0^{t/2}(t-s)^{-3/2(2+\sigma)}[[(\eta_1,\mathbf{u}_1)(\cdot,s)]]\,ds\\ &\le C(t/2)^{-3/2(2+\sigma)}\int_0^{t/2} < s >^{-b} < s >^b [[(\eta_1,\mathbf{u}_1)(\cdot,s)]]\,ds\\ &\le Ct^{-3/2(2+\sigma)}\Big(\int_0^\infty < s >^{-p'b}\,ds\Big)^{1/p'}\tilde{E}_T(\eta_1,\mathbf{u}_1).\end{aligned}$$

Noting that $(3/2(2+\sigma))p' > bp' > 1$ and using (68), we have

$$\begin{aligned}II_{q,0}(t) &\le C\int_{t/2}^{t-1}(t-s)^{-3/2(2+\sigma)}\|(\eta_1,\mathbf{u}_1)(\cdot,s)]]\,ds\\ &\le C\Big(\int_{t/2}^{t-1}((t-s)^{-3/2(2+\sigma)} < s >^{-b})^{p'}\,ds\Big)^{1/p'}\Big(\int_{t/2}^{t-1}(< s >^b [[(\eta_1,\mathbf{u}_1)(\cdot,s)]])^p\,ds\Big)^{1/p}\\ &\le C < t >^{-b} \tilde{E}_T(\eta_1,\mathbf{u}_1).\end{aligned}$$

By (69), we have

$$\begin{aligned}III_{q,0}(t) &\le C\int_{t-1}^t [[(\eta_1,\mathbf{u}_1)(\cdot,s)]]\,ds\\ &\le C < t >^{-b}\int_{t-1}^t < s >^b [[(\eta_1,\mathbf{u}_1)(\cdot,s)]]\,ds\\ &\le C < t >^{-b}\Big(\int_{t-1}^t ds\Big)^{1/p'}\tilde{E}_T(\eta_1,\mathbf{u}_1).\end{aligned}$$

Since $b < 3/2(2+\sigma)$, combining these estimates yields that

$$\sup_{2<t<T} < t >^b \|(\eta_2^2, \mathbf{u}_2^2)(\cdot,t)\|_{L_q(\Omega)} \le C\tilde{E}_T(\eta_1,\mathbf{u}_1). \tag{75}$$

For $0 < t < \min(2,T)$, by standard estimate (69) of continuous analytic semigroup, we have

$$\sup_{0<t<\min(2,T)} < t >^b \|(\eta_2^2, \mathbf{u}_2^2)(\cdot,t)\|_{L_q(\Omega)} \le C\tilde{E}_T(\eta_1,\mathbf{u}_1)$$

which, combined with (75), yields that

$$\| < t >^b (\eta_2^2, \mathbf{u}_2^2)(\cdot,t)\|_{L_\infty((0,T),L_q(\Omega)} \le C\tilde{E}_T(\eta_1,\mathbf{u}_1) \tag{76}$$

for $q = 2$ and 6.

Recalling that $\eta = \eta_1 + \eta_2$ and $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, noting that $E_T(\eta_1,\mathbf{u}_1) \le C(\tilde{E}_T(\eta_1,\mathbf{u}_1) + \|(\theta_0,\mathbf{v}_0)\|_{\mathcal{I}})$ as follows from (66), and combining (73), (74), (76), and (71) yield that

$$E_T(\eta,\mathbf{u}) \le C(\epsilon^2 + \epsilon^3 + \epsilon^4). \tag{77}$$

If we choose $\epsilon > 0$ so small that $C(\epsilon + \epsilon^2 + \epsilon^3) < 1$ in (77), we have $E_T(\eta, \mathbf{u}) \le \epsilon$. Moreover, by (43)

$$\sup_{t\in(0,T)} \|\eta(\cdot,t)\|_{L_\infty(\Omega)} \le C(\|\eta_0\|_{H^1_6} + \|\partial_t\eta\|_{L_p((0,T),H^1_6(\Omega))}) \le C(\epsilon^2 + \epsilon^3 + \epsilon^4).$$

Thus, choosing $\epsilon > 0$ so small that $C(\epsilon^2 + \epsilon^3 + \epsilon^4) \le \rho_*/2$, we see that $\sup_{t\in(0,T)} \|\eta(\cdot,t)\|_{L_\infty(\Omega)} \le \rho_*/2$. And also,

$$\int_0^T \|\nabla\mathbf{u}(\cdot,s)\|_{L_\infty(\Omega)}\,ds \le \Big(\int_0^\infty <s>^{-p'b}\,ds\Big)^{1/p'} \| <t>^b \nabla\mathbf{u}\|_{L_p((0,T),H^1_6(\Omega))} \le C_{p',b}(\epsilon^2 + \epsilon^3 + \epsilon^4).$$

