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Equilibrium Indeterminacy under Forward-Looking Interest Rate Rules†

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Abstract

Why is the rational expectations equilibrium locally indeterminate if the central bank raises the nominal interest rate too actively in response to a rise in expected inflation? This is because although the bank succeeds in stabilizing the expectations of the future economy, it allows the current economy to fluctuate arbitrarily fluctuate. This indeterminacy expands in the dimension as the forecast horizon of the rule becomes long.

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1 Introduction

The literature has found that for the rational expectations equilibrium to be uniquely determined, the central bank should raise the nominal interest rate by more than one-for-one in response to a rise in the current inflation rate (i.e., *the Taylor principle*). On the other hand, Bernanke and Woodford (1997) argue that the bank should follow the Taylor principle in response to the rate of *expected* inflation but *should not raise* the interest rate too actively. The reason for this is that the equilibrium also becomes indeterminate under a too active forward-looking interest rate rule.[1] Moreover, Batini and Haldane (1999) argue that if the forecast horizon of the rule is long, the forward-looking rule makes the economy fluctuating.

However, the related literature contains no work that explains the reason why the forward-looking rule makes the equilibrium indeterminate, although it does contain work in which the determinacy conditions on policy parameters have been derived analytically. Considering that the effects of monetary policy tend to have a lag, it is essential for the central bank to formulate a policy rule that is endowed with the forward-looking perspective. In addition, it is necessary to investigate the performance of a forward-looking rule with a long forecast horizon.

This paper presents two results. First, indeterminacy emerges under an active forward-looking rule because although the central bank succeeds in stabilizing the expectations of the future economy, it allows the current economy to fluctuate arbitrarily. Second, as the forecast horizon of the forward-looking rule becomes long, there is an increase in the dimension of indeterminacy that makes the economy more fluctuating.

The next section presents a basic NK model and the conditions for equilibrium determinacy. Section 3 clarifies the mechanism of indeterminacy under simple assumptions. Section 4 simulates the impulse responses of stable sunspot equilibria that are possible under a forward-looking rule. The final section contains the main results.

2 The Model

We use a basic NK model to obtain the conditions for the determinate rational expectations equilibrium as described by Gali (2008, chapter 3):[2]

$$\tilde{y}_t = E_t\tilde{y}_{t+1} - 1/\sigma\,(i_t - E_t\pi_{t+1}), \tag{1}$$

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1) Similar arguments are offered by Clarida, Gali, and Gertler (2000, chapter 4) and Woodford (2003).

2) Gali (2008) introduces the natural rate r^n in Eq. (1) and the steady state nominal interest rate ρ in Eq. (3). We assume $r^n = \rho = 0$ to simplify the analysis. However, our analysis is unchanged.

$$\pi_t = \beta E_t \pi_{t+1} + \kappa \tilde{y}_t, \tag{2}$$
$$i_t = \phi E_t \pi_{t+1} + \varepsilon_t. \tag{3}$$

Here, $\tilde{y}_t$ denotes the output gap from its natural level in period t, π_t is the rate of inflation, and i_t is the nominal interest rate. E is the expectation operator. The last equation describes a forward-looking nominal interest rate rule, where $\phi > 0$ is the policy parameter representing the magnitude of the central bank's response to expected inflation. ε_t is a single exogenous fundamental shock that satisfies $E_{t-1}\varepsilon_t = 0$ for all t. $\sigma > 0$, $0 < \beta < 1$, and $\kappa > 0$ are parameters.

Gali (2008) provides the following sufficient and necessary condition for determinacy:

$$1 < \phi < \bar{\phi},$$

where $\bar{\phi} \equiv 1 + \frac{2\sigma(1+\beta)}{\kappa}$. This condition suggests that in addition to following the Taylor principle ($\phi > 1$), the central bank *should not raise* the nominal interest rate too actively in response to a rise in the rate of expected inflation.

3 Mechanism of Indeterminacy

To establish theoretical reasons for the indeterminacy that appears under a forward-looking rule, we consider equilibrium solutions under an active rule $\phi = +\infty$ as a simple case.

3.1 Under an active rule

Proposition 1 *For any value of fundamental shock ε in period t, if $\phi = +\infty$ under Eq. (3), the current equilibrium is indeterminate while the future equilibria are determinate as*

$$\begin{cases} \pi_t = \kappa \tilde{y}_t = -\frac{\kappa}{\sigma} i_t, \\ E_t \pi_{t+j} = E_t \tilde{y}_{t+j} = E_t i_{t+j} = 0 \text{ for } j \geq 1. \end{cases} \tag{4}$$

The proof is in Appendix A.

This indeterminacy stems from the characteristic of the forward-looking rule. This rule implies that the central bank primarily focuses on stabilizing the expectations of future variables. Then, to the extent that a forward-looking rule is active, the bank allows the current economy to arbitrarily fluctuate while satisfying Eq. (4). As a result, the too active forward-looking rule makes the current equilibrium indeterminate.[3]

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3) This characteristic resembles that of a passive rule ($\phi < 1$), in that the bank's response to the fluctuations in the current variables is weak.

This result is in contrast with the equilibrium dynamics under a *current* -looking rule, $i_t = \phi\pi_t$, instead of Eq. (3). Under the same assumptions in Proposition 1, the stable equilibrium is uniquely determined as

$$E_t\pi_{t+j} = E_t\tilde{y}_{t+j} = E_t i_{t+j} = 0 \text{ for } \forall j.$$

The derivation is in Appendix B.

The current-looking rule differs from Proposition 1 in that it makes the current equilibrium determinate. This is because the central bank always focuses on stabilizing variables in the same period. Then both the current economy and future economies are uniquely determined only if $\phi > 1$.

As our simulations will show later, the mechanism shown in Proposition 1 works under the forward-looking rule of $\bar{\phi} < \phi < +\infty$ in general.

3.2 A long forecast horizon

The implication of Proposition 1 can be applied toward understanding the economy in which the central bank responds to the expected inflation of a *long forecast horizon*. As a simple example, we consider the following rule instead of Eq. (3):

$$i_t = \phi E_t\pi_{t+2} + \varepsilon_t. \tag{5}$$

Proposition 2 *For any value of fundamental shock ε in period t, if $\phi = +\infty$ under Eq. (5), the equilibrium solution has two-dimensional indeterminacy.*

The proof is in Appendix C.

The intuition is similar. Under this rule, the central bank gives top priority to stabilizing the expectations of future variables from period $t + 2$ onward. Then, to the extent that a forward-looking rule is active, the bank allows both the current and next period's economies to fluctuate arbitrarily. This leads to two-dimensional indeterminacy.