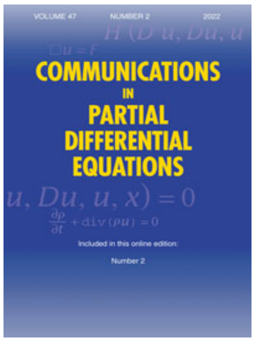

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Dissipative structure for symmetric hyperbolic-parabolic systems with Korteweg-type dispersion

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ABSTRACT
In this paper, we are concerned with generally symmetric hyperbolic-parabolic systems with Korteweg-type dispersion. Referring to those classical efforts by Kawashima et al., we formulate new structural conditions for the Korteweg-type dispersion and develop the dissipative mechanism of "regularity-gain type." As an application, it is checked that several concrete model systems (e.g., the compressible Navier-Stokes(-Fourier)-Korteweg system) satisfy the general structural conditions. In addition, the optimality of our general theory on the dissipative structure is also verified by calculating the asymptotic expansions of eigenvalues.

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1. Introduction

We consider linear symmetric hyperbolic-parabolic systems with Korteweg-type dispersion:

$$A^0 u_t + \sum_{j=1}^{n} A^j u_{x_j} = \sum_{j,k=1}^{n} B^{jk} u_{x_j x_k} + \sum_{j,k,\ell=1}^{n} D^{jk\ell} u_{x_j x_k x_\ell}, \tag{1.1}$$

where $u = u(t,x) \in \mathbb{R}^m$ is the unknown function of $t > 0$ and $x \in \mathbb{R}^n$, and A^0, A^j, B^{jk}, and $D^{jk\ell}$ are $m \times m$ real constant matrices. Here A^0 is real symmetric and positive definite, $A(\omega) = \sum_{j=1}^{n} A^j \omega_j$ is real symmetric, and $B(\omega) = \sum_{j,k=1}^{n} B^{jk} \omega_j \omega_k$ is real symmetric and nonnegative definite, where $\omega \in S^{n-1}$. We study the dissipative structure and decay property of the system (1.1) in a general framework which contains the following linearized compressible Navier-Stokes-Fourier-Korteweg system as an example:

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$$\begin{cases} \rho_t + \bar{\rho}\mathrm{div}\, u = 0, \\ \bar{\rho}u_t + \bar{p}_\rho \nabla\rho + \bar{p}_\theta \nabla\theta = \mu\Delta u + (\mu + \mu')\nabla \mathrm{div}\ \ u + \kappa\nabla\Delta\rho, \\ \bar{\rho}\bar{e}_\theta\theta_t + \bar{\theta}\bar{p}_\theta \mathrm{div}\, u = \nu\Delta\theta. \end{cases} \tag{1.2}$$

Here ρ, u, and θ denote the perturbations of the density, velocity and absolute temperature from the corresponding constant states $\bar{\rho} > 0$, $\bar{u} = 0 \in \mathbb{R}^3$ and $\bar{\theta} > 0$, respectively; we used the abbreviations $\bar{p}_\rho = p_\rho(\bar{\rho}, \bar{\theta})$, $\bar{p}_\theta = p_\theta(\bar{\rho}, \bar{\theta})$ and $\bar{e}_\theta = e_\theta(\bar{\rho}, \bar{\theta})$, where $p = p(\rho, \theta)$ is the pressure and $e = e(\rho, \theta)$ is the internal energy, and we assume that $\bar{p}_\rho > 0$ and $\bar{e}_\theta > 0$; μ and μ' are the coefficients of viscosity satisfying $\mu > 0$ and $2\mu + \mu' > 0$, $\nu > 0$ is the coefficient of heat-conduction, and $\kappa > 0$ is the coefficient of capillarity.

So far there are lots of efforts dedicated to the nonlinear Navier-Stokes(-Fourier)-Korteweg system. The rigorous derivation of the corresponding equations is due to Dunn and Serrin [1], which can be used to the phase transition. The existence of smooth solutions was known since the works by Hattori and Li [2, 3]. In contrast with the local existence, global solutions are obtained only for initial data close enough to the stable equilibrium $(\bar{\varrho}, 0)$ with convex pressure profiles. Danchin and Desjardins [4] established the global existence of strong solutions in so-called critical Besov spaces which are invariant by the scaling of Korteweg system, for initial data close enough to stable equilibria. Bresch, Desjardins, and Lin [5] proved the global existence of weak solutions in a periodic or strip domain. Later, their results were improved by Haspot in [6]. Antonelli and Spirito [7] established the global existence of finite energy weak solutions for large initial data, where vacuum regions are allowed in the definition of weak solutions. Kotschote [8] considered the initial-boundary value problem in bounded domain and proved the local existence and uniqueness of strong solutions in maximal L^p-regularity class, which are treated by Dore-Venni Theory, real interpolation and $\mathcal{H}^\infty$ calculus. Based on [2, 3], Tan and Wang [9] deduced various optimal L^2 and $L^p (p \geq 2)$ time-decay rates of solutions and their spatial derivatives. Chikami and Kobayashi [10] established the global existence and decay of strong solutions in the critical Besov spaces, where the assumption on the pressure law is not necessary monotone increasing. Charve, Danchin, and the third author [11] investigated the global existence and Gevrey analyticity in more general critical L^p framework, which is the first effort that exhibits Gevrey analyticity for a model of *compressible* fluids. Furthermore, the authors [12] developed the L^p energy methods (independent of spectral analysis), which leads to the optimal time-decay estimates of L^q-L^r type. Recently, Murata and the second author [13] addressed a totally different statement on the global existence and decay estimates, provided that the initial data belong to $\Omega_{q,p} \triangleq B^{3-2/p}_{q,p} \times B^{2(1-1/p)}_{q,p}$ whose regularity is independent of the spatial dimensions. The maximal L^p-L^q regularity to the linearized equation in $\mathbb{R}^n$ is mainly employed. See also [14] where the maximal L^p-L^q regularity for compressible fluids of Korteweg type has been established on general domains. Inspired by the work of the second author and Tanaka [15], Chen and Zhao [16] studied the global existence and nonlinear stability of stationary solutions to the compressible Navier-Stokes-Korteweg system with the external force. Bian, Yao, and Zhu [17] performed the vanishing capillarity limit of smooth solutions to the initial value problem. The convergence rate estimates are also presented for any positive time. Li and Yong [18] investigated the zero Mach number limit in the regime of smooth solutions. It was shown that smooth solutions of Navier-Stokes-Korteweg equations converged to those for incompressible Navier-Stokes equations at some convergence rate. Charve [19] studied three capillary compressible models (the classical local systems and non-local models) for large initial data bounded away from zero. He proved that these systems had a unique local in time solution and studied the corresponding convergence rate of the solutions of the non-local models toward the local Korteweg model. Germain and Lefloch [20] validated the zero viscosity-capillarity limit associated with the Navier-Stokes-Korteweg system in one dimension. Specifically, they established the existence of finite energy solutions as well as their convergence toward entropy solutions to the Euler system.

In the present paper, the main task is to study the dissipative structure and decay property of the system (1.1). To this end, as structural conditions, we formulate Craftsmanship conditions (S) and (K) (see Section 2). Under these structural conditions we prove that the dissipative structure of the system (1.1) is described by

$$\lambda(i\xi) \leq -c|\xi|^2/(1+|\xi|^2), \qquad \xi \in \mathbb{R}^n, \tag{1.3}$$

where $\lambda(i\xi)$ denotes the eigenvalue of the system (1.1) in the Fourier space. See Theorem 2.1 and Remark 2.3. Moreover, in another case where Additional condition (A) (see Section 2) is satisfied, the dissipative structure can be improved, which is described by

$$\lambda(i\xi) \leq -c|\xi|^2, \qquad \xi \in \mathbb{R}^n. \tag{1.4}$$

This dissipativity is the same as that of heat kernel and is of the "regularity-gain type." See Theorem 2.2 and Remark 2.5. We verify that the compressible Navier-Stokes-Fourier-Korteweg system (1.2) in the three-dimensional case and in the one-dimensional case are both fall into the framework of (1.4), while the one-dimensional compressible Euler-Fourier-Korteweg system (see (4.12)) is in the framework of (1.3). Furthermore, we derive the asymptotic expansions of the eigenvalues of the one-dimensional compressible Navier-Stokes-Fourier-Korteweg system (resp. the one-dimensional compressible Euler-Fourier-Korteweg system) for $\xi \to 0$ and $|\xi| \to \infty$ and show that our characterization (1.4) (resp. (1.3)) is optimal in each general framework.

The dissipative structure and the corresponding decay property were first studied in [21] for symmetric hyperbolic-parabolic systems (with symmetric diffusion and/or symmetric relaxation). It was proved in [21] that under the original Craftsmanship conditions (K) the dissipative structure of the systems is described by (1.3). Moreover, it was shown in [22] that this dissipative structure (1.3) is completely characterized by Stability condition formulated in [22]. Later, another characterization was given in [23] by using the Kalman rank condition. Rather recently, symmetric hyperbolic systems with non-symmetric relaxation were studied in [24,25] and some weaker dissipative structures of the regularity-loss type were investigated. Very recently, the dissipative structure of symmetric hyperbolic systems with memory-type symmetric diffusion or memory-type symmetric relaxation were studied in [26–28]. The dissipative structure and the decay property for systems with memory-type dissipation are interesting problems with long history.

The paper is organized as follows. In Section 2, we formulate structural conditions and state the main results on the dissipative structure of the system (1.1). The proofs are given in Section 3, which is based on the energy method in the Fourier space. In Section 4, we investigate the compressible Navier-Stokes-Fourier-Korteweg system and the compressible Euler-Fourier-Korteweg system as applications of our main results. In the final section, we verify the optimality of our general theory by calculating the asymptotic expansions of the eigenvalues of the model systems.

1.1. Notations

We denote by $\langle \cdot, \cdot \rangle$ the standard inner product in $\mathbb{R}^m$ or $\mathbb{C}^m$. For $u \in \mathbb{R}^m$ (column vector), we denote by u^T (row vector) the transpose of u. For an $m \times m$ real matrix X, we denote by X^T the transpose of X. We denote by X^{sy} and X^{asy} the symmetric part and skew-symmetric part of X, respectively. Namely, $X^{sy} = \frac{1}{2}(X + X^T)$ and $X^{asy} = \frac{1}{2}(X - X^T)$. Let $1 \le p \le \infty$. Then $L^p = L^p(\mathbb{R}^n)$ denotes the usual Lebesgue space over $\mathbb{R}^n$ with the norm $\| \cdot \|_{L^p}$. For $s \ge 0$, $H^s = H^s(\mathbb{R}^n)$ denotes the sth order Sobolev space over $\mathbb{R}^n$ in the L^2 sense, equipped with the norm $\| \cdot \|_{H^s}$. We note that $L^2 = H^0$. Finally, in this paper, we use C and c to denote generic positive constants, which may change from line to line, when the exact value of the constant is not essential.

