Mutations of cluster algebras and discrete integrable systems

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Abstract

In [1], Okubo presented a direct connection between time evolutions of the discrete Toda lattice and quiver mutations of some cluster algebra. Since the discrete Toda lattice of type $A_N^{(1)}$ is arising from the point addition on the hyperelliptic curve as its spectral curve, this fact leads us to a geometric interpretation of the quiver mutation. In particular, for the discrete Toda lattice of type $A_1^{(1)}$, which is the non-trivial lowest dimensional Toda lattice with periodic boundary, a small quiver consisting of four vertices emulates the time evolution. Thus the quiver mutation is regarded as the point addition $P \mapsto P+T$ on the elliptic curve, where T is a fixed point on the curve. Moreover, if we restrict the time evolution alternately, which corresponds to the point addition $P \mapsto P+2T$ on the curve, then we find that all sub-quivers of rank two appearing in the evolutions are of type $A_1^{(1)}$. (The type of a quiver is defined to be the one of the Cartan matrix of the corresponding anti-symmetric integral matrix. It is known that the cluster algebra associated with the quiver of type $A_1^{(1)}$ is of infinite type, that is, the set of all seeds obtained by applying its mutations to the initial seed is an infinite set.) This observation suggests a direct link between the quiver mutation and the time evolution of the Toda lattice both of which are of type $A_1^{(1)}$, which enables us to give a geometric interpretation of the quiver mutation of type $A_1^{(1)}$.

We show that we can realize the mutation of the quiver of type $A_1^{(1)}$ with tropical coefficients as the QRT map by using a variable transformation. We also realize the time evolution of the Toda lattice of type $A_1^{(1)}$ as the QRT map by applying an appropriate birational transformation which simultaneously sends the unit of addition and the adding point on the spectral curve to the ones on the invariant curve of the QRT map, respectively. We then construct a direct connection between the alternate time evolution of the Toda lattice and the quiver mutation both of which are of type $A_1^{(1)}$ by appropriate choices of the parameters and the initial points. This connection gives a geometric interpretation of the quiver mutation of type $A_1^{(1)}$ as a degenerate limit of the point addition on the elliptic curve [2].

References

[1] N. Okubo, Discrete Integrable Systems and Cluster Algebras, *RIMS Kôkyûroku Bessatsu* **B41** (2013) 25-42.

[2] A. Nobe, Mutations of the cluster algebra of type $A_1^{(1)}$ and the periodic discrete Toda lattice, *J. Phys. A: Math. Theor.*, **49** (2016), 285201.