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Falling Drop in an Unbounded Liquid Reservoir: Steady-state Solutions

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Abstract. The equations governing the motion of a three-dimensional liquid drop moving freely in an unbounded liquid reservoir under the influence of a gravitational force are investigated. Provided the (constant) densities in the two liquids are sufficiently close, existence of a steady-state solution is shown. The proof is based on a suitable linearization of the equations. A setting of function spaces is introduced in which the corresponding linear operator acts as a homeomorphism.

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1. Introduction

Consider a drop of liquid with density ρ_1 submerged into an unbounded reservoir of liquid with density ρ_2. Assume the liquids are immiscible. We investigate the motion of the drop under the influence of a constant gravitational force and surface tension on the interface. Specifically, we shall show existence of a steady-state solution to the governing equations of motion, provided the difference $|\rho_1 - \rho_2|$ of the densities is sufficiently small.

The dynamics of a falling (or rising) drop in a quiescent fluid has attracted a lot of attention in the field of fluid mechanics. Such flows have been studied extensively both experimentally and numerically with truly fascinating outcomes (see [3] for a comprehensive overview and further references), but it remains an intriguing task to analytically validate the observations. The observed dynamics can be characterized as a series of bifurcations with respect to the Reynolds number as parameter. Broadly speaking, steady-state solutions are observed for small Reynolds numbers, with bifurcations into oscillating motions as the Reynolds number increases. Bifurcations into more complex solutions can be observed as the Reynolds number increases even further.

In the following we investigate steady-state solutions for small Reynolds number, which corresponds to a small density difference $|\rho_1 - \rho_2|$. One of our aims is to develop a framework of function spaces that can be used not only to study steady states, but also as a foundation for further investigations into the dynamics described above. In particular, the framework should facilitate a stability analysis of the steady states, and an investigation of the Hopf-type bifurcations (into oscillating motions) observed in experiments. For this purpose, it should satisfy certain properties. First and foremost, it should be possible to identify function spaces within the framework such that the differential operator of the linearized equations of motion acts as a homeomorphism. Second, the framework should have a natural extension to a suitable time-periodic framework (recall that a steady-state solution is trivially also time-periodic). Third, the framework should adequately facilitate a spectral analysis of the operators obtained by linearizing the equations of motion around a steady state. To meet these criteria, we propose a framework of Sobolev spaces. Although a setting of Sobolev spaces seems natural, and the most convenient to work with, it is by no means trivial to identify one that conforms to the problem of a freely falling (or rising) drop. Indeed, one of the novelties of this article is the introduction of such a Sobolev-space setting that meets at least the first and most important criteria, and possibly also the other two, mentioned above, and in which existence of steady-state solutions can be shown effortlessly for small data. The investigation of steady-state solutions is not new, though. It was initiated by Bemelmans [5] and advanced by Solonnikov [14,15]. However, the analysis carried out by Bemelmans and Solonnikov does not lead to a framework of Sobolev spaces. Indeed, for reasons that will be explained in detail below, the approaches of both Bemelmans and Solonnikov *cannot* be adapted to a Sobolev-space setting with the desired properties.

We shall consider the most commonly used model for two-phase flows with surface tension on the interface. It is assumed both fluids are Navier–Stokes liquids, which are incompressible, viscous, and Newtonian. It is further assumed that the fluids are immiscible with surface tension on their interface, which acts in normal direction proportional to the mean curvature. Moreover, we consider a system in which the drop is a ball B_{R_0} of radius R_0 when no external forces act on the system, that is, in its stress free configuration. If we choose a coordinate system attached to the falling drop, these assumptions lead to the following equations of motion for a steady state (see Sect. 2 for details on the derivation):

$$\begin{cases} -\operatorname{div}\mathrm{T}(u,\mathfrak{p})+\rho\,(u\cdot\nabla u+\lambda\partial_3 u)=-\rho\,g\,e_3 & \text{in } \mathbb{R}^3\setminus\Gamma_\eta,\\ \operatorname{div}u=0 & \text{in } \mathbb{R}^3\setminus\Gamma_\eta,\\ [\![\mathrm{T}(u,\mathfrak{p})\mathrm{n}]\!]=\sigma\,\mathrm{H}(\eta)\mathrm{n} & \text{on } \Gamma_\eta,\\ [\![u]\!]=0 & \text{on } \Gamma_\eta,\\ u\cdot\mathrm{n}=-\lambda e_3\cdot\mathrm{n} & \text{on } \Gamma_\eta,\\ \big|\Omega_\eta^{(1)}\big|=\dfrac{4\pi}{3}R_0^3, \qquad \lim\limits_{|x|\to\infty}u(x)=0. \end{cases} \tag{1.1}$$

Here Γ_η denotes the interface between the two liquids, which we assume to be a closed manifold parameterized by a “height” function $\eta\colon\partial\mathrm{B}_{R_0}\to\mathbb{R}$ describing the displacement of the drop’s boundary points in normal direction. The domain $\Omega_\eta^{(1)}\subset\mathbb{R}^3$ bounded by Γ_η describes the domain occupied by the drop, and the exterior domain $\Omega_\eta^{(2)}:=\mathbb{R}^3\backslash\overline{\Omega}_\eta^{(1)}$ the region of the liquid reservoir. The drop velocity $-\lambda e_3$, $\lambda\in\mathbb{R}$, is assumed to be directed along the axis of the (constant) gravitational force ge_3. The first two equations in (1.1) are the Navier–Stokes equations written in a moving frame of reference, where $u\colon\mathbb{R}^3\setminus\Gamma_\eta\to\mathbb{R}^3$ denotes the Eulerian velocity field of the liquids, $\mathfrak{p}\colon\mathbb{R}^3\setminus\Gamma_\eta\to\mathbb{R}$ the scalar pressure field, and $\mathrm{T}(u,\mathfrak{p})$ the corresponding Cauchy stress tensor. The density function $\rho\colon\mathbb{R}^3\setminus\Gamma_\eta\to\mathbb{R}$ is constant in both components of $\mathbb{R}^3\setminus\Gamma_\eta$. The third equation states that the surface tension in normal direction on the interface Γ_η is proportional to the mean curvature H, with $\sigma>0$ a constant. The notation $[\![\cdot]\!]$ is used to denote the jump of a quantity across Γ_η. Immiscibility of the two liquids under a no-slip assumption at the interface is expressed via the fourth and fifth equation. Observe that the normal velocity on the interface then coincides with that of the moving frame, which moves with the same velocity $-\lambda e_3$ as the falling drop. The equations are augmented with a volume condition for the drop and the requirement that the liquid in the reservoir is at rest at spatial infinity in the sixth and seventh equation, respectively.

A key part of our investigation is directed towards finding an appropriate linearization of (1.1) with respect to the unknowns u, $\mathfrak{p}$, λ and η. The canonical linearization, i.e., around the trivial state $(0,0,0,0)$, leads to the Navier–Stokes equations $(1.1)_{1-2}$ being replaced with the Stokes system

$$\begin{cases} -\operatorname{div}\mathrm{T}(u,\mathfrak{p})=f & \text{in } \mathbb{R}^3\setminus\partial\mathrm{B}_{R_0},\\ \operatorname{div}u=0 & \text{in } \mathbb{R}^3\setminus\partial\mathrm{B}_{R_0}. \end{cases} \tag{1.2}$$

An analysis based on this linearization would have to be carried out in a setting of function spaces conforming to the properties of the Stokes problem. The Stokes setting of function spaces, however, is not suitable for an investigation of the exterior domain Navier–Stokes equations in a moving frame. Since the falling drop, and thus the frame of reference, moves with a nonzero velocity $-\lambda e_3$, the appropriate linearization of the Navier–Stokes equations $(1.1)_{1-2}$ is an Oseen system. At least in a setting of Sobolev spaces, the steady-state exterior-domain Navier–Stokes equations in a moving frame can only be solved in a framework of spaces conforming to the Oseen linearization. To resolve this issue, we propose to rewrite the system (1.1) as a perturbation around a state $(u_0, \mathfrak{p}_0, \lambda_0, \eta_0)$ with $\lambda_0 \neq 0$. A subsequent linearization of (1.1) then yields the Oseen problem

$$\begin{cases} -\operatorname{div} \mathrm{T}(u, \mathfrak{p}) + \lambda_0 \partial_3 u = f & \text{in } \mathbb{R}^3 \setminus \partial \mathrm{B}_R, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \setminus \partial \mathrm{B}_R. \end{cases} \tag{1.3}$$

The main challenge, and indeed novelty of this article, is the introduction of a suitable state $(u_0, \mathfrak{p}_0, \lambda_0, \eta_0)$ that renders the problem well posed in a framework of classical Sobolev spaces. To this end, we employ the auxiliary fields introduced by Happel and Brenner [11]. A similar utilization of these auxiliary fields to linearize a free boundary Navier–Stokes problem set in an exterior domain was first carried out by Bemelmans, Galdi and Kyed [6].

The starting point of our investigation were the articles [14,15] by Solonnikov, which contain a number of truly outstanding ideas on how to analyze (1.1). However, Solonnikov overlooks the necessity of an Oseen linearization as described above. Instead, he employs a Stokes linearization and consequently a setting of function spaces in which the nonlinear term $\lambda \partial_3 u$ cannot be correctly treated on the right-hand side. Our approach resolves this issue.

We derive the steady-state equations of motion for the falling drop and state the main theorem in the following Sect. 2. In Sect. 3 we collect the basic notation. The aforementioned framework of Sobolev spaces is then introduced in Sect. 4. Fundamental L^r estimates are established in Sect. 5, and a reformulation of (1.1) in a fixed reference configuration in Sect. 6. The linearization around a non-trivial state is carried out in Sect. 7. In Sect. 8 we show in Theorem 8.1 that the operator corresponding to this linearization is a homeomorphism in our framework of Sobolev spaces, which finally enables us to establish a proof of the main theorem, namely the existence of a steady-state solution for $|\rho_1 - \rho_2|$ sufficiently small.

2. Equations of Motion and Statement of the Main Theorem

We derive the system of equations governing the motion of a freely falling drop in a liquid under the influence of a constant gravitational force. We shall express these equations in a frame of reference with origin in the barycenter of the drop. More specifically, we denote by $\xi(t)$ the barycenter of the falling drop with respect to an inertial frame, whose coordinates we denote by y, and express the equations of motion in barycentric coordinates $x(t, y) := y - \xi(t)$. In these coordinates, the domain $\Omega_t^{(1)} \subset \mathbb{R}^3$ occupied by the drop at time t satisfies

$$\int_{\Omega_t^{(1)}} x \, \mathrm{d}x = 0. \tag{2.1}$$

We let $\Omega_t^{(2)} := \mathbb{R}^3 \backslash \overline{\Omega}_t^{(1)}$ denote the domain of the surrounding liquid reservoir, and put $\Omega_t := \Omega_t^{(1)} \cup \Omega_t^{(2)}$. The surface $\Gamma_t := \partial \Omega_t^{(1)}$ describes the interface between the two liquids. Moreover, we let μ_1, μ_2 and ρ_1, ρ_2 denote the constant viscosities and densities of the drop and the liquid reservoir, respectively. The functions

$$\mu : \bigcup_{t \in \mathbb{R}_+} \{t\} \times \Omega_t \to \mathbb{R}, \quad \mu(t, x) := \begin{cases} \mu_1, & x \in \Omega_t^{(1)}, \\ \mu_2, & x \in \Omega_t^{(2)}, \end{cases}$$

$$\rho : \bigcup_{t \in \mathbb{R}_+} \{t\} \times \Omega_t \to \mathbb{R}, \quad \rho(t, x) := \begin{cases} \rho_1, & x \in \Omega_t^{(1)}, \\ \rho_2, & x \in \Omega_t^{(2)}, \end{cases}$$

then describe the viscosity and density of the liquid occupying the point x at a given time t. Expressed in a frame of reference attached to the barycenter ξ, the conservation of momentum and mass of both liquids is described by the Navier–Stokes system

$$\begin{cases} \rho(\partial_t v + v\cdot\nabla v - \dot{\xi}\cdot\nabla v) = \operatorname{div}\mathrm{T}(v,p) + \rho b \\ \operatorname{div} v = 0 \end{cases} \quad \text{in } \bigcup_{t\in\mathbb{R}_+}\{t\}\times\Omega_t, \tag{2.2}$$

where v denotes the Eulerian velocity field in the liquids, p the pressure,

$$\mathrm{T}(v,p) := 2\mu\,\mathrm{S}(v) - p\mathrm{I}, \quad \mathrm{S}(v) := \frac{1}{2}\big(\nabla v + \nabla v^\top\big)$$

the Cauchy stress tensor, and $b\in\mathbb{R}^3$ a constant gravitational acceleration. One can decompose the velocity field and pressure term into

$$v^{(1)} : \bigcup_{t\in\mathbb{R}_+}\{t\}\times\Omega_t^{(1)} \to \mathbb{R}^3, \quad p^{(1)} : \bigcup_{t\in\mathbb{R}_+}\{t\}\times\Omega_t^{(1)} \to \mathbb{R}$$

describing the liquid flow in the drop, and another part

$$v^{(2)} : \bigcup_{t\in\mathbb{R}_+}\{t\}\times\Omega_t^{(2)} \to \mathbb{R}^3, \quad p^{(2)} : \bigcup_{t\in\mathbb{R}_+}\{t\}\times\Omega_t^{(2)} \to \mathbb{R}$$

describing the flow in the reservoir. We employ the notation

$$[\![v]\!] := v^{(1)}\big|_\Gamma - v^{(2)}\big|_\Gamma$$

to denote the jump in a quantity on the interface between the two liquids. Concerning the physical nature of the interface, we make the basic assumption that slippage between the two liquids cannot occur, i.e., a no-slip boundary condition, and that liquid cannot be absorbed in the interface. Consequently, there is no jump in the velocity field neither in tangential nor in normal direction:

$$[\![v]\!] = 0 \quad \text{on } \bigcup_{t\in\mathbb{R}_+}\{t\}\times\Gamma_t. \tag{2.3}$$

Since the liquids are immiscible, the normal component of the liquid velocity at the interface coincides with the velocity of the interface itself. If Φ_Γ denotes a Lagrangian description of the interface in barycentric coordinates, the immiscibility condition therefore takes the form

$$v\cdot\mathrm{n} = \partial_t\Phi_\Gamma\cdot\mathrm{n} + \dot{\xi}\cdot\mathrm{n} \qquad \text{on } \bigcup_{t\in\mathbb{R}_+}\{t\}\times\Gamma_t. \tag{2.4}$$

In a classical two-phase flow model, surface tension on the interface, *i.e.*, the difference in normal stresses of the two liquids, is proportional to the mean curvature in normal direction and in balance in tangential direction:

$$\mathrm{n}\cdot[\![\mathrm{T}(v,p)\mathrm{n}]\!] = \sigma\mathrm{H} \quad \text{on } \bigcup_{t\in\mathbb{R}_+}\{t\}\times\Gamma_t, \tag{2.5}$$

$$(\mathrm{I}-\mathrm{n}\otimes\mathrm{n})[\![\mathrm{T}(v,p)\mathrm{n}]\!] = 0 \qquad \text{on } \bigcup_{t\in\mathbb{R}_+}\{t\}\times\Gamma_t. \tag{2.6}$$

Since we consider the motion of a drop in a *quiescent* liquid, the velocity in the reservoir vanishes at spatial infinity

$$\lim_{|x|\to\infty} v(t,x) = 0. \tag{2.7}$$

Due to the incompressibility of the liquid drop, its volume is constant. Since we consider a drop that takes the shape of the ball B_{R_0} in its stress free configuration, this volume is prescribed by

$$|\Omega_t^{(1)}| = \frac{4\pi}{3}R_0^3. \tag{2.8}$$

In conclusion, the system obtained by combining (2.1)–(2.8) governs the motion of a liquid drop falling freely in a liquid reservoir under the influence of a constant gravitational force.

In this article we will establish existence of a steady-state solution, that is, a time-independent solution to (2.1)–(2.8). Such a solution is of course only steady with respect to the chosen frame of reference; in our case the frame attached to the barycenter. Other types of steady states can be investigated by analyzing time-independent solutions in other frames. For example, it is conceivable that falling drops can perform steady rotating motions, which should be investigated by considering the equations of motion in a rotating frame of reference.

The unknowns in (2.1)–(2.8) are the functions $v, p, \dot{\xi}, \Phi_\Gamma$. The mean curvature H can be computed from Φ_Γ. The viscosities $\mu_1, \mu_2 > 0$, surface tension $\sigma > 0$ and the prescribed volume $\frac{4\pi}{3}R_0^3$ of the drop are constants, which may be chosen arbitrarily. Also the gravitational force $b \in \mathbb{R}^3$ is an arbitrary constant, but upon a re-orientation of the coordinates we may assume without loss of generality that it is directed along the negative e_3 axis, *i.e.*, $b = -ge_3$ with $g > 0$. The constant densities $\rho_1, \rho_2 > 0$ shall be restricted to pairs whose difference $|\rho_1 - \rho_2|$ is sufficiently small. In this sense, we treat $\rho_1 - \rho_2$ as the data of the system. Since the geometry $\big(\Omega_t^{(1)}, \Omega_t^{(2)}, \Gamma_t\big)$ of the problem is determined by the unknown description Φ_Γ of the interface, (2.1)–(2.8) is a free boundary problem.

As mentioned above, we will establish existence of a steady-state, that is, time-independent, solution $(v, p, \dot{\xi}, \Phi_\Gamma)$ to (2.1)–(2.8). In this case the velocity $\dot{\xi}$ is a constant vector. We focus on solutions with $\dot{\xi}$ directed along the axis of gravity, *i.e.*, $\dot{\xi} = -\lambda e_3$. The steady-state equations of motion then read

$$\begin{cases} \rho(v\cdot\nabla v + \lambda\partial_3 v) = \operatorname{div}\mathrm{T}(v,p) - ge_3 & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ [\![v]\!] = 0 & \text{on } \Gamma, \\ v\cdot \mathrm{n} = -\lambda e_3 \cdot \mathrm{n} & \text{on } \Gamma, \\ \mathrm{n}\cdot[\![\mathrm{T}(v,p)\mathrm{n}]\!] = \sigma \mathrm{H} & \text{on } \Gamma, \\ (\mathrm{I}-\mathrm{n}\otimes\mathrm{n})[\![\mathrm{T}(v,p)\mathrm{n}]\!] = 0 & \text{on } \Gamma, \\ \lim_{|x|\to\infty} v(x) = 0, \quad |\Omega^{(1)}| = \dfrac{4\pi}{3}R_0^3, \quad \displaystyle\int_{\Omega^{(1)}} x\,\mathrm{d}x = 0, \end{cases} \tag{2.9}$$

where the interface Γ is an unknown computed from the parameterization Φ_Γ. The unknowns in (2.9) are $v, p, \lambda, \Phi_\Gamma$.

At the outset, it is clear that (2.9) can have multiple solutions. This is best illustrated by considering $\rho_1 = \rho_2$, in which case the trivial solution with $v = 0$, $\lambda = 0$ and constant pressures $p^{(1)}$, $p^{(2)}$ is a steady-state solution if σH equals the constant hydrostatic pressure difference $p^{(1)} - p^{(2)}$ between the drop and the reservoir. Since a constant mean curvature H is realized whenever $\Omega^{(1)}$ is a multiple of disjoint balls, we obtain for each Φ_Γ describing one or more spheres a trivial solution by adjusting the hydrostatic pressure difference accordingly (depending on the fixed volume $|\Omega^{(1)}|$). In the case (2.9) above, the fixed volume of $|\Omega^{(1)}|$ coincides with the volume of the ball B_{R_0}. With constant pressures satisfying $p^{(1)} - p^{(2)} = \frac{2}{R_0}$, the ball B_{R_0} therefore becomes an admissible steady-state drop configuration when $\rho_1 = \rho_2$. We shall single out this configuration for further investigation in the sense that we investigate non-trivial steady-states with a configuration close to the ball B_{R_0} for $\rho_1 \neq \rho_2$ with $|\rho_1 - \rho_2|$ sufficiently small.

From a physical perspective, a smallness condition is only meaningful when expressed in a non-dimensional form. In order to obtain a dimensionless formulation of (2.9), we choose R_0 as characteristic length scale, $V_0 := \sqrt{gR_0}$ as the characteristic velocity, $\rho_1 + \rho_2$ as characteristic density, $(\rho_1 + \rho_2)R_0V_0$ as the characteristic viscosity, and $(\rho_1 + \rho_2)R_0V_0^2$ as the characteristic surface tension. Investigating the resulting non-dimensional equations of motion, we will establish existence of a non-trivial steady-state solution with drop configuration close to the unit ball B_1. For this purpose, it is convenient to introduce (in the non-dimensionalized coordinates) the normalized pressures

$$\mathfrak{p}^{(1)}(x) : \Omega^{(1)} \to \mathbb{R}, \quad \mathfrak{p}^{(1)}(x) := p^{(1)}(x) + \rho_1\, e_3\cdot x - 2\sigma$$
$$\mathfrak{p}^{(2)}(x) : \Omega^{(2)} \to \mathbb{R}, \quad \mathfrak{p}^{(2)}(x) := p^{(2)}(x) + \rho_2\, e_3\cdot x.$$

We then obtain the following system of non-dimensional equations:

$$\begin{cases} \rho(v\cdot\nabla v+\lambda\partial_3 v)=\operatorname{div}\mathrm{T}(v,\mathfrak{p}) & \text{in } \Omega,\\ \operatorname{div} v=0 & \text{in } \Omega,\\ [\![v]\!]=0 & \text{on } \Gamma,\\ v\cdot\mathrm{n}=-\lambda e_3\cdot\mathrm{n} & \text{on } \Gamma,\\ \mathrm{n}\cdot[\![\mathrm{T}(v,\mathfrak{p})\mathrm{n}]\!]=\sigma(\mathrm{H}+2)+(\rho_1-\rho_2)e_3\cdot x & \text{on } \Gamma,\\ (\mathrm{I}-\mathrm{n}\otimes\mathrm{n})[\![\mathrm{T}(v,\mathfrak{p})\mathrm{n}]\!]=0 & \text{on } \Gamma,\\ \lim_{|x|\to\infty} v(x)=0, \quad |\Omega^{(1)}|=\dfrac{4\pi}{3}, \quad \displaystyle\int_{\Omega^{(1)}} x\,\mathrm{d}x=0. \end{cases} \tag{2.10}$$

Observe that the mean curvature now appears in the form $(\mathrm{H}+2)$ that vanishes if Γ is the unit sphere, which means that $(v,\mathfrak{p},\lambda)=(0,0,0)$ is a trivial solution when $\rho_1-\rho_2=0$.