2. Main results

In [24,25], the first author and his collaborators formulated the uniform dissipativity of type (k, l) for the dissipative system:

$$\text{Re } \lambda(i\xi) \le -c \frac{|\xi|^{2k}}{(1 + |\xi|^2)^l}, \quad \xi \in \mathbb{R}^n, \tag{2.1}$$

for some constant $c > 0$, where k, l are nonnegative integers. It calls the system is of "standard type" when $k = l$ and calls the one is of "regularity-loss type" when $k < l$.

The main purpose of this paper is to introduce a new dissipative mechanism with $k > l$, which is the so-called "regularity-gain type". To this end, we first take the Fourier transform of (1.1) to have

$$A^0 \hat{u}_t + i|\xi|A(\omega)\hat{u} + |\xi|^2 B(\omega)\hat{u} + i|\xi|^3 D(\omega)\hat{u} = 0, \tag{2.2}$$

where

$$A(\omega) = \sum_{j=1}^{n} A^j \omega_j, \quad B(\omega) = \sum_{j,k=1}^{n} B^{jk} \omega_j \omega_k, \quad D(\omega) = \sum_{j,k,\ell=1}^{n} D^{jk\ell} \omega_j \omega_k \omega_\ell$$

for $\omega = \xi/|\xi| \in \mathbb{S}^{n-1}$. We introduce the following basic assumption on the matrices A^0, $A(\omega)$ and $B(\omega)$.

Basic assumption (B).

- A^0 is real symmetric and positive definite.
- $A(\omega)$ is real symmetric for each $\omega \in \mathbb{S}^{n-1}$.

- $B(\omega)$ is real symmetric and nonnegative definite for each $\omega \in \mathbb{S}^{n-1}$ such that $\ker(B(\omega))$ is independent of $\omega \in \mathbb{S}^{n-1}$.

Also we formulate the following structural conditions.

Craftsmanship condition (S). There is an $m \times m$ real matrix $S(\omega)$ with $S(\cdot) \in C^\infty(\mathbb{S}^{n-1})$ that satisfies the following properties:

- $S(\omega)A^0$ is real symmetric and nonnegative definite for each $\omega \in \mathbb{S}^{n-1}$ such that $\ker(S(\omega)A^0)$ is independent of $\omega \in \mathbb{S}^{n-1}$.
- $S(\omega)A(\omega) + D(\omega)$ is real symmetric for each $\omega \in \mathbb{S}^{n-1}$.
- $(S(\omega)B(\omega))^{sy}$ is (real symmetric and) nonnegative definite for each $\omega \in \mathbb{S}^{n-1}$.
- $S(\omega)D(\omega)$ is real symmetric for each $\omega \in \mathbb{S}^{n-1}$.

Here and in what follows X^{sy} denotes the symmetric part of the real matrix X.

Craftsmanship condition (K). There is an $m \times m$ real matrix $K(\omega)$ with $K(\cdot) \in C^\infty(\mathbb{S}^{n-1})$ that satisfies the following properties:

- $K(\omega)A^0$ is real skew-symmetric for each $\omega \in \mathbb{S}^{n-1}$.
- $(K(\omega)A(\omega))^{sy} + B(\omega)$ is (real symmetric and) positive definite for each $\omega \in \mathbb{S}^{n-1}$.
- $(K(\omega)D(\omega))^{sy} + B(\omega)$ is (real symmetric and) nonnegative definite for each $\omega \in \mathbb{S}^{n-1}$. Moreover, $\ker((K(\omega)D(\omega))^{sy} + B(\omega))$ is independent of $\omega \in \mathbb{S}^{n-1}$ and satisfies

$$\ker((K(\omega)D(\omega))^{sy} + B(\omega)) \subseteq \ker(S(\omega)A^0),$$

where $S(\omega)$ is the matrix in Craftsmanship condition (S).

We denote by P, Q_0 and Q the orthogonal projections onto $\ker(B(\omega))$, $\ker(S(\omega)A^0)$ and $\ker((K(\omega)D(\omega))^{sy} + B(\omega))$, respectively. Then our first main result is stated as follows.

Theorem 2.1. *Assume (B), (S), and (K) hold. Then the solution u of (1.1) with the initial data u_0 satisfies the following pointwise estimate:*

$$|\hat{u}(t,\xi)|^2 + |\xi|^2|(I - Q_0)\hat{u}(t,\xi)|^2 \le Ce^{-c\rho(\xi)t}\{|\hat{u}_0(\xi)|^2 + |\xi|^2|(I - Q_0)\hat{u}_0(\xi)|^2\} \tag{2.3}$$

for $t \ge 0$ and $\xi \in \mathbb{R}^n$, where $\rho(\xi) = |\xi|^2/(1 + |\xi|^2)$.

Remark 2.1. The system (1.1) is uniformly dissipative of the type (1,1). Namely, the eigenvalues $\lambda = \lambda(i\xi)$ of the system (2.2) satisfy

$$\text{Re } \lambda(i\xi) \le -c\rho(\xi), \qquad \xi \in \mathbb{R}^n, \tag{2.4}$$

where $\rho(\xi) = |\xi|^2/(1 + |\xi|^2)$. This dissipativity is of the standard type in (2.1).

Next we consider another case by modifying Craftsmanship condition (K) as follows.

Craftsmanship condition (K)'. There is an $m \times m$ real matrix $K(\omega)$ with $K(\cdot) \in C^\infty(\mathbb{S}^{n-1})$ that satisfies the following properties:

- $K(\omega)A^0$ is real skew-symmetric for each $\omega \in \mathbb{S}^{n-1}$.
- $(K(\omega)A(\omega))^{sy} + B(\omega)$ is (real symmetric and) positive definite for each $\omega \in \mathbb{S}^{n-1}$.
- $(K(\omega)D(\omega))^{sy}$ is (real symmetric and) nonnegative definite for each $\omega \in \mathbb{S}^{n-1}$. Moreover, $\ker((K(\omega)D(\omega))^{sy})$ is independent of $\omega \in \mathbb{S}^{n-1}$ and satisfies

$$\ker((K(\omega)D(\omega))^{sy}) \subseteq \ker(S(\omega)A^0),$$

where $S(\omega)$ is the matrix in Craftsmanship condition (S).

We would like to mention that the last assumption in Conditions (K) (resp. (K)') gives the relationship between (S) and (K) (resp. (K)'). The nonnegativity is used for energy estimate (3.11) (resp. (3.18)) and the inclusion is used only for decay estimate (2.3) (resp. (2.5)). Also, we assume the following additional condition for another case.

Additional condition (A).

- $Q_0 K(\omega)A^0 Q_0 = 0$ for each $\omega \in \mathbb{S}^{n-1}$.
- $Q' K(\omega)B(\omega) = 0$ for each $\omega \in \mathbb{S}^{n-1}$.

Here we denote by P, Q_0 and Q' the orthogonal projections onto $\ker(B(\omega))$, $\ker(S(\omega)A^0)$ and $\ker((K(\omega)D(\omega))^{sy})$, respectively. Then our second main result is stated as follows.

Theorem 2.2. *Assume (B), (S), (K)' and (A) hold. Then the solution u of (1.1) with the initial data u_0 satisfies the following pointwise estimate:*

$$|\hat{u}(t,\xi)|^2 + |\xi|^2|(I-Q_0)\hat{u}(t,\xi)|^2 \le Ce^{-c|\xi|^2 t}\{|\hat{u}_0(\xi)|^2 + |\xi|^2|(I-Q_0)\hat{u}_0(\xi)|^2\} \tag{2.5}$$

for $t \ge 0$ and $\xi \in \mathbb{R}^n$.

Remark 2.2. The system (1.1) is uniformly dissipative of the type (1,0). Namely, the eigenvalues $\lambda = \lambda(i\xi)$ of the system (2.2) satisfy

$$\text{Re } \lambda(i\xi) \le -c|\xi|^2, \qquad \xi \in \mathbb{R}^n. \tag{2.6}$$

This dissipativity is essentially the same as that of the heat equation and is of the "regularity-gain type" in (2.1).

3. Energy method in the Fourier space

The aim of this section is to prove those main results. As is well known, the notion of entropy for hyperbolic conservation laws was introduced by Godunov [29] and Friedrichs-Lax [30]. The first author and his collaborator performed the entropy extension to generally hyperbolic-parabolic composite systems and developed the energy method in Fourier spaces, see [21,22]. Based on the same spirt of energy methods, we shall deal with the new context arising from the Koreteweg-type dispersion term $D(\omega)$ and get the following results on the energy estimates.

Proposition 3.1. *Let those conditions in Theorem 2.1 fulfill. Then the solution u of (1.1) with the initial data u_0 satisfies the following energy estimate:*

$$\begin{aligned} &\|u(t)\|_{H^s}^2+\|(I-Q_0)\partial_x u(t)\|_{H^s}^2+\int_0^t\|\partial_x u(\tau)\|_{H^{s-1}}^2+\|(I-Q)\partial_x^2 u(\tau)\|_{H^{s-1}}^2\,d\tau \\ &+\int_0^t\|(I-P)\partial_x u(\tau)\|_{H^s}^2\,d\tau\le C\{\|u_0\|_{H^s}^2+\|(I-Q_0)\partial_x u_0\|_{H^s}^2\} \end{aligned} \tag{3.1}$$

for $t\ge 0$, *provided that* $u_0\in H^s$ *and* $(I-Q_0)\partial_x u_0\in H^s$ *for* $s\ge 0$.

In addition, one can have the "regularity-gain type" energy estimate under different conditions.

Proposition 3.2. *Let those conditions in Theorem 2.2 fulfill. Then the solution* u *of* (1.1) *with the initial data* u_0 *satisfies the following energy estimate:*

$$\begin{aligned} &\|u(t)\|_{H^s}^2+\|(I-Q_0)\partial_x u(t)\|_{H^s}^2+\int_0^t\|\partial_x u(\tau)\|_{H^s}^2+\|(I-Q')\partial_x^2 u(\tau)\|_{H^s}^2\,d\tau \\ &\le C\{\|u_0\|_{H^s}^2+\|(I-Q_0)\partial_x u_0\|_{H^s}^2\} \end{aligned} \tag{3.2}$$

for $t\ge 0$, *provided that* $u_0\in H^s$ *and* $(I-Q_0)\partial_x u_0\in H^s$ *for* $s\ge 0$.