Thus, choosing $\epsilon > 0$ so small that $C_{p',b}(\epsilon^2 + \epsilon^3 + \epsilon^4) \le \delta$, we see that $\int_0^T \|\nabla\mathbf{u}(\cdot,s)\|_{L_\infty(\Omega)}\,ds \le \delta$. From consideration above, it follows that $(\eta,\mathbf{u}) \in \mathcal{V}_{T,\epsilon}$. Let $\mathcal{S}$ be an operator defined by $\mathcal{S}(\theta,\mathbf{v}) = (\eta,\mathbf{u})$ for $(\theta,\mathbf{v}) \in \mathcal{V}_{T,\epsilon}$, and then $\mathcal{S}$ maps $\mathcal{V}_{T,\epsilon}$ into itself.

We now show that $\mathcal{S}$ is a contraction map. Let $(\theta_i,\mathbf{v}_i) \in \mathcal{V}_{T,\epsilon}$ $(i = 1, 2)$ and set $(\eta,\mathbf{u}) = (\eta_1,\mathbf{u}_1) - (\eta_2,\mathbf{u}_2) = \mathcal{S}(\theta_1,\mathbf{v}_1) - \mathcal{S}(\theta_2,\mathbf{v}_2)$, and $F = F(\theta_1,\mathbf{v}_1) - F(\theta_2,\mathbf{v}_2)$ and $\mathbf{G} = \mathbf{G}(\theta_1,\mathbf{v}_1) - \mathbf{G}(\theta_2,\mathbf{v}_2)$. And then, from (55) it follows that

$$\begin{aligned} \partial_t\eta + \rho_*\mathrm{div}\,\mathbf{u} = F \qquad & \text{in } \Omega\times(0,T),\\ \rho_*\partial_t\mathbf{u} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}) + \nu\mathrm{div}\,\mathbf{u}\mathbf{I} - \mathfrak{p}'(\rho_*)\eta) = \mathbf{G} \qquad & \text{in } \Omega\times(0,T),\\ \mathbf{u}|_\Gamma = 0, \quad (\eta,\mathbf{u})|_{t=0} = (0,0) \qquad & \text{in } \Omega. \end{aligned} \tag{78}$$

By (34), (45), (47), (52), and (54), we have

$$\|(F,\mathbf{G})\|_{L_p((0,T),H^{1,0}_r(\Omega))} + \sum_{q=2,2+\sigma,6} \|(F,\mathbf{G})\|_{L_p((0,T),H^{1,0}_q(\Omega))} \le C(\epsilon + \epsilon^2 + \epsilon^3)E_T((\theta_1,\mathbf{v}_1) - (\theta_2,\mathbf{v}_2)).$$

Applying the same argument as in proving (77) to Eq. (78) and recalling $(\eta,\mathbf{u}) = \mathcal{S}(\theta_1,\mathbf{v}_1) - S(\theta_2,\mathbf{v}_2)$, we have

$$E_T(\mathcal{S}(\theta_1,\mathbf{v}_1) - S(\theta_2,\mathbf{v}_2)) \le C(\epsilon + \epsilon^2 + \epsilon^3)E_T((\theta_1,\mathbf{v}_1) - (\theta_2,\mathbf{v}_2)),$$

for some constant $C$ independent of $\epsilon$ and $T$. Thus, choosing $\epsilon > 0$ so small that $C(\epsilon + \epsilon^2 + \epsilon^3) < 1$, we have that $\mathcal{S}$ is a contraction map on $\mathcal{V}_{T,\epsilon}$, which proves Theorem 6. Since the contraction mapping principle yields the uniqueness of solutions in $\mathcal{V}_{T,\epsilon}$, we have completed the proof of Theorem 6.

## 6. A Proof of Theorem 3

We shall prove Theorem 3 with the help of Theorem 6. In what follows, let $b$ and $p$ be the constants given in Theorem 6, and $q = 2$ and 6. As was stated in Sect. 2, the Lagrange transform (7) gives a $C^{1+\omega}$ $(\omega \in (0,1/2))$ diffeomorphism on $\Omega$ and $dx = \det(\mathbf{I} + \mathbf{k})\,dy$, where $\{x\}$ and $\{y\}$ denote respective Euler coordinates and Lagrange coordinates on $\Omega$ and $\mathbf{k} = \int_0^t \nabla\mathbf{u}(\cdot,s)\,ds$. By (8), $\|\mathbf{k}\|_{L_\infty(\Omega)} \le \delta < 1$. In particular, choosing $\delta > 0$ smaller if necessary, we may assume that $C^{-1} \le \det(\mathbf{I} + \int_0^t \nabla\mathbf{u}(\cdot,s)\,ds) \le C$ with some constant $C > 0$ for any $(x,t) \in \Omega\times(0,T)$. Let $y = X_t(x)$ be an inverse map of Lagrange transform (7), and set $\theta(x,t) = \eta(X_t(x),t)$ and $\mathbf{v}(x,t) = \mathbf{u}(X_t(x),t)$. We have

$$\|(\theta,\mathbf{v})\|_{L_q(\Omega)} \le C\|(\eta,\mathbf{u})\|_{L_q(\Omega)}.$$