We shall employ a parameterization of Γ over the unit sphere $\mathbb{S}^2\subset\mathbb{R}^3$ and subsequently linearize (2.10). The linearization of the operator $\sigma(\mathrm{H}+2)$, however, has a non-trivial kernel. To circumvent an introduction of corresponding compatibility conditions, we employ an idea from [15] and replace the two equations

$$\mathrm{n}\cdot[\![\mathrm{T}(v,\mathfrak{p})\mathrm{n}]\!]=\sigma(\mathrm{H}+2)+(\rho_1-\rho_2)e_3\cdot x, \qquad \int_{\Omega^{(1)}} x\,\mathrm{d}x=0 \tag{2.11}$$

in (2.10) with the equations

$$\begin{aligned} &\mathrm{n}\cdot[\![\mathrm{T}(v,\mathfrak{p})\mathrm{n}]\!]=\sigma(\mathrm{H}+2)+\frac{1}{4\pi}\mathrm{n}\cdot\int_{\Omega^{(1)}} x\,\mathrm{d}x+(\rho_1-\rho_2)e_3\cdot x,\\ &\int_\Gamma [\![\mathrm{T}(v,\mathfrak{p})\mathrm{n}]\!]\,\mathrm{d}S=(\rho_1-\rho_2)\frac{4\pi}{3}e_3. \end{aligned} \tag{2.12}$$

The resulting system then reads

$$\begin{cases} \rho(v\cdot\nabla v+\lambda\partial_3 v)=\operatorname{div}\mathrm{T}(v,\mathfrak{p}) & \text{in } \Omega,\\ \operatorname{div} v=0 & \text{in } \Omega,\\ [\![v]\!]=0 & \text{on } \Gamma,\\ v\cdot\mathrm{n}=-\lambda e_3\cdot\mathrm{n} & \text{on } \Gamma,\\ (\mathrm{I}-\mathrm{n}\otimes\mathrm{n})[\![\mathrm{T}(v,\mathfrak{p})\mathrm{n}]\!]=0 & \text{on } \Gamma,\\ \mathrm{n}\cdot[\![\mathrm{T}(v,\mathfrak{p})\mathrm{n}]\!]=\sigma(\mathrm{H}+2)+\dfrac{1}{4\pi}\mathrm{n}\cdot\displaystyle\int_{\Omega^{(1)}} x\,\mathrm{d}x+(\rho_1-\rho_2)e_3\cdot x & \text{on } \Gamma,\\ \displaystyle\int_\Gamma [\![\mathrm{T}(v,\mathfrak{p})\mathrm{n}]\!]\,\mathrm{d}S=(\rho_1-\rho_2)\frac{4\pi}{3}e_3, & \\ \lim_{|x|\to\infty} v(x)=0, \quad |\Omega^{(1)}|=\dfrac{4\pi}{3}. & \end{cases} \tag{2.13}$$

The systems (2.10) and (2.13) are equivalent. Clearly, (2.10) implies (2.13). To verify the reverse implication, observe that $(2.13)_{5-7}$ imply

$$\begin{aligned}(\rho_1-\rho_2)\frac{4\pi}{3}e_3 &= \int_\Gamma [\![\mathrm{T}(v,\mathfrak{p})\mathrm{n}]\!]\,\mathrm{d}S = \int_\Gamma \big(\mathrm{n}\cdot[\![\mathrm{T}(v,\mathfrak{p})\mathrm{n}]\!]\big)\mathrm{n}\,\mathrm{d}S\\ &= \int_\Gamma \sigma(\mathrm{H}+2)\mathrm{n}\,\mathrm{d}S + \frac{1}{4\pi}\int_\Gamma \mathrm{n}\otimes\mathrm{n}\,\mathrm{d}S\int_{\Omega^{(1)}} x\,\mathrm{d}x + (\rho_1-\rho_2)\frac{4\pi}{3}e_3\\ &= \int_\Gamma \sigma\Delta_\Gamma x\,\mathrm{d}S + 2\sigma\int_\Gamma \mathrm{n}\,\mathrm{d}S + \frac{1}{4\pi}\int_\Gamma \mathrm{n}\otimes\mathrm{n}\,\mathrm{d}S\int_{\Omega^{(1)}} x\,\mathrm{d}x + (\rho_1-\rho_2)\frac{4\pi}{3}e_3\\ &= 0+0+\frac{1}{4\pi}\int_\Gamma \mathrm{n}\otimes\mathrm{n}\,\mathrm{d}S\int_{\Omega^{(1)}} x\,\mathrm{d}x + (\rho_1-\rho_2)\frac{4\pi}{3}e_3.\end{aligned}$$

The matrix $\int_\Gamma \mathrm{n}\otimes\mathrm{n}\,\mathrm{d}S$ is symmetric positive definite and thus invertible. Consequently, the equation above implies $\int_{\Omega^{(1)}} x\,\mathrm{d}x = 0$. We conclude that (2.13) implies (2.10).

Since we investigate existence of non-trivial steady-states in a drop configuration close to the ball B_1 (in non-dimensionalized coordinates) under the restriction that the difference in densities of the two liquids is sufficiently small, it is convenient to introduce

$$\widetilde{\rho} := \rho_1 - \rho_2$$

as smallness parameter. Moreover, it is convenient to parameterize the interface Γ via a height function $\eta : \mathbb{S}^2 \to \mathbb{R}$ that describes the drop's displacement in normal direction with respect to its unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ stress-free configuration. The geometry then becomes a function of η:

$$\begin{aligned}&\Omega^{(1)} = \Omega^{(1)}_\eta := \big\{r\zeta \;\big|\; \zeta\in\mathbb{S}^2,\ 0\le r < 1+\eta(\zeta)\big\},\\ &\Omega^{(2)} = \Omega^{(2)}_\eta := \big\{r\zeta \;\big|\; \zeta\in\mathbb{S}^2,\ 1+\eta(\zeta) < r\big\},\\ &\Gamma = \Gamma_\eta := \big\{(1+\eta(\zeta))\zeta \;\big|\; \zeta\in\mathbb{S}^2\big\}, \quad \Omega = \Omega_\eta := \Omega^{(1)}_\eta \cup \Omega^{(2)}_\eta.\end{aligned}$$

The system of steady-state equations of motion finally takes the form

$$\begin{cases} \rho(v\cdot\nabla v + \lambda\partial_3 v) = \operatorname{div}\mathrm{T}(v,\mathfrak{p}) & \text{in } \Omega_\eta,\\ \operatorname{div} v = 0 & \text{in } \Omega_\eta,\\ [\![v]\!] = 0 & \text{on } \Gamma_\eta,\\ v\cdot\mathrm{n} = -\lambda e_3\cdot\mathrm{n} & \text{on } \Gamma_\eta,\\ (\mathrm{I}-\mathrm{n}\otimes\mathrm{n})[\![\mathrm{T}(v,\mathfrak{p})\mathrm{n}]\!] = 0 & \text{on } \Gamma_\eta,\\ \mathrm{n}\cdot[\![\mathrm{T}(v,\mathfrak{p})\mathrm{n}]\!] = \sigma(\mathrm{H}+2) + \frac{1}{16\pi}\mathrm{n}\cdot\int_{\mathbb{S}^2}\zeta\big((1+\eta(\zeta))^4-1\big)\,\mathrm{d}S + \widetilde{\rho}e_3\cdot x & \text{on } \Gamma_\eta,\\ \int_{\Gamma_\eta}[\![\mathrm{T}(v,\mathfrak{p})\mathrm{n}]\!]\,\mathrm{d}S = \widetilde{\rho}\frac{4\pi}{3}e_3, \quad \int_{\mathbb{S}^2}\big((1+\eta(\zeta))^3-1\big)\,\mathrm{d}S = 0, \quad \lim_{|x|\to\infty} v(x) = 0 \end{cases} \tag{2.14}$$

with respect to unknowns $(v,\mathfrak{p},\lambda,\eta)$.

As the main result in the article we prove existence of a solution to the steady-state equations of motion (2.14) under a smallness condition on the density difference $\widetilde{\rho}$.

Theorem 2.1 (Main Theorem). *There is an $\varepsilon > 0$ such that for $0 < |\widetilde{\rho}| \le \varepsilon$ there is a solution*

$$(v,\mathfrak{p},\lambda,\eta) \in \mathrm{C}^\infty(\Omega_\eta)^3 \times \mathrm{C}^\infty(\Omega_\eta) \times \mathbb{R} \times \mathrm{C}^\infty(\mathbb{S}^2)$$

to (2.14). The solution is smooth up to the interface, that is,

$$v\big|_{\Omega^{(2)}_\eta},\, \mathfrak{p}\big|_{\Omega^{(2)}_\eta} \in \mathrm{C}^\infty\big(\overline{\Omega}^{(2)}_\eta\big), \qquad v\big|_{\Omega^{(1)}_\eta},\, \mathfrak{p}\big|_{\Omega^{(1)}_\eta} \in \mathrm{C}^\infty\big(\overline{\Omega}^{(1)}_\eta\big). \tag{2.15}$$

Moreover, it possesses the integrability properties

$$\forall q\in(1,2): \quad v\in\mathrm{L}^{\frac{2q}{2-q}}(\Omega_\eta), \quad \nabla v\in\mathrm{L}^{\frac{4q}{4-q}}(\Omega_\eta), \quad \partial_3 v, \nabla^2 v, \nabla\mathfrak{p} \in \mathrm{L}^q(\Omega_\eta), \tag{2.16}$$

and admits the representation

$$v(x) = \frac{4\pi}{3}\widetilde{\rho}\,\Gamma^{\lambda}_{\mathrm{Oseen}}(x)\,e_3 + O\big(|x|^{-\frac{3}{2}+\varepsilon}\big) \quad \text{as } |x| \to \infty \tag{2.17}$$

for all $\varepsilon > 0$*, where* $\Gamma^{\lambda}_{\mathrm{Oseen}}$ *denotes the Oseen fundamental solution.*[1] *The solution is symmetric with respect to rotations leaving* e_3 *invariant:*

$$\forall R \in SO(3),\ Re_3 = e_3: \quad R^{\top}v(Rx) = v(x),\ p(Rx) = p(x),\ \eta(Rx) = \eta(x), \tag{2.18}$$

and the velocity λ *of the drop's barycenter is non-vanishing.*

By far the most challenging part of proving Theorem 2.1 is to establish the existence of a solution. As mentioned in the introduction, via a perturbation around a non-trivial state we are able to solve the system in a setting of Sobolev spaces adopted from the 3D exterior-domain Oseen linearization of the Navier–Stokes equations. Consequently, we are led to a solution with the integrability properties (2.16). The symmetry (2.18) follows from the observation that (2.14) is invariant with respect to rotations leaving e_3 invariant. Higher-order regularity is obtained via a standard approach utilizing the ellipticity of (2.14), while the asymptotic profile (2.17) is a direct consequence of (2.16) and a celebrated result of Babenko [4] and Galdi [7]. Observe that the coefficient vector in the asymptotic expansion, which at the outset is given by

$$\int_{\Gamma_\eta} \mathrm{T}(v,\mathfrak{p})\big|_{\Omega^{(2)}_\eta}\mathrm{n}\,\mathrm{d}S,$$

coincides with the net force $\frac{4\pi}{3}\widetilde{\rho}e_3 = \widetilde{\rho}|\Omega^{(1)}_\eta|e_3$ acting on the liquid drop, that is, the difference of the gravitational force and the buoyancy force.

The solution obtained in Theorem 2.1 is locally unique. Specifically, a radius r can be quantified in terms of the density difference $\widetilde{\rho}$ such that the solution is unique in a ball B_r in a suitable Banach space; see Theorem 8.4. The local uniqueness follows directly from Banach's Fixed Point Theorem. Global uniqueness for small data is expected, but the energy type estimates required to show this goes beyond the scope of this article.

3. Notation

We use capital letters to denote global constants in the proofs and theorems, and small letters for local constants appearing in the proofs.

By $\mathrm{B}_R := \mathrm{B}_R(0)$ we denote a ball in $\mathbb{R}^n$ centered at 0 with radius R. Moreover, we let

$$\mathrm{B}^R := \mathbb{R}^3 \setminus \overline{\mathrm{B}_R}, \quad \mathrm{B}_{R,r} := \mathrm{B}_R \setminus \overline{\mathrm{B}}_r, \quad \Omega_R := \Omega \cap \mathrm{B}_R, \quad \Omega^R := \Omega \cap \mathrm{B}^R$$

for a domain $\Omega \subset \mathbb{R}^n$. Additionally, we use $\mathbb{S}^2 := \partial\mathrm{B}_1$ to denote the unit sphere. By

$$\dot{\mathbb{R}}^3 := \big\{(x_1,x_2,x_3) \in \mathbb{R}^3 \bigm| x_3 \neq 0\big\}$$

we denote the twofold half space, which is the union of the two domains

$$\dot{\mathbb{R}}^3_+ := \big\{(x_1,x_2,x_3) \in \mathbb{R}^3 \bigm| x_3 > 0\big\}, \qquad \dot{\mathbb{R}}^3_- := \big\{(x_1,x_2,x_3) \in \mathbb{R}^3 \bigm| x_3 < 0\big\}.$$

We use the notation (x', x_3) for a vector $x = (x_1,x_2,x_3) \in \mathbb{R}^3$.

Lebesgue spaces are denoted by $\mathrm{L}^q(\Omega)$ with associated norms $\|\cdot\|_{q,\Omega}$. By $\mathrm{W}^{k,q}(\Omega)$ we denote the corresponding Sobolev space of order $k \in \mathbb{N}_0$ with norm $\|\cdot\|_{k,q,\Omega}$, and we introduce the subspaces

$$\mathrm{W}^{k,q}_0(\Omega) := \overline{\mathrm{C}^\infty_0(\Omega)}^{\|\cdot\|_{k,q,\Omega}}.$$

[1] An explicit formula for $\Gamma^{\lambda}_{\mathrm{Oseen}}$ can be found in [8, Sect. VII.3] for example.

Moreover, $\mathrm{W}^{-k,q}(\Omega)$ and $\mathrm{W}_0^{-k,q}(\Omega)$ denote the dual spaces of $\mathrm{W}^{k,q'}(\Omega)$ and $\mathrm{W}_0^{k,q'}(\Omega)$, respectively, where $q' := \frac{q}{q-1}$. We further introduce homogeneous Sobolev spaces $\mathrm{D}^{k,q}(\Omega)$ defined by

$$\mathrm{D}^{k,q}(\Omega) := \{u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega) \ \mid \ \nabla^k u \in \mathrm{L}^q(\Omega)\},$$

and the corresponding seminorm

$$|u|_{k,q,\Omega} := \|\nabla^k u\|_{q,\Omega} := \sum_{|\alpha|=k} \|\partial^\alpha u\|_{q,\Omega}.$$

In general, $\mathrm{D}^{k,p}(\Omega)$ is not a Banach space. However, $|\cdot|_{k,q,\Omega}$ defines a norm on $\mathrm{C}_0^\infty(\Omega)$, and the completion

$$\mathrm{D}_0^{k,q}(\Omega) := \overline{\mathrm{C}_0^\infty(\Omega)}^{|\cdot|_{k,q,\Omega}}$$

is therefore a Banach space. By Sobolev's Embedding Theorem, $\mathrm{D}_0^{k,q}(\Omega)$ can be identified with a subspace of $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$ if $kq < 3$. We denote its dual space by $\mathrm{D}_0^{-k,q'}(\Omega)$. For a sufficiently smooth manifold $\Gamma \subset \mathbb{R}^3$ and $s > 0$, $s \notin \mathbb{N}$, we let $\mathrm{W}^{s,q}(\Gamma)$ denote the Sobolev–Slobodeckij space of order s with norm $\|\cdot\|_{s,q,\Gamma}$.

4. Preliminaries

In this section we introduce a bespoke framework of Sobolev spaces for the investigation of (2.14). For this purpose we let $\Omega \subset \mathbb{R}^3$ denote an open set of the same type as in Sect. 2, that is, we assume

$$\begin{aligned}&\Omega^{(1)} \subset \mathbb{R}^3 \text{ is a bounded domain such that } \Omega^{(2)} := \mathbb{R}^3 \setminus \overline{\Omega^{(1)}} \text{ is a domain,}\\ &\Gamma := \partial\Omega^{(1)}, \qquad \Omega := \Omega^{(1)} \cup \Omega^{(2)} = \mathbb{R}^3 \setminus \Gamma.\end{aligned} \tag{4.1}$$

For a function $u : \Omega \to \mathbb{R}$ we use the abbreviations

$$u^{(1)} := u\big|_{\Omega^{(1)}}, \qquad u^{(2)} := u\big|_{\Omega^{(2)}}.$$

The function $\mathrm{n} = \mathrm{n}_\Gamma$ denotes the unit outer normal at Γ. If u is sufficiently regular, we set

$$[\![u]\!] := u^{(1)}\big|_\Gamma - u^{(2)}\big|_\Gamma,$$

where the restrictions to Γ have to be understood in the trace sense. Furthermore, $\delta(\Omega)$ denotes the diameter of $\Omega^{(1)}$.

When considering a function $u : \Omega \to \mathbb{R}$, we often have to distinguish between its properties on the disjoint sub-domains $\Omega^{(1)}$ and $\Omega^{(2)}$. To this end, function spaces of the type

$$X := \big\{u : \Omega \to \mathbb{R} \ \big| \ u^{(1)} \in X^{(1)},\, u^{(2)} \in X^{(2)}\big\}$$

are introduced. Equipped with the norm

$$\|u\|_X := \|u^{(1)}\|_{X^{(1)}} + \|u^{(2)}\|_{X^{(2)}},$$

such a space X is isomorphic to the direct sum of the spaces $X^{(1)}$ and $X^{(2)}$. Clearly, X is a Banach space if $X^{(1)}$ and $X^{(2)}$ are so.

Let $q \in \big(1, \frac{3}{2}\big)$ and $r \in (3, \infty)$. For $\lambda_0 \in \mathbb{R}, \lambda_0 \neq 0$ the space

$$\begin{aligned}\mathrm{X}^{q,r,\lambda_0}_{\mathrm{Oseen}} := \mathrm{X}^{q,r,\lambda_0}_{\mathrm{Oseen}}(\Omega^{(2)}) := \big\{u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega^{(2)})^3 \ \big| \ &u \in \mathrm{L}^{\frac{2q}{2-q}} \cap \mathrm{D}^{1,\frac{4q}{4-q}} \cap \mathrm{D}^{2,q} \cap \mathrm{D}^{2,r},\\ &\partial_3 u \in \mathrm{L}^q \cap \mathrm{L}^r\big\}\end{aligned}$$

equipped with the norm

$$\begin{aligned}\|u\|_{\lambda_0,\mathrm{Oseen}} := |\lambda_0|^{\frac{1}{2}} \|u\|_{\frac{2q}{2-q}} + |\lambda_0|^{\frac{1}{4}} \|\nabla u\|_{\frac{4q}{4-q}} + \|\nabla^2 u\|_q\\ + \|\nabla^2 u\|_r + |\lambda_0|\, \|\partial_3 u\|_q + |\lambda_0|\, \|\partial_3 u\|_r\end{aligned}$$

is the canonical solution space for solutions to the exterior domain Oseen problem

$$\begin{cases} -\operatorname{div}\mathrm{T}(u,\mathfrak{p})+\lambda_0\partial_3 u=f & \text{in } \Omega^{(2)},\\ \operatorname{div} u=g & \text{in } \Omega^{(2)} \end{cases} \tag{4.2}$$

for forcing terms f in $\mathrm{L}^q\big(\Omega^{(2)}\big)\cap\mathrm{L}^r\big(\Omega^{(2)}\big)$; see for example [8, Chapter VII.7]. Let

$$\begin{aligned}
\mathbf{X}_{1,\lambda_0}^{q,r} &:= \big\{u\in\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^3)^3 \;\big|\; u^{(1)}\in\mathrm{W}^{2,r},\ u^{(2)}\in\mathrm{X}^{q,r,\lambda_0}_{\mathrm{Oseen}},\ [\![u]\!]=0\big\},\\
\mathbf{X}_2^{q,r} &:= \big\{\mathfrak{p}\in\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^3) \;\big|\; \mathfrak{p}^{(1)}\in\mathrm{W}^{1,r},\ \mathfrak{p}^{(2)}\in\mathrm{D}^{1,q}\cap\mathrm{D}^{1,r}\cap\mathrm{L}^{\frac{3q}{3-q}}\big\},\\
\mathbf{X}_3^{q,r} &:= \mathbb{R},\\
\mathbf{X}_4^{q,r} &:= \mathrm{W}^{3-1/r,r}(\Gamma),
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{Y}_1^{q,r} &:= \mathrm{L}^q(\mathbb{R}^3)^3\cap\mathrm{L}^r(\mathbb{R}^3)^3,\\
\mathbf{Y}_2^{q,r} &:= \big\{g\in\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^3) \;\big|\; g^{(1)}\in\mathrm{W}^{1,r},\ g^{(2)}\in\mathrm{D}^{1,q}\cap\mathrm{D}^{1,r}\cap\mathrm{L}^{\frac{3q}{3-q}}\big\},\\
\mathbf{Y}_3^{q,r} &:= \mathrm{W}^{2-1/r,r}(\Gamma),\\
\mathbf{Y}_{2,3}^{q,r} &:= \left\{(g,h)\in\mathbf{Y}_2^{q,r}\times\mathbf{Y}_3^{q,r} \;\middle|\; \int_{\Omega^{(1)}} g\,\mathrm{d}x=\int_\Gamma h\,\mathrm{d}S\right\},\\
\mathbf{Y}_4^{q,r} &:= \big\{h\in\mathrm{W}^{1-1/r,r}(\Gamma)^3 \;\big|\; h\cdot\mathrm{n}=0\big\},\\
\mathbf{Y}_5^{q,r} &:= \mathbf{Y}_6^{q,r} := \mathbb{R},\\
\mathbf{Y}_7^{q,r} &:= \mathrm{W}^{1-1/r,r}(\Gamma).
\end{aligned}$$

The bespoke framework of Sobolev spaces we shall employ in our investigation of (2.14) is then given by

$$\begin{aligned}
\mathbf{X}_{\lambda_0}^{q,r} &:= \mathbf{X}_{\lambda_0}^{q,r}(\Omega) := \mathbf{X}_{1,\lambda_0}^{q,r}\times\mathbf{X}_2^{q,r}\times\mathbf{X}_3^{q,r}\times\mathbf{X}_4^{q,r},\\
\mathbf{Y}^{q,r} &:= \mathbf{Y}^{q,r}(\Omega) := \mathbf{Y}_1^{q,r}\times\mathbf{Y}_{2,3}^{q,r}\times\mathbf{Y}_4^{q,r}\times\mathbf{Y}_5^{q,r}\times\mathbf{Y}_6^{q,r}\times\mathbf{Y}_7^{q,r}.
\end{aligned}$$

In Theorem 8.1 we show that the operator corresponding to the appropriate linearization of (2.14) maps $\mathbf{X}_{\lambda_0}^{q,r}$ homeomorphically onto $\mathbf{Y}^{q,r}$.

The following embedding is valid:

Proposition 4.1. *Let $u\in\mathbf{X}_{1,\lambda_0}^{q,r}$ with $q\in\big(1,\frac{3}{2}\big)$, $r\in(3,\infty)$, and consider $s\in\big[\frac{2q}{2-q},\infty\big]$ and $t\in\big[\frac{4q}{4-q},\infty\big]$. Then $u\in\mathrm{L}^s(\Omega)\cap\mathrm{D}^{1,t}(\Omega)$. If $s\geq\frac{3q}{3-2q}$ and $t\geq\frac{3q}{3-q}$, then*

$$\|u\|_s+\|\nabla u\|_t\leq C\|u\|_{\mathbf{X}_{1,\lambda_0}^{q,r}}. \tag{4.3}$$

If $\frac{2q}{2-q}\leq s<\frac{3q}{3-2q}$, $\frac{4q}{4-q}\leq t<\frac{3q}{3-q}$, $\theta_s:=2+\frac{3}{s}-\frac{3}{q}$ and $\theta_t:=1+\frac{3}{t}-\frac{3}{q}$, then

$$|\lambda_0|^{\theta_s}\|u\|_s+|\lambda_0|^{\theta_t}\|\nabla u\|_t\leq C\|u\|_{\mathbf{X}_{1,\lambda_0}^{q,r}}. \tag{4.4}$$

Here $C=C(q,r,s,t,\Omega)>0$.

Proof. The above estimates for the part $u^{(1)}$ of u defined on a bounded domain follows directly from Sobolev embedding theorems. Concerning the part $u^{(2)}$ defined on an exterior domain, it follows from [8, Lemma II.6.1] and the Sobolev inequality that

$$\|\nabla u^{(2)}\|_\infty\leq c\big(|\nabla u^{(2)}|_{1,r}+\|\nabla u^{(2)}\|_{\frac{3q}{3-q}}\big)\leq c\big(|u^{(2)}|_{2,r}+|u^{(2)}|_{2,q}\big)\leq c\|u\|_{\mathbf{X}_{1,\lambda_0}^{q,r}}.$$

Interpolation with the Sobolev-type inequality

$$\|\nabla u\|_{\frac{3q}{3-q}}\leq c\|\nabla^2 u\|_q\leq c\|u\|_{\mathbf{X}_{1,\lambda_0}^{q,r}} \tag{4.5}$$

yields estimate (4.3) of ∇u. Estimate (4.4) of ∇u follows by interpolating (4.5) with the trivial estimate $|\lambda_0|^{\frac{1}{4}}\|\nabla u^{(2)}\|_{\frac{4q}{4-q}}\leq\|u\|_{\mathbf{X}_{1,\lambda_0}^{q,r}}$. The estimates (4.3)–(4.4) of u can be verified in a similar manner. □

5. Auxiliary Linear Problem

Let Ω be a domain of the same type as in Sect. 4, *i.e.*, satisfying (4.1). We further assume that the boundary Γ is at least Lipschitz. The linear system

$$\begin{cases} -\operatorname{div} \mathrm{T}(u,\mathfrak{p}) + \lambda_0 \partial_3 u = f & \text{in } \Omega, \\ \operatorname{div} u = g & \text{in } \Omega, \\ [\![u]\!] = 0 & \text{on } \Gamma, \\ u \cdot \mathrm{n} = h_1 & \text{on } \Gamma, \\ (\mathrm{I}-\mathrm{n}\otimes\mathrm{n})[\![\mathrm{T}(u,\mathfrak{p})\mathrm{n}]\!] = h_2 & \text{on } \Gamma \end{cases} \tag{5.1}$$

is an integral part of the linearization of (2.14). It is a two-phase strongly coupled Oseen ($\lambda_0 \neq 0$) or Stokes ($\lambda_0 = 0$) system. Since the coupling is strong, the question of existence and uniqueness of solutions as well as *a priori* estimates hereof cannot be investigated by means of a simple decomposition into two classical Oseen/Stokes problems (one for each phase). In the following, we carry out an analysis of (5.1) in the framework of the Sobolev spaces introduced in the previous section. Existence and uniqueness of solutions is first shown in a setting of weak solutions. Strong *a priori* estimates of Agmon–Douglis–Nirenberg type are subsequently established, first in the half space, and then in the general case via a localization technique. The main result of the section is contained in Theorems 5.9 and 5.10.

