Complete exceptional surgeries on two-bridge links

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base on a joint work with

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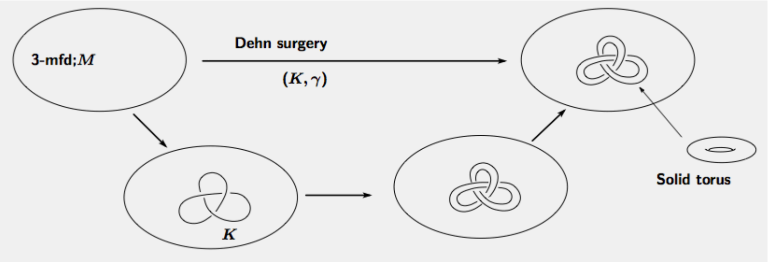
Dehn surgery on a knot

K : a knot in a 3-manifold M

Dehn surgery on K

1) remove the open tubular neighborhood of K from M
(to obtain the exterior $E(K)$ of K)

2) glue a solid torus V back (along a slope γ)



Surgery slope

K : a knot in the 3-sphere S^3

For $f : \partial V \to \partial E(K)$ and the meridian m of V,
the slope (i.e., isotopy class) γ of the loop $f(m)$ on $\partial E(K)$
is called the surgery slope.
Such a slope on $\partial E(K)$ can be regarded as $r \in \mathbb{Q} \cup \{1/0\}$.

Dehn surgery on links

Dehn surgery & surgery slope for a LINK are defined in the same way.

Hyperbolic Dehn Surgery Theorem

Only finitely many Dehn surgeries on a hyperbolic knot (i.e., knot with hyperbolic complement) yield non-hyperbolic manifolds. [Thurston]

Recall:

Every closed orientable 3-manifold is;

- Reducible (containing essential sphere)
- Toroidal (containing essential torus)
- Seifert fibered (admitting a foliation by circles)
- Hyperbolic (admitting Riem.metric of const.curv. -1)

as a consequence of the Geometrization Conjecture
including famous Poincaré Conjecture (1904)
conjectured by Thurston (late '70s)
established by Perelman (2002-03)

Exceptional surgery

Exceptional surgery

A Dehn surgery on a hyperbolic knot or a link is called exceptional if it yields a non-hyperbolic manifold.

Due to Hyperbolic Dehn Surgery Theorem, each hyperbolic knot has only finitely many exceptional surgeries.

Ultimate Goal

Classify all the exceptional surgeries

on hyperbolic knots and links in the 3-sphere S^3.

Today's target: two-bridge links in S^3.

2-bridge link

A link admitting a diagram with two maxima and minima.

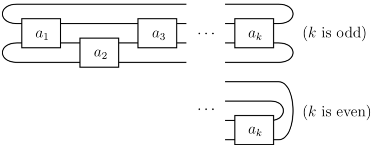


FIGURE 1. A diagram of a two-bridge link $L_{[a_1,\ldots,a_k]}$.

$$[a_1, \ldots, a_k] := \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cdots - \cfrac{1}{a_k}}}$$

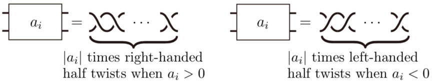


FIGURE 2. The right-handed twists when $a_i > 0$, and the left-handed twists when $a_i < 0$.

Let $L_{[a_1,\ldots,a_k]}$ denote the two-bridge link in S^3, represented by the diagram above.

Known facts (2-bridge links)

Let M_L be a 3-mfd obtained by a Dehn surgery on a component of a 2-bridge link L.

Theorem [Wu (1999)]

If M_L contains an essential disk, annulus, or 2-sphere, then L is equivalent to $L_{[b_1,b_2]}$.

A key ingredient used in [Wu (1999)] is a construction of an essential branched surface, originally given by [Delman].

Theorem [Goda-Hayashi-Song (2009)]

A complete classification of L for which M_L is a non-trivial, non-core torus knot exterior, or a cable knot exterior.
A necessary condition of L for which M_L is a prime satellite knot exterior.

Known facts (2-bridge links)

Theorem [I. 2012]

A complete classification of two-bridge links & surgery slopes admitting exceptional surgery on one component of the link.

$L(r)$ contains neither essential D nor S.
$L(r)$ contains an essential torus if and only if $L \cong L_{[2w,v,2u]}$ & $r = -w - u$ with

1. $w = 1, u = -1, |v| \geq 2$
2. $w \geq 2, |u| \geq 2, |v| = 1$
3. $w \geq 2, |u| \geq 2, |v| \geq 2$

In all the cases, $L(r)$ is never Seifert fibered, and $L(r)$ is a graph manifold if and only if u, v, w satisfy 1st & 2nd conditions.
If $L(r)$ contains an essential annulus, but contains no essential tori, then $L(r)$ is a Seifert fibered space.
$L(r)$ is a Seifert fibered space if and only if, for $w \geq 1, u \neq 0, -1$,

1. $L \cong L_{[3,2u+1]}$ & $r = u$
2. $L \cong L_{[2w+1,3]}$ & $r = -w - 1$
3. $L \cong L_{[3,-3]}$ & $r = -1$
4. $L \cong L_{[2w+1,2u+1]}$ & $r = -w + u$

Key ingredient: W. Floyd and A. Hatcher, The space of incompressible surfaces in a 2-bridge link complement, Trans. Amer. Math. Soc. 305 (1988), 575-599.

Main result

Complete exceptional surgery on a hyperbolic link

Dehn surgery on whole components of the link to obtain a closed non-hyperbolic 3-mfd, and all its proper sub-fillings are hyperbolic.