Noting that $(\eta,\mathbf{u})(y,t) = (\theta,\mathbf{v})(y + \int_0^t \mathbf{u}(y,s)\,ds, t)$, the chain rule of composite functions yields that

$$\begin{aligned} &\|(\nabla(\theta,\mathbf{v})\|_{L_q(\Omega)} \le C(1 - \|\mathbf{k}\|_{L_\infty(\Omega)})^{-1}\|\nabla(\eta,\mathbf{u})\|_{L_q(\Omega)};\\ &\quad\|\nabla^2\mathbf{v}\|_{L_q(\Omega)} \le C((1 - \|\mathbf{k}\|_{L_\infty(\Omega)})^{-2}\|\nabla^2\mathbf{u}\|_{L_q(\Omega)} + (1 - \|\mathbf{k}\|_{L_\infty(\Omega)})^{-1}\|\nabla\mathbf{k}\|_{L_q(\Omega)}\|\nabla\mathbf{u}\|_{L_\infty(\Omega)}). \end{aligned}$$

Thus, using $\|\nabla\mathbf{k}\|_{L_q(\Omega)} \le C\| <t>^b \nabla^2\mathbf{u}\|_{L_p((0,T),L_q(\Omega))}$ and $\|\nabla\mathbf{u}\|_{L_\infty(\Omega)} \le C\|\nabla\mathbf{u}\|_{H^1_6(\Omega)}$, we have

$$\begin{aligned} &\| <t>^b \nabla(\theta,\mathbf{v})\|_{L_\infty((0,T),L_2(\Omega)\cap L_6(\Omega))} \le C\| <t>^b \nabla(\theta,\mathbf{v})\|_{L_\infty((0,T),L_2(\Omega)\cap L_6(\Omega))};\\ &\| <t>^b (\theta,\mathbf{v})\|_{L_p((0,T),L_6(\Omega))} \le C\| <t>^b (\theta,\mathbf{v})\|_{L_p((0,T),L_6(\Omega))}; \end{aligned}$$

$$\| <t>^b (\theta, \mathbf{v})\|_{L_\infty((0,T),L_2(\Omega)\cap L_6(\Omega))} \leq C\| <t>^b (\theta, \mathbf{v})\|_{L_p((0,T),L_2(\Omega)\cap L_6(\Omega))};$$
$$\| <t>^b \nabla^2\mathbf{v}\|_{L_p((0,T),L_2(\Omega)\cap L_6(\Omega))} \leq C(\| <t>^b \nabla^2\mathbf{u}\|_{L_p((0,T),L_2(\Omega)\cap L_6(\Omega))}$$
$$+ \| <t>^b \nabla^2\mathbf{u}\|_{L_p((0,T),L_q(\Omega))}\| <t>^b \nabla\mathbf{u}\|_{L_p((0,T),H^1_6(\Omega))}).$$

Since $\partial_t(\eta, \mathbf{u})(y,t) = \partial_t[(\theta, \mathbf{v})(y + \int_0^t \mathbf{u}(y,s)\,ds, t)] = \partial_t(\theta, \mathbf{v})(x,t) + \mathbf{u}\cdot\nabla(\theta, \mathbf{v})(x,t)$, we have

$$\|\partial_t(\theta, \mathbf{v})\|_{L_q(\Omega)} \leq C(\|\partial_t(\eta, \mathbf{u})\|_{L_q(\Omega)} + \|\mathbf{u}\|_{L_\infty(\Omega)}\|\nabla\eta\|_{L_q(\Omega)} + \|\mathbf{u}\|_{L_q(\Omega)}\|\nabla\mathbf{u}\|_{L_\infty(\Omega)}).$$