5.1. Weak Solutions

We introduce a weak formulation of (5.1) in the setting of the function spaces:

$$\begin{aligned} C &:= \{\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^3)^3 \mid \varphi\cdot\mathrm{n} = 0 \text{ on } \Gamma\}, \\ \mathscr{C} &:= \{\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^3)^3 \mid \varphi\cdot\mathrm{n} = 0 \text{ on } \Gamma,\ \operatorname{div}\varphi = 0\}, \\ H &:= \overline{C}^{|\cdot|_{1,2}} = \{\varphi \in \mathrm{D}_0^{1,2}(\mathbb{R}^3)^3 \mid \varphi\cdot\mathrm{n} = 0 \text{ on } \Gamma\}, \\ \mathscr{H} &:= \overline{\mathscr{C}}^{|\cdot|_{1,2}} = \{\varphi \in \mathrm{D}_0^{1,2}(\mathbb{R}^3)^3 \mid \varphi\cdot\mathrm{n} = 0 \text{ on } \Gamma,\ \operatorname{div}\varphi = 0\}, \\ \mathrm{L}_0^2(\mathbb{R}^3) &:= \left\{p \in \mathrm{L}^2(\mathbb{R}^3) \;\middle|\; \int_{\Omega^{(1)}} p \,\mathrm{d}x = 0\right\}. \end{aligned}$$

In the following, we establish existence and uniqueness as well as higher-order regularity of weak solutions to (5.1) in this framework. We start with the definition of a weak solution:

Definition 5.1. Let $f \in \mathscr{H}'$, $g \in \mathrm{L}^2(\mathbb{R}^3)$, $h_1 \in \mathrm{W}^{\frac{1}{2},2}(\Gamma)$ and $h_2 \in \mathrm{W}^{-\frac{1}{2},2}(\Gamma)^3$. A vector field $u \in \mathrm{D}_0^{1,2}(\mathbb{R}^3)^3$ is called a *weak solution to* (5.1) if

$$\forall \varphi \in \mathscr{C}: \quad \int_{\mathbb{R}^3} 2\mu\, \mathrm{S}(u) : \mathrm{S}(\varphi)\,\mathrm{d}x + \lambda_0 \int_{\mathbb{R}^3} \partial_3 u\cdot\varphi\,\mathrm{d}x = \langle f,\varphi\rangle + \langle h_2,\varphi\rangle \tag{5.2}$$

as well as $\operatorname{div} u = g$ in Ω and $u\cdot\mathrm{n} = h_1$ on Γ.

Existence of a weak solution u can be shown by standard techniques; we sketch a proof below.

Theorem 5.2. *Assume that the boundary Γ is Lipschitz. For every $f \in \mathscr{H}'$, $g \in \mathrm{L}^2(\mathbb{R}^3)$, $h_1 \in \mathrm{W}^{\frac{1}{2},2}(\Gamma)$ and $h_2 \in \mathrm{W}^{-\frac{1}{2},2}(\Gamma)^3$ satisfying*

$$\int_{\Omega^{(1)}} g\,\mathrm{d}x = \int_\Gamma h_1\,\mathrm{d}S \tag{5.3}$$

there is a weak solution $u \in \mathrm{D}_0^{1,2}(\mathbb{R}^3)^3$ to (5.1) satisfying

$$|u|_{1,2} \le C\big(\|f\|_{\mathscr{H}'} + \|g\|_2 + \|h_1\|_{\frac{1}{2},2} + \|h_2\|_{-\frac{1}{2},2}\big), \tag{5.4}$$

where $C = C(\Gamma,\lambda_0)$.

Proof. We sketch a proof of existence following [8, Proof of Theorem VII.2.1] based on a Galerkin approximation. To this end, a Schauder basis $\{\varphi_k\}_{k=1}^{\infty} \subset \mathscr{C}$ for the function space $\{\varphi \in \mathrm{W}^{1,2}(\mathbb{R}^3) \mid \varphi\cdot \mathrm{n} = 0 \text{ on } \Gamma,\ \operatorname{div}\varphi = 0\}$ satisfying $\int_\Omega 2\mu\,\mathrm{S}(\varphi_k) : \mathrm{S}(\varphi_l)\,\mathrm{d}x = \delta_{k,l}$ is constructed. This function space is clearly separable, whence such a basis can be constructed via a Gram–Schmidt procedure. We consider first the case $(g, h_1) = (0, 0)$. Existence of an approximate solution of order $m \in \mathbb{N}$, that is, a vector field $u_m := \sum_{l=1}^{m} \xi_l\varphi_l$ satisfying the equation in (5.2) for all test functions in $\operatorname{span}\{\varphi_1, \ldots, \varphi_m\}$, then follows directly from the fact that the matrix $A \in \mathbb{R}^{m\times m}$, $A_{kl} := \int_{\mathbb{R}^3} \partial_3\varphi_l \cdot \varphi_k$, is skew symmetric and $I + \lambda_0 A$ therefore invertible. Specifically, the coefficient vector $\xi := (I + \lambda_0 A)^{-1}F$ with $F_k := \langle f, \varphi_k\rangle + \langle h_2, \varphi_k\rangle$ induces an approximate solution u_m. Employing u_m itself as a test function in the weak formulation, one obtains a uniform bound on $\|\mathrm{S}(u_m)\|_2$, which, since u_m is solenoidal, also implies a uniform bound as in (5.4) on $|u_m|_{1,2}$. A weak solution to (5.1) is now obtained as the limit $u := \lim_{m\to\infty} u_m$ in $\mathscr{H}$. The general case of non-vanishing g and h_1 follows by a lifting argument. Employing a right inverse of the trace operator $\mathrm{W}^{1,2}(\mathbb{R}^3) \to \mathrm{W}^{\frac{1}{2},2}(\Gamma)$, we find $u_1 \in \mathrm{W}^{1,2}(\mathbb{R}^3)$ with $u_1 = h_1$ on Γ satisfying $\|u_1\|_{1,2} \le c\|h_1\|_{\frac{1}{2},2}$. The compatibility condition (5.3) ensures that $\int_{\Omega^{(1)}} g - \operatorname{div} u_1\,\mathrm{d}x = 0$ so that we can find $u_2 \in \mathrm{D}_0^{1,2}(\mathbb{R}^3)$ with $\operatorname{div} u_2 = g - \operatorname{div} u_1$ and satisfying $u_2 = 0$ on Γ as well as (5.4); see for example [8, Theorem III.3.1 and III.3.6]. The ansatz $u = v + u_1 + u_2$ now reduces the problem to the case above with respect to the unknown v. We thus conclude existence of a weak solution. □

A pressure $\mathfrak{p}$ can be associated to a weak solution u such that $(u, \mathfrak{p})$ becomes a solution to (5.1) in the sense of distributions:

Theorem 5.3. *Assume $f \in H'$. To every weak solution $u \in \mathrm{D}_0^{1,2}(\mathbb{R}^3)$ to (5.1) there is a unique $\mathfrak{p} \in \mathrm{L}_0^2(\mathbb{R}^3)$ such that*

$$\forall\varphi \in C:\ \int_{\mathbb{R}^3} 2\mu\,\mathrm{S}(u) : \mathrm{S}(\varphi)\,\mathrm{d}x + \lambda_0\int_{\mathbb{R}^3} \partial_3 u\cdot\varphi\,\mathrm{d}x = \int_{\mathbb{R}^3} \mathfrak{p}\ \operatorname{div}\varphi\,\mathrm{d}x + \langle f, \varphi\rangle + \langle h_2, \varphi\rangle \tag{5.5}$$

and

$$\|\mathfrak{p}\|_2 \le C\big(\|f\|_{H'} + \|g\|_2 + \|h_1\|_{\frac{1}{2},2} + \|h_2\|_{-\frac{1}{2},2}\big) \tag{5.6}$$

with $C = C(\Gamma) > 0$.

Proof. The proof is modification of [8, Lemma VII.1.1]. For arbitrary $M \in \mathbb{N}$ with $M > \delta(\Omega)$ we let $H_M := \{\varphi \in H \mid \operatorname{supp}\varphi \subset B_M\}$ and consider the functional

$$F_M : H_M \to \mathbb{R},\ F_M(\varphi) := \int_{B_M} 2\mu\,\mathrm{S}(u) : \mathrm{S}(\varphi)\,\mathrm{d}x + \lambda_0\int_{B_M} \partial_3 u\cdot\varphi\,\mathrm{d}x - \langle f, \varphi\rangle - \langle h_2, \varphi\rangle,$$

which is continuous on H_M by Sobolev embedding. We further introduce the space

$$\mathrm{L}^2_{0,M} := \left\{p \in \mathrm{L}^2(B_M)\ \middle|\ \int_{\Omega^{(2)}\cap B_M} p\,\mathrm{d}x = \int_{\Omega^{(1)}} p\,\mathrm{d}x = 0\right\}$$

and the operator $\operatorname{div} : H_M \to \mathrm{L}^2_{0,M}$. The operator is surjective, which is seen by solving for arbitrary $p \in \mathrm{L}^2_{0,M}$ the two equations

$$\begin{cases} \operatorname{div} u^{(1)} = p & \text{in } \Omega^{(1)},\\ u^{(1)} = 0 & \text{on } \Gamma, \end{cases} \qquad \begin{cases} \operatorname{div} u^{(2)} = p & \text{in } \Omega^{(2)}\cap B_M,\\ u^{(2)} = 0 & \text{on } \Gamma\cup\partial B_M, \end{cases}$$

according to [8, Theorem III.3.1]. It follows that the operator and hence also its adjoint div^* are both closed. Since u is a weak solution, F_M vanishes on the kernel of div and consequently belongs to the image of div^*. We thus obtain a function $\mathfrak{p}_M \in \mathrm{L}^2_{0,M}$ such that $\langle F_M, \varphi\rangle := \int_{B_R} \mathfrak{p}_M \operatorname{div}\varphi\,\mathrm{d}x$. After possibly adding a constant to $\mathfrak{p}^{(2)}_{M+1}$, we may assume $\mathfrak{p}_{M+1} = \mathfrak{p}_M$ in B_M. The sequence $\{\mathfrak{p}_M\}_{M=1}^{\infty}$ then induces a pressure $\mathfrak{p} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$ satisfying (5.5) and $\int_{\Omega^{(1)}} \mathfrak{p}\,\mathrm{d}x = 0$. It remains to establish $\mathrm{L}^2(\mathbb{R}^3)$ integrability of $\mathfrak{p}$. If $\lambda_0 = 0$, the functional F_M remains continuous if H_M is replaced with H. In this case the argument above directly yields a pressure $\mathfrak{p} \in \mathrm{L}^2_0(\mathbb{R})^3$ satisfying (5.5). Subsequently choosing a function $\varphi \in H$ with $\operatorname{div}\varphi = \mathfrak{p}$ in

$\mathbb{R}^3$ and $|\varphi|_{1,2} \leq c\|\mathfrak{p}\|_2$, which can be done via [8, Theorem III.3.1 and Theorem III.3.6], one obtains (5.6) by inserting φ into (5.5). If $\lambda_0 \neq 0$, it suffices to observe that $(u, \mathfrak{p})$ is a weak solution to an Oseen problem in the exterior domain $\Omega^{(2)}$, whence [8, Theorem VII.7.2] yields $\mathfrak{p} \in \mathrm{L}^2_0(\mathbb{R})^3$ satisfying (5.5)–(5.6). $\square$

Provided u and $\mathfrak{p}$ are sufficiently regular, integration by parts in (5.5) reveals that $(u, \mathfrak{p})$ is a classical solution to (5.1). Higher-order regularity of $(u, \mathfrak{p})$ can be obtained via a classical approach under appropriate regularity assumptions on the data:

Theorem 5.4. *Let $k \in \mathbb{N}_0$ and assume that Γ is a C^{k+3}-smooth closed surface. If*

$$f \in \mathrm{W}^{k,2}(\Omega)^3, \quad g \in \mathrm{W}^{k+1,2}(\Omega), \quad h_1 \in \mathrm{W}^{k+3/2,2}(\Gamma), \quad h_2 \in \mathrm{W}^{k+1/2,2}(\Gamma)^3,$$

then a weak solution $u \in \mathrm{D}^{1,2}_0(\mathbb{R}^3)^3$ to (5.1) with associated pressure $\mathfrak{p} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$ satisfying (5.5) also satisfies

$$u \in \bigcap_{\ell=0}^{k} \mathrm{D}^{\ell+2,2}(\Omega), \qquad \mathfrak{p} \in \bigcap_{\ell=0}^{k} \mathrm{D}^{\ell+1,2}(\Omega). \tag{5.7}$$

Proof. The proof is a standard application of a well-known technique based on difference quotients. In fact, with only minimal modification it is similar to a proof of higher-order regularity for solutions to the Stokes system with prescribed normal velocity and tangential stress on the boundary; see [16, Proof of Theorem 2]. For the sake of completeness, we sketch the proof. We include only the case $h_1 = 0$ and $k = 0$. The general case $h_1 \neq 0$ and $k > 0$ follows by a simple lifting technique and iteration procedure, respectively. Since higher-order regularity in Ω away from the boundary Γ is well known for Stokes systems (see for example [8, Sect. IV.2]), we focus on regularity up to the boundary Γ. To this end, consider an arbitrary $\tilde{x} \in \Gamma$ and choose a cube $Q_r(\tilde{x}) \subset \mathbb{R}^3$, centered at $\tilde{x}$ with side length r, such that $\Gamma \cap Q_r(\tilde{x})$ can be parameterized by a C^3 function ω. Without loss of generality, we may assume that $\tilde{x} = 0$ and

$$\Gamma \cap Q_r(\tilde{x}) = \Gamma \cap Q_r(0) = \big\{\big(x_1, x_2, \omega(x_1, x_2)\big) \bigm| (x_1, x_2) \in Q'_r(0)\big\},$$

where $Q'_r(0) \subset \mathbb{R}^2$ is the two-dimensional cube centered around 0, and that $\nabla\omega(0) = 0$ as well as $\|\nabla\omega\|_\infty \to 0$ as $r \to 0$. Let $\chi \in \mathrm{C}^\infty_0(\mathbb{R}^3)$ be a cut-off function with $\chi = 1$ on $Q_{\frac{r}{2}}(0)$ and put

$$\begin{aligned}&\Phi(x) := \big(x_1, x_2, x_3 - \omega(x_1, x_2)\big),\\ &U : Q_r(0) \to \mathbb{R}^3,\ U := \big[\nabla\Phi(\chi u)\big] \circ \Phi^{-1}, \quad \mathfrak{P} : Q_r(0) \to \mathbb{R},\ \mathfrak{P} := [\chi\mathfrak{p}] \circ \Phi^{-1}.\end{aligned}$$

We introduce test functions

$$\mathrm{W}^{1,2}_{0,\Gamma_0}\big(Q_r(0)\big) := \{\psi \in \mathrm{W}^{1,2}\big(Q_r(0)\big)^3 \ \mid\ \psi = 0 \text{ on } \partial Q_r(0),\ \psi \cdot \mathrm{n} = 0 \text{ on } \Gamma_0\}$$

with

$$\Gamma_0 := \{x \in Q_r(0) \ \mid\ x_3 = 0\}.$$

The transformed fields $(U, \mathfrak{P})$ satisfy the weak formulation

$$\forall \psi \in \mathrm{W}^{1,2}_{0,\Gamma_0}\big(Q_r(0)\big) : \int_{\mathbb{R}^3} 2\mu\, \mathrm{S}(U) : \mathrm{S}(\psi)\, \mathrm{d}x - \int_{\mathbb{R}^3} \mathfrak{P} \operatorname{div}\psi \,\mathrm{d}x = \langle F_0, \psi\rangle + \langle F_1, \nabla\psi\rangle, \tag{5.8}$$

where F_0 contains up to first-order terms of u and zeroth-order terms of $\mathfrak{p}$, and F_1 contains first-order terms of U multiplied with components of $\nabla\omega$. The magnitude of the latter terms can be made small by choosing r small. Difference quotients are denoted by $D^h_l U(x) := \frac{1}{h}\big(U(x + he_l) - U(x)\big)$. Importantly, difference quotients $D^{-h}_l D^h_l U$ in tangential direction $l = 1, 2$ are admissible as test functions in $\mathrm{W}^{1,2}_{0,\Gamma_0}\big(Q_r(0)\big)$ and can therefore be inserted into (5.8), which yields an estimate of $\|\mathrm{S}(D^h_l U)\|_2$ in terms of lower-order norms of u and $\mathfrak{p}$ as well as $\|D^h_l \mathfrak{P}\|_2$. A similar bound on $\|\nabla D^h_l U\|_2$ follows from Korn's inequality. Choosing in (5.8) a test function $D^{-h}_l \psi \in \mathrm{W}^{1,2}_{0,\Gamma_0}\big(Q_r(0)\big)$ with $\operatorname{div}\psi = D^h_l\mathfrak{P}$, a bound on $\|D^h_l\mathfrak{P}\|_2$ in terms of lower-order norms of u and $\mathfrak{p}$ is obtained. Such a test function is constructed by setting $\psi := \psi^+$ in $Q^+ := \{x \in Q_r(0) \mid x_3 > 0\}$ and $\psi := \psi^-$ in $Q^- := \{x \in Q_r(0) \mid x_3 < 0\}$ where

$$\begin{cases} \operatorname{div} \psi^+ = D_l^h \mathfrak{P} & \text{in } Q^+, \\ \quad \psi^+ = 0 & \text{on } \partial Q^+, \end{cases} \qquad \begin{cases} \operatorname{div} \psi^- = D_l^h \mathfrak{P} & \text{in } Q^-, \\ \quad \psi^- = 0 & \text{on } \partial Q^-. \end{cases}$$

Existence of solutions to the two equations above and the estimates $\|\psi^\pm\|_{1,2} \leq c\|D_l^h \mathfrak{P}\|_2$ are secured by [8, Corollary III.5.1]. It follows that $\|\nabla D_l^h U\|_2 + \|D_l^h \mathfrak{P}\|_2$ is uniformly bounded in h, which implies $\partial_l \nabla U, \partial_l \mathfrak{P} \in \mathrm{L}^2(Q_r(0))$ for $l = 1, 2$. Since $\operatorname{div} U = G$ with G containing only zeroth-order terms of u, $\partial_3^2 U \in \mathrm{L}^2(Q_r(0))$ follows as a combination of $\partial_3 \operatorname{div} U = \partial_3 G$ and the regularity of U's tangential derivatives. Finally, the distributional derivative $\partial_3 \mathfrak{P}$ can now be isolated in (5.8) to deduce in each half of the cube $\mathfrak{P} \in \mathrm{W}^{1,2}(Q^+)$ and $\mathfrak{P} \in \mathrm{W}^{1,2}(Q^-)$. It follows that $(u, \mathfrak{p}) \in \mathrm{W}^{2,2}(\mathcal{O}^{(1)}(\tilde{x})) \times \mathrm{W}^{1,2}(\mathcal{O}^{(1)}(\tilde{x}))$ as well as $(u, \mathfrak{p}) \in \mathrm{W}^{2,2}(\mathcal{O}^{(2)}(\tilde{x})) \times \mathrm{W}^{1,2}(\mathcal{O}^{(2)}(\tilde{x}))$, where $\mathcal{O}(\tilde{x})$ is a neighborhood of $\tilde{x}$ and $\mathcal{O}^{(1)}(\tilde{x}) := \mathcal{O}(\tilde{x}) \cap \Omega^{(1)}$, $\mathcal{O}^{(2)}(\tilde{x}) := \mathcal{O}(\tilde{x}) \cap \Omega^{(2)}$. Higher-order regularity of $(u, \mathfrak{p})$ up to the boundary Γ is thereby established. □

Finally, uniqueness of a weak solution to (5.1) can be established. In fact, uniqueness can be obtained in a much larger class of distributional solutions with even less summability at spatial infinity than $u \in \mathrm{L}^6(\mathbb{R}^3)$ satisfied by a weak solution via Sobolev embedding. The theorem below is not optimal in this respect, but suffices for the purposes of this article.

Theorem 5.5. *Let Γ be a C^2-smooth closed surface, and let $(u, \mathfrak{p}) \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^3)^3 \times \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$ be a solution to (5.1) in the sense of (5.5) with $u \in \mathrm{L}^q(\mathbb{R}^3)^3$ and $\mathfrak{p} \in \mathrm{L}^r(\mathbb{R}^3)$ for some $q, r \in (1, \infty)$. If $(f, g, h_1, h_2) = (0, 0, 0, 0)$, then $u = 0$.*

Proof. The integrability assumption $u \in \mathrm{L}^q(\mathbb{R}^3)$ combined with the fact that $(u, \mathfrak{p})$ solves a classical Stokes ($\lambda_0 = 0$) or Oseen ($\lambda_0 \neq 0$) problem with homogeneous right-hand side in the exterior domain $\Omega^{(2)}$ implies that u exhibits the same pointwise rate of decay as the three-dimensional Stokes fundamental solution ($\lambda_0 = 0$) or the three-dimensional Oseen fundamental solution ($\lambda_0 \neq 0$); see [8, Theorem V.3.2 and Theorem VII.6.2] for example. This means that $u(x) = O(|x|^{-1})$ as $|x| \to \infty$. Moreover, we obtain $\mathfrak{p} \in O(|x|^{-2})$. Let $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$ be a cut-off function with $\chi = 1$ for $|x| < 1$ and $\chi = 0$ for $|x| > 2$, and put $\chi_R := \chi\big(\frac{|x|}{R}\big)$. Then $\chi_R u$ is admissible as a test function in (5.5), which implies

$$\int_{\mathbb{R}^3} 2\mu\, \mathrm{S}(u) : \big(\chi_R \mathrm{S}(u) + \nabla\chi_R \otimes u + u \otimes \nabla\chi_R\big) \mathrm{d}x + \lambda_0 \int_{\mathbb{R}^3} (\partial_3 u \cdot u)\chi_R \,\mathrm{d}x \\ = \int_{\mathbb{R}^3} \mathfrak{p}(\nabla\chi_R \cdot u) \mathrm{d}x.$$

Utilizing that $u = O\big(|x|^{-1}\big)$, we use Hölder's inequality to estimate

$$\Big|\int_{\mathbb{R}^3} \mathrm{S}(u) : \nabla\chi_R \otimes u \,\mathrm{d}x\Big| \leq c\|\mathrm{S}(u)\|_2 \bigg(\int_{B_{2R,R}} \frac{|u|^2}{R^2} \,\mathrm{d}x\bigg)^{\frac{1}{2}} \leq cR^{-1/2} \overset{R\to\infty}{\longrightarrow} 0.$$

Furthermore, in the Oseen case ($\lambda_0 \neq 0$) we even have the better averaged decay estimate

$$\int_{\partial \mathrm{B}_r} |u|^2 \,\mathrm{d}S \leq cr^{-1};$$

see [8, Exercise VII.6.1]; which leads to

$$\Big|\int_{\mathbb{R}^3} (\partial_3 u \cdot u)\chi_R \,\mathrm{d}x\Big| = \Big|\frac{1}{2}\int_{\mathbb{R}^3} |u|^2 \partial_3\chi_R \,\mathrm{d}x\Big| \leq c\int_R^{2R} \int_{\partial \mathrm{B}_r} \frac{|u|^2}{R} \,\mathrm{d}S\mathrm{d}r \leq cR^{-1} \overset{R\to\infty}{\longrightarrow} 0.$$

Since also

$$\Big|\int_{\mathbb{R}^3} \mathfrak{p}(\nabla\chi_R \cdot u)\mathrm{d}x\Big| \leq c\int_{\mathrm{B}_{2R,R}} R^{-4}\,\mathrm{d}x \leq cR^{-1} \overset{R\to\infty}{\longrightarrow} 0,$$

we deduce $\|\mathrm{S}(u)\|_2 = 0$ and thus $u = 0$. □

5.2. Twofold Half Space

The main challenge towards L^r estimates of solutions to (5.1), *i.e.*, *a priori* estimates of Agmon–Douglis–Nirenberg type, is to obtain such estimates in the half-space case under disregard of the lower-order terms in the equations. The general case then follows via a localization argument. We therefore first consider the system

$$\begin{cases} \operatorname{div}\mathrm{T}(u,\mathfrak{p}) = f & \text{in } \dot{\mathbb{R}}^3,\\ \operatorname{div} u = g & \text{in } \dot{\mathbb{R}}^3,\\ [\![u]\!] = h_0 & \text{on } \partial\dot{\mathbb{R}}^3,\\ u\big|_{\mathbb{R}^3_+}\cdot \mathrm{n} = h_1 & \text{on } \partial\dot{\mathbb{R}}^3,\\ (\mathrm{I}-\mathrm{n}\otimes\mathrm{n})[\![\mathrm{T}(u,\mathfrak{p})\mathrm{n}]\!] = h_2 & \text{on } \partial\dot{\mathbb{R}}^3, \end{cases} \tag{5.9}$$

where $\mathrm{n} = -e_3$. We shall implicitly identify $\partial\dot{\mathbb{R}}^3$ with $\mathbb{R}^2$. In Theorem 5.8 we establish the *a priori* L^r estimate (5.21) for solutions to (5.9).