Proof of Theorem 2.1 and Proposition 3.1

We apply the energy method to the system (2.2) in the Fourier space. We first take the $\mathbb{C}^m$ inner product of (2.2) with $\hat{u}$. From the real part we have

$$\frac{1}{2}\frac{\partial}{\partial t}\langle A^0\hat{u},\hat{u}\rangle+|\xi|^2\langle B(\omega)\hat{u},\hat{u}\rangle+\mathrm{Re}\{i|\xi|^3\langle D(\omega)\hat{u},\hat{u}\rangle\}=0, \tag{3.3}$$

where we used the fact that A^0, $A(\omega)$, and $B(\omega)$ are real symmetric by Basic assumption. Secondly, letting $S(\omega)$ be the matrix in Craftsmanship condition (S), we multiply (2.2) by $|\xi|^2S(\omega)$ and take the inner product of the resulting equation with $\hat{u}$. From the real part we have

$$\frac{1}{2}\frac{\partial}{\partial t}\{|\xi|^2\langle S(\omega)A^0\hat{u},\hat{u}\rangle\}+\mathrm{Re}\{i|\xi|^3\langle S(\omega)A(\omega)\hat{u},\hat{u}\rangle\}+|\xi|^4\langle (S(\omega)B(\omega))^{sy}\hat{u},\hat{u}\rangle=0, \tag{3.4}$$

where we used the fact that $S(\omega)A^0$ and $S(\omega)D(\omega)$ are real symmetric by Craftsmanship condition (S). Now we add (3.3) and (3.4). Since $S(\omega)A(\omega)+D(\omega)$ is real symmetric by Craftsmanship condition (S), we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\{\langle A^0\hat{u},\hat{u}\rangle+|\xi|^2\langle S(\omega)A^0\hat{u},\hat{u}\rangle\}+|\xi|^2\langle B(\omega)\hat{u},\hat{u}\rangle+|\xi|^4\langle (S(\omega)B(\omega))^{sy}\hat{u},\hat{u}\rangle=0. \tag{3.5}$$

Thirdly, we use the matrix $K(\omega)$ in Craftsmanship condition (K). We multiply (2.2) by $-i|\xi|K(\omega)$ and take the inner product of the resulting equation with $\hat{u}$. Taking the real part, we have

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\{-|\xi|\langle iK(\omega)A^0\hat{u},\hat{u}\rangle\}+|\xi|^2\langle (K(\omega)A(\omega))^{sy}\hat{u},\hat{u}\rangle \\ &+|\xi|^4\langle (K(\omega)D(\omega))^{sy}\hat{u},\hat{u}\rangle=\mathrm{Re}\{i|\xi|^3\langle K(\omega)B(\omega)\hat{u},\hat{u}\rangle\}. \end{aligned} \tag{3.6}$$

Here we used the fact that $iK(\omega)A^0$ is Hermitian by Craftsmanship condition (K).

We now combine (3.5) and (3.6) such that $(3.5)+(3.6)\times\alpha/(1+|\xi|^2)$, where $\alpha>0$ is a constant to be determined. Then we obtain the following energy equality

$$\frac{1}{2}\frac{\partial}{\partial t}E+D=F, \tag{3.7}$$

where our Lyapunov function E, the corresponding dissipation term D and the error term F are given, respectively, by

$$\begin{cases} E=\langle A^0\hat{u},\hat{u}\rangle+|\xi|^2\langle S(\omega)A^0\hat{u},\hat{u}\rangle-\frac{\alpha|\xi|}{1+|\xi|^2}\langle iK(\omega)A^0\hat{u},\hat{u}\rangle, \\ D=|\xi|^2\langle B(\omega)\hat{u},\hat{u}\rangle+|\xi|^4\langle (S(\omega)B(\omega))^{sy}\hat{u},\hat{u}\rangle \\ \qquad +\frac{\alpha|\xi|^2}{1+|\xi|^2}\langle (K(\omega)A(\omega))^{sy}\hat{u},\hat{u}\rangle+\frac{\alpha|\xi|^4}{1+|\xi|^2}\langle (K(\omega)D(\omega))^{sy}\hat{u},\hat{u}\rangle \\ F=\frac{\alpha|\xi|^3}{1+|\xi|^2}\mathrm{Re}\{i\langle K(\omega)B(\omega)\hat{u},\hat{u}\rangle\}. \end{cases} \tag{3.8}$$

We first estimate E. Since A^0 is positive definite and $S(\omega)A^0$ is nonnegative definite, we find a positive constant α_0 such that for $\alpha\in(0,\alpha_0]$, we have

$$c_0\{|\hat{u}|^2+|\xi|^2|(I-Q_0)\hat{u}|^2\}\le E\le C_0\{|\hat{u}|^2+|\xi|^2|(I-Q_0)\hat{u}|^2\}, \tag{3.9}$$

where c_0 and C_0 are positive constants independent of α and Q_0 is the orthogonal projection onto $\ker(S(\omega)A^0)$. Next we estimate D. We have

$$\begin{aligned} D&=\alpha\rho(\xi)\{\langle(((K(\omega)A(\omega))^{sy}+B(\omega))\hat{u},\hat{u}\rangle+|\xi|^2\langle(((K(\omega)D(\omega))^{sy}+B(\omega))\hat{u},\hat{u}\rangle\} \\ &\quad+(1-\alpha)|\xi|^2\langle B(\omega)\hat{u},\hat{u}\rangle+|\xi|^4\langle(S(\omega)B(\omega))^{sy}\hat{u},\hat{u}\rangle \\ &\ge\alpha c_1\rho(\xi)\{|\hat{u}|^2+|\xi|^2|(I-Q)\hat{u}|^2\}+c_1|\xi|^2|(I-P)\hat{u}|^2 \end{aligned}$$

for $\alpha\in(0,1/2]$, where $\rho(\xi)=|\xi|^2/(1+|\xi|^2)$ and c_1 is a positive constant independent of α. Here we used the fact that $(K(\omega)A(\omega))^{sy}+B(\omega)$ is positive definite, and $(K(\omega)D(\omega))^{sy}+B(\omega)$, $B(\omega)$ and $(S(\omega)B(\omega))^{sy}$ are nonnegative definite; Q and P are the orthogonal projections onto $\ker((K(\omega)D(\omega))^{sy}+B(\omega))$ and $\ker B(\omega)$, respectively. Finally, we estimate F as

$$\begin{aligned} |F|&\le\alpha C\rho(\xi)|\xi||\hat{u}||(I-P)\hat{u}| \\ &\le\alpha\epsilon\rho(\xi)|\hat{u}|^2+\alpha C_\epsilon\rho(\xi)|\xi|^2|(I-P)\hat{u}|^2 \end{aligned}$$

for any α, $\epsilon>0$, where C_ϵ is a constant depending on ϵ but is independent of α. We take $\epsilon>0$ and $\alpha>0$ such that $\epsilon\le c_1/2$ and $\alpha C_\epsilon\le c_1/2$. Then we find that

$$D-F\ge c\rho(\xi)\{|\hat{u}|^2+|\xi|^2|(I-Q)\hat{u}|^2\}+c|\xi|^2|(I-P)\hat{u}|^2. \tag{3.10}$$

We now integrate (3.7) with respect to t. Then, using (3.9) and (3.10), we obtain the following energy estimate in the Fourier space:

$$\begin{aligned} &|\hat{u}(t,\xi)|^2+|\xi|^2|(I-Q_0)\hat{u}(t,\xi)|^2+\rho(\xi)\int_0^t|\hat{u}(\tau,\xi)|^2+|\xi|^2|(I-Q)\hat{u}(\tau,\xi)|^2\ d\tau \\ &\quad+\int_0^t|\xi|^2|(I-P)\hat{u}(\tau,\xi)|^2\ d\tau\le C\{|\hat{u}_0(\xi)|^2+|\xi|^2|(I-Q_0)\hat{u}_0(\xi)|^2\}, \end{aligned} \tag{3.11}$$

where $\hat{u}_0$ is the initial data. We multiply this estimate by $(1+|\xi|^2)^s$ and integrate over $\xi \in \mathbb{R}^n$. This yields the desired energy estimate (3.1) and hence the proof of Proposition 3.1 is complete.

Finally, we prove Theorem 2.1. Since $\ker((K(\omega)D(\omega))^{sy}+B(\omega)) \subseteq \ker(S(\omega)A^0)$ by Craftsmanship condition (K), we have $|(I-Q)\hat{u}| \geq |(I-Q_0)\hat{u}|$. Therefore we have from (3.10) that

$$D - F \geq c\rho(\xi)\{|\hat{u}|^2 + |\xi|^2|(I-Q_0)\hat{u}|^2\} \geq c\rho(\xi)E, \tag{3.12}$$

where we used (3.9) in the last inequality. We substitute (3.12) into (3.7) to get

$$\frac{\partial}{\partial t}E + c\rho(\xi)E \leq 0.$$

Solving this differential inequality, we obtain $E(t,\xi) \leq e^{-c\rho(\xi)t}E(0,\xi)$, which together with (3.9) yields the desired pointwise estimate (2.3). This completes the proof of Theorem 2.1.

We remark that the dissipativity (2.4) can be shown in the same way. In fact, we apply the same energy method to the eigenvalue problem associated with (2.2):

$$\lambda A^0\phi + i|\xi|A(\omega)\phi + |\xi|^2B(\omega)\phi + i|\xi|^3D(\omega)\phi = 0, \tag{3.13}$$

where $\lambda \in \mathbb{C}$ and $\phi \in \mathbb{C}^m$ with $\phi \neq 0$. Then, as the counterpart of $\frac{\partial}{\partial t}E + c\rho(\xi)E \leq 0$, we obtain $\mathrm{Re}\ \lambda + c\rho(\xi) \leq 0$, which shows (2.4). □

Proof of Theorem 2.2 and Proposition 3.2

In this case, we also have the energy equalities (3.5) and (3.6), where $S(\omega)$ and $K(\omega)$ are the matrices in Craftsmanship conditions (S) and (K)', respectively. We combine (3.5) and (3.6) such that $(3.5) + (3.6) \times \alpha$, where $\alpha > 0$ is a constant to be determined. Then, as the counterpart of (3.7), we obtain the following energy equality

$$\frac{1}{2}\frac{\partial}{\partial t}\tilde{E} + \tilde{D} = \tilde{F}, \tag{3.14}$$

where the Lyapunov function $\tilde{E}$, the corresponding dissipation term $\tilde{D}$ and the error term $\tilde{F}$ in this case are given respectively by

$$\begin{cases} \tilde{E} = \langle A^0\hat{u},\hat{u}\rangle + |\xi|^2\langle S(\omega)A^0\hat{u},\hat{u}\rangle - \alpha|\xi|\langle iK(\omega)A^0\hat{u},\hat{u}\rangle, \\ \tilde{D} = |\xi|^2\langle B(\omega)\hat{u},\hat{u}\rangle + |\xi|^4\langle (S(\omega)B(\omega))^{sy}\hat{u},\hat{u}\rangle \\ \qquad + \alpha|\xi|^2\langle (K(\omega)A(\omega))^{sy}\hat{u},\hat{u}\rangle + \alpha|\xi|^4\langle (K(\omega)D(\omega))^{sy}\hat{u},\hat{u}\rangle \\ \tilde{F} = \alpha|\xi|^3\mathrm{Re}\{i\langle K(\omega)B(\omega)\hat{u},\hat{u}\rangle\}. \end{cases} \tag{3.15}$$

This (3.15) is the counterpart of (3.8).