Since $\|\nabla\eta\|_{L_\infty((0,T),L_q(\Omega))} \leq \|\nabla\theta_0\|_{L_q(\Omega)} + C\| <t>^b \partial_t\eta\|_{L_p((0,T),H^1_q(\Omega))}$, we have

$$\| <t>^b \partial_t(\theta, \mathbf{v})\|_{L_p((0,T),L_q(\Omega))} \leq C(\| <t>^b \partial_t(\eta, \mathbf{u})\|_{L_p((0,T),L_q(\Omega))}$$
$$+ (\|\nabla\theta_0\|_{L_q(\Omega)} + \| <t>^b \partial_t\eta\|_{L_p((0,T),H^1_q(\Omega))})\| <t>^b \mathbf{u}\|_{L_p((0,T),H^1_6(\Omega))}$$
$$+ \| <t>^b \mathbf{u}\|_{L_\infty((0,T),L_q(\Omega))}\| <t>^b \nabla\mathbf{u}\|_{L_p((0,T),H^1_6(\Omega))}).$$

By Theorem 6, we see that there exists a small constant $\epsilon > 0$ such that if initial data $(\theta_0, \mathbf{v}_0) \in \mathcal{I}$ satisifes the compatibility condition: $\mathbf{v}_0|_\Gamma = 0$ and the smallness condition: $\|(\theta_0, \mathbf{v}_0)\|_{\mathcal{I}} \leq \epsilon^2$ then problem (1) admits unique solutions $\rho = \rho_* + \theta$ and $\mathbf{v}$ satisfying the regularity conditions (4) and $\mathcal{E}(\theta, \mathbf{v}) \leq \epsilon$. This completes the proof of Theorem 3.

## 7. Comment on the Proof

Let $N \geq 3$ and $\Omega$ be an exterior domain in $\mathbb{R}^N$. Assume that $L_p$-$L_q$ decay estmates for continuous analytic semigroup like (68) are valid. We choose $q_1 = 2$, $q_2 = 2 + \sigma$, and $q_3$ in such a way that $q_3 > N$ and

$$\frac{1}{2} + \frac{N}{2(2+\sigma)} \leq \frac{N}{2}\Big(\frac{1}{2} + \frac{1}{2+\sigma} - \frac{1}{q_3}\Big).$$

Namely, $q_3 = 6$ $(N = 3)$ and $q_3 > N \geq 2N/(N-2)$ for $N \geq 4$. If $L_1$ in space estimates hold, then the global well-posedness is established with $q_1 = q_2 = 2$. But, so far $L_1$ in space estimates does not hold, and so we have chosen $q_1 = 2$ and $q_2 = 2 + \sigma$. Let $p$ and $b$ be chosen in such a way that

$$\Big(\frac{1}{2} + \frac{N}{2(2+\sigma)} - b\Big)p > 1, \quad bp' > 1.$$

If we write equations as

$$\partial_t u - Au = f, \quad Bu = g \quad (t > 0), \quad u|_{t=0} = u_0.$$

Here, $Bu = g$ is corresponding to boundary conditions, and $f$ and $g$ are corresponding to nonlinear terms. The first reduction is that $u_1$ is a solution to equations:

$$\partial_t u_1 + \lambda_1 u_1 - Au_1 = f, \quad Bu_1 = g \quad (t \in \mathbb{R}).$$

Then, $u_1$ has the same decay properties as nonlinear terms $f$ and $g$ have. If $u_1$ does not belong to the domain of the operator $(A, B)$ (free boundary conditions or slip boundary conditions cases)), in addition we choose $u_2$ as a solution of equations:

$$\partial_t u_2 + \lambda_1 u_2 - Au_2 = \lambda_1 u_1, \quad Bu_2 = 0 \quad (t \in \mathbb{R})$$

with very large constant $\lambda_1 > 0$. Since $u_2$ belongs to the domain of operator $A$ for any $t > 0$, we choose $u_3$ as a solution of equations:

$$\partial_t u_3 - Au_3 = \lambda_1 u_2, \quad Bu_3 = 0 \quad (t > 0), \quad u_3|_{t=0} = u_0 - (u_1 + u_2)|_{t=0}.$$

And then, by the Duhamel principle, we have

$$u_3 = T(t)(u_0 - (u_1 + u_2)|_{t=0}) + \lambda_1 \int_0^t T(t-s)u_2(s)\,ds,$$

and we use $L_p$-$L_q$ decay estimate like (68) for $0 < s < t-1$ and a standard semigroup estimate for $t-1 < s < t$, that is $\|T(t-s)u_2(s)\|_{D(A)} \leq C\|u(s)\|_{D(A)}$ for $t-1 < s < t$, where $\|\cdot\|_{D(A)}$ is a domain norm.

When $N = 2$, the method above is fail, because

$$\frac{1}{2} + \frac{2}{2(2+\sigma)} < 1.$$

And so, Matsumura–Nishida method seems to be only the way to prove the global wellposedness in two dimensional exterior domains.

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