In the celebrated work [2] of Agmon, Douglis and Nirenberg, *a priori* L^r estimates for strong solutions to elliptic systems with boundary values of a certain type were established. Since (5.9) can be decomposed into two Stokes systems, with a slip and a Dirichlet boundary condition, respectively, that both fall into the category covered by [2], it might seem at the outset as if L^r estimates such as (5.21) can be derived from [2]. However, since the two Stokes systems would be strongly coupled, there is no direct way to derive L^r estimates for the full system (5.9) from [2]. Instead, one can turn to existing L^r estimates for the resolvent problem corresponding to (5.9). For large resolvent parameters such estimates were established in [13, Theorem 3.1.4]. In the following, we only utilize L^r estimates for (5.9) in a localization argument in the proof of Theorem 5.9. In this application, the estimates in [13] would be sufficient, since in the context of a localization argument the resolvent term is irrelevant. Nevertheless, we choose to establish in Theorem 5.8 below the classical Agmon–Douglis–Nirenberg L^r estimate for (5.9), which is also interesting in its own right. We present a new type of proof based on Fourier multipliers and real interpolation that seems particularly well suited for coupled systems such as (5.9).

The proof of Theorem 5.8 is divided into Lemma 5.6 and Lemma 5.7. For technical reasons, it is convenient to decompose both the data and the solution to (5.9) into one part with lower frequency support and another part with higher frequency support in tangential directions e_1, e_2. We shall repeatedly employ the Fourier transform $\mathscr{F}_{\mathbb{R}^2}$ with respect to these two directions. To this end, observe that $\mathscr{F}_{\mathbb{R}^2}\big[u(\cdot,x_3)\big](\xi')$ is well-defined in the sense of distributions $\mathscr{S}'(\mathbb{R}^3)$ when $u \in \mathrm{L}^r(\mathbb{R}^3)$ for some $r \in (1,\infty)$, which will be the case whenever such an expression appears below.

Lemma 5.6. *Let* $r \in (1,\infty)$ *and* $b \in \mathrm{W}^{2-1/r,r}(\mathbb{R}^2)^3$ *with* $\operatorname{supp}\mathscr{F}_{\mathbb{R}^2}[b] \subset \mathbb{R}^2\setminus\mathrm{B}_1(0)$. *Then there is a solution* $(u,\mathfrak{p}) \in \mathrm{W}^{2,r}(\dot{\mathbb{R}}^3)^3 \times \mathrm{W}^{1,r}(\dot{\mathbb{R}}^3)$ *to*

$$\begin{cases} \operatorname{div}\mathrm{T}(u,\mathfrak{p}) = 0 & \text{in } \dot{\mathbb{R}}^3,\\ \operatorname{div} u = 0 & \text{in } \dot{\mathbb{R}}^3,\\ u = b & \text{on } \partial\dot{\mathbb{R}}^3, \end{cases} \tag{5.10}$$

which satisfies

$$\|u\|_{2,r} + \|\mathfrak{p}\|_{1,r} \leq C\,\|b\|_{2-1/r,r}, \tag{5.11}$$

where $C = C(r)$. Moreover, $\mathscr{F}_{\mathbb{R}^2}\big[u(\cdot,x_3)\big](\xi')$ and $\mathscr{F}_{\mathbb{R}^2}\big[\mathfrak{p}(\cdot,x_3)\big](\xi')$ are supported away from $(\xi',x_3) \in B_{1/2}(0)\times\mathbb{R}$.

Proof. A solution to (5.10) can be constructed explicitly. To this end, consider first a sufficiently smooth right-hand side $b \in \mathscr{S}(\mathbb{R}^2)^3$ with $\operatorname{supp}\mathscr{F}_{\mathbb{R}^2}[b] \subset \mathbb{R}^2 \setminus \mathrm{B}_{1/2}(0)$. We employ the notation $\widehat{b} := \mathscr{F}_{\mathbb{R}^2}[b]$ and $v := (u_1,u_2)$, $w := u_3$ as well as $b_v := (b_1,b_2)$ and $b_w := b_3$. An application of the Fourier transform $\mathscr{F}_{\mathbb{R}^2}$ with respect to $x' \in \mathbb{R}^2$ in (5.10) yields

$$\begin{cases} -\mu|\xi'|^2\widehat{v} + \mu\partial_3^2\widehat{v} - i\xi'\widehat{\mathfrak{p}} = 0 & \text{in } \dot{\mathbb{R}}^3, \\ -\mu|\xi'|^2\widehat{w} + \mu\partial_3^2\widehat{w} - \partial_3\widehat{\mathfrak{p}} = 0 & \text{in } \dot{\mathbb{R}}^3, \\ i\xi'\cdot\widehat{v} + \partial_3\widehat{w} = 0 & \text{in } \dot{\mathbb{R}}^3, \\ (\widehat{v},\widehat{w}) = (\widehat{b}_v,\widehat{b}_w) & \text{on } \partial\dot{\mathbb{R}}^3. \end{cases} \tag{5.12}$$

Therefore $\mathfrak{p}$ satisfies $|\xi'|^2\,\widehat{\mathfrak{p}} - \partial_3^2\,\widehat{\mathfrak{p}} = 0$ and thus

$$\widehat{\mathfrak{p}}(\xi',x_3) = \begin{cases} A_1(\xi')e^{-|\xi'|x_3} & \text{if } x_3 > 0, \\ A_2(\xi')e^{|\xi'|x_3} & \text{if } x_3 < 0. \end{cases}$$

We insert $\mathfrak{p}$ into $(5.12)_1$ and $(5.12)_2$ and solve the resulting differential equations. Taking into account the boundary conditions $(5.12)_4$, we obtain

$$\widehat{u} = \begin{cases} \left[\dfrac{A_1(\xi')x_3}{2\mu|\xi'|}\begin{pmatrix}-i\xi'\\|\xi'|\end{pmatrix} + \begin{pmatrix}\widehat{b}_v\\\widehat{b}_w\end{pmatrix}\right]e^{-|\xi'|x_3} & \text{if } x_3 > 0, \\ \left[\dfrac{A_2(\xi')x_3}{2\mu|\xi'|}\begin{pmatrix}i\xi'\\|\xi'|\end{pmatrix} + \begin{pmatrix}\widehat{b}_v\\\widehat{b}_w\end{pmatrix}\right]e^{|\xi'|x_3} & \text{if } x_3 < 0. \end{cases}$$

Inserting the above formula for $\widehat{u}$ into $(5.12)_3$, we find that

$$A_1(\xi') = A_2(\xi') = 2\mu\big(\operatorname{sgn}(x_3)|\xi'|\widehat{b}_w - i\xi'\cdot\widehat{b}_v\big).$$

Consequently, a solution to (5.10) is given by

$$\begin{aligned} u(x',x_3) &:= \mathscr{F}^{-1}_{\mathbb{R}^2}\big[M_b(\xi',x_3)\,e^{-|\xi'||x_3|}\big], \\ \mathfrak{p}(x',x_3) &:= \mathscr{F}^{-1}_{\mathbb{R}^2}\big[m_b(\xi',x_3)\,e^{-|\xi'||x_3|}\big], \end{aligned} \tag{5.13}$$

where

$$\begin{aligned} M_b(\xi',x_3) &:= \frac{\big(\operatorname{sgn}(x_3)|\xi'|\widehat{b}_w - i\xi'\cdot\widehat{b}_v\big)|x_3|}{|\xi'|}\begin{pmatrix}-i\xi'\\\operatorname{sgn}(x_3)|\xi'|\end{pmatrix} + \begin{pmatrix}\widehat{b}_v\\\widehat{b}_w\end{pmatrix}, \\ m_b(\xi',x_3) &:= 2\mu\big(\operatorname{sgn}(x_3)|\xi'|\widehat{b}_w - i\xi'\cdot\widehat{b}_v\big). \end{aligned}$$

Although M_b has a singularity, $(u,\mathfrak{p})$ as defined above is a well-defined solution, smooth on $\dot{\mathbb{R}}^3$ even, due to the assumption that $\widehat{b}(\xi')$ has support away from 0. In order to provide an estimate for the solution, we let $\kappa_{1/4} \in \mathrm{C}_0^\infty(\mathbb{R}^2)$ with $\kappa_{1/4} = 0$ on $\mathrm{B}_{1/4}(0)$ and $\kappa_{1/4} = 1$ on $\mathbb{R}^2\setminus\mathrm{B}_{1/2}(0)$, and consider the truncation

$$\mathcal{K} : \mathscr{S}(\mathbb{R}^2)^3 \to \mathscr{S}(\mathbb{R}^3)^3, \quad \mathcal{K}(\varphi) := \mathscr{F}^{-1}_{\mathbb{R}^2}\Big[\kappa_{1/4}(\xi')M_\varphi(\xi',x_3)\,e^{-|\xi'||x_3|}\Big] \tag{5.14}$$

of the solution operator. The singularity of M_φ makes it necessary to employ the truncation $\kappa_{1/4}$ to ensure that $\mathcal{K}$ is well-defined. We shall use real interpolation to show that $\mathcal{K}$ extends to a bounded operator $\mathcal{K} : \mathrm{W}^{2-1/r,r}(\mathbb{R}^2) \to \mathrm{W}^{2,r}(\dot{\mathbb{R}}^3)$. To this end, we observe for $m \in \mathbb{N}_0$ and any $x_3 \in \mathbb{R}$ that the symbol $\xi' \mapsto (|\xi'||x_3|)^m e^{-|\xi'||x_3|}$ is an $\mathrm{L}^r(\mathbb{R}^2)$-multiplier. Specifically, one may verify that

$$\sup_{x_3\in\mathbb{R}}\ \sup_{\varepsilon\in\{0,1\}^2}\ \sup_{\xi'\in\mathbb{R}^2}\Big|{\xi'_1}^{\varepsilon_1}{\xi'_2}^{\varepsilon_2}\partial_{\xi'_1}^{\varepsilon_1}\partial_{\xi'_2}^{\varepsilon_2}\big[(|\xi'||x_3|)^m e^{-|\xi'||x_3|}\big]\Big| < \infty,$$

whence it follows from the Marcinkiewicz Multiplier Theorem (see for example [10, Corollary 6.2.5]) that the Fourier-multiplier operator with symbol $\xi' \mapsto (|\xi'||x_3|)^m e^{-|\xi'||x_3|}$ is a bounded operator on $\mathrm{L}^r(\mathbb{R}^2)$ with operator norm independent of x_3, that is,

$$\sup_{x_3\in\mathbb{R}} \Big\|\varphi \mapsto \mathscr{F}^{-1}_{\mathbb{R}^2}\Big[(|\xi'||x_3|)^m e^{-|\xi'||x_3|}\mathscr{F}_{\mathbb{R}^2}[\varphi]\Big]\Big\|_{\mathscr{L}(\mathrm{L}^r(\mathbb{R}^2),\mathrm{L}^r(\mathbb{R}^2))} < \infty. \tag{5.15}$$

We return to (5.14) and employ (5.15) to deduce

$$\begin{aligned}
\|\nabla^2_{x'}\mathcal{K}(\varphi)\|_{\mathrm{L}^\infty_{x_3}(\mathbb{R};\mathrm{L}^r(\mathbb{R}^2))} &\le c\,\|\nabla^2\varphi\|_{\mathrm{L}^r(\mathbb{R}^2)},\\
\|\partial^2_{x_3}\mathcal{K}(\varphi)\|_{\mathrm{L}^\infty_{x_3}(\dot{\mathbb{R}};\mathrm{L}^r(\mathbb{R}^2))} &\le c\,\|\nabla^2\varphi\|_{\mathrm{L}^r(\mathbb{R}^2)},\\
\|\mathcal{K}(\varphi)\|_{\mathrm{L}^\infty_{x_3}(\mathbb{R};\mathrm{L}^r(\mathbb{R}^2))} &\le c\,\|\varphi\|_{\mathrm{L}^r(\mathbb{R}^2)},
\end{aligned}$$

where the restriction in the norm of the left-hand side to the twofold real line $\dot{\mathbb{R}}$ in the second estimate is required since $\partial_{x_3}\mathcal{K}(\varphi)$ has a singularity at $x_3 = 0$. It follows that

$$\|\nabla^2\mathcal{K}(\varphi)\|_{\mathrm{L}^\infty_{x_3}(\dot{\mathbb{R}};\mathrm{L}^r(\mathbb{R}^2))} + \|\mathcal{K}(\varphi)\|_{\mathrm{L}^\infty_{x_3}(\dot{\mathbb{R}};\mathrm{L}^r(\mathbb{R}^2))} \le c\,\|\varphi\|_{\mathrm{W}^{2,r}(\mathbb{R}^2)}. \tag{5.16}$$

This estimate shall serve as an interpolation endpoint. To obtain the opposite endpoint, we again employ (5.15) to infer

$$\begin{aligned}
\sup_{x_3\in\mathbb{R}} \||x_3|\,\nabla^2_{x'}\mathcal{K}(\varphi)\|_{\mathrm{L}^r(\mathbb{R}^2)} &\le c\,\|\nabla\varphi\|_{\mathrm{L}^r(\mathbb{R}^2)},\\
\sup_{x_3\in\dot{\mathbb{R}}} \||x_3|\,\partial^2_{x_3}\mathcal{K}(\varphi)\|_{\mathrm{L}^r(\mathbb{R}^2)} &\le c\,\|\nabla\varphi\|_{\mathrm{L}^r(\mathbb{R}^2)},\\
\sup_{x_3\in\mathbb{R}} \||x_3|\,\mathcal{K}(\varphi)\|_{\mathrm{L}^r(\mathbb{R}^2)} &\le c\,\|\varphi\|_{\mathrm{L}^r(\mathbb{R}^2)},
\end{aligned}$$

where the last estimate relies on the truncation introduced in $\mathcal{K}$. It follows that

$$\|\nabla^2\mathcal{K}(\varphi)\|_{\mathrm{L}^{1,\infty}_{x_3}(\dot{\mathbb{R}};\mathrm{L}^r(\mathbb{R}^2))} + \|\mathcal{K}(\varphi)\|_{\mathrm{L}^{1,\infty}_{x_3}(\dot{\mathbb{R}};\mathrm{L}^r(\mathbb{R}^2))} \le c\,\|\varphi\|_{\mathrm{W}^{1,r}(\mathbb{R}^2)}. \tag{5.17}$$

Real interpolation yields

$$\begin{aligned}
\Big(\mathrm{L}^{1,\infty}\big(\dot{\mathbb{R}};\mathrm{L}^r(\mathbb{R}^2)\big),\mathrm{L}^\infty\big(\dot{\mathbb{R}};\mathrm{L}^r(\mathbb{R}^2)\big)\Big)_{1-1/r,r} &= \mathrm{L}^r\big(\dot{\mathbb{R}},\mathrm{L}^r(\mathbb{R}^2)\big),\\
\Big(\mathrm{W}^{2,r}(\mathbb{R}^2),\mathrm{W}^{1,r}(\mathbb{R}^2)\Big)_{1-1/r,r} &= \mathrm{W}^{2-1/r,r}(\mathbb{R}^2).
\end{aligned}$$

Consequently, (5.16) and (5.17) imply

$$\|\mathcal{K}(\varphi)\|_{\mathrm{W}^{2,r}(\dot{\mathbb{R}}^3)} \le c\,\|\varphi\|_{\mathrm{W}^{2-1/r,r}(\mathbb{R}^2)},$$

whence $\mathcal{K}$ extends to a bounded operator $\mathcal{K} : \mathrm{W}^{2-1/r,r}(\mathbb{R}^2) \to \mathrm{W}^{2,r}(\dot{\mathbb{R}}^3)$. Recalling the formula (5.13) for the solution u to (5.10) and that $\operatorname{supp}\mathscr{F}_{\mathbb{R}^2}[b] \subset \mathbb{R}^2 \setminus \mathrm{B}_{1/2}(0)$, we clearly have $u = \mathcal{K}(b)$. It follows that $\|u\|_{2,r} \le c\,\|b\|_{2-1/r,r}$. In a completely similar manner, one shows that also $\|\mathfrak{p}\|_{1,r} \le c\,\|b\|_{2-1/r,r}$. Thus the lemma follows for this particular choice of $b \in \mathscr{S}(\mathbb{R}^2)$. Since any $b \in \mathrm{W}^{2-1/r,r}(\mathbb{R}^2)$ with $\operatorname{supp}\mathscr{F}_{\mathbb{R}^2}[b] \subset \mathbb{R}^2 \setminus \mathrm{B}_1(0)$ can be approximated in $\mathrm{W}^{2-1/r,r}(\mathbb{R}^2)$ by a sequence $\{b_k\}_{k=1}^\infty \subset \mathscr{S}(\mathbb{R}^2)$ with $\operatorname{supp}\mathscr{F}_{\mathbb{R}^2}[b] \subset \mathbb{R}^2 \setminus \mathrm{B}_{1/2}(0)$ via a standard mollifier procedure, we conclude the lemma in its entirety. □

Lemma 5.7. *Let $r \in (1,\infty)$. For all $H_1 \in \mathrm{W}^{2-1/r,r}(\mathbb{R}^2)$ and $H_2 \in \mathrm{W}^{1-1/r,r}(\mathbb{R}^2)^3$ with $\operatorname{supp}\mathscr{F}_{\mathbb{R}^2}[H_1] \subset \mathbb{R}^2\setminus\mathrm{B}_1(0)$, $\operatorname{supp}\mathscr{F}_{\mathbb{R}^2}[H_2] \subset \mathbb{R}^2\setminus\mathrm{B}_1(0)$ and $H_2\cdot e_3 = 0$ there exists a solution $(u,\mathfrak{p}) \in \mathrm{W}^{2,r}(\dot{\mathbb{R}}^3)^3\times\mathrm{W}^{1,r}(\dot{\mathbb{R}}^3)$*

to

$$\begin{cases} \operatorname{div}\mathrm{T}(u,\mathfrak{p})=0 & \textit{in } \dot{\mathbb{R}}^3,\\ \operatorname{div}u=0 & \textit{in } \dot{\mathbb{R}}^3,\\ [\![u]\!]=0 & \textit{on } \partial\dot{\mathbb{R}}^3,\\ u\cdot\mathrm{n}=H_1 & \textit{on } \partial\dot{\mathbb{R}}^3,\\ (\mathrm{I}-\mathrm{n}\otimes\mathrm{n})[\![\mathrm{T}(u,\mathfrak{p})\mathrm{n}]\!]=H_2 & \textit{on } \partial\dot{\mathbb{R}}^3 \end{cases} \tag{5.18}$$

that satisfies

$$\|u\|_{2,r}+\|\mathfrak{p}\|_{1,r}\leq C\,(\|H_1\|_{2-1/r,r}+\|H_2\|_{1-1/r,r}), \tag{5.19}$$

where $C=C(r)$. *Moreover,* $\mathscr{F}_{\mathbb{R}^2}\big[u(\cdot,x_3)\big](\xi')$ *and* $\mathscr{F}_{\mathbb{R}^2}\big[\mathfrak{p}(\cdot,x_3)\big](\xi')$ *are supported away from* $(\xi',x_3)\in B_{1/2}(0)\times\mathbb{R}$.

Proof. Put

$$b:=\begin{pmatrix}b_v\\ b_w\end{pmatrix}:=\mathscr{F}^{-1}_{\mathbb{R}^2}\Bigg[\begin{pmatrix}0 & -\frac{1}{2\mu|\xi'|}\big(\mathrm{I}-\frac{\xi'\otimes\xi'}{2|\xi'|^2}\big)\\ 1 & 0\end{pmatrix}\begin{pmatrix}\widehat{H_1}\\ \widehat{H_2}\end{pmatrix}\Bigg]. \tag{5.20}$$

Let $\kappa_{1/4}\in\mathrm{C}_0^\infty(\mathbb{R}^2)$ with $\kappa_{1/4}=0$ on $\mathrm{B}_{1/4}(0)$ and $\kappa_{1/4}=1$ on $\mathbb{R}^2\setminus\mathrm{B}_{1/2}(0)$. Clearly, the truncated operator

$$\mathcal{M}:\mathscr{S}(\mathbb{R}^2)^3\to\mathscr{S}(\mathbb{R}^2)^3,\quad \mathcal{M}(\varphi):=\mathscr{F}^{-1}_{\mathbb{R}^2}\Bigg[\kappa_{1/4}(\xi')\frac{-1}{2|\xi'|}\bigg(\mathrm{I}-\frac{\xi'\otimes\xi'}{2|\xi'|^2}\bigg)\widehat{\varphi}\Bigg]$$

corresponding to the Fourier multiplier appearing in (5.20) extends to a bounded operator $\mathcal{M}:\mathrm{W}^{1-1/r,r}(\mathbb{R}^2)^3\to\mathrm{W}^{2-1/r,r}(\mathbb{R}^2)^3$. The assumption $\widehat{H_2}\subset\mathbb{R}^2\setminus\mathrm{B}_1(0)$ implies that $b_v=\mathcal{M}(H_2)$. It follows that $b\in\mathrm{W}^{2-1/r,r}(\mathbb{R}^2)^3$, and we can therefore introduce the corresponding solution $(u,\mathfrak{p})\in\mathrm{W}^{2,r}(\dot{\mathbb{R}}^3)^3\times\mathrm{W}^{1,r}(\dot{\mathbb{R}}^3)$ to (5.10) from Lemma 5.6. By construction, $u\cdot\mathrm{n}=H_1$ on $\partial\dot{\mathbb{R}}^3$. Moreover, recalling (5.13) we compute

$$(\mathrm{I}-\mathrm{n}\otimes\mathrm{n})[\![\mathrm{T}(u,\mathfrak{p})\mathrm{n}]\!]=\mathscr{F}^{-1}_{\mathbb{R}^2}\Bigg[\begin{pmatrix}-2\mu\big(|\xi'|\mathrm{I}+\frac{\xi'\otimes\xi'}{|\xi'|}\big)\widehat{b}_v\\ 0\end{pmatrix}\Bigg]=H_2.$$

Consequently, $(u,\mathfrak{p})$ is a solution to (5.18). Employing (5.11) we deduce

$$\|u\|_{2,r}+\|\mathfrak{p}\|_{1,r}\leq c\,\|b\|_{2-1/r,r}\leq c\,(\|H_1\|_{2-1/r,r}+\|H_2\|_{1-1/r,r})$$

and conclude the lemma. □

Theorem 5.8. *Let* $r\in(1,\infty)$ *and*

$$\begin{aligned} &f\in\mathrm{L}^r(\mathbb{R}^3)^3,\ g\in\mathrm{W}^{1,r}(\dot{\mathbb{R}}^3),\\ &h_0\in\mathrm{W}^{2-1/r,r}(\mathbb{R}^2)^3,\ h_1\in\mathrm{W}^{2-1/r,r}(\mathbb{R}^2),\ h_2\in\mathrm{W}^{1-1/r,r}(\mathbb{R}^2)^3. \end{aligned}$$

Then all solutions $(u,\mathfrak{p})\in\mathrm{W}^{2,r}(\dot{\mathbb{R}}^3)^3\times\mathrm{W}^{1,r}(\dot{\mathbb{R}}^3)$ *to* (5.9) *satisfy*

$$\|u\|_{2,r}+\|\mathfrak{p}\|_{1,r}\leq C\big(\|f\|_r+\|g\|_{1,r}+\|h_0\|_{2-1/r,r}+\|h_1\|_{2-1/r,r}+\|h_2\|_{1-1/r,r}+\|u\|_r\big), \tag{5.21}$$

where $C=C(r,k)>0$.