We estimate each term in (3.14). We recall that A^0 is positive definite and $S(\omega)A^0$ is nonnegative definite. Also we have $Q_0K(\omega)A^0Q_0 = 0$ by Condition (A), where Q_0 is the orthogonal projection onto $\ker(S(\omega)A^0)$. This gives

$$\begin{aligned} |\alpha|\xi|\langle iK(\omega)A^0\hat{u},\hat{u}\rangle| &\leq \alpha C|\xi||\hat{u}||(I-Q_0)\hat{u}| \\ &\leq \alpha C\{|\hat{u}|^2 + |\xi|^2|(I-Q_0)\hat{u}|^2\}. \end{aligned}$$

Consequently, we find a positive constant α_0 such that for $\alpha \in (0,\alpha_0]$, we have

$$c_0\{|\hat{u}|^2+|\xi|^2|(I-Q_0)\hat{u}|^2\}\leq\tilde{E}\leq C_0\{|\hat{u}|^2+|\xi|^2|(I-Q_0)\hat{u}|^2\}, \tag{3.16}$$

where c_0 and C_0 are positive constants independent of α. This is the counterpart of (3.9). Next we estimate $\tilde{D}$. We have

$$\begin{aligned}\tilde{D}&=\alpha|\xi|^2\{\langle((K(\omega)A(\omega))^{sy}+B(\omega))\hat{u},\hat{u}\rangle+|\xi|^2\langle(K(\omega)D(\omega))^{sy}\hat{u},\hat{u}\rangle\}\\&\quad+(1-\alpha)|\xi|^2\langle B(\omega)\hat{u},\hat{u}\rangle+|\xi|^4\langle(S(\omega)B(\omega))^{sy}\hat{u},\hat{u}\rangle\\&\geq\alpha c_1|\xi|^2\{|\hat{u}|^2+|\xi|^2|(I-Q')\hat{u}|^2\}+c_1|\xi|^2|(I-P)\hat{u}|^2\end{aligned}$$

for $\alpha\in(0,1/2]$, where c_1 is a positive constant independent of α. Here we used the fact that $(K(\omega)A(\omega))^{sy}+B(\omega)$ is positive definite, and $(K(\omega)D(\omega))^{sy}$ and $B(\omega)$ are nonnegative definite by Condition (K)'; Q' and P are the orthogonal projections onto $\ker(K(\omega)D(\omega))^{sy}$ and $\ker B(\omega)$, respectively. Finally, we estimate $\tilde{F}$ as

$$\begin{aligned}|\tilde{F}|&\leq\alpha C|\xi|^3|(I-Q')\hat{u}||(I-P)\hat{u}|\\&\leq\alpha\epsilon|\xi|^4|(I-Q')\hat{u}|^2+\alpha C_\epsilon|\xi|^2|(I-P)\hat{u}|^2\end{aligned}$$

for any $\alpha,\ \epsilon>0$, where C_ϵ is a constant depending on ϵ but is independent of α. Here we used $Q'K(\omega)B(\omega)=0$ in Condition (A). We take $\epsilon>0$ and $\alpha>0$ such that $\epsilon\leq c_1/2$ and $\alpha C_\epsilon\leq c_1$. Then we find that

$$\tilde{D}-\tilde{F}\geq c|\xi|^2\{|\hat{u}|^2+|\xi|^2|(I-Q')\hat{u}|^2\}, \tag{3.17}$$

which is the counterpart of (3.10).

We integrate (3.14) with respect to t and substitute (3.16)–(3.17) into the resultant equality. Then, as the counterpart of (3.11), we obtain the following energy estimate in the Fourier space:

$$\begin{aligned}&|\hat{u}(t,\xi)|^2+|\xi|^2|(I-Q_0)\hat{u}(t,\xi)|^2+|\xi|^2\int_0^t|\hat{u}(\tau,\xi)|^2+|\xi|^2|(I-Q')\hat{u}(\tau,\xi)|^2\ d\tau\\&\leq C\{|\hat{u}_0(\xi)|^2+|\xi|^2|(I-Q_0)\hat{u}_0(\xi)|^2\},\end{aligned} \tag{3.18}$$

where $\hat{u}_0$ is the initial data. We multiply (3.18) by $(1+|\xi|^2)^s$ and integrate over $\xi\in\mathbb{R}^n$. This yields the desired energy estimate (3.2) and hence the proof of Proposition 3.2 is complete.

Finally, we prove Theorem 2.2. Since $\ker((K(\omega)D(\omega))^{sy})\subseteq\ker(S(\omega)A^0)$ by Condition (K)', we have $|(I-Q')\hat{u}|\geq|(I-Q_0)\hat{u}|$. Therefore it follows from (3.17) that

$$\tilde{D}-\tilde{F}\geq c|\xi|^2\{|\hat{u}|^2+|\xi|^2|(I-Q_0)\hat{u}|^2\}\geq c|\xi|^2\tilde{E}, \tag{3.19}$$

where we used (3.16) in the last inequality. We substitute (3.19) into (3.14) to get

$$\frac{\partial}{\partial t}\tilde{E}+c|\xi|^2\tilde{E}\leq 0.$$

Solving this differential inequality, we obtain $\tilde{E}(t,\xi)\leq e^{-c|\xi|^2t}\tilde{E}(0,\xi)$, which together with (3.16) yields the desired pointwise estimate (2.5). This completes the proof of Theorem 2.1. We remark that the dissipativity (2.6) can be shown in the same way as before and we omit the details. □

4. Applications

In this section, we treat several concrete model systems as applications of Theorems 2.1 and 2.2 and Propositions 3.1 and 3.2.

4.1. Navier-Stokes-Fourier-Korteweg system

We first consider the compressible Navier-Stokes-Fourier-Korteweg system (1.2), where the coefficients satisfy

$$\bar\rho > 0,\ \bar\theta > 0; \quad \bar p_\rho > 0,\ \bar e_\theta > 0; \quad \mu > 0,\ 2\mu + \mu' > 0,\ \nu > 0; \quad \kappa > 0. \tag{4.1}$$

We apply the Fourier transform to (1.2) and obtain

$$\begin{cases} \hat\rho_t + i|\xi|\bar\rho(\omega\cdot\hat u) = 0, \\ \bar\rho\hat u_t + i|\xi|(\bar p_\rho\omega\hat\rho + \bar p_\theta\omega\hat\theta) + |\xi|^2(\mu\hat u + (\mu+\mu')\omega(\omega\cdot\hat u)) + i|\xi|^3\kappa\omega\hat\rho = 0, \\ \bar\rho\bar e_\theta\hat\theta_t + i|\xi|\bar\theta\bar p_\theta(\omega\cdot\hat u) + |\xi|^2\nu\hat\theta = 0, \end{cases} \tag{4.2}$$

where $\omega = (\omega_1, \omega_2, \omega_3)^T = \xi/|\xi| \in \mathbb{S}^2$ is regarded as a column vector and $\omega\cdot\hat u$ denotes the $\mathbb{C}^3$ inner product of ω and $\hat u$. The system (4.2) is written as

$$A^0\hat U_t + i|\xi|A(\omega)\hat U + |\xi|^2B(\omega)\hat U + i|\xi|^3D(\omega)\hat U = 0, \tag{4.3}$$

where $U = (\rho, u^T, \theta)^T \in \mathbb{R}^5$ ($u\in\mathbb{R}^3$ is regarded as a column vector) and

$$A^0 = \begin{pmatrix} \bar p_\rho/\bar\rho & 0 & 0 \\ 0 & \bar\rho I & 0 \\ 0 & 0 & \bar\rho\bar e_\theta/\bar\theta \end{pmatrix}, \qquad A(\omega) = \begin{pmatrix} 0 & \bar p_\rho\omega^T & 0 \\ \bar p_\rho\omega & 0 & \bar p_\theta\omega \\ 0 & \bar p_\theta\omega^T & 0 \end{pmatrix},$$
$$B(\omega) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu I + (\mu+\mu')\omega\omega^T & 0 \\ 0 & 0 & \nu/\bar\theta \end{pmatrix}, \qquad D(\omega) = \begin{pmatrix} 0 & 0 & 0 \\ \kappa\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.4}$$

We want to verify that the system (1.2) satisfies the conditions assumed in Theorem 2.2. We see that A^0, $A(\omega)$ and $B(\omega)$ in (4.4) are real symmetric. Also A^0 is positive definite and $B(\omega)$ is nonnegative definite such that $\ker(B(\omega)) = \mathrm{span}\{e_1\}$ in $\mathbb{C}^5$, where $\{e_1, ..., e_5\}$ is the standard orthogonal basis in $\mathbb{R}^5$. In fact, we have

$$\langle B(\omega)U, U\rangle = \mu|u|^2 + (\mu+\mu')|\omega\cdot u|^2 + (\nu/\bar\theta)|\theta|^2$$

for $U = (\rho, u^T, \theta)^T \in \mathbb{C}^5$, where $\rho\in\mathbb{C}$, $u\in\mathbb{C}^3$ (column vector) and $\theta\in\mathbb{C}$. When $\mu+\mu'\geq 0$, we have $\mu|u|^2 + (\mu+\mu')|\omega\cdot u|^2 \geq \mu|u|^2$. On the other hand, when $\mu+\mu'\leq 0$, we have

$$\mu|u|^2 + (\mu+\mu')|\omega\cdot u|^2 = (2\mu+\mu')|u|^2 - (\mu+\mu')(|u|^2 - |\omega\cdot u|^2) \geq (2\mu+\mu')|u|^2.$$

Thus, we arrive at

$$\langle B(\omega)U, U\rangle \geq \min\{\mu, 2\mu+\mu'\}|u|^2 + (\nu/\bar\theta)|\theta|^2. \tag{4.5}$$

Therefore, our system (1.2) satisfies Assumption (B).