Theorem [I.-Jong-Masai]

If a hyperbolic two-bridge link L admits a complete exceptional surgery with the surgery slopes (γ_1, γ_2), then L & (γ_1, γ_2) are equivalent to one of those given in Table 1–8 (omitted):

By essential branched surface

Theorem.

If a non-torus two-bridge link L admits a complete exceptional surgery, then L is equivalent to one of the followings:

(a-1) $L_{[2m+1,2n-1]}$ with $m \geq 1$, $n \neq 0, 1$.

(b-1) $L_{[2m,2n,2l]}$ with $m \geq 1$, $|n| \geq 2$, $|l| \geq 2$.

(b-2) $L_{[2m,2n-1,-2l]}$ with $m \geq 1$, $|n| \geq 2$, $l \geq 1$.

(b-3) $L_{[2m,2n+1,2l]}$ with $m \geq 1$, $|n| \geq 2$, $l \geq 1$.

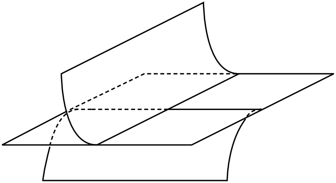
(b-4) $L_{[2m+1,2n,2l-1]}$ with $m \geq 1$, $n \neq 0$, $l \neq 0, 1$.

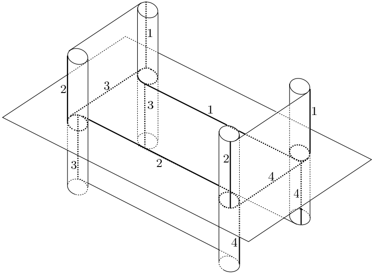
(c-1) $L_{[2m+1,2n,-2\,\mathrm{sgn}(l),2l-1]}$ with $m \geq 1$, $n \neq 0$, $l \neq 0, 1$.

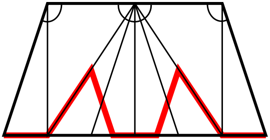
(c-2) $L_{[2m+1,2n-1,-2\,\mathrm{sgn}(l),2l]}$ with $m \geq 1$, $n \neq 0, 1$, $l \neq 0$.

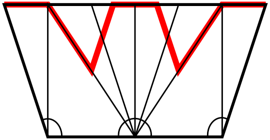
Here $\mathrm{sgn}(l)$ denotes 1 (resp. -1) when l is positive (resp. negative). In addition, in (b-2) and (b-3), if $m = 1$, then $n \leq -2$ holds.

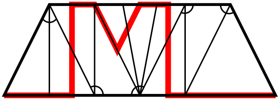
Essential branched surface & Allowable edge-path

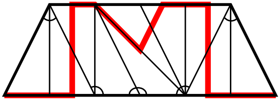












By using computer

Theorem.

If a non-torus two-bridge link L admits a complete exceptional surgery with the surgery slopes (γ_1, γ_2), then $L\&(\gamma_1, \gamma_2)$ are equivalent to one of those given in Table1–8:

Link	slopes
$L_{[2,-4,4]}$	(8,0) (8,1) (7,0) (7,1) (1,1)
$L_{[2,4,4]}$	(7,1) (6,0) (6,1)
$L_{[2,4,-4]}$	(3,-3)
$L_{[2,2*B,-2*C]}$	(3 - C,-C - 1)

Link	slopes
$L_{[3,-2,-3]}$	(-4,-3) (-4,0) (-3,-4) (-3,0) (-3,1) (0,3) (1,2) (1,3) (2,1) (2,2) (3,0) (3,1)
$L_{[3,-2,3]}$	(-3,-3) (-3,-2) (-3,-1) (-2,-3) (-2,-2) (-2,-1) (-1,-4) (-1,-3) (-1,-2) (-1,-1) (3,-1) (3,6) (4,-1) (4,5) (5,4) (-4,-1) (6,3)
$L_{[3,-2,-2*C-1]}$	(-C - 2,-C - 2)
$L_{[3,2,-3]}$	(-3,-1) (-3,0) (-2,-2) (-2,-1) (-1,-3) (-1,-2) (0,-3) (3,-1) (3,0) (3,4) (4,0) (4,3)
$L_{[3,2,3]}$	(-1,1) (-1,2) (-1,3) (0,1) (0,2) (0,3) (1,2) (2,0) (2,1) (3,0) (5,0) (5,1) (5,2) (5,3)
$L_{[3,-4,-3]}$	(0,2) (0,3) (1,1) (1,2) (1,4) (2,1) (2,3) (3,0) (3,2) (3,3) (4,1) (4,2) (4,3)
$L_{[3,-4,3]}$	(2,6) (3,4) (3,5) (3,6) (4,5)

Using Computer

To study exceptional surgeries on the links,
we further used a computer program developed in;

B.Martelli, C.Petronio, F.Roukema (2014)
Exceptional Dehn surgery on the minimally twisted five-chain link

The program relies upon

- SnapPy (based on SnapPea): computer software calculates various hyperbolic invariants for 3-manifolds.
  http://www.math.uic.edu/t3m/SnapPy/

Ingredients

We modified the original codes to use interval arithmetics, and applied the program **hikmot** developed in

Hoffman, Ichihara, Kashiwagi, Masai, Oishi, and Takayasu (2016)
Verified computations for hyperbolic 3-manifolds
http://www.oishi.info.waseda.ac.jp/~takayasu/hikmot/

It can possibly give us a rigorous complete classification of exceptional surgeries on a given hyperbolic link.

Fin.