Proof. We decompose both the solution and the data into one part with lower and another part with higher frequency support in tangential directions e_1,e_2. For this purpose, we introduce cut-off functions $\kappa_\alpha\in\mathrm{C}_0^\infty(\mathbb{R}^2)$ with $\kappa_\alpha=0$ on $\mathrm{B}_\alpha(0)$ and $\kappa_\alpha=1$ on $\mathbb{R}^2\setminus\mathrm{B}_{2\alpha}(0)$, and put

$$u_\#(x',x_3):=\mathscr{F}^{-1}_{\mathbb{R}^2}\big[\kappa_1(\xi')\mathscr{F}_{\mathbb{R}^2}[u(\cdot,x_3)]\big](x')\in\mathrm{W}^{2,r}(\dot{\mathbb{R}}^3)^3,\qquad u_\perp:=u-u_\#.$$

Similarly, we introduce $\mathfrak{p}_\#,\mathfrak{p}_\perp$ and $f_\#,g_\#,h_{0\#},h_{1\#},h_{2\#}$. Observe that $(u_\#,\mathfrak{p}_\#)$ solves (5.9) with respect to data $(f_\#,g_\#,h_{0\#},h_{1\#},h_{2\#})$. We shall construct another solution satisfying estimate (5.21), and subsequently show that it coincides with $(u_\#,\mathfrak{p}_\#)$. To this end, we let $g^+_\#\in\mathrm{W}^{1,r}(\mathbb{R}^3)$ denote an extension of $g_\#|_{\mathbb{R}^3_+}$ to $\mathrm{W}^{1,r}(\mathbb{R}^3)$. Specifically employing Heesten's extension operator (see for example [1, Theorem 4.26]) one readily verifies that the extension retains the property that the Fourier transform (in tangential directions) $\mathscr{F}_{\mathbb{R}^2}\big[g_\#^+(\cdot,x_3)\big](\xi')$ is supported away from $(\xi',x_3)\in B_1(0)\times\mathbb{R}$. Consequently,

$$G_\#^+ := \mathscr{F}^{-1}_{\mathbb{R}^3}\left[\frac{-i\xi}{|\xi|^2}\mathscr{F}_{\mathbb{R}^3}\left[g_\#^+\right]\right] \in \mathrm{W}^{2,r}(\mathbb{R}^3)^3$$

is well defined. Similarly, we introduce an extension of $g_\#|_{\mathbb{R}^3_-}$ to $\mathrm{W}^{1,r}(\mathbb{R}^3)$ and construct a field $G_\#^- \in \mathrm{W}^{2,r}(\mathbb{R}^3)^3$ as above. Letting

$$G_\# := \begin{cases} G_\#^+ & \text{in } \mathbb{R}^3_+,\\ G_\#^- & \text{in } \mathbb{R}^3_-,\end{cases}$$

we then obtain a field $G_\# \in \mathrm{W}^{2,r}(\dot{\mathbb{R}}^3)^3$ with $\operatorname{div} G_\# = g_\#$ in $\dot{\mathbb{R}}^3$. Moreover, a straight-forward application of Marcinkiewicz's Multiplier Theorem (see for example [9, Corollary 5.2.5]) yields

$$\|G_\#\|_{2,r} \le c\|g\|_{1,r}.$$

Now put

$$\begin{aligned} V_\# &:= \mathscr{F}^{-1}_{\mathbb{R}^3}\left[\frac{1}{|\xi|^2}\left(\mathrm{I}-\frac{\xi\otimes\xi}{|\xi|^2}\right)\mathscr{F}_{\mathbb{R}^3}\left[f_\# - \operatorname{div}\mathrm{S}(G_\#)\right]\right],\\ Q_\# &:= \mathscr{F}^{-1}_{\mathbb{R}^3}\left[\frac{\xi}{|\xi|^2}\cdot\mathscr{F}_{\mathbb{R}^3}\left[f_\# - \operatorname{div}\mathrm{S}(G_\#)\right]\right]. \end{aligned}$$

Owing to the fact that $G_\#, \operatorname{div}\mathrm{S}(G_\#) \in \mathrm{L}^r(\mathbb{R}^3)^3$ with $\mathscr{F}_{\mathbb{R}^3}[G_\#]$ and $\mathscr{F}_{\mathbb{R}^3}[\operatorname{div}\mathrm{S}(G_\#)]$ supported away from 0, the expressions above are well defined and yield functions with $V_\# \in \mathrm{W}^{2,r}(\mathbb{R}^3)^3$ and $Q_\# \in \mathrm{W}^{1,r}(\mathbb{R}^3)$ satisfying

$$\begin{cases} \operatorname{div}\mathrm{T}(V_\#,Q_\#) = f_\# - \operatorname{div}\mathrm{S}(G_\#) & \text{in } \mathbb{R}^3,\\ \operatorname{div} V_\# = 0 & \text{in } \mathbb{R}^3.\end{cases}$$

Moreover, another straight-forward application of Marcinkiewicz's Multiplier Theorem yields

$$\|V_\#\|_{2,r} + \|Q_\#\|_{1,r} \le c\big(\|f_\#\|_r + \|\operatorname{div}\mathrm{S}(G_\#)\|_r\big) \le c\big(\|f_\#\|_r + \|g_\#\|_{1,r}\big).$$

Utilizing Lemma 5.6, we construct a solution $(W_\#,\Pi_\#) \in \mathrm{W}^{2,r}(\dot{\mathbb{R}}^3)^3 \times \mathrm{W}^{1,r}(\dot{\mathbb{R}}^3)$ to

$$\begin{cases} \operatorname{div}\mathrm{T}(W_\#,\Pi_\#) = 0 & \text{in } \dot{\mathbb{R}}^3,\\ \operatorname{div} W_\# = 0 & \text{in } \dot{\mathbb{R}}^3,\\ [\![W_\#]\!] = h_{0\#} - [\![V_\#]\!] - [\![G_\#]\!] & \text{on } \partial\dot{\mathbb{R}}^3 \end{cases}$$

satisfying

$$\|W_\#\|_{2,r} + \|\Pi_\#\|_{1,r} \le c\big(\|h_{0\#}\|_{2-1/r,r} + \|V_\#\|_{2-1/r,r} + \|G_\#\|_{2-1/r,r}\big).$$

Finally, by Lemma 5.7 there is a solution $(\widetilde{W}_\#,\widetilde{\Pi}_\#) \in \mathrm{W}^{2,r}(\dot{\mathbb{R}}^3)^3 \times \mathrm{W}^{1,r}(\dot{\mathbb{R}}^3)$ to

$$\begin{cases} \operatorname{div}\mathrm{T}(\widetilde{W}_\#,\widetilde{\Pi}_\#) = 0 & \text{in } \dot{\mathbb{R}}^3,\\ \operatorname{div}\widetilde{W}_\# = 0 & \text{in } \dot{\mathbb{R}}^3,\\ [\![\widetilde{W}_\#]\!] = 0 & \text{on } \partial\dot{\mathbb{R}}^3,\\ \widetilde{W}_\#\cdot\mathrm{n} = h_{1\#} - \big(W_\# + V_\# + G_\#\big)\cdot\mathrm{n} & \text{on } \partial\dot{\mathbb{R}}^3,\\ (\mathrm{I}-\mathrm{n}\otimes\mathrm{n})[\![\mathrm{T}(\widetilde{W}_\#,\widetilde{\Pi}_\#)\mathrm{n}]\!] = h_{2\#} - (\mathrm{I}-\mathrm{n}\otimes\mathrm{n})[\![\mathrm{T}(W_\#+V_\#+G_\#,\Pi_\#+Q_\#)\mathrm{n}]\!] & \text{on } \partial\dot{\mathbb{R}}^3, \end{cases}$$

which obeys

$$\|\widetilde{W}_\#\|_{2,r} + \|\widetilde{\Pi}_\#\|_{1,r} \leq c\big(\|h_{1\#}\|_{2-1/r,r} + \|h_{2\#}\|_{1-1/r,r} + \|W_\#\|_{2-1/r,r} + \|\Pi_\#\|_{1-1/r,r} + \|V_\#\|_{2-1/r,r} + \|Q_\#\|_{1-1/r,r} + \|G_\#\|_{2-1/r,r}\big).$$

It follows that

$$U_\# := \widetilde{W}_\# + W_\# + V_\# + G_\#, \qquad \mathfrak{P}_\# := \widetilde{\Pi}_\# + \Pi_\# + Q_\#$$

is a solution to (5.9) with $(f_\#, g_\#, h_{0\#}, h_{1\#}, h_{2\#})$ as the right-hand side, and that $(U_\#, \mathfrak{P}_\#) \in \mathrm{W}^{2,r}(\dot{\mathbb{R}}^3)^3 \times \mathrm{W}^{1,r}(\dot{\mathbb{R}}^3)$ satisfies

$$\|U_\#\|_{2,r} + \|\mathfrak{P}_\#\|_{1,r} \leq c\big(\|f_\#\|_r + \|g_\#\|_{1,r} + \|h_{0\#}\|_{2-1/r,r} + \|h_{1\#}\|_{2-1/r,r} + \|h_{2\#}\|_{1-1/r,r}\big). \tag{5.22}$$

Consequently, $(U_\#, \mathfrak{P}_\#)$ and $(u_\#, \mathfrak{p}_\#)$ solve the same equations. Using a classical duality argument, we shall show that they coincide. To this end, let $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^3)$ and put $\varphi_\# := \mathscr{F}^{-1}_{\mathbb{R}^2}\big[\kappa_{1/4}(\xi')\mathscr{F}_{\mathbb{R}^2}[\varphi(\cdot, x_3)]\big]$. Employing the same procedure as above, we construct a solution $(\mathfrak{z}_\#, \mathfrak{q}_\#) \in \mathrm{W}^{2,r'}(\dot{\mathbb{R}}^3)^3 \times \mathrm{W}^{1,r'}(\dot{\mathbb{R}}^3)$ to (5.9) with right-hand side $(\varphi_\#, 0, 0, 0, 0)$. Since by construction both $\mathscr{F}_{\mathbb{R}^2}\big[U_\#(\cdot, x_3)\big](\xi')$ and $\mathscr{F}_{\mathbb{R}^2}\big[u_\#(\cdot, x_3)\big](\xi')$ are supported away from $(\xi', x_3) \in B_{1/2}(0) \times \mathbb{R}$, we compute

$$\begin{aligned}\int_{\mathbb{R}^3} (u_\# - U_\#) \cdot \varphi \,\mathrm{d}x &= \int_{\mathbb{R}^3} (u_\# - U_\#) \cdot \varphi_\# \,\mathrm{d}x \\ &= \int_{\dot{\mathbb{R}}^3} (u_\# - U_\#) \cdot \operatorname{div} \mathrm{T}(\mathfrak{z}_\#, \mathfrak{q}_\#) \,\mathrm{d}x \\ &= \int_{\dot{\mathbb{R}}^3} \operatorname{div} \mathrm{T}(u_\# - U_\#, \mathfrak{p}_\# - \mathfrak{P}_\#) \cdot \mathfrak{z}_\# \,\mathrm{d}x = 0.\end{aligned}$$

Since φ can be taken arbitrarily, we obtain $u_\# = U_\#$, and in turn from (5.9) also $\mathfrak{p}_\# = \mathfrak{P}_\#$. It follows that also $(u_\#, \mathfrak{p}_\#)$ satisfies (5.22) and thus

$$\|u_\#\|_{2,r} + \|\mathfrak{p}_\#\|_{1,r} \leq c\big(\|f\|_r + \|g\|_{1,r} + \|h_0\|_{2-1/r,r} + \|h_1\|_{2-1/r,r} + \|h_2\|_{1-1/r,r}\big). \tag{5.23}$$

Finally, from $\mathscr{F}_{\mathbb{R}^2}\big[u_\perp(\cdot, x_3)\big](\xi') \subset \mathrm{B}_1(0) \times \mathbb{R}$ and $\mathscr{F}_{\mathbb{R}^2}\big[\mathfrak{p}_\perp(\cdot, x_3)\big](\xi') \subset \mathrm{B}_1(0) \times \mathbb{R}$ it follows via the Marcinkiewicz Multiplier Theorem that $\|\nabla_{x'}\nabla u_\perp\|_r + \|\nabla_{x'}\mathfrak{p}_\perp\| \leq c\,\|u_\perp\|_r$. Introducing the decomposition $u = u_\# + u_\perp$ in (5.9) and isolating $\partial_3 u_{\perp 3}$ on the left-hand side in $(5.9)_2$, we then infer after differentiation that $\|\partial_3^2 u_{\perp 3}\|_r \leq c\|u_\perp\|_r$. Subsequently isolating $\partial_3 \mathfrak{p}_\perp$ in the third coordinate equation of $(5.9)_1$, we deduce $\|\partial_3 \mathfrak{p}_\perp\|_{1,r} \leq c\|u_\perp\|_r$. Lastly isolating $\partial_3^2 u_{\perp 1}$ and $\partial_3^2 u_{\perp 2}$ in the first and second coordinate equation of $(5.9)_1$, respectively, we further deduce $\|\partial_3^2 u_{\perp 1}\|_r + \|\partial_3^2 u_{\perp 2}\|_r \leq c\|u_\perp\|_r$. In conclusion,

$$\|u_\perp\|_{2,r} + \|\nabla \mathfrak{p}_\perp\|_r \leq c\|u_\perp\|_r. \tag{5.24}$$

Combining (5.23) and (5.24) we conclude (5.21) and thus the theorem. □

5.3. A Priori Estimates for Strong Solutions

We return to the linearized two-phase-flow Navier–Stokes problem (5.1), where Ω is an open set of the same type as in Sect. 4, *i.e.*, satisfying (4.1). Based on the estimates obtained in the twofold-half-space case in Theorem 5.8, we shall establish L^r estimates of solutions to (5.1). The Oseen case ($\lambda_0 \neq 0$) and Stokes case ($\lambda_0 = 0$) are treated separately in Theorem 5.9 and Theorem 5.10, respectively.

Theorem 5.9. *Let Γ be a C^5-smooth surface, $q \in (1, \frac{3}{2})$, $r \in (3, \infty)$ and $\overline{\lambda} > 0$. For every $0 < \lambda_0 \leq \overline{\lambda}$ and $(f, g, h_1, h_2) \in \mathbf{Y}_1^{q,r} \times \mathbf{Y}_{2,3}^{q,r} \times \mathbf{Y}_4^{q,r}$ there exists a unique solution $(u, \mathfrak{p}) \in \mathbf{X}_{1,\lambda_0}^{q,r} \times \mathbf{X}_2^{q,r}$ to (5.1) satisfying*

$$\int_{\Omega^{(1)}} \mathfrak{p}^{(1)} \,\mathrm{d}x = 0. \tag{5.25}$$

Moreover,

$$\|u\|_{\mathbf{X}^{q,r}_{1,\lambda_0}} + \|\mathfrak{p}\|_{\mathbf{X}^{q,r}_2} \le C\|(f,g,h_1,h_2)\|_{\mathbf{Y}^{q,r}_1\times\mathbf{Y}^{q,r}_{2,3}\times\mathbf{Y}^{q,r}_4}, \tag{5.26}$$

where $C = C(\Omega, q, r, \overline{\lambda}) > 0$.

Proof. We first consider data

$$\begin{aligned} &(f,g,h_1,h_2) \in \mathrm{C}^\infty(\Omega)\times\mathrm{C}^\infty(\Omega)\times\mathrm{C}^5(\Gamma)\times\mathrm{C}^5(\Gamma),\\ &\operatorname{supp} f \text{ and } \operatorname{supp} g \text{ compact in } \mathbb{R}^3,\\ &\int_{\Omega^{(1)}} g\,\mathrm{d}x = \int_\Gamma h_1\,\mathrm{d}S, \end{aligned} \tag{5.27}$$

so that the theorems from Sect. 5.1 can be applied. Recalling the regularity of Γ, Theorems 5.2, 5.3 and 5.4 yield a solution $(u,\mathfrak{p}) \in \mathrm{D}^{1,2}_0(\mathbb{R}^3)^3 \times \mathrm{L}^2_0(\mathbb{R}^3)$ to (5.1) satisfying

$$u \in \bigcap_{\ell=0}^{2} \mathrm{D}^{\ell+2,2}(\Omega), \qquad \mathfrak{p} \in \bigcap_{\ell=0}^{2} \mathrm{D}^{\ell+1,2}(\Omega).$$

We fix an $R > \delta(\Omega)$ and observe that $(u,\mathfrak{p}) \in \mathrm{W}^{2,r}(\Omega_{2R})^3 \times \mathrm{W}^{1,r}(\Omega_{2R})$ by Sobolev embedding. According to the regularity assumptions, Γ can be covered by a finite number of balls $\Gamma \subset \bigcup_{i=1}^m \mathrm{B}_{r_i}(x_i)$ each of which upon a rotation R_i can be mapped to $\mathrm{B}_{r_i}(0)$ by a C^5-diffeomorphism Φ_i, that is, $\Phi_i\circ\mathrm{R}_i : \mathrm{B}_{r_i}(x_i) \to \mathrm{B}_{r_i}(0)$, in such a way that $\Phi_i \circ \mathrm{R}_i\big(\Gamma \cap \mathrm{B}_{r_i}(x_i)\big) = \{x \in \mathrm{B}_{r_i}(0) \mid x_3 = 0\}$ and with $\|\nabla\Phi_i\|_\infty$ arbitrarily small for sufficiently small radii r_i, $i = 1,\dots,m$. The covering can clearly be augmented with bounded open sets $O_1 \subset\subset \Omega^{(1)}$ and $O_2 \subset\subset \Omega^{(2)}_{2R}$ so that $\overline{\Omega_R} \subset \cup_{i=1}^m \mathrm{B}_{r_i}(x_i) \cup O_1 \cup O_2$. Employing a partition of unity subordinate to such a covering, we can decompose and transform the solution $(u,\mathfrak{p})$ into m solutions $(u_i,\mathfrak{p}_i) \in \mathrm{W}^{2,r}(\dot{\mathbb{R}}^3)^3 \times \mathrm{W}^{1,r}(\dot{\mathbb{R}}^3)$, $i = 1,\dots,m$, to the twofold half-space Stokes problem (5.9), two solutions $(u_{m+1},\mathfrak{p}_{m+1}), (u_{m+2},\mathfrak{p}_{m+2}) \in \mathrm{W}^{2,r}(\mathbb{R}^3)^3 \times \mathrm{W}^{1,r}(\mathbb{R}^3)$ to a whole-space Stokes problem, and finally one solution $(w,\mathfrak{q}) \in \mathrm{D}^{1,2}_0(\mathbb{R}^3)^3 \times \mathrm{L}^2_0(\mathbb{R}^3)$ to the whole-space Oseen problem

$$\begin{cases} -\operatorname{div}\mathrm{T}(w,\mathfrak{q}) + \lambda_0\partial_3 w = F & \text{in } \mathbb{R}^3,\\ \operatorname{div} w = G & \text{in } \mathbb{R}^3. \end{cases} \tag{5.28}$$

In all three cases, the data contain lower-order terms of u and $\mathfrak{p}$ supported in B_{2R}. Furthermore, the data in the twofold half-space Stokes equations satisfied by $(u_i,\mathfrak{p}_i)$, $i = 1,\dots,m$, also contain higher-order terms of u and $\mathfrak{p}$ supported in B_{2R} and multiplied with components of $\nabla\Phi_i$. By Sobolev embeddings, we have $w \in \mathrm{D}^{1,2}_0(\mathbb{R}^3) \hookrightarrow \mathrm{L}^6(\mathbb{R}^3)$, and it is therefore easy to verify, for example by applying the Fourier transform in (5.28), that $(w,\mathfrak{q})$ coincides with the solution from [8, Theorem VII.4.1] and therefore satisfies

$$\begin{aligned} \|w\|_{\mathbf{X}^{q,r}_{1,\lambda_0}} + \|\mathfrak{q}\|_{\mathbf{X}^{q,r}_2} &\le c\big(\|F\|_{\mathrm{L}^q(\mathbb{R}^3)\cap\mathrm{L}^r(\mathbb{R}^3)} + \|G\|_{\mathrm{D}^{1,q}(\mathbb{R}^3)\cap\mathrm{L}^{3q/3-q}(\mathbb{R}^3)\cap\mathrm{D}^{1,r}(\mathbb{R}^3)}\big)\\ &\le c\big(\|f\|_{\mathbf{Y}^{q,r}_1} + \|g\|_{\mathbf{Y}^{q,r}_2} + \|u\|_{\mathrm{W}^{1,r}(\Omega_{2R})} + \|\mathfrak{p}\|_{\mathrm{L}^r(\Omega_{2R})}\big) \end{aligned} \tag{5.29}$$

with a constant $c = c(q,r,\overline{\lambda})$ independent of λ_0. A similar estimate is satisfied by the solutions $(u_{m+1},\mathfrak{p}_{m+1})$ and $(u_{m+2},\mathfrak{p}_{m+2})$ to the whole-space Stokes problems by [8, Theorem IV.2.1]. Moreover, Theorem 5.8 implies that $(u_i,\mathfrak{p}_i)$, $i = 1,\dots,m$, also satisfies the estimate, provided a covering is chosen with $\|\nabla\Phi_i\|_\infty$ sufficiently small, so that the higher-order terms can be absorbed on the left-hand side. We thus conclude

$$\|u\|_{\mathbf{X}^{q,r}_{1,\lambda_0}} + \|\mathfrak{p}\|_{\mathbf{X}^{q,r}_2} \le c\big(\|(f,g,h_1,h_2)\|_{\mathbf{Y}^{q,r}_1\times\mathbf{Y}^{q,r}_{2,3}\times\mathbf{Y}^{q,r}_4} + \|u\|_{\mathrm{W}^{1,r}(\Omega_{2R})} + \|\mathfrak{p}\|_{\mathrm{L}^r(\Omega_{2R})}\big), \tag{5.30}$$

where $c = c(\Gamma,q,r,\overline{\lambda}) > 0$. It remains to show that the lower-order terms of u and $\mathfrak{p}$ on the right-hand side can be neglected. This can be achieved by a standard contradiction argument. Assuming that

$$\begin{aligned} &\exists c > 0\ \forall 0 < |\lambda_0| < \overline{\lambda}\ \forall \text{solutions } (u,\mathfrak{p}) \in \mathbf{X}^{q,r}_{1,\lambda_0} \times \mathbf{X}^{q,r}_2 \text{w.r.t. data (5.27.)}:\\ &\|u\|_{\mathrm{W}^{1,r}(\Omega_{2R})} + \|\mathfrak{p}\|_{\mathrm{L}^r(\Omega_{2R})} \le c\|(f,g,h_1,h_2)\|_{\mathbf{Y}^{q,r}_1\times\mathbf{Y}^{q,r}_{2,3}\times\mathbf{Y}^{q,r}_4} \end{aligned} \tag{5.31}$$

does *not* hold, one can utilize (5.30) to construct a sequence $(\lambda_n, u_n, \mathfrak{p}_n)$ normalized such that $\|u_n\|_{\mathrm{W}^{1,r}(\Omega_{2R})} + \|\mathfrak{p}_n\|_{\mathrm{L}^r(\Omega_{2R})} = 1$ and with $\lambda_n \to \lambda$ and $(u_n, \mathfrak{p}_n)$ weakly convergent in the Banach space

$$\mathrm{D}^{2,r}(\Omega) \cap \mathrm{D}^{2,q}(\Omega) \cap \mathrm{L}^{\frac{3q}{3-2q}}(\Omega) \times \mathrm{D}^{1,q}(\Omega) \cap \mathrm{L}^{\frac{3q}{3-q}}(\Omega)$$

to a solution $(u, \mathfrak{p})$ to (5.1) with parameter $\lambda \in [0, \overline{\lambda}]$ and homogeneous right-hand side. The restriction $q < \frac{3}{2}$ is critical in this step. Theorem 5.5 implies $(u, \mathfrak{p}) = (0, 0)$, contradicting $\|u\|_{\mathrm{W}^{1,r}(\Omega_{2R})} + \|\mathfrak{p}\|_{\mathrm{L}^r(\Omega_{2R})} = 1$ obtained due to the compactness of the embeddings $\mathrm{D}^{2,r}(\Omega) \cap \mathrm{L}^{\frac{3q}{3-2q}}(\Omega) \hookrightarrow \mathrm{W}^{1,r}(\Omega_{2R})$ and $\mathrm{D}^{1,r}(\Omega) \cap \mathrm{L}^{\frac{3q}{3-q}}(\Omega) \hookrightarrow \mathrm{L}^r(\Omega_{2R})$. We conclude (5.31). Therefore, the lower-order terms of u and $\mathfrak{p}$ on the right-hand side in (5.30) can be neglected, which yields (5.26). Uniqueness of the solution follows from Theorem 5.5, and the theorem is thereby established for data satisfying (5.27). However, it is easy to verify that data satisfying (5.27) are dense in the space $\mathbf{Y}_1^{q,r} \times \mathbf{Y}_{2,3}^{q,r} \times \mathbf{Y}_4^{q,r}$. Consequently, the general case follows by a density argument. □

Theorem 5.10. *Let Γ be a C^5-smooth closed surface, $q \in (1, \frac{3}{2})$, $r \in (3, \infty)$ and $\lambda_0 = 0$. For every $(f, g, h_1, h_2) \in \mathbf{Y}_1^{q,r} \times \mathbf{Y}_{2,3}^{q,r} \times \mathbf{Y}_4^{q,r}$ there exists a unique solution $(u, \mathfrak{p})$ to (5.1) with*

$$\begin{aligned} &u^{(1)} \in \mathrm{W}^{2,r}(\Omega^{(1)})^3, \quad u^{(2)} \in \left(\mathrm{D}^{2,q}(\Omega^{(2)}) \cap \mathrm{D}^{2,r}(\Omega^{(2)}) \cap \mathrm{D}^{1,\frac{3q}{3-q}}(\Omega^{(2)}) \cap \mathrm{L}^{\frac{3q}{3-2q}}(\Omega^{(2)})\right)^3, \\ &\mathfrak{p}^{(1)} \in \mathrm{W}^{1,r}(\Omega^{(1)}), \quad \mathfrak{p}^{(2)} \in \mathrm{D}^{1,q}(\Omega^{(2)}) \cap \mathrm{D}^{1,r}(\Omega^{(2)}) \cap \mathrm{L}^{\frac{3q}{3-q}}(\Omega^{(2)}), \end{aligned} \tag{5.32}$$

that satisfies (5.25) *and*

$$\begin{aligned} \|\nabla^2 u\|_q + \|\nabla^2 u\|_r + \|\nabla u\|_{\frac{3q}{3-q}} + \|u\|_{\frac{3q}{3-2q}} + \|\nabla \mathfrak{p}\|_q + \|\nabla \mathfrak{p}\|_r \\ \leq C\|(f, g, h_1, h_2)\|_{\mathbf{Y}_1^{q,r} \times \mathbf{Y}_{2,3}^{q,r} \times \mathbf{Y}_4^{q,r}}, \end{aligned} \tag{5.33}$$

where $C = C(q, r, \Omega) > 0$.