Next we verify Craftsmanship condition (S) for the system (1.2). We take the matrix $S(\omega)$ as

$$S(\omega) = S := \begin{pmatrix} \kappa/\bar{p}_\rho & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.6}$$

Then we see that

$$S(\omega)A^0 = \begin{pmatrix} \kappa/\bar{\rho} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is real symmetric and nonnegative definite such that $\ker(S(\omega)A^0) = \operatorname{span}\{e_2, ..., e_5\}$. Also we have

$$S(\omega)A(\omega) = \begin{pmatrix} 0 & \kappa\omega^T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S(\omega)A(\omega) + D(\omega) = \begin{pmatrix} 0 & \kappa\omega^T & 0 \\ \kappa\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $S(\omega)A(\omega) + D(\omega)$ is real symmetric. Moreover a simple computation shows that $S(\omega)B(\omega) = 0$ and $S(\omega)D(\omega) = 0$. Thus we have verified that our system (1.2) satisfies Condition (S).

To verify Condition (K)', as in [31], we take the matrix $K(\omega)$ as follows.

$$K(\omega) = \alpha \begin{pmatrix} 0 & \omega^T & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (A^0)^{-1} = \alpha \begin{pmatrix} 0 & (1/\bar{\rho})\omega^T & 0 \\ -(\bar{\rho}/\bar{p}_\rho)\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.7}$$

where $\alpha > 0$ is a constant to be determined. Then $K(\omega)A^0$ is real skew-symmetric. Also we see that

$$K(\omega)A(\omega) = \alpha \begin{pmatrix} \bar{p}_\rho/\bar{\rho} & 0 & \bar{p}_\theta/\bar{\rho} \\ 0 & -\bar{\rho}\omega\omega^T & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, we have

$$\begin{aligned} \langle (K(\omega)A(\omega))^{sy} U, U \rangle &\geq \alpha\{(\bar{p}_\rho/\bar{\rho})|\rho|^2 - \bar{\rho}|\omega \cdot u|^2 - (|\bar{p}_\theta|/\bar{\rho})|\rho||\theta|\} \\ &\geq \alpha\{(\bar{p}_\rho/2\bar{\rho})|\rho|^2 - C(|u|^2 + |\theta|^2)\} \end{aligned}$$

for $U = (\rho, u^T, \theta)^T \in \mathbb{C}^5$, where C is a positive constant independent of α. This together with (4.5) shows that $(K(\omega)A(\omega))^{sy} + B(\omega)$ is positive definite for $\alpha > 0$ satisfying $2\alpha C \leq \min\{\mu,\ 2\mu + \mu',\ \nu/\bar{\theta}\}$. Moreover, a simple computation gives

$$K(\omega)D(\omega) = \alpha \begin{pmatrix} \kappa/\bar{\rho} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is real symmetric and nonnegative definite such that

$$\ker((K(\omega)D(\omega))^{sy}) = \operatorname{span}\{e_2, ..., e_5\} = \ker(S(\omega)A^0),$$

where $S(\omega)$ is in (4.6). Thus we have verified that our system (1.2) satisfies Condition (K)'.

Finally we check Condition A. Let Q_0 and Q' be the orthogonal projections onto $\ker(S(\omega)A^0)$ and $\ker((K(\omega)D(\omega))^{sy})$, respectively, Then

$$Q_0 = Q' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By direct computations we find that $Q_0K(\omega)A^0Q_0 = 0$ and $Q'K(\omega)B(\omega) = 0$. Therefore, our system (1.2) satisfies Additional condition. Consequently, we conclude that Theorem 2.2 and Proposition 3.2 are applicable to the system (1.2). Namely, we have

Proposition 4.1. *Let (4.1) fulfill. Then the compressible Navier-Stokes-Fourier-Korteweg system (1.2) satisfies (B), (S), (K)' and (A); in particular, the matrices $S(\omega)$ and $K(\omega)$ are given by (4.6) and (4.7), respectively. Therefore, Theorem 2.2 and Proposition 3.2 are applicable to the system (1.2). As the consequence we have:*

$$|(\hat{\rho}, \hat{u}, \hat{\theta})(t, \xi)|^2 + |\xi|^2|\hat{\rho}(t, \xi)|^2 \le Ce^{-c|\xi|^2 t}\{|(\hat{\rho}_0, \hat{u}_0, \hat{\theta}_0)(\xi)|^2 + |\xi|^2|\hat{\rho}_0(\xi)|^2\}$$

for $t \ge 0$ and $\xi \in \mathbb{R}^3$, and

$$\begin{aligned} &\|(\rho, u, \theta)(t)\|_{H^s}^2 + \|\partial_x\rho(t)\|_{H^s}^2 + \int_0^t \|\partial_x(\rho, u, \theta)(\tau)\|_{H^s}^2 + \|\partial_x^2\rho(\tau)\|_{H^s}^2 \, d\tau \\ &\le C\{\|(\rho_0, u_0, \theta_0)\|_{H^s}^2 + \|\partial_x\rho_0\|_{H^s}^2\} \end{aligned}$$

for $t \ge 0$, provided that $(\rho_0, u_0, \theta_0) \in H^s$ and $\partial_x\rho_0 \in H^s$ for $s \ge 0$.

Remark 4.1. (i) The same result holds true also for the one-dimensional compressible Navier-Stokes-Fourier-Korteweg system (4.8) given below. (ii) We have the same result also for the barotropic model of (1.2) and the corresponding one-dimensional model.

We briefly verify the above remark (i). The one-dimensional version of the system (1.2) is written in the form

$$\begin{cases} \rho_t + \bar{\rho}u_x = 0, \\ \bar{\rho}u_t + \bar{p}_\rho\rho_x + \bar{p}_\theta\theta_x = \tilde{\mu}u_{xx} + \kappa\rho_{xxx}, \\ \bar{\rho}\bar{e}_\theta\theta_t + \bar{\theta}\bar{p}_\theta u_x = \nu\theta_{xx}, \end{cases} \tag{4.8}$$

where $\tilde{\mu} := 2\mu + \mu'$ and the coefficients satisfy (4.1). This system can be rewritten as

$$A^0U_t + AU_x = BU_{xx} + DU_{xxx}, \tag{4.9}$$

where $U = (\rho, u, \theta)^T \in \mathbb{R}^3$ and

$$\begin{aligned} A^0 &= \begin{pmatrix} \bar{p}_\rho/\bar{\rho} & 0 & 0 \\ 0 & \bar{\rho} & 0 \\ 0 & 0 & \bar{\rho}\bar{e}_\theta/\bar{\theta} \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & \bar{p}_\rho & 0 \\ \bar{p}_\rho & 0 & \bar{p}_\theta \\ 0 & \bar{p}_\theta & 0 \end{pmatrix}, \\ B &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tilde{\mu} & 0 \\ 0 & 0 & \nu/\bar{\theta} \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 0 & 0 \\ \kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{4.10}$$

Compare (4.10) with (4.4). This one-dimensional system (4.8) satisfies Basic assumption. In particular, we see that $\ker(B) = \mathrm{span}\{e_1\}$ in $\mathbb{C}^3$, where $\{e_1, e_2.e_3\}$ is the standard orthogonal basis in $\mathbb{R}^3$. Also we can verify Conditions (S) and (K)' together with (A) by taking the matrices S, K, Q_0 and Q' as follows.

$$S = \begin{pmatrix} \kappa/\bar{p}_\rho & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_0 = Q' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$K = \alpha \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (A^0)^{-1} = \alpha \begin{pmatrix} 0 & 1/\bar{\rho} & 0 \\ -\bar{\rho}/\bar{p}_\rho & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.11}$$

where $\alpha > 0$ is a small constant as in Proposition 4.1. In particular, we see that $\ker((KD)^{sy}) = \ker(SA^0) = \mathrm{span}\{e_2, e_3\}$ in $\mathbb{C}^3$. We omit the detailed discussion.

4.2. Euler-Fourier-Korteweg system

In this subsection, we treat the following one-dimensional Euler-Fourier-Korteweg system:

$$\begin{cases} \rho_t + \bar{\rho} u_x = 0, \\ \bar{\rho} u_t + \bar{p}_\rho \rho_x + \bar{p}_\theta \theta_x = \kappa \rho_{xxx}, \\ \bar{\rho}\bar{e}_\theta \theta_t + \bar{\theta}\bar{p}_\theta u_x = \nu \theta_{xx}. \end{cases} \tag{4.12}$$

This system is obtained from (4.8) by putting $\tilde{\mu} := 2\mu + \mu' = 0$. Here the coefficients satisfy

$$\bar{\rho} > 0,\ \bar{\theta} > 0; \quad \bar{p}_\rho > 0,\ \bar{p}_\theta \neq 0,\ \bar{e}_\theta > 0; \quad \nu > 0; \quad \kappa > 0. \tag{4.13}$$

Compare this (4.13) with (4.1). The system (4.12) is written in the form of (4.9) with $U = (\rho, u, \theta)^T \in \mathbb{R}^3$ and the following coefficient matrices:

$$A^0 = \begin{pmatrix} \bar{p}_\rho/\bar{\rho} & 0 & 0 \\ 0 & \bar{\rho} & 0 \\ 0 & 0 & \bar{\rho}\bar{e}_\theta/\bar{\theta} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \bar{p}_\rho & 0 \\ \bar{p}_\rho & 0 & \bar{p}_\theta \\ 0 & \bar{p}_\theta & 0 \end{pmatrix},$$
$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \nu/\bar{\theta} \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ \kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.14}$$

Compare this (4.14) with (4.10).

We want to verify that the one-dimensional system (4.12) satisfies the conditions assumed in Theorem 2.1. We see that A^0, A and B in (4.14) are real symmetric. Also A^0 is positive definite and B is nonnegative definite such that $\ker(B) = \mathrm{span}\{e_1, e_2\}$ in $\mathbb{C}^3$, where $\{e_1, e_2, e_3\}$ is the standard orthogonal basis in $\mathbb{R}^3$. Therefore our system (4.12) satisfies Basic assumption.

Next we verify Condition (S) for the system (4.12) similarly as in Subsection 4.1. We take the matrix S as

$$S = \begin{pmatrix} \kappa/\bar{p}_\rho & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.15}$$

which is the same as in (4.11) (cf. (4.6)). Then we see that

$$SA^0 = \begin{pmatrix} \kappa/\bar{\rho} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is real symmetric and nonnegative definite such that $\ker(SA^0) = \operatorname{span}\{e_2, e_3\}$. Also we have

$$SA = \begin{pmatrix} 0 & \kappa & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad SA + D = \begin{pmatrix} 0 & \kappa & 0 \\ \kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $SA + D$ is real symmetric. Moreover a simple computation shows that $SB = 0$ and $SD = 0$. Thus we have verified that our system (4.12) satisfies Condition (S).