Proof. The proof is similar to that of Theorem 5.9, the only difference being that $\lambda_0 = 0$ in (5.28). This implies that $(w, \mathfrak{q})$ solves a whole-space Stokes problem instead of an Oseen problem. Therefore, we use [8, Theorem IV.2.1] in this case to obtain estimate (5.33). The rest of the proof is identical to that of Theorem 5.9. □

6. Reformulation on a Fixed Domain

The steady-state equations of motion as expressed in (2.14) in a frame attached to the barycenter of the falling drop form a classical free boundary problem. Specifically, the boundary Γ depends on the unknown height function η. For further analysis it is necessary to refer all unknowns in this so-called *current configuration* to a fixed domain *reference configuration*. This section is devoted to such a reformulation.

As mentioned in the introduction and further elaborated on in Sect. 2, we investigate a falling drop whose stress-free configuration, *i.e.*, the configuration when the density in the two liquids is the same, is the unit ball B_1 in non-dimensionalized coordinates. Our aim is to establish existence of steady-state configurations close to the stress-free configuration B_1 for small density differences. Canonically, we therefore choose

$$\Omega_0 := \mathbb{R}^3 \setminus \mathbb{S}^2$$

as the fixed liquid reference domain.

In order to refer the equations of motion to Ω_0, we first construct a suitable coordinate transformation Φ^η based on the height function η. For technical reasons, it is important that Φ^η retains any rotational symmetry possessed by η.

Lemma 6.1. *Let $r \in (3, \infty)$. There is an extension operator*

$$E : \mathrm{W}^{3-1/r,r}(\mathbb{S}^2) \to \mathrm{W}^{3,r}(\mathbb{R}^3 \setminus \mathbb{S}^2)^3$$

satisfying $\operatorname{Tr}_{\mathbb{S}^2} E(\eta) = \eta\, \mathrm{Id}$, $\operatorname{supp} E(\eta) \subset \mathrm{B}_4$ *and*

$$\|E(\eta)\|_{\mathrm{W}^{3,r}} \le C\|\eta\|_{\mathrm{W}^{3-1/r,r}}. \tag{6.1}$$

The extension operator is invariant with respect to rotations, that is, for all $R \in SO(3)$*:*

$$E\big(\eta(R\,\cdot)\big)(x) = R^\top E(\eta)(Rx). \tag{6.2}$$

If $r > 3$*, there is a* $\delta_0 > 0$ *such that for any* $\eta \in \mathrm{W}^{3-1/r,r}(\mathbb{S}^2)$ *with* $\|\eta\|_{\mathrm{W}^{3-1/r,r}} < \delta_0$ *the mapping*

$$\Phi^\eta : \mathbb{R}^3 \to \mathbb{R}^3, \quad \Phi^\eta(x) = x + E(\eta)(x)$$

is continuous and maps Ω_0 C^2*-diffeomorphically onto* $\Omega = \Omega_\eta$ *with*

$$\Phi^\eta(\mathbb{S}^2) = \Gamma_\eta, \quad \Phi^\eta(\mathrm{B}_1) = \Omega_\eta^{(1)}, \quad \Phi^\eta(\mathrm{B}^1) = \Omega_\eta^{(2)}.$$

Proof. For $\eta \in \mathrm{W}^{3-1/r,r}(\mathbb{S}^2)$ let $H_\eta \in \mathrm{W}^{3,r}(\mathrm{B}_4 \setminus \mathbb{S}^2)$ denote the unique solution to

$$\begin{cases} \Delta H_\eta = 0 & \text{in } \mathrm{B}_1, \\ H_\eta = \eta & \text{on } \mathbb{S}^2, \end{cases} \qquad \begin{cases} \Delta H_\eta = 0 & \text{in } \mathrm{B}_4 \setminus \overline{\mathrm{B}_1}, \\ H_\eta = \eta & \text{on } \mathbb{S}^2, \\ H_\eta = 0 & \text{on } \partial\mathrm{B}_4. \end{cases} \tag{6.3}$$

Since the Laplace operator is rotational invariant, also the solution H_η is invariant with respect to rotations of the data η. Let $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$ be a cut-off function with $\chi(s) = 1$ for $|s| \le 2$ and $\chi(s) = 1$ for $|s| \ge 3$. Putting

$$E(\eta)(x) := \chi(|x|)\, H_\eta(x)\, x,$$

we obtain an operator with the desired properties. Observe that $E(\eta) \in \mathrm{W}^{1,r}(\mathbb{R}^3)$. Therefore, $\Phi^\eta(x) := x + E(\eta)(x)$ is a well-defined pointwise mapping $\Phi^\eta\colon \mathbb{R}^3 \to \mathbb{R}^3$. Since $r > 3$, the Sobolev embedding $\mathrm{W}^{3,r}(\mathbb{R}^3 \setminus \mathbb{S}^2) \hookrightarrow \mathrm{C}^2(\mathbb{R}^3 \setminus \mathbb{S}^2)$ implies that $\Phi^\eta \in \mathrm{C}^2(\Omega_0)$. Moreover, by (6.1) we clearly have $\det \nabla\Phi^\eta = \det\big(\mathrm{I} + \nabla E(\eta)\big) > 0$ when $\|\eta\|_{\mathrm{W}^{3-1/r,r}(\mathbb{S}^2)}$ is sufficiently small. In this case, Φ^η is a C^2-diffeomorphism onto its image Ω by the global inverse function theorem of Hadamard. □

We shall use Φ^η to change the coordinates and consequently express (2.14) in the reference configuration Ω_0. To this end, we set

$$w := v \circ \Phi^\eta, \qquad \mathfrak{q} := p \circ \Phi^\eta. \tag{6.4}$$

In order to simplify the notation, we put

$$F_\eta := \nabla\Phi^\eta = \mathrm{I} + \nabla E(\eta), \tag{6.5}$$

$$J_\eta := \det F_\eta = 1 + \operatorname{div} E(\eta) + \sum_{i=1}^{3} \prod_{\substack{j=1 \\ j\neq i}}^{3} \partial_j E(\eta) + \det(\nabla E(\eta)), \tag{6.6}$$

$$A_\eta := (\operatorname{cof} F_\eta)^\top = \big(1 + \operatorname{div} E(\eta)\big)\mathrm{I} - \nabla E(\eta) + \operatorname{cof}(\nabla E(\eta))^\top, \tag{6.7}$$

and introduce the transformed stress tensor

$$\mathrm{T}^\eta(w, \mathfrak{q}) := \big[\mu(\nabla w F_\eta^{-1} + F_\eta^{-\top}\nabla w^\top) - \mathfrak{q}\mathrm{I}\big] A_\eta^\top = \big(\mathrm{T}(v,p) \circ \Phi^\eta\big) A_\eta^\top. \tag{6.8}$$

Observe that an application of the Piola identity yields

$$\operatorname{div} \mathrm{T}^\eta(w, \mathfrak{q}) = J_\eta\big(\operatorname{div} \mathrm{T}(v,p)\big) \circ \Phi^\eta \quad \text{and} \quad \operatorname{div}(A_\eta w) = J_\eta(\operatorname{div} w) \circ \Phi^\eta.$$

The normal vector n_Γ at Γ expressed in the coordinates of the reference configuration is given by

$$\mathrm{n}_\Gamma \circ \Phi^\eta = \frac{A_\eta^\top \mathrm{n}_{\mathbb{S}^2}}{|A_\eta^\top \mathrm{n}_{\mathbb{S}^2}|},$$

and the transformed tangential projection by

$$\mathcal{P}^\eta := \mathrm{I} - |A_\eta^\top \mathrm{n}_{\mathbb{S}^2}|^{-2} A_\eta^\top (\mathrm{n}_{\mathbb{S}^2} \otimes \mathrm{n}_{\mathbb{S}^2}) A_\eta = (\mathrm{I} - \mathrm{n}_\Gamma \otimes \mathrm{n}_\Gamma) \circ \Phi^\eta.$$

With this notation, the steady-state equations of motion (2.14) take the following form in the reference configuration:

$$
\begin{cases}
\rho\big((A_\eta w)\cdot\nabla w+\lambda\nabla w A_\eta e_3\big)=\operatorname{div}\mathrm{T}^\eta(w,\mathfrak{q}) & \text{in } \Omega_0,\\
\operatorname{div}(A_\eta w)=0 & \text{in } \Omega_0,\\
[\![w]\!]=0 & \text{on } \mathbb{S}^2,\\
J_\eta w\cdot\dfrac{A_\eta^\top \mathrm{n}_{\mathbb{S}^2}}{|A_\eta^\top \mathrm{n}_{\mathbb{S}^2}|}=-J_\eta\lambda e_3\cdot\dfrac{A_\eta^\top \mathrm{n}_{\mathbb{S}^2}}{|A_\eta^\top \mathrm{n}_{\mathbb{S}^2}|} & \text{on } \mathbb{S}^2,\\
A_\eta\,\mathcal{P}^\eta[\![\mathrm{T}^\eta(w,\mathfrak{q})\mathrm{n}_{\mathbb{S}^2}]\!]=0 & \text{on } \mathbb{S}^2,\\
\dfrac{A_\eta^\top \mathrm{n}_{\mathbb{S}^2}}{|A_\eta^\top \mathrm{n}_{\mathbb{S}^2}|^2}\cdot[\![\mathrm{T}^\eta(w,\mathfrak{q})\mathrm{n}_{\mathbb{S}^2}]\!]=\dfrac{1}{16\pi}\dfrac{A_\eta^\top \mathrm{n}_{\mathbb{S}^2}}{|A_\eta^\top \mathrm{n}_{\mathbb{S}^2}|}\cdot\displaystyle\int_{\mathbb{S}^2}\zeta\big[(1+\eta(\zeta))^4-1\big]\,\mathrm{d}S & \\
\qquad\qquad+\sigma\big(\mathrm{H}+2\big)\circ\Phi^\eta+\widetilde{\rho}(1+\eta)e_3\cdot\mathrm{n}_{\mathbb{S}^2} & \text{on } \mathbb{S}^2,\\
\displaystyle\int_{\mathbb{S}^2}[\![\mathrm{T}^\eta(w,\mathfrak{q})\mathrm{n}_{\mathbb{S}^2}]\!]\,\big|A_\eta^\top \mathrm{n}_{\mathbb{S}^2}\big|^{-1}J_\eta\,\mathrm{d}S=\widetilde{\rho}\frac{4\pi}{3}e_3, & \\
\displaystyle\int_{\mathbb{S}^2}\big[(1+\eta)^3-1\big]\,\mathrm{d}S=0, & \\
\lim_{|x|\to\infty}w(x)=0 &
\end{cases}
\tag{6.9}
$$

with respect to unknowns $(w,\mathfrak{q},\lambda,\eta)$. We use the notation $\mathrm{n}=\mathrm{n}_{\mathbb{S}^2}$ in the following.

In the next step, we exploit an inherent symmetry in (6.9) and simplify the system by replacing $(6.9)_7$ with

$$
e_3\cdot\int_{\mathbb{S}^2}[\![\mathrm{T}^\eta(w,\mathfrak{q})\mathrm{n}]\!]\,\big|A_\eta^\top \mathrm{n}\big|^{-1}J_\eta\,\mathrm{d}S=\widetilde{\rho}\frac{4\pi}{3}.
$$

We shall *a posteriori* verify that a solution to the simplified system exhibits axial symmetry around e_3 and consequently satisfies

$$
e_j\cdot\int_{\mathbb{S}^2}[\![\mathrm{T}^\eta(w,\mathfrak{q})\mathrm{n}]\!]\,\big|A_\eta^\top \mathrm{n}\big|^{-1}J_\eta\,\mathrm{d}S=0\quad\text{for } j=1,2.
$$

Consequently, a solution to the simplified system

$$
\begin{cases}
\rho\big((A_\eta w)\cdot\nabla w+\lambda\nabla w A_\eta e_3\big)=\operatorname{div}\mathrm{T}^\eta(w,\mathfrak{q}) & \text{in } \Omega_0,\\
\operatorname{div}(A_\eta w)=0 & \text{in } \Omega_0,\\
[\![w]\!]=0 & \text{on } \mathbb{S}^2,\\
J_\eta w\cdot\dfrac{A_\eta^\top \mathrm{n}_{\mathbb{S}^2}}{|A_\eta^\top \mathrm{n}_{\mathbb{S}^2}|}=-J_\eta\lambda e_3\cdot\dfrac{A_\eta^\top \mathrm{n}_{\mathbb{S}^2}}{|A_\eta^\top \mathrm{n}_{\mathbb{S}^2}|} & \text{on } \mathbb{S}^2,\\
A_\eta\,\mathcal{P}^\eta[\![\mathrm{T}^\eta(w,\mathfrak{q})\mathrm{n}_{\mathbb{S}^2}]\!]=0 & \text{on } \mathbb{S}^2,\\
\dfrac{A_\eta^\top \mathrm{n}_{\mathbb{S}^2}}{|A_\eta^\top \mathrm{n}_{\mathbb{S}^2}|^2}\cdot[\![\mathrm{T}^\eta(w,\mathfrak{q})\mathrm{n}_{\mathbb{S}^2}]\!]=\dfrac{1}{16\pi}\dfrac{A_\eta^\top \mathrm{n}_{\mathbb{S}^2}}{|A_\eta^\top \mathrm{n}_{\mathbb{S}^2}|}\cdot\displaystyle\int_{\mathbb{S}^2}\zeta\big[(1+\eta(\zeta))^4-1\big]\,\mathrm{d}S & \\
\qquad\qquad+\sigma\big(\mathrm{H}+2\big)\circ\Phi^\eta+\widetilde{\rho}(1+\eta)e_3\cdot\mathrm{n}_{\mathbb{S}^2} & \text{on } \mathbb{S}^2,\\
e_3\cdot\displaystyle\int_{\mathbb{S}^2}[\![\mathrm{T}^\eta(w,\mathfrak{q})\mathrm{n}]\!]\,\big|A_\eta^\top \mathrm{n}\big|^{-1}J_\eta\,\mathrm{d}S=\widetilde{\rho}\frac{4\pi}{3}, & \\
\displaystyle\int_{\mathbb{S}^2}\big[(1+\eta)^3-1\big]\,\mathrm{d}S=0, & \\
\lim_{|x|\to\infty}w(x)=0 &
\end{cases}
\tag{6.10}
$$

with unknowns $(w, \mathfrak{q}, \lambda, \eta)$ is also a solution to (6.9). The analysis in the remaining part of the article is carried out on the system (6.10).

7. Linearization

A main challenge is to identify a suitable linearization of (6.10) such that the fully nonlinear system can be solved via a perturbation technique. Indeed, as explained in the introduction, the trivial linearization obtained by neglecting all nonlinear terms is not suitable since it leads to a Stokes-type rather than an Oseen-type problem. Instead, we shall linearize the equations around a non-trivial first-order approximation.

In order to identify the first-order approximation, we utilize an idea going back to Happel and Brenner [11] and introduce as *auxiliary field* a solution to the system

$$\begin{cases} \operatorname{div} \mathrm{T}(U,\mathfrak{P}) = 0 & \text{in } \Omega_0, \\ \operatorname{div} U = 0 & \text{in } \Omega_0, \\ [\![U]\!] = 0 & \text{on } \mathbb{S}^2, \\ U\cdot \mathrm{n} = -e_3\cdot \mathrm{n} & \text{on } \mathbb{S}^2, \\ (\mathrm{I}-\mathrm{n}\otimes\mathrm{n})[\![\mathrm{T}(U,\mathfrak{P})\mathrm{n}]\!] = 0 & \text{on } \mathbb{S}^2, \\ \lim_{|x|\to\infty} U(x) = 0. \end{cases} \tag{7.1}$$

By Theorem 5.10, a solution $(U,\mathfrak{P})$ to (7.1) exists with

$$\begin{aligned} &\forall s\in(3,\infty]: \quad U\in \mathrm{L}^s(\Omega_0),\\ &\forall s\in\left(\tfrac{3}{2},\infty\right]: \quad \nabla U,\ \mathfrak{P}\in \mathrm{L}^s(\Omega_0),\\ &\forall s\in(1,\infty): \quad \nabla^2 U,\ \nabla\mathfrak{P}\in \mathrm{L}^s(\Omega_0). \end{aligned} \tag{7.2}$$

Moreover, standard regularity theory for the Stokes problem implies that both U and $\mathfrak{P}$ are smooth in Ω_0, and well-known decay estimates for the 3D exterior domain Stokes problem (see for example [8, Theorem V.3.2]) yield

$$U = O(|x|^{-1}),\quad \nabla U = O(|x|^{-2})\quad \text{and}\quad \mathfrak{P} = O(|x|^{-2})\quad \text{as } |x|\to\infty. \tag{7.3}$$

Additionally, both the Stokes operator and the boundary operator on the left-hand side of (7.1) are invariant with respect to rotations. Since the data on the right-hand side is clearly invariant with respect to rotations $R\in SO(3)$ leaving e_3 invariant, the solution $(U,\mathfrak{P})$ retains this symmetry:

$$\forall R\in SO(3),\ Re_3 = e_3:\quad R^\top U(Rx) = U(x),\ \mathfrak{P}(Rx) = \mathfrak{P}(x). \tag{7.4}$$

By adding a constant to $\mathfrak{P}^{(1)}$, that is, replacing $\mathfrak{P}$ with

$$\widetilde{\mathfrak{P}} := \begin{cases} \mathfrak{P}+C & \text{in } B_1,\\ \mathfrak{P} & \text{in } B^1, \end{cases}$$

we may assume, by choosing the constant C appropriately, that

$$\int_{\mathbb{S}^2} \mathrm{n}\cdot[\![\mathrm{T}(U,\mathfrak{P})\mathrm{n}]\!]\,\mathrm{d}S = 0. \tag{7.5}$$

Moreover, we utilize $(7.1)_5$ to compute

$$
\begin{aligned}
-e_3 \cdot \int_{\mathbb{S}^2} [\![\mathrm{T}(U,\mathfrak{P})\mathrm{n}]\!]\,\mathrm{d}S &= -\int_{\mathbb{S}^2} \big(e_3\cdot \mathrm{n}\big)\,\mathrm{n}\cdot [\![\mathrm{T}(U,\mathfrak{P})\mathrm{n}]\!]\,\mathrm{d}S \\
&= \int_{\mathbb{S}^2} \big(U\cdot \mathrm{n}\big)\,\mathrm{n}\cdot [\![\mathrm{T}(U,\mathfrak{P})\mathrm{n}]\!]\,\mathrm{d}S = \int_{\mathbb{S}^2} [\![U\cdot \mathrm{T}(U,\mathfrak{P})\mathrm{n}]\!]\,\mathrm{d}S \\
&= \int_{\Omega_0} \nabla U : \mathrm{T}(U,\mathfrak{P}) + U\cdot \operatorname{div}\mathrm{T}(U,\mathfrak{P})\,\mathrm{d}x \\
&= \int_{\Omega_0} 2\mu\,\big|\mathrm{S}(U)\big|^2\,\mathrm{d}x > 0.
\end{aligned}
\tag{7.6}
$$

We can therefore choose

$$
\lambda_0(\widetilde{\rho}) := \left(e_3\cdot \int_{\mathbb{S}^2} [\![\mathrm{T}(U,\mathfrak{P})\mathrm{n}]\!]\,\mathrm{d}S\right)^{-1} \widetilde{\rho}\,\frac{4\pi}{3}. \tag{7.7}
$$

This choice of $\lambda_0(\widetilde{\rho})$ combined with the fact that the symmetry (7.4) implies

$$
e_j\cdot \int_{\mathbb{S}^2} [\![\mathrm{T}(U,\mathfrak{P})\mathrm{n}]\!]\,\mathrm{d}S = 0 \quad (j=1,2)
$$

means that $(\lambda_0(\widetilde{\rho})U, \lambda_0(\widetilde{\rho})\mathfrak{P}, \lambda_0(\widetilde{\rho}), 0)$ is a solution to the trivial linearization of (6.10) around the zero state, that is, to the system obtained by neglecting in (6.10) all nonlinear terms with respect to $(w, \mathfrak{q}, \lambda, \eta)$. The state $(\lambda_0(\widetilde{\rho})U, \lambda_0(\widetilde{\rho})\mathfrak{P}, \lambda_0(\widetilde{\rho}), 0)$ can therefore be seen as a first-order approximation of the solution to (6.10).

We shall seek to linearize (6.10) around $(\lambda_0(\widetilde{\rho})U, \lambda_0(\widetilde{\rho})\mathfrak{P}, \lambda_0(\widetilde{\rho}), 0)$. Since $\widetilde{\rho}\neq 0$ implies $\lambda_0(\widetilde{\rho})\neq 0$, a linearization around $(\lambda_0(\widetilde{\rho})U, \lambda_0(\widetilde{\rho})\mathfrak{P}, \lambda_0(\widetilde{\rho}), 0)$ would result in an Oseen-type problem. However, a direct linearization around $(\lambda_0(\widetilde{\rho})U, \lambda_0(\widetilde{\rho})\mathfrak{P}, \lambda_0(\widetilde{\rho}), 0)$ is still precarious since $(U,\mathfrak{P})$ is a solution to a Stokes problem, whence a linearization around this state would bring about right-hand side terms inadmissible in an Oseen setting. Instead, we introduce a truncation of the state. More specifically, we let $\chi\in \mathrm{C}_0^\infty(\mathbb{R})$ be a cut-off function with $\chi(r)=1$ for $|x|\leq 1$ and $\chi(r)=0$ for $|r|\geq 2$, and define $\chi_R\in \mathrm{C}_0^\infty(\mathbb{R}^3)$ by $\chi_R(x) := \chi\big(R^{-1}|x|\big)$ for $R>4$. Via the truncated auxiliary fields

$$
U_R := \chi_R U, \quad \mathfrak{P}_R := \chi_R \mathfrak{P}, \tag{7.8}
$$

we finally obtain the state $(\lambda_0(\widetilde{\rho})U_R, \lambda_0(\widetilde{\rho})\mathfrak{P}_R, \lambda_0(\widetilde{\rho}), 0)$ around which we shall linearize the system (6.10). Specifically, we let

$$
\kappa := \lambda - \lambda_0(\widetilde{\rho}), \quad u := w - \lambda_0(\widetilde{\rho})U_R - \kappa U_R, \quad \mathfrak{p} := \mathfrak{q} - \lambda_0(\widetilde{\rho})\mathfrak{P}_R - \kappa \mathfrak{P}_R \tag{7.9}
$$

and investigate (6.10) with respect to the unknowns $(u, \mathfrak{p}, \kappa, \eta)$.

To conclude the linearization, we express the mean curvature H on Γ as a function of η. As in [12, Sect. 2.2.5], we obtain

$$
\mathrm{H}\circ \Phi^\eta = \frac{1}{1+\eta}\left(\frac{\Delta_{\mathbb{S}^2}\eta}{\sqrt{g}} + \nabla_{\mathbb{S}^2}\frac{1}{\sqrt{g}}\cdot \nabla_{\mathbb{S}^2}\eta - \frac{2(1+\eta)}{\sqrt{g}}\right),
$$

where $\Delta_{\mathbb{S}^2}$ and $\nabla_{\mathbb{S}^2}$ denote the Laplace–Beltrami operator and the surface gradient on the unit sphere $\mathbb{S}^2$, respectively, and

$$
g := (1+\eta)^2 + |\nabla_{\mathbb{S}^2}\eta|^2.
$$

Then we have

$$
(\mathrm{H}+2)\circ \Phi^\eta = \Delta_{\mathbb{S}^2}\eta + 2\eta - \mathcal{G}_{\mathrm{H}}(\eta)
$$

with

$$
\mathcal{G}_{\mathrm{H}}(\eta) := -\frac{1}{1+\eta}\,\frac{1-(1+\eta)\sqrt{g}}{\sqrt{g}}\Delta_{\mathbb{S}^2}\eta - \frac{1}{1+\eta}\nabla_{\mathbb{S}^2}\frac{1}{\sqrt{g}}\cdot \nabla_{\mathbb{S}^2}\eta + \frac{2-2(1-\eta)\sqrt{g}}{\sqrt{g}}
$$

containing all the nonlinear terms.

We are now in a position to express (6.10) as a suitable perturbation of a linear problem with respect to the unknowns $(u, \mathfrak{p}, \kappa, \eta)$. Indeed, in a setting of velocity fields satisfying $[\![u]\!] = 0$ and $\lim_{|x|\to\infty} u(x) = 0$ we can express (6.10) equivalently as

$$\mathcal{L}^{\lambda_0(\widetilde{\rho})}(u, \mathfrak{p}, \kappa, \eta) = \mathcal{N}^{R,\widetilde{\rho}}(u, \mathfrak{p}, \kappa, \eta), \tag{7.10}$$

where the linear operator $\mathcal{L}^{\lambda_0(\widetilde{\rho})}$ is given by

$$\begin{aligned}\mathcal{L}^{\lambda_0(\widetilde{\rho})}(u, \mathfrak{p}, \kappa, \eta) &:= \begin{pmatrix} -\operatorname{div} \mathrm{T}(u, \mathfrak{p}) + \rho\lambda_0(\widetilde{\rho})\,\partial_3 u \\ \operatorname{div} u \\ u \cdot \mathrm{n} \\ (\mathrm{I} - \mathrm{n} \otimes \mathrm{n})[\![\mathrm{T}(u, \mathfrak{p})\mathrm{n}]\!] \\ \kappa e_3 \cdot \int_{\mathbb{S}^2} [\![\mathrm{T}(U, \mathfrak{P})\mathrm{n}]\!]\,\mathrm{d}S + e_3 \cdot \int_{\mathbb{S}^2} [\![\mathrm{T}(u, \mathfrak{p})\mathrm{n}]\!]\,\mathrm{d}S \\ \int_{\mathbb{S}^2} \eta\,\mathrm{d}S \\ \sigma(\Delta_{\mathbb{S}^2} + 2)\eta + \frac{1}{4\pi}\mathrm{n} \cdot \int_{\mathbb{S}^2} \eta \mathrm{n}\,\mathrm{d}S - \kappa \mathrm{n} \cdot [\![\mathrm{T}(U, \mathfrak{P})\mathrm{n}]\!] - \mathrm{n} \cdot [\![\mathrm{T}(u, \mathfrak{p})\mathrm{n}]\!] \end{pmatrix} \\ &=: \begin{pmatrix} \mathcal{L}_{1-4}(u, \mathfrak{p}) \\ \mathcal{L}_5(u, \mathfrak{p}, \kappa) \\ \mathcal{L}_6(\eta) \\ \mathcal{L}_7(u, \mathfrak{p}, \kappa, \eta) \end{pmatrix}\end{aligned} \tag{7.11}$$

and the nonlinear operator $\mathcal{N}^{R,\widetilde{\rho}} = (\mathcal{N}_1, \ldots, \mathcal{N}_7)$ consists of the components

$$\begin{aligned}
\mathcal{N}_1(u, \mathfrak{p}, \kappa, \eta) &:= (\lambda_0(\widetilde{\rho}) + \kappa) \operatorname{div} \mathrm{T}^\eta(U_R, \mathfrak{P}_R) + \operatorname{div} \mathrm{T}^\eta(u, \mathfrak{p}) - \operatorname{div} \mathrm{T}(u, \mathfrak{p}) - \rho A_\eta u \cdot \nabla u \\
&\quad - \rho(\lambda_0(\widetilde{\rho}) + \kappa)\big(A_\eta U_R \cdot \nabla u + A_\eta u \cdot \nabla U_R\big) - \rho(\lambda_0(\widetilde{\rho}) + \kappa)^2 A_\eta U_R \cdot \nabla U_R \\
&\quad - \rho\kappa \nabla u A_\eta e_3 - \rho\lambda_0(\widetilde{\rho})\nabla u(A_\eta - \mathrm{I})e_3 - \rho(\lambda_0(\widetilde{\rho}) + \kappa)^2 \nabla U_R A_\eta e_3, \\
\mathcal{N}_2(u, \mathfrak{p}, \kappa, \eta) &:= \operatorname{div}((\mathrm{I} - A_\eta)u) - (\lambda_0(\widetilde{\rho}) + \kappa) \operatorname{div}(A_\eta U_R), \\
\mathcal{N}_3(u, \mathfrak{p}, \kappa, \eta) &:= \big(u + (\lambda_0(\widetilde{\rho}) + \kappa)(U + e_3)\big) \cdot \big(\mathrm{I} - J_\eta |A_\eta^\top \mathrm{n}|^{-1} A_\eta^\top\big)\mathrm{n}, \\
\mathcal{N}_4(u, \mathfrak{p}, \kappa, \eta) &:= \mathcal{P}^0[\![\mathrm{T}(u, \mathfrak{p})\mathrm{n}]\!] - A_\eta\, \mathcal{P}^\eta[\![\mathrm{T}^\eta(u, \mathfrak{p})\mathrm{n}]\!] - (\lambda_0(\widetilde{\rho}) + \kappa)A_\eta\, \mathcal{P}^\eta[\![\mathrm{T}^\eta(U, \mathfrak{P})\mathrm{n}]\!], \\
\mathcal{N}_5(u, \mathfrak{p}, \kappa, \eta) &:= (\lambda_0(\widetilde{\rho}) + \kappa)e_3 \cdot \int_{\mathbb{S}^2} \big([\![\mathrm{T}(U, \mathfrak{P})\mathrm{n}]\!] - [\![\mathrm{T}^\eta(U, \mathfrak{P})\mathrm{n}]\!]|A_\eta^\top \mathrm{n}|J_\eta\big)\,\mathrm{d}S \\
&\quad + e_3 \cdot \int_{\mathbb{S}^2} \big([\![\mathrm{T}(u, \mathfrak{p})\mathrm{n}]\!] - [\![\mathrm{T}^\eta(u, \mathfrak{p})\mathrm{n}]\!]|A_\eta^\top \mathrm{n}|J_\eta\big)\,\mathrm{d}S, \\
\mathcal{N}_6(u, \mathfrak{p}, \kappa, \eta) &:= -\int_{\mathbb{S}^2} \eta^2 + \frac{1}{3}\eta^3\,\mathrm{d}S, \\
\mathcal{N}_7(u, \mathfrak{p}, \kappa, \eta) &:= \frac{A_\eta^\top \mathrm{n}}{|A_\eta^\top \mathrm{n}|^2} \cdot [\![\mathrm{T}^\eta(u, \mathfrak{p})\mathrm{n}]\!] - \mathrm{n} \cdot [\![\mathrm{T}(u, \mathfrak{p})\mathrm{n}]\!] + \lambda_0(\widetilde{\rho})\frac{A_\eta^\top \mathrm{n}}{|A_\eta^\top \mathrm{n}|^2} \cdot [\![\mathrm{T}^\eta(U, \mathfrak{P})\mathrm{n}]\!] \\
&\quad + \kappa\Big(\frac{A_\eta^\top \mathrm{n}}{|A_\eta^\top \mathrm{n}|^2} \cdot [\![\mathrm{T}^\eta(U, \mathfrak{P})\mathrm{n}]\!] - \mathrm{n} \cdot [\![\mathrm{T}(U, \mathfrak{P})\mathrm{n}]\!]\Big) \\
&\quad - \frac{1}{4\pi}\frac{A_\eta^\top \mathrm{n}}{|A_\eta^\top \mathrm{n}|} \cdot \int_{\mathbb{S}^2} \big(\frac{3}{2}\eta^2 + \eta^3 + \frac{1}{4}\eta^4\big)\,\mathrm{n}\,\mathrm{d}S + \frac{1}{4\pi}\big(\mathrm{n} - \frac{A_\eta^\top \mathrm{n}}{|A_\eta^\top \mathrm{n}|}\big) \cdot \int_{\mathbb{S}^2} \eta\,\mathrm{n}\,\mathrm{d}S \\
&\quad - \widetilde{\rho}(1 + \eta)e_3 \cdot \mathrm{n} + \sigma\mathcal{G}_{\mathrm{H}}(\eta).
\end{aligned}$$

8. Main Theorems

The formulation (7.10) is compatible with the framework of function spaces introduced in Sect. 4. More specifically, we shall show that $\mathcal{L}^{\lambda_0}$ maps $\mathbf{X}^{q,r}_{\lambda_0}(\Omega_0)$ homeomorphically onto $\mathbf{Y}^{q,r}(\Omega_0)$, and a solution to the fully nonlinear problem (7.10) can be established via the contraction mapping principle. We start with the first assertion:

Theorem 8.1. *Let $q \in (1, \frac{3}{2})$, $r \in (3, \infty)$ and $0 < |\lambda_0| \le \overline{\lambda}$. Then*

$$\mathcal{L}^{\lambda_0} : \mathbf{X}^{q,r}_{\lambda_0}(\Omega_0) \to \mathbf{Y}^{q,r}(\Omega_0)$$

is a homeomorphism with $\|(\mathcal{L}^{\lambda_0})^{-1}\| \le C$ and $C = C(q, r, \overline{\lambda})$ independent of λ_0.

Proof. We first show that $\mathcal{L}^{\lambda_0}$ is onto. To this end, we consider $(f, g, h_1, h_2, a_1, a_2, h_3) \in \mathbf{Y}^{q,r}(\Omega_0)$ and establish existence of $(u, \widetilde{\mathfrak{p}}, \kappa, \eta) \in \mathbf{X}^{q,r}_{\lambda_0}(\Omega_0)$ such that $\mathcal{L}^{\lambda_0}(u, \widetilde{\mathfrak{p}}, \kappa, \eta) = (f, g, h_1, h_2, a_1, a_2, h_3)$. By Theorem 5.9 there is a solution $(u, \mathfrak{p}) \in \mathbf{X}^{q,r}_{1,\lambda_0}(\Omega_0) \times \mathbf{X}^{q,r}_2(\Omega_0)$ to (5.1) with $\Omega = \Omega_0$. We put

$$c_{\mathfrak{p}} := \frac{1}{|\mathbb{S}^2|}\left(2\sigma a_2 - \int_{\mathbb{S}^2} \mathrm{n} \cdot [\![\mathrm{T}(u, \mathfrak{p})\mathrm{n}]\!] - \int_{\mathbb{S}^2} h_3 \,\mathrm{d}S\right) \tag{8.1}$$

and replace $\mathfrak{p}$ with

$$\widetilde{\mathfrak{p}} := \begin{cases} \mathfrak{p} + c_{\mathfrak{p}} & \text{in } B_1, \\ \mathfrak{p} & \text{in } B^1. \end{cases}$$

Then $(u, \widetilde{\mathfrak{p}})$ still solves (5.1), whence

$$\mathcal{L}_{1-4}(u, \widetilde{\mathfrak{p}}) = (f, g, h_1, h_2). \tag{8.2}$$

Recalling (7.6), we can define

$$\kappa := \left(e_3 \cdot \int_{\mathbb{S}^2} [\![\mathrm{T}(U, \mathfrak{P})\mathrm{n}]\!] \,\mathrm{d}S\right)^{-1}\left(a_1 - e_3 \cdot \int_{\mathbb{S}^2} [\![\mathrm{T}(u, \widetilde{\mathfrak{p}})\mathrm{n}]\!] \,\mathrm{d}S\right) \tag{8.3}$$

and thus obtain

$$\mathcal{L}_5(u, \widetilde{\mathfrak{p}}, \kappa) = a_1. \tag{8.4}$$

It remains to solve $\mathcal{L}_6(\eta) = a_2$ and $\mathcal{L}_7(u, \mathfrak{p}, \kappa, \eta) = h_3$ with respect to η. We briefly recall some properties of the operator $\Delta_{\mathbb{S}} + 2$. In particular, it is Fredholm in the setting $\Delta_{\mathbb{S}} + 2 : \mathrm{W}^{3-1/r,r}(\mathbb{S}^2) \to \mathrm{W}^{1-1/r,r}(\mathbb{S}^2)$ (see for example [17, Theorem 7.4.3]). It is well known, and easy to verify by a direct computation, that the components of the outer normal n on $\mathbb{S}^2$ span its kernel, that is, $\ker(\Delta_{\mathbb{S}} + 2) = \operatorname{span}\{\mathrm{n}_1, \mathrm{n}_2, \mathrm{n}_3\}$. We denote the projection onto this kernel and the corresponding complementary projection by

$$\mathcal{P}\psi := \frac{1}{4\pi}\mathrm{n} \cdot \int_{\mathbb{S}^2} \psi \mathrm{n} \,\mathrm{d}S \quad \text{and} \quad \mathcal{P}_\perp := \mathrm{Id} - \mathcal{P}.$$

The self-adjoint nature of $\Delta_{\mathbb{S}} + 2$ implies that $\mathcal{P}$ is also a projection onto the kernel of its adjoint $(\Delta_{\mathbb{S}} + 2)^*$. The Fredholm property thus implies that

$$\Delta_{\mathbb{S}} + 2 : \mathcal{P}_\perp \mathrm{W}^{3-1/r,r}(\mathbb{S}^2) \to \mathcal{P}_\perp \mathrm{W}^{1-1/r,r}(\mathbb{S}^2) \quad \text{homeomorphically.} \tag{8.5}$$

We can therefore introduce

$$\begin{aligned} \eta_\| &:= \mathcal{P}\big(h_3 + \kappa \mathrm{n} \cdot [\![\mathrm{T}(U, \mathfrak{P})\mathrm{n}]\!] + \mathrm{n} \cdot [\![\mathrm{T}(u, \widetilde{\mathfrak{p}})\mathrm{n}]\!]\big), \\ \eta_\perp &:= \sigma^{-1}(\Delta_{\mathbb{S}} + 2)^{-1} \mathcal{P}_\perp \big(h_3 + \kappa \mathrm{n} \cdot [\![\mathrm{T}(U, \mathfrak{P})\mathrm{n}]\!] + \mathrm{n} \cdot [\![\mathrm{T}(u, \widetilde{\mathfrak{p}})\mathrm{n}]\!]\big), \end{aligned}$$

and obtain a solution $\eta := \eta_\| + \eta_\perp \in \mathrm{W}^{3-1/r,r}(\mathbb{S}^2)$ to

$$\mathcal{L}_7(u, \widetilde{\mathfrak{p}}, \kappa, \eta) = h_3. \tag{8.6}$$

Moreover, integrating (8.6) over $\mathbb{S}^2$ and recalling both the choice of $c_{\mathfrak{p}}$ in (8.1) and (7.5), we observe that

$$\mathcal{L}_6(\eta) = a_2. \tag{8.7}$$

From (8.2), (8.4), (8.7) and (8.6) we deduce $\mathcal{L}^{\lambda_0}(u, \widetilde{\mathfrak{p}}, \kappa, \eta) = (f, g, h_1, h_2, a_1, a_2, h_3)$ and consequently that $\mathcal{L}^{\lambda_0}$ is onto. Uniqueness of the solution $(u, \widetilde{\mathfrak{p}}, \kappa, \eta)$ is a direct consequence of Theorem 5.9 and (8.5), which means that $\mathcal{L}^{\lambda_0}$ is also injective. The operator is clearly continuous and therefore a homeomorphism. Furthermore, from Theorem 5.9 we deduce the estimate

$$\begin{aligned}\|(u,\widetilde{\mathfrak{p}})\|_{\mathbf{X}^{q,r}_{1,\lambda_0}\times\mathbf{X}^{q,r}_2} &\le c\big(\|(f,g,h_1,h_2)\|_{\mathbf{Y}^{q,r}_1\times\mathbf{Y}^{q,r}_{2,3}\times\mathbf{Y}^{q,r}_4}+|c_{\mathfrak{p}}|\big)\\ &\le c\big(\|(f,g,h_1,h_2)\|_{\mathbf{Y}^{q,r}_1\times\mathbf{Y}^{q,r}_{2,3}\times\mathbf{Y}^{q,r}_4}+\|a_2\|_{\mathbf{Y}^{q,r}_6}+\|h_3\|_{\mathbf{Y}^{q,r}_7}\big)\end{aligned}$$

with $c=c(q,r,\overline{\lambda})$ *independent* of λ_0. In turn, we estimate in (8.3)

$$|\kappa|=\|\kappa\|_{\mathbf{X}^{q,r}_3}\le c\big(\|(f,g,h_1,h_2)\|_{\mathbf{Y}^{q,r}_1\times\mathbf{Y}^{q,r}_{2,3}\times\mathbf{Y}^{q,r}_4}+\|a_1\|_{\mathbf{Y}^{q,r}_5}+\|a_2\|_{\mathbf{Y}^{q,r}_6}+\|h_3\|_{\mathbf{Y}^{q,r}_7}\big)$$

with $c=c(q,r,\overline{\lambda})$ *independent* of λ_0. Since additionally

$$\begin{aligned}\|\eta\|_{\mathbf{X}^{q,r}_4} &\le \|\eta_\parallel\|_{\mathrm{W}^{3-1/r,r}}+\|\eta_\perp\|_{\mathrm{W}^{3-1/r,r}}\\ &\le c\big(\|\mathcal{P}\big(h_3+\kappa\mathrm{n}\cdot[\![\mathrm{T}(U,\mathfrak{P})\mathrm{n}]\!]+\mathrm{n}\cdot[\![\mathrm{T}(u,\widetilde{\mathfrak{p}})\mathrm{n}]\!]\big)\|_{\mathrm{W}^{3-1/r,r}}\big)\\ &\qquad+\|(\Delta_{\mathbb{S}}+2)^{-1}\|\|\mathcal{P}_\perp\big(h_3+\kappa\mathrm{n}\cdot[\![\mathrm{T}(U,\mathfrak{P})\mathrm{n}]\!]+\mathrm{n}\cdot[\![\mathrm{T}(u,\widetilde{\mathfrak{p}})\mathrm{n}]\!]\big)\|_{\mathrm{W}^{1-1/r,r}}\\ &\le c\big(\|h_3\|_{\mathrm{W}^{1-1/r,r}}+|\kappa|+\|[\![\mathrm{T}(u,\widetilde{\mathfrak{p}})\mathrm{n}]\!]\|_{\mathrm{W}^{1-1/r,r}}\\ &\qquad+\Big|\int_{\mathbb{S}^2}h_3\,\mathrm{d}S\Big|+\Big|\int_{\mathbb{S}^2}\mathrm{n}\cdot[\![\mathrm{T}(u,\widetilde{\mathfrak{p}})\mathrm{n}]\!]\,\mathrm{d}S\Big|\big)\\ &\le c\big(\|h_3\|_{\mathrm{W}^{1-1/r,r}}+|\kappa|+\|(u,\widetilde{\mathfrak{p}})\|_{\mathbf{X}^{q,r}_{1,\lambda_0}\times\mathbf{X}^{q,r}_2}\big),\end{aligned}$$

we conclude

$$\|(u,\widetilde{\mathfrak{p}},\kappa,\eta)\|_{\mathbf{X}^{q,r}_{\lambda_0}(\Omega_0)}\le c\|(f,g,h_1,h_2,a_1,a_2,h_3)\|_{\mathbf{Y}^{q,r}(\Omega_0)}$$

with $c=c(q,r,\overline{\lambda})$ *independent* of λ_0. It follows that $\|(\mathcal{L}^{\lambda_0})^{-1}\|\le c$ with $c=c(q,r,\overline{\lambda})$ *independent* of λ_0. □

The proof that the composition $(\mathcal{L}^{\lambda_0})^{-1}\circ\mathcal{N}^{R,\widetilde{\rho}}$ is a contraction is prepared in the following two lemmas. We first establish estimates of the change-of-coordinate matrices.

Lemma 8.2. *Let $r\in(3,\infty)$. There is $\delta_1>0$ such that for all $\eta_1,\eta_2\in\mathrm{W}^{3-1/r,r}(\mathbb{S}^2)$ with $\|\eta_j\|_{\mathrm{W}^{3-1/r,r}}\le\delta_1$ $(j=1,2)$ the following estimates are valid:*

$$\begin{aligned}&\|\mathrm{I}-A_{\eta_1}\|_{\mathrm{W}^{1,\infty}}\le C\|\eta_1\|_{\mathrm{W}^{3-1/r,r}}, &&\|A_{\eta_1}-A_{\eta_2}\|_{\mathrm{W}^{1,\infty}}\le C\|\eta_1-\eta_2\|_{\mathrm{W}^{3-1/r,r}},\\ &\|\mathrm{I}-F^{-1}_{\eta_1}\|_{\mathrm{W}^{1,\infty}}\le C\|\eta_1\|_{\mathrm{W}^{3-1/r,r}}, &&\|F^{-1}_{\eta_1}-F^{-1}_{\eta_2}\|_{\mathrm{W}^{1,\infty}}\le C\|\eta_1-\eta_2\|_{\mathrm{W}^{3-1/r,r}},\\ &\|1-J_{\eta_1}\|_{\mathrm{W}^{1,\infty}}\le C\|\eta_1\|_{\mathrm{W}^{3-1/r,r}}, &&\|J_{\eta_1}-J_{\eta_2}\|_{\mathrm{W}^{1,\infty}}\le C\|\eta_1-\eta_2\|_{\mathrm{W}^{3-1/r,r}}\end{aligned}$$

where $C=C(\delta_1,r)$.

Proof. Recalling (6.7), we observe that $\mathrm{I}-A_{\eta_1}$ contains only terms of first and second order with respect to components of $\nabla E(\eta_1)$. Utilizing that $\mathrm{W}^{2,r}(\mathbb{R}^3\setminus\mathbb{S}^2)$ is an algebra for $r>3$, and the Sobolev embedding $\mathrm{W}^{2,r}(\mathbb{R}^3\setminus\mathbb{S}^2)\hookrightarrow\mathrm{W}^{1,\infty}(\mathbb{R}^3\setminus\mathbb{S}^2)$, we deduce

$$\|\mathrm{I}-A_{\eta_1}\|_{\mathrm{W}^{1,\infty}}\le c\|\mathrm{I}-A_{\eta_1}\|_{\mathrm{W}^{2,r}}\le c\big(1+\|\nabla E(\eta_1)\|_{\mathrm{W}^{2,r}}\big)\|\nabla E(\eta_1)\|_{\mathrm{W}^{2,r}}.$$

The first assertion of the lemma then follows from (6.1) in Lemma 6.1. The next assertions follows in a similar manner. Concerning the estimates involving $F^{-1}_{\eta_1}$, we recall from (6.5)–(6.7) that $F^{-1}_{\eta_1}=J^{-1}_{\eta_1}A_{\eta_1}$. Consequently, we obtain an estimate of $\|\mathrm{I}-F^{-1}_{\eta_1}\|_{\mathrm{W}^{1,\infty}}$ as above, provided J_{η_1} is bounded away from 0. To this end, we recall (6.6) and choose δ_1 so small that $J_{\eta_1}>\frac{1}{2}$ for $\|\eta_1\|_{\mathrm{W}^{3-1/r,r}}\le\delta_1$. One may now verify the rest of the assertions analogously. □

The linearization (7.10) is a result of expressing the velocity field and pressure term as a perturbation (7.9) around a truncated auxiliary field $(U_R,\mathfrak{P}_R)$. The truncation is necessary to avoid right-hand side terms in (7.10) with inadmissible decay properties. Instead, compactly supported right-hand side terms appear. Suitable estimates of these terms are established in the following lemma. In particular, the magnitude of their norms are estimated in terms of the distance R of the truncation χ_R from the drop domain:

Lemma 8.3. *Let $q \in \left(1, \frac{3}{2}\right)$, $r \in (3,\infty)$ and δ_1 be the constant from Lemma 8.2. For all $\eta_1, \eta_2 \in \mathrm{W}^{3-1/r,r}(\mathbb{S}^2)$ with $\|\eta_j\|_{\mathrm{W}^{3-1/r,r}} \le \delta_1$ $(j=1,2)$*

$$\begin{aligned}
&\|\operatorname{div} \mathrm{T}^{\eta_1}(U_R,\mathfrak{P}_R)\|_{\mathbf{Y}_1^{q,r}} \le C(R^{-3+3/q} + \|\eta_1\|_{\mathrm{W}^{3-1/r,r}}),\\
&\|\operatorname{div} \mathrm{T}^{\eta_1}(U_R,\mathfrak{P}_R) - \operatorname{div} \mathrm{T}^{\eta_2}(U_R,\mathfrak{P}_R)\|_{\mathbf{Y}_1^{q,r}} \le C\|\eta_1-\eta_2\|_{\mathrm{W}^{3-1/r,r}},\\
&\|\operatorname{div}(A_{\eta_1}U_R)\|_{\mathbf{Y}_2^{q,r}} \le C(R^{-3+3/q} + \|\eta_1\|_{\mathrm{W}^{3-1/r,r}}),\\
&\|\operatorname{div}(A_{\eta_1}U_R) - \operatorname{div}(A_{\eta_2}U_R)\|_{\mathbf{Y}_2^{q,r}} \le C\|\eta_1-\eta_2\|_{\mathrm{W}^{3-1/r,r}},
\end{aligned} \tag{8.8}$$

where $C = C(q,r,\delta_1)$. Moreover,

$$\begin{aligned}
&\|A_{\eta_1}U_R\cdot\nabla U_R\|_{\mathbf{Y}_1^{q,r}} \le C,\\
&\|A_{\eta_1}U_R\cdot\nabla U_R - A_{\eta_2}U_R\cdot\nabla U_R\|_{\mathbf{Y}_1^{q,r}} \le C\|\eta_1-\eta_2\|_{\mathrm{W}^{3-1/r,r}},\\
&\|\nabla U_R A_{\eta_1}e_3\|_{\mathbf{Y}_1^{q,r}} \le C,\\
&\|\nabla U_R A_{\eta_1}e_3 - \nabla U_R A_{\eta_2}e_3\|_{\mathbf{Y}_1^{q,r}} \le C\|\eta_1-\eta_2\|_{\mathrm{W}^{3-1/r,r}},
\end{aligned} \tag{8.9}$$

where $C = C(q,r,\delta_1)$.