To verify Condition (K), as in [31], we take the matrix K as follows

$$K = \alpha \begin{pmatrix} 0 & \beta & 0 \\ -\beta & 0 & 1/\bar{p}_\theta \\ 0 & -1/\bar{p}_\theta & 0 \end{pmatrix} (A^0)^{-1} = \alpha \begin{pmatrix} 0 & \beta(1/\bar{\rho}) & 0 \\ -\beta(\bar{\rho}/\bar{p}_\rho) & 0 & \bar{\theta}/\bar{\rho}\bar{e}_\theta\bar{p}_\theta \\ 0 & -1/\bar{\rho}\bar{p}_\theta & 0 \end{pmatrix}, \tag{4.16}$$

where $\alpha,\ \beta > 0$ are constants to be determined; here we used $\bar{p}_\theta \neq 0$. Then KA^0 is real skew-symmetric. Also we see that

$$KA = \alpha \begin{pmatrix} \beta(\bar{p}_\rho/\bar{\rho}) & 0 & \beta(\bar{p}_\theta/\bar{\rho}) \\ 0 & \bar{\theta}/\bar{\rho}\bar{e}_\theta - \beta\bar{\rho} & 0 \\ -\bar{p}_\rho/\bar{\rho}\bar{p}_\theta & 0 & -1/\bar{\rho} \end{pmatrix}.$$

Therefore, we have

$$\begin{aligned} &\langle (KA)^{sy} U, U \rangle \\ &\geq \alpha\{\beta(\bar{p}_\rho/\bar{\rho})|\rho|^2 + (\bar{\theta}/\bar{\rho}\bar{e}_\theta - \beta\bar{\rho})|u|^2 - (1/\bar{\rho})|\theta|^2 - (\beta|\bar{p}_\theta|/\bar{\theta} + \bar{p}_\rho/\bar{\rho}|\bar{p}_\theta|)|\rho||\theta|\} \\ &\geq \alpha\{\beta(\bar{p}_\rho/2\bar{\rho})|\rho|^2 + (\bar{\theta}/2\bar{\rho}\bar{e}_\theta)|u|^2 - C_\beta|\theta|^2\} \end{aligned}$$

for $U = (\rho, u, \theta)^T \in \mathbb{C}^3$, where we have taken $\beta > 0$ such that $2\beta\bar{\rho} \leq \bar{\theta}/\bar{\rho}\bar{e}_\theta$. Here the constant C_β is depending on the above choice of β but is independent of $\alpha > 0$. Now we choose $\alpha > 0$ such that $2\alpha C_\beta \leq \nu/\bar{\theta}$. Then we find that $(KA)^{sy} + B$ is positive definite. Moreover, by a simple computation, we see that

$$KD = \alpha \begin{pmatrix} \beta(\kappa/\bar{\rho}) & 0 & 0 \\ 0 & 0 & 0 \\ -\kappa/\bar{\rho}\bar{p}_\theta & 0 & 0 \end{pmatrix}.$$

Therefore, we have

$$\begin{aligned}\langle (KD)^{sy}U, U\rangle &\geq \alpha\{\beta(\kappa/\bar{\rho})|\rho|^2 - (\kappa/\bar{\rho}|\bar{p}_\theta|)|\rho||\theta|\}\\ &\geq \alpha\{\beta(\kappa/2\bar{\rho})|\rho|^2 - \tilde{C}_\beta|\theta|^2\}\end{aligned}$$

for $U = (\rho, u, \theta)^T \in \mathbb{C}^3$, where $\tilde{C}_\beta$ is a constant depending on the above choice of β but not on α. We take $\alpha > 0$ so small that $2\alpha\tilde{C}_\beta \leq \nu/\bar{\theta}$. Then we see that $(KD)^{sy} + B$ is (real symmetric and) nonnegative definite such that

$$\ker((KD)^{sy} + B) = \text{span}\{e_2\} \subseteq \text{span}\{e_2, e_3\} = \ker(SA^0),$$

where S is in (4.15). Thus we have verified that our system (4.12) satisfies Craftsmanship condition (K). Consequently, we conclude that Theorem 2.1 and Proposition 3.1 are applicable to the system (4.12). Namely, we have

Proposition 4.2. *Assume (4.13). Then the one-dimensional compressible Euler-Fourier-Korteweg system (4.12) satisfies (B), (S) and (K); in particular, the matrices S and K are given by (4.15) and (4.16), respectively. Therefore, Theorem 2.1 and Proposition 3.1 are applicable to the system (4.12). As the consequence we have*

$$|(\hat{\rho}, \hat{u}, \hat{\theta})(t, \xi)|^2 + |\xi|^2|\hat{\rho}(t, \xi)|^2 \leq Ce^{-c\rho(\xi)t}\{|(\hat{\rho}_0, \hat{u}_0, \hat{\theta}_0)(\xi)|^2 + |\xi|^2|\hat{\rho}_0(\xi)|^2\}$$

for $t \geq 0$ and $\xi \in \mathbb{R}$, where $\rho(\xi) = |\xi|^2/(1 + |\xi|^2)$. Also we have

$$\|(\rho, u, \theta)(t)\|_{H^s}^2 + \|\partial_x\rho(t)\|_{H^s}^2 + \int_0^t \|\partial_x(\rho, u, \theta)(\tau)\|_{H^{s-1}}^2 + \|\partial_x^2(\rho, \theta)(\tau)\|_{H^{s-1}}^2 \, d\tau$$
$$+ \int_0^t \|\partial_x\theta(\tau)\|_{H^s}^2 \, d\tau \leq C\{\|(\rho_0, u_0, \theta_0)\|_{H^s}^2 + \|\partial_x\rho_0\|_{H^s}^2\}$$

for $t \geq 0$, provided that $(\rho_0, u_0, \theta_0) \in H^s$ and $\partial_x\rho_0 \in H^s$ for $s \geq 0$.

5. Dispersion effect and optimality of dissipative structure

In this last section, we consider the eigenvalues of the one-dimensional systems (4.8) and (4.12) and derive their asymptotic expansions for $\xi \to 0$ and $|\xi| \to \infty$. Then we want to verify the optimality of the dissipative structures characterized by (2.4) (in Remark 2.1) and (2.6) (in Remark 2.2), respectively. Also we want to verify the possibility of dispersion effects for those systems.

We consider the system (4.9) with (4.10) or (4.14). We apply the Fourier transform to (4.9) and obtain

$$A^0\hat{U}_t + i\xi A\hat{U} - (i\xi)^2B\hat{U} - (i\xi)^3D\hat{U} = 0,$$

where $\xi \in \mathbb{R}$. The corresponding eigenvalue problem is

$$\{\lambda A^0 + i\xi A - (i\xi)^2B - (i\xi)^3D\}\phi = 0, \tag{5.1}$$

where $\lambda \in \mathbb{C}$ and $\phi \in \mathbb{C}^3$ with $\phi \neq 0$. Before considering this eigenvalue problem, we study the eigenvalue problems for the hyperbolic part system (compressible Euler system) $A^0U_t + AU_x = 0$ and the dispersion system (Korteweg system) $A^0U_t = DU_{xxx}$. Let $\alpha \in \mathbb{R}$ be an eigenvalue of the matrix $(A^0)^{-1}A$ with the corresponding eigenvector $r \in \mathbb{R}^3$. Then $(\alpha A^0 - A)r = 0$ with $r \neq 0$. By direct computation, using (4.10) (or (4.14)), we know that

$$\det(\alpha A^0 - A) = \frac{\bar\rho\bar p_\rho \bar e_\theta}{\bar\theta}\ \alpha\left\{\alpha^2 - \left(\bar p_\rho + \frac{\bar\theta \bar p_\theta^2}{\bar\rho^2 \bar e_\theta}\right)\right\} = 0.$$

Thus we find that there are three different eigenvalues $\alpha = \alpha_j,\ j = 1,\ 2,\ 3$, where

$$\begin{aligned} &\alpha_1 = -\alpha_*, \qquad \alpha_2 = 0, \qquad \alpha_3 = \alpha_*, \\ &\alpha_* = \sqrt{\bar p_\rho + c_*}, \qquad c_* = \frac{\bar\theta \bar p_\theta^2}{\bar\rho^2 \bar e_\theta}. \end{aligned} \tag{5.2}$$

Here α_* denotes the sound speed. Let $r = r_j$ be the eigenvector corresponding to $\alpha = \alpha_j,\ j = 1,\ 2,\ 3$. By direct computations we find that

$$r_1 = \begin{pmatrix} \bar\rho \\ -\alpha_* \\ \bar\theta \bar p_\theta / \bar\rho \bar e_\theta \end{pmatrix}, \qquad r_2 = \begin{pmatrix} \bar p_\theta \\ 0 \\ -\bar p_\rho \end{pmatrix}, \qquad r_3 = \begin{pmatrix} \bar\rho \\ \alpha_* \\ \bar\theta \bar p_\theta / \bar\rho \bar e_\theta \end{pmatrix}. \tag{5.3}$$

Next we compute the eigenvalues of the matrix $(A^0)^{-1}D$. Let λ be an eigenvalue of $(A^0)^{-1}D$ with the corresponding eigenvector q. Then $(\lambda A^0 - D)q = 0$ with $q \neq 0$. By direct computation, using (4.10) (or (4.14)), we see that

$$\det(\lambda A^0 - D) = \frac{\bar\rho\bar p_\rho \bar e_\theta}{\bar\theta}\ \lambda^3 = 0.$$

Therefore, we have $\lambda = 0$ (with multiplicity 3). The corresponding eigenvectors q satisfy $Dq = 0$. Hence we have:

$$\lambda = 0 \ (\text{multipcility}\ 3), \qquad q \in \ker(D) = \text{span}\{e_2, e_3\}, \tag{5.4}$$

where $\{e_1, e_2, e_3\}$ is the standard orthogonal basis in $\mathbb{R}^3$.

Asymptotic expansions for $\xi \to 0$. We consider the eigenvalue problem (5.1). By applying the perturbation theory of one-parameter family of matrices (see [32]), we know that the eigenvalues $\lambda = \lambda(i\xi)$ and the corresponding eigenvectors $\phi = \phi(i\xi)$ have the following asymptotic expansions as $\xi \to 0$.

$$\lambda(i\xi) = \sum_{n=1}^{\infty} (i\xi)^n \lambda^{(n)}, \qquad \phi(i\xi) = \sum_{n=0}^{\infty} (i\xi)^n \phi^{(n)}, \tag{5.5}$$

where $\lambda^{(n)} \in \mathbb{C}$ and $\phi^{(n)} \in \mathbb{C}^3$. We substitute (5.5) into (5.1) and arrange the result according to the powers of $i\xi$. From the terms with the powers $(i\xi)^n,\ n = 1,\ 2$, we have

$$(\lambda^{(1)} A^0 + A)\phi^{(0)} = 0, \tag{5.6}$$

$$(\lambda^{(2)} A^0 - B)\phi^{(0)} + (\lambda^{(1)} A^0 + A)\phi^{(1)} = 0. \tag{5.7}$$

The Eq. (5.6) implies that $-\lambda^{(1)}$ is an eigenvalue of $(A^0)^{-1}A$ and $\phi^{(0)}$ is the corresponding eigenvector. Therefore, noting (5.2) and (5.3), we can determine $\lambda^{(1)}$ and $\phi^{(0)}$ as

$$\lambda^{(1)} = -\alpha_j, \qquad \phi^{(0)} = r_j, \qquad j = 1, 2, 3. \tag{5.8}$$

We substitute these relations into (5.7) and get $(\lambda^{(2)} A^0 - B)r_j = (\alpha_j A^0 - A)\phi^{(1)}$. This equation can be solved with respect to $\phi^{(1)}$ if and only if $(\lambda^{(2)} A^0 - B)r_j \in \text{range}(\alpha_j A^0 -$

$A) = \ker(\alpha_j A^0 - A)^\perp$. Since $\ker(\alpha_j A^0 - A) = \mathrm{span}\{r_j\}$, we have $\langle(\lambda^{(2)}A^0 - B)r_j, r_j\rangle = 0$. Thus we obtain (cf. [28, 31, 33])

$$\lambda^{(2)} = \frac{\langle Br_j, r_j\rangle}{\langle A^0 r_j, r_j\rangle} =: \beta_j, \qquad j = 1, 2, 3. \tag{5.9}$$

We compute the values of β_j. For the compressible Navier-Stokes-Fourier-Korteweg system (4.8), by using (4.10) and (5.3), we have

$$\langle A^0 r_2, r_2\rangle = \frac{\bar\rho \bar p_\rho \bar e_\theta}{\bar\theta}\alpha_*^2, \qquad \langle Br_2, r_2\rangle = \frac{\bar p_\rho^2}{\bar\theta}\nu,$$
$$\langle A^0 r_j, r_j\rangle = 2\bar\rho\alpha_*^2, \qquad \langle Br_j, r_j\rangle = \left(\tilde\mu + \frac{c_*}{\bar e_\theta \alpha_*^2}\nu\right)\alpha_*^2, \qquad j = 1, 3,$$

where α_* and c_* are given in (5.2). Therefore we obtain

$$\beta_2 = \frac{\bar p_\rho}{\alpha_*^2}\nu_*, \qquad \beta_j = \frac{1}{2}\left(\frac{\tilde\mu}{\bar\rho} + \frac{c_*}{\alpha_*^2}\nu_*\right), \quad j = 1, 3, \qquad \nu_* := \frac{\nu}{\bar\rho \bar e_\theta}. \tag{5.10}$$

The values β_j for the compressible Euler-Fourier-Korteweg system (4.12) are calculated by using (4.14) and (5.3), and we have

$$\beta_2 = \frac{\bar p_\rho}{\alpha_*^2}\nu_*, \qquad \beta_j = \frac{c_*}{2\alpha_*^2}\nu_*, \quad j = 1, 3, \qquad \nu_* := \frac{\nu}{\bar\rho \bar e_\theta}, \tag{5.11}$$

which are formally obtained by putting $\tilde\mu = 0$ in (5.10).

Asymptotic expansions for $|\xi| \to \infty$. Next we consider the eigenvalue problem (5.1) for $|\xi| \to \infty$. By applying the perturbation theory of one-parameter family of matrices (see [32]), we know that the eigenvalues $\lambda = \lambda(i\xi)$ and the corresponding eigenvectors $\phi = \phi(i\xi)$ have the following asymptotic expansions as $|\xi| \to \infty$

$$\lambda(i\xi) = \sum_{n=1}^{3} (i\xi)^n \lambda^{(n)} + \sum_{n=0}^{\infty} (i\xi)^{-n}\lambda^{(-n)}, \qquad \phi(i\xi) = \sum_{n=0}^{\infty} (i\xi)^{-n}\phi^{(-n)}, \tag{5.12}$$

where $\lambda^{(n)} \in \mathbb{C}$ and $\phi^{(n)} \in \mathbb{C}^3$. We substitute (5.12) into (5.1) and arrange the result according to the powers of $i\xi$. From the terms with the powers $(i\xi)^n$, $n = 3, 2, 1, 0, -1$, we have

$$(\lambda^{(3)}A^0 - D)\phi^{(0)} = 0, \tag{5.13}$$

$$(\lambda^{(3)}A^0 - D)\phi^{(-1)} + (\lambda^{(2)}A^0 - B)\phi^{(0)} = 0, \tag{5.14}$$

$$(\lambda^{(3)}A^0 - D)\phi^{(-2)} + (\lambda^{(2)}A^0 - B)\phi^{(-1)} + (\lambda^{(1)}A^0 + A)\phi^{(0)} = 0, \tag{5.15}$$

$$(\lambda^{(3)}A^0 - D)\phi^{(-3)} + (\lambda^{(2)}A^0 - B)\phi^{(-2)} + (\lambda^{(1)}A^0 + A)\phi^{(-1)} + \lambda^{(0)}A^0\phi^{(0)} = 0, \tag{5.16}$$

$$\begin{aligned}(\lambda^{(3)}A^0 - D)\phi^{(-4)} &+ (\lambda^{(2)}A^0 - B)\phi^{(-3)} \\ &+ (\lambda^{(1)}A^0 + A)\phi^{(-2)} + \lambda^{(0)}A^0\phi^{(-1)} + \lambda^{(-1)}A^0\phi^{(0)} = 0.\end{aligned} \tag{5.17}$$

It follows from (5.13) that $\lambda^{(3)}$ is an eigenvalue of $(A^0)^{-1}D$ and $\phi^{(0)}$ is the corresponding eigenvector. Therefore, noting (5.4), we have

$$\lambda^{(3)} = 0 \text{ (multipcility 3)}, \qquad \phi^{(0)} = q \in \ker(D) = \mathrm{span}\{e_2, e_3\}. \tag{5.18}$$

Next we determine $\lambda^{(2)}$. We first treat the compressible Navier-Stoke-Fourier-Korteweg system (4.8). We substitute (5.18) into (5.14) and (5.15) and get

$$(\lambda^{(2)}A^0 - B)q = D\phi^{(-1)},$$
$$(\lambda^{(2)}A^0 - B)\phi^{(-1)} + (\lambda^{(1)}A^0 + A)q = D\phi^{(-2)}.$$

We put $q = (0, u_0, \theta_0)^T \in \text{span}\{e_2, e_3\}$ and $\phi^{(-n)} = (\rho_n, u_n, \theta_n)^T$, $n=1$, 2. By using (4.10), we rewrite the above equalities explicitly as

$$(\lambda^{(2)}\bar{\rho} - \tilde{\mu})u_0 = \kappa\rho_1, \qquad (1/\bar{\theta})(\lambda^{(2)}\bar{\rho}\bar{e}_\theta - \nu)\theta_0 = 0, \tag{5.19}$$

$$\begin{aligned} &(\bar{p}_\rho/\bar{\rho})(\lambda^{(2)}\rho_1 + \bar{\rho}u_0) = 0, \\ &(\lambda^{(2)}\bar{\rho} - \tilde{\mu})u_1 + (\lambda^{(1)}\bar{\rho}u_0 + \bar{p}_\theta\theta_0) = \kappa\rho_2, \\ &(1/\bar{\theta})\{(\lambda^{(2)}\bar{\rho}\bar{e}_\theta - \nu)\theta_1 + (\bar{\theta}\bar{p}_\theta u_0 + \lambda^{(1)}\bar{\rho}\bar{e}_\theta\theta_0)\} = 0. \end{aligned} \tag{5.20}$$

We use the first equation of (5.19) and the first equation of (5.20). We eliminate ρ_1 from these two equations and obtain $\{(\lambda^{(2)})^2 - (\tilde{\mu}/\bar{\rho})\lambda^{(2)} + \kappa\}u_0 = 0$. When $u_0 \neq 0$, we have $(\lambda^{(2)})^2 - (\tilde{\mu}/\bar{\rho})\lambda^{(2)} + \kappa = 0$. Thus we have

$$\lambda^{(2)} = \frac{\tilde{\mu}}{2\bar{\rho}} \pm i\left\{\kappa - \left(\frac{\tilde{\mu}}{2\bar{\rho}}\right)^2\right\}^{1/2},$$

where we assumed $(\tilde{\mu}/2\bar{\rho})^2 \leq \kappa$ (small viscosity) for simplicity. In this case we have $\theta_0 = 0$ from the second equation of (5.19) and hence we can take $q = e_2$. On the other hand, when $u_0 = 0$, we have $\theta_0 \neq 0$ so that we can take $q = e_3$. Also we have $\lambda^{(2)} = \nu/\bar{\rho}\bar{e}_\theta = \nu_*$ from the second equation of (5.19). Consequently, we obtain:

$$\begin{aligned} &\lambda^{(2)} = \sigma_k, \qquad \phi^{(0)} = q_k, \qquad k = 1, 2, 3, \\ &\sigma_{1,2} := \frac{\tilde{\mu}}{2\bar{\rho}} \pm i\left\{\kappa - \left(\frac{\tilde{\mu}}{2\bar{\rho}}\right)^2\right\}^{1/2}, \quad q_{1,2} := e_2; \qquad \sigma_3 := \frac{\nu}{\bar{\rho}\bar{e}_\theta} =: \nu_*, \quad q_3 := e_3. \end{aligned} \tag{5.21}$$

Next we consider the compressible Euler-Fourier-Korteweg system (4.12). For this system, we use (4.14) instead of (4.10) and obtain (5.21) with $\tilde{\mu} = 0$. Namely, we have:

$$\begin{aligned} &\lambda^{(2)} = \sigma_k, \qquad \phi^{(0)} = q_k, \qquad k = 1, 2, 3, \\ &\sigma_{1,2} := \pm i\sqrt{\kappa}, \quad q_{1,2} := e_2; \qquad \sigma_3 := \frac{\nu}{\bar{\rho}\bar{e}_\theta} =: \nu_*, \quad q_3 := e_3. \end{aligned} \tag{5.22}$$