Proof. Let $\eta=\eta_1$. Recalling from Lemma 6.1 that $A_\eta(x) = F_\eta(x) = \mathrm{I}$ for $|x|\ge 4$, we utilize Lemma 8.2 to estimate

$$\begin{aligned}
&\|\operatorname{div} \mathrm{T}^{\eta_1}(U_R,\mathfrak{P}_R)\|_{\mathbf{Y}_1^{q,r}}\\
&\quad\le \|\operatorname{div} \mathrm{T}^{\eta_1}(U_R,\mathfrak{P}_R) - \operatorname{div} \mathrm{T}(U_R,\mathfrak{P}_R)\|_{\mathrm{L}^q\cap\mathrm{L}^r} + \|\operatorname{div} \mathrm{T}(U_R,\mathfrak{P}_R)\|_{\mathrm{L}^q\cap\mathrm{L}^r}\\
&\quad\le \|\mu(\nabla U_R F_\eta^{-1}A_\eta^\top - \nabla U_R) + \mu(F_\eta^{-\top}\nabla U_R^\top A_\eta^\top - \nabla U_R^\top) - (\mathfrak{P}_R A_\eta^\top - \mathfrak{P}_R\mathrm{I})\|_{\mathrm{D}^{1,q}(\mathrm{B}_4)\cap\mathrm{D}^{1,r}(\mathrm{B}_4)}\\
&\quad+ \|\operatorname{div} \mathrm{T}(U_R,\mathfrak{P}_R)\|_{\mathrm{L}^q\cap\mathrm{L}^r}\\
&\quad\le c(\|\eta\|^2_{\mathrm{W}^{3-1/r,r}} + \|\eta\|_{\mathrm{W}^{3-1/r,r}})(\|\nabla U_R\|_{\mathrm{W}^{1,q}(\mathrm{B}_4)\cap\mathrm{W}^{1,r}(\mathrm{B}_4)} + \|\mathfrak{P}_R\|_{\mathrm{W}^{1,q}(\mathrm{B}_4)\cap\mathrm{W}^{1,r}(\mathrm{B}_4)}\\
&\quad+ \|\operatorname{div} \mathrm{T}(U_R,\mathfrak{P}_R)\|_{\mathrm{L}^q\cap\mathrm{L}^r}).
\end{aligned}$$

Recalling the truncation (7.8), the pointwise decay of the auxiliary fields (7.3), and that $\operatorname{supp}\nabla\chi_R \subset \mathrm{B}_{2R,R}$ with $|\nabla\chi_R(x)| \le cR^{-1}$ as well as $|\nabla^2\chi_R(x)| \le cR^{-2}$, we further obtain

$$\begin{aligned}
\|\operatorname{div} \mathrm{T}(U_R,\mathfrak{P}_R)\|_q &= \|\operatorname{div}\left(\mu(\nabla[\chi_R U] + \nabla[\chi_R U]^\top) - \chi_R\mathfrak{P}\right)\|_q\\
&\le c\big(\|R^{-2}U\|_{\mathrm{L}^q(\mathrm{B}_{2R,R})} + \|R^{-1}\nabla U\|_{\mathrm{L}^q(\mathrm{B}_{2R,R})} + \|R^{-1}\mathfrak{P}\|_{\mathrm{L}^q(\mathrm{B}_{2R,R})}\big)\\
&\le cR^{-3+3/q}.
\end{aligned}$$

Since $r>q$, we obtain an even better estimate for $\|\operatorname{div} \mathrm{T}(U_R,\mathfrak{P}_R)\|_r$ with respect to decay in R, and thus conclude the first assertion of the lemma. The other inequalities in (8.8) follow in a similar manner.

The most critical estimate in (8.9) is the second one. Employing Lemma 8.2 together with the integrability properties (7.2) and the pointwise decay (7.3) of the auxiliary fields, we conclude

$$\begin{aligned}
&\|A_{\eta_1}U_R\cdot\nabla U_R - A_{\eta_2}U_R\cdot\nabla U_R\|_{\mathbf{Y}_1^{q,r}}\\
&\quad\le \|A_{\eta_1} - A_{\eta_2}\|_\infty \|\chi_R(U\cdot\nabla\chi_R)U + \chi_R^2 U\cdot\nabla U\|_{\mathrm{L}^q\cap\mathrm{L}^r}\\
&\quad\le c\|\eta_1-\eta_2\|_{\mathrm{W}^{3-1/r,r}}\big(\|R^{-1}|U|^2\|_{\mathrm{L}^q(\mathrm{B}_{2R,R})\cap\mathrm{L}^r(\mathrm{B}_{2R,R})} + \|U\|_{\mathrm{L}^{3q}\cap\mathrm{L}^{3r}}\|\nabla U\|_{\mathrm{L}^{3q/2}\cap\mathrm{L}^{3r/2}}\big)\\
&\quad\le c\|\eta_1-\eta_2\|_{\mathrm{W}^{3-1/r,r}}\big(R^{-3+1/q} + R^{-3+1/r} + c\big) \le c\|\eta_1-\eta_2\|_{\mathrm{W}^{3-1/r,r}}
\end{aligned}$$

since $R>4$. The remaining estimates in (8.9) are verified in a similar fashion. □

We are now in a position to show existence of a solution to (7.10).

Theorem 8.4. *Let $q \in \left(1, \frac{4}{3}\right]$, $r\in(3,\infty)$ and $\frac{3}{4} < \alpha < 1$. There is an $\varepsilon>0$ such that for all $0<|\widetilde{\rho}|<\varepsilon$ there is an $R>0$ and a solution $(u,\mathfrak{p},\kappa,\eta) \in \mathbf{X}^{q,r}_{\lambda_0(\widetilde{\rho})}(\Omega_0)$ to*

$$\mathcal{L}^{\lambda_0(\widetilde{\rho})}(u,\mathfrak{p},\kappa,\eta) = \mathcal{N}^{R,\widetilde{\rho}}(u,\mathfrak{p},\kappa,\eta), \tag{8.10}$$

which satisfies

$$\|(u,\mathfrak{p},\kappa,\eta)\|_{\mathbf{X}^{q,r}_{\lambda_0}} \le |\widetilde{\rho}|^{\alpha}. \tag{8.11}$$

This solution is unique in the class of elements in $\mathbf{X}^{q,r}_{\lambda_0(\widetilde{\rho})}(\Omega_0)$ *satisfying* (8.11).

Proof. We let $R := R(\widetilde{\rho}) := |\widetilde{\rho}|^{-\alpha}$ and show (8.10) by establishing existence of a fixed point of the mapping

$$\mathcal{M} : \mathbf{X}^{q,r}_{\lambda_0(\widetilde{\rho})}(\Omega_0) \to \mathbf{X}^{q,r}_{\lambda_0(\widetilde{\rho})}(\Omega_0), \quad \mathcal{M}(u,\mathfrak{p},\kappa,\eta) := \big(\mathcal{L}^{\lambda_0(\widetilde{\rho})}\big)^{-1} \circ \mathcal{N}^{R(\widetilde{\rho}),\widetilde{\rho}}(u,\mathfrak{p},\kappa,\eta)$$

for sufficiently small $\widetilde{\rho}$. To ensure that $\mathcal{M}$ is well defined, observe that

$$\operatorname{div}(A_\eta u) = J_\eta(\operatorname{div}(u\circ(\Phi^\eta)^{-1}))\circ\Phi^\eta$$

and

$$0 = \int_{\mathbb{S}^2} \big(U + e_3\big)\cdot \mathrm{n}\,\mathrm{d}S, \qquad 0 = \int_{\mathbb{S}^2} e_3\cdot J_\eta |A_\eta^\top \mathrm{n}|^{-1} A_\eta^\top\,\mathrm{d}S,$$

which implies

$$\int_{\mathrm{B}_1} \mathcal{N}_2(u,\mathfrak{p},\kappa,\eta)\,dx = \int_{\mathbb{S}^2} \mathcal{N}_3(u,\mathfrak{p},\kappa,\eta)\,\mathrm{d}S.$$

Moreover, a change of coordinates yields $\mathcal{N}_4(u,\mathfrak{p},\kappa,\eta)\cdot\mathrm{n} = 0$, and we conclude that $\mathcal{N}^{R(\widetilde{\rho}),\widetilde{\rho}}(u,\mathfrak{p},\kappa,\eta) \in \mathbf{Y}^{q,r}(\Omega_0)$ after establishing the corresponding estimates below. By fixing some $\overline{\lambda}$ and choosing ε so small that $|\lambda_0(\widetilde{\rho})| \le \overline{\lambda}$, Theorem 8.1 ensures that $\mathcal{L}^{\lambda_0(\widetilde{\rho})}$ is invertible from $\mathbf{Y}^{q,r}(\Omega_0)$ onto $\mathbf{X}^{q,r}_{\lambda_0(\widetilde{\rho})}(\Omega_0)$, and $\mathcal{M}$ therefore well defined. In the next step, we show that $\mathcal{M}$ is a contractive self-mapping on the ball $\mathrm{B}_{\widetilde{\rho}^\alpha}(0) \subset \mathbf{X}^{q,r}_{\lambda_0(\widetilde{\rho})}(\Omega_0)$. To this end, consider $(u,\mathfrak{p},\kappa,\eta)\in \mathrm{B}_{\widetilde{\rho}^\alpha}(0)$. The most critical part of the proof is to obtain a suitable estimate of $\mathcal{N}^{R(\widetilde{\rho}),\widetilde{\rho}}(u,\mathfrak{p},\kappa,\eta)$. We first utilize Lemma 8.3 and recall from (7.7) that $\lambda_0(\widetilde{\rho})$ depends linearly on $\widetilde{\rho}$ to estimate

$$\|(\lambda_0(\widetilde{\rho})+\kappa)\operatorname{div}\mathrm{T}^\eta(U_R,\mathfrak{P}_R)\|_q \le c(|\widetilde{\rho}| + |\widetilde{\rho}|^\alpha)\big(|\widetilde{\rho}|^{(3-3/q)\alpha} + |\widetilde{\rho}|^\alpha\big) = o\big(|\widetilde{\rho}|^\alpha\big) \text{ as } |\widetilde{\rho}|\to 0. \tag{8.12}$$

An application of Lemma 8.2 yields

$$\begin{aligned}\|\operatorname{div}\mathrm{T}^\eta(u,\mathfrak{p}) - \operatorname{div}\mathrm{T}(u,\mathfrak{p})\|_q &\le c\|\eta\|_{\mathrm{W}^{3-1/r,r}}(\|u\|_{\mathbf{X}^{q,r}_{1,\lambda_0}} + \|\mathfrak{p}\|_{\mathbf{X}^{q,r}_2})\\ &\le c|\widetilde{\rho}|^{2\alpha} = o\big(|\widetilde{\rho}|^\alpha\big) \text{ as } |\widetilde{\rho}|\to 0.\end{aligned} \tag{8.13}$$

Lemma 8.2 also implies $\|A_\eta\|_\infty \le c(\delta_1)$. Employing first Hölder's inequality and then estimate (4.4) from Proposition 4.1 with $t=2$, we obtain

$$\begin{aligned}\|\rho A_\eta u\cdot\nabla u\|_q &\le c\|A_\eta\|_\infty \|u\|_{\frac{2q}{2-q}}\|\nabla u\|_2\\ &\le c|\widetilde{\rho}|^{-\frac12-(1+\frac32-\frac3q)}\|u\|^2_{\mathbf{X}^{q,r}_{1,\lambda_0}} \le c|\widetilde{\rho}|^{\frac3q-3+2\alpha} = o\big(|\widetilde{\rho}|^\alpha\big) \text{ as } |\widetilde{\rho}|\to 0\end{aligned} \tag{8.14}$$

since $\frac34 < \alpha$. Further applications of Hölder's inequality in combination with the integrability properties (7.2) of U yield

$$\begin{aligned}&\|\rho(\lambda_0+\kappa)\big(A_\eta U_R\cdot\nabla u + A_\eta u\cdot\nabla U_R\big)\|_q\\ &\quad\le \big(|\lambda_0|+|\kappa|\big)\|A_\eta\|_\infty\big(\|U_R\|_4\|\nabla u\|_{\frac{4q}{4-q}} + \|u\|_{\frac{2q}{2-q}}\|\nabla U_R\|_2\big)\\ &\quad\le c\big(|\widetilde{\rho}|+|\widetilde{\rho}|^\alpha\big)\big(|\widetilde{\rho}|^{-\frac14}\|u\|_{\mathbf{X}^{q,r}_{1,\lambda_0}} + |\widetilde{\rho}|^{-\frac12}\|u\|_{\mathbf{X}^{q,r}_{1,\lambda_0}}\big)\\ &\quad\le c\big(|\widetilde{\rho}|+|\widetilde{\rho}|^\alpha\big)\big(|\widetilde{\rho}|^{-\frac14}+|\widetilde{\rho}|^{-\frac12}\big)|\widetilde{\rho}|^\alpha = o\big(|\widetilde{\rho}|^\alpha\big) \text{ as } |\widetilde{\rho}|\to 0\end{aligned} \tag{8.15}$$

since $\frac12 < \alpha$. From the integrability properties (7.2) we also obtain $U_R\cdot\nabla U_R \in \mathrm{L}^s(\mathbb{R}^3)$ for all $s>1$ and thus

$$\|\rho(\lambda_0(\widetilde{\rho})+\kappa)^2 A_\eta U_R\cdot\nabla U_R\|_q \le c\|A_\eta\|_\infty\big(|\widetilde{\rho}|+|\widetilde{\rho}|^\alpha\big)^2 = o\big(|\widetilde{\rho}|^\alpha\big) \text{ as } |\widetilde{\rho}|\to 0. \tag{8.16}$$

We move on to the so-called drift terms. Recalling that $A_\eta = \mathrm{I}$ on B_4^c, we estimate

$$\|\rho\kappa\nabla u A_\eta e_3\|_q \leq c|\widetilde{\rho}|^\alpha(\|\nabla u\|_{\mathrm{L}^q(\mathrm{B}_4)} + \|\partial_3 u\|_q) \leq c|\widetilde{\rho}|^\alpha|\widetilde{\rho}|^\alpha = o\big(|\widetilde{\rho}|^\alpha\big) \text{ as } |\widetilde{\rho}| \to 0 \tag{8.17}$$

and similarly

$$\|\rho\lambda_0(\widetilde{\rho})\nabla u(A_\eta - \mathrm{I})e_3\|_q \leq c|\widetilde{\rho}|^{2\alpha} = o\big(|\widetilde{\rho}|^\alpha\big) \text{ as } |\widetilde{\rho}| \to 0. \tag{8.18}$$

Finally, we once more employ Lemma 8.3 to deduce

$$\begin{aligned}\|\rho(\lambda_0(\widetilde{\rho}) + \kappa)^2\nabla U_R A_\eta e_3\|_q &\leq c\big(|\widetilde{\rho}| + |\widetilde{\rho}|^\alpha\big)^2 R^{-2+3/q}\\ &= c\big(|\widetilde{\rho}| + |\widetilde{\rho}|^\alpha\big)^2|\widetilde{\rho}|^{(2-3/q)\alpha} = o\big(|\widetilde{\rho}|^\alpha\big) \text{ as } |\widetilde{\rho}| \to 0.\end{aligned} \tag{8.19}$$

Summarizing (8.12)–(8.19), we conclude $\|\mathcal{N}_1(u, \mathfrak{p}, \kappa, \eta)\|_q = o\big(|\widetilde{\rho}|^\alpha\big)$ as $|\widetilde{\rho}| \to 0$, which is the most critical estimate of the proof. With less effort, the same estimate can be established for $\|\mathcal{N}_1(u, \mathfrak{p}, \kappa, \eta)\|_r$. Hence, $\|\mathcal{N}_1(u, \mathfrak{p}, \kappa, \eta)\|_{\mathbf{Y}_1^{q,r}} = o\big(|\widetilde{\rho}|^\alpha\big)$ as $|\widetilde{\rho}| \to 0$. The other components $\mathcal{N}_2, \dots, \mathcal{N}_7$ of $\mathcal{N}^{R(\widetilde{\rho}),\widetilde{\rho}}(u, \mathfrak{p}, \kappa, \eta)$ are estimated similarly. In particular, employing that $\mathrm{W}^{1,r}(\mathbb{S}^2)$ is an algebra due to $r > 3$, the nonlinear term $\|\mathcal{G}_{\mathrm{H}}(\eta)\|_{\mathrm{W}^{1-1/r,r}(\mathbb{S}^2)}$ can be estimated such that we obtain

$$\|\mathcal{N}_7(u, \mathfrak{p}, \kappa, \eta)\|_{\mathbf{Y}_7^{q,r}} \leq c\big(|\widetilde{\rho}| + |\widetilde{\rho}|^2 + |\widetilde{\rho}|^3 + |\widetilde{\rho}|^4\big).$$

Since $\alpha < 1$, we deduce $\|\mathcal{N}_7(u, \mathfrak{p}, \kappa, \eta)\|_{\mathbf{Y}_7^{q,r}} = o\big(|\widetilde{\rho}|^\alpha\big)$ and thus $\|\mathcal{N}(u, \mathfrak{p}, \kappa, \eta)\|_{\mathbf{Y}^{q,r}} = o\big(|\widetilde{\rho}|^\alpha\big)$ as $|\widetilde{\rho}| \to 0$. Recalling from Theorem 8.1 that $\|\big(\mathcal{L}^{\lambda_0(\widetilde{\rho})}\big)^{-1}\|$ is independent of $\lambda_0(\widetilde{\rho})$, we conclude that also $\|\mathcal{M}\|_{\mathbf{X}^{q,r}_{\lambda_0(\widetilde{\rho})}} = o\big(|\widetilde{\rho}|^\alpha\big)$ as $|\widetilde{\rho}| \to 0$. Consequently, $\mathcal{M}$ is a self-mapping on the ball $\mathrm{B}_{|\widetilde{\rho}|^\alpha}(0) \subset \mathbf{X}^{q,r}_{\lambda_0(\widetilde{\rho})}(\Omega_0)$ for sufficiently small $|\widetilde{\rho}|$. Estimates completely similar to the ones above can be used to verify that $\mathcal{M}$ is also a contraction on $\mathrm{B}_{|\widetilde{\rho}|^\alpha}(0) \subset \mathbf{X}^{q,r}_{\lambda_0(\widetilde{\rho})}(\Omega_0)$ for sufficiently small $|\widetilde{\rho}|$. Therefore, the contraction mapping principle (or Banach's Fixed Point Theorem) yields a unique fixed point $(u, \mathfrak{p}, \kappa, \eta)$ in $\mathrm{B}_{|\widetilde{\rho}|^\alpha}(0)$ of $\mathcal{M}$, which is clearly a solution to (8.10) satisfying (8.11). □

Finally, we are able to prove the main theorem of the article.

Proof of Theorem 2.1. Choosing the parameters as in Theorem 8.4, we let $(u, \mathfrak{p}, \kappa, \eta) \in \mathbf{X}^{q,r}_{\lambda_0(\widetilde{\rho})}(\Omega_0)$ denote the corresponding solution to (8.10).

A boot-strapping argument based on coercive L^r estimates in the whole and half space for the principle part of the operators $\mathcal{L}_{1-4}$ and $\mathcal{L}_7$, furnished by Theorem 5.8 in the former case and well-know estimates for the classical Laplace operator in the latter case, yields higher-order regularity. More specifically, after smoothing out the boundary in the $\mathcal{L}_{1-4}$ part of equation (8.10), difference quotients of $(u, \mathfrak{p})$ can be estimated using Theorem 5.8, which implies additional regularity of $(u, \mathfrak{p})$. In turn, classical L^r estimates for the Laplace operator in the 2D whole space yields bounds on difference quotients for η after smoothing out the interface in the $\mathcal{L}_7$ part of equation (8.10). In both cases, we choose ε and thus $|\widetilde{\rho}|$ sufficiently small in order to absorb higher-order terms from the right-hand side. Bootstrapping this procedure, we conclude regularity of arbitrary order for both $(u, \mathfrak{p})$ and η, and thereby deduce that the solution is smooth up to the boundary.

We further claim that the solution is invariant with respect to rotations that leave the e_3-axis invariant. To this end, consider an arbitrary $R \in SO(3)$ with $Re_3 = e_3$. Define

$$\widetilde{u}(x) := R^\top u(Rx), \quad \widetilde{\mathfrak{p}}(x) := \mathfrak{p}(Rx), \quad \widetilde{\kappa} := \kappa, \quad \widetilde{\eta}(x) := \eta(Rx).$$

Utilizing that (7.4) leads to rotation invariance of $(U_R, \mathfrak{P}_R)$, and that (6.2) implies $\Phi^{\widetilde{\eta}}(x) = R^\top\Phi^\eta(Rx)$, one readily verifies that $(\widetilde{u}, \widetilde{\mathfrak{p}}, \widetilde{\kappa}, \widetilde{\eta}) \in \mathbf{X}^{q,r}_{\lambda_0(\widetilde{\rho})}(\Omega_0)$ is another solution to (8.10) satisfying (8.11). The uniqueness assertion of Theorem 8.4 therefore yields $(u, \mathfrak{p}, \kappa, \eta) = (\widetilde{u}, \widetilde{\mathfrak{p}}, \widetilde{\kappa}, \widetilde{\eta})$, and we conclude the claimed rotational symmetry of the solution.

Now recall from (7.9) that a solution to (7.10) yields a solution $(w, \mathfrak{q}, \lambda, \eta)$ to (6.10). Due to the rotation symmetry of $(w, \mathfrak{q}, \lambda, \eta)$, we thereby obtain a solution in $\mathbf{X}^{q,r}_{\lambda_0(\widetilde{\rho})}(\Omega_0)$ to (6.9). Finally recalling (6.4), we deduce existence of a solution $(v, \mathfrak{p}, \lambda, \eta)$ to (2.14) satisfying (2.15) and (2.18).

Since $v \in X^{q,r,\lambda_0}_{\text{Oseen}}(\Omega^{(2)}_\eta)$, we may "test" the system with v, *i.e.*, multiplication of $(2.14)_1$ by v and subsequent integration by parts is a valid computation. Under the assumption $\lambda = 0$ this computation yields $\widetilde{\rho} = 0$. Since we are assuming $\widetilde{\rho} \neq 0$, we conclude that also $\lambda \neq 0$.

Finally, since $(v, \mathfrak{p})$ solves the classical Navier–Stokes equations in a 3D exterior domain with $v \in X^{q,r,\lambda_0}_{\text{Oseen}}(\Omega^{(2)}_\eta)$ and $\lambda \neq 0$, the integrability properties (2.16) and asymptotic structure (2.17) follow from [8, Theorem X.6.4] and [8, Theorem X.8.1], respectively. $\square$

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Declarations

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