In this case, for $k=1$, 2, we have to further compute $\lambda^{(1)}$ and $\lambda^{(0)}$. To this end, we set $\gamma := \pm\sqrt{\kappa}$ and write $\lambda^{(2)} = \sigma_k = i\gamma$ for $k=1$, 2. We put $\tilde{\mu} = 0$, $\lambda^{(2)} = i\gamma$ and $(u_0, \theta_0) = (1, 0)$ in (5.20). Then (5.20) is reduced to

$$i\gamma\rho_1 + \bar{\rho} = 0, \qquad \bar{\rho}(i\gamma u_1 + \lambda^{(1)}) = \kappa\rho_2, \qquad (i\gamma\bar{\rho}\bar{e}_\theta - \nu)\theta_1 + \bar{\theta}\bar{p}_\theta = 0. \tag{5.23}$$

From the first and the third equations of (5.23) we determine ρ_1 and θ_1 as

$$\rho_1 = i\bar{\rho}/\gamma, \qquad \theta_1 = \bar{\theta}\bar{p}_\theta/(\nu - i\gamma\bar{\rho}\bar{e}_\theta). \tag{5.24}$$

To compute $\lambda^{(1)}$ and $\lambda^{(0)}$, we use (5.16) and (5.17). We put $\lambda^{(3)} = 0$, $\lambda^{(2)} = i\gamma$ and $\phi^{(0)} = q = e_2$ in (5.16) and (5.17). Then we have

$$\begin{aligned}&(i\gamma A^0 - B)\phi^{(-2)} + (\lambda^{(1)}A^0 + A)\phi^{(-1)} + \lambda^{(0)}A^0 e_2 = D\phi^{(-3)},\\ &(i\gamma A^0 - B)\phi^{(-3)} + (\lambda^{(1)}A^0 + A)\phi^{(-2)} + \lambda^{(0)}A^0\phi^{(-1)} + \lambda^{(-1)}A^0 e_2 = D\phi^{(-4)}\end{aligned} \tag{5.25}$$

By using (4.14), we write the first equation of (5.25) explicitly as

$$\begin{aligned}&(\bar{p}_\rho/\bar{\rho})\{i\gamma\rho_2 + (\lambda^{(1)}\rho_1 + \bar{\rho}u_1)\} = 0,\\ &i\gamma\bar{\rho}u_2 + (\bar{p}_\rho\rho_1 + \lambda^{(1)}\bar{\rho}u_1 + \bar{p}_\theta\theta_1) + \lambda^{(0)}\bar{\rho} = \kappa\rho_3,\\ &(1/\bar{\theta})\{(i\gamma\bar{\rho}\bar{e}_\theta - \nu)\theta_2 + (\bar{\theta}\bar{p}_\theta u_1 + \lambda^{(1)}\bar{\rho}\bar{e}_\theta\theta_1)\} = 0,\end{aligned} \tag{5.26}$$

where $\phi^{(-n)} = (\rho_n, u_n, \theta_n)^T$, $n = 1, 2, 3$. Similarly, the first component of the second equation of (5.25) is given as follows:

$$(\bar{p}_\rho/\bar{\rho})\{i\gamma\rho_3 + (\lambda^{(1)}\rho_2 + \bar{\rho}u_2) + \lambda^{(0)}\rho_1\} = 0. \tag{5.27}$$

We use $\rho_1 = i\bar{\rho}/\gamma$ (in (5.24)) together with $\gamma^2 = \kappa$ and find that the first equation of (5.26) is rewritten in the form $\bar{\rho}(i\gamma u_1 - \lambda^{(1)}) = \kappa\rho_2$. This equality combined with the second equation of (5.23) shows that

$$\lambda^{(1)} = 0. \tag{5.28}$$

Next we substitute (5.28) into (5.27) and use $\rho_1 = i\bar{\rho}/\gamma$ together with $\gamma^2 = \kappa$. Then we see that $\bar{\rho}(i\gamma u_2 - \lambda^{(0)}) = \kappa\rho_3$. This equation and the second equation of (5.26) with $\lambda^{(1)} = 0$ yield

$$(\bar{p}_\rho\rho_1 + \bar{p}_\theta\theta_1) + 2\bar{\rho}\lambda^{(0)} = 0.$$

Consequently, we obtain

$$\lambda^{(0)} = -\frac{1}{2\bar{\rho}}(\bar{p}_\rho\rho_1 + \bar{p}_\theta\theta_1) = -\frac{1}{2}\left\{\frac{c_*\nu_*}{\nu_*^2 + \kappa} + i\gamma\left(\frac{\bar{p}_\rho}{\kappa} + \frac{c_*}{\nu_*^2 + \kappa}\right)\right\},$$

where we used (5.24); c_* and ν_* are given in (5.2) and (5.11), respectively. Thus we have

$$\lambda^{(0)} = \eta_k, \quad k = 1, 2, \qquad \eta_{1,2} := -\frac{1}{2}\left\{\frac{c_*\nu_*}{\nu_*^2 + \kappa} \pm i\sqrt{\kappa}\left(\frac{\bar{p}_\rho}{\kappa} + \frac{c_*}{\nu_*^2 + \kappa}\right)\right\}. \tag{5.29}$$

Above observations are summarized as follows.

Proposition 5.1 **(Navier-Stokes-Fourier-Korteweg).** *Assume (4.1). Then the eigenvalues $\lambda = \lambda_j(i\xi)$, $j = 1, 2, 3$, of the compressible Navier-Stokes-Fourier-Korteweg system (4.8) have the following asymptotic expansions as $\xi \to 0$ and $|\xi| \to \infty$:*

$$\begin{aligned}&\lambda_j(i\xi) = \sum_{n=1}^{\infty}(i\xi)^n\lambda_j^{(n)}, \qquad \xi \to 0,\\ &\lambda_j(i\xi) = \sum_{n=0}^{3}(i\xi)^{3-n}\tilde{\lambda}_j^{(3-n)} + \sum_{n=1}^{\infty}(i\xi)^{-n}\lambda_j^{(-n)}, \qquad |\xi| \to \infty,\end{aligned}$$

where

$$\lambda_{1,3}^{(1)} = \pm\alpha_*, \quad \lambda_2^{(1)} = 0; \qquad \lambda_{1,3}^{(2)} = \frac{1}{2}\left(\frac{\tilde{\mu}}{\bar{\rho}} + \frac{c_*}{\alpha_*^2}\nu_*\right), \quad \lambda_2^{(2)} = \frac{\bar{p}_\rho}{\alpha_*^2}\nu_*,$$

$$\tilde{\lambda}_{1,2,3}^{(3)} = 0; \qquad \tilde{\lambda}_{1,2}^{(2)} = \frac{\tilde{\mu}}{2\bar{\rho}} \pm i\left\{\kappa - \left(\frac{\tilde{\mu}}{2\bar{\rho}}\right)^2\right\}^{1/2}, \quad \tilde{\lambda}_3^{(2)} = \nu_*.$$

Here α_* *and* c_* *are given in* (5.2), *and* ν_* *is in* (5.10). *Notice that* $\lambda_j^{(2)} > 0$ *and* Re $\tilde{\lambda}_j^{(2)} > 0$ *for* $j = 1, 2, 3$. *Therefore these asymptotic expansions suggest the optimality of the characterization* (2.6) *for the dissipative structure in the framework of Theorem 2.2.*

Proposition 5.2 (Euler-Fourier-Korteweg). *Assume* (4.13). *Then the eigenvalues* $\lambda = \lambda_j(i\xi)$, $j = 1, 2, 3$, *of the compressible Euler-Fourier-Korteweg system* (4.12) *have the following asymptotic expansions as* $\xi \to 0$ *and* $|\xi| \to \infty$:

$$\lambda_j(i\xi) = \sum_{n=1}^{\infty}(i\xi)^n\lambda_j^{(n)}, \qquad \xi \to 0,$$

$$\lambda_j(i\xi) = \sum_{n=0}^{3}(i\xi)^{3-n}\tilde{\lambda}_j^{(3-n)} + \sum_{n=1}^{\infty}(i\xi)^{-n}\lambda_j^{(-n)}, \qquad |\xi| \to \infty,$$

where

$$\lambda_{1,3}^{(1)} = \pm\alpha_*, \quad \lambda_2^{(1)} = 0; \qquad \lambda_{1,3}^{(2)} = \frac{c_*}{2\alpha_*^2}\nu_*, \quad \lambda_2^{(2)} = \frac{\bar{p}_\rho}{\alpha_*^2}\nu_*,$$

$$\tilde{\lambda}_{1,2,3}^{(3)} = 0; \qquad \tilde{\lambda}_{1,2}^{(2)} = \pm i\sqrt{\kappa}, \quad \tilde{\lambda}_3^{(2)} = \nu_*;$$

$$\tilde{\lambda}_{1,2}^{(1)} = 0; \qquad \tilde{\lambda}_{1,2}^{(0)} = -\frac{1}{2}\left\{\frac{c_*\nu_*}{\nu_*^2+\kappa} \pm i\sqrt{\kappa}\left(\frac{\bar{p}_\rho}{\kappa} + \frac{c_*}{\nu_*^2+\kappa}\right)\right\}.$$

Here α_* *and* c_* *are given in* (5.2), *and* ν_* *is in* (5.11). *Notice that* $\lambda_j^{(2)} > 0$ *for* $j = 1, 2, 3$. *Also,* $\tilde{\lambda}_3^{(2)} > 0$, Re $\tilde{\lambda}_k^{(2)} = 0$ *and* Re $\tilde{\lambda}_k^{(0)} < 0$ *for* $k = 1, 2$. *Therefore these asymptotic expansions suggest the optimality of the characterization* (2.4) *for the dissipative structure in the framework of Theorem 2.1.*

Finally in this last section, we briefly consider the dispersive effect for the one-dimensional systems (4.8) and (4.12). From Proposition 5.1 (resp. Proposition 5.2) we observe that the solution of (4.8) (resp. (4.12)) contains the function

$$e^{-(\beta+i\alpha)\xi^2 t} \qquad (\text{resp. } e^{\eta t - i\gamma\xi^2 t})$$

as an approximation form in the high frequency region, where $\beta > 0$ and $\alpha \in \mathbb{R}$ with $\alpha \neq 0$ (resp. Re $\eta < 0$ and $\gamma \in \mathbb{R}$ with $\gamma \neq 0$). Thus we expect that the dispersive effect appears in the solutions but it is not verified quantitatively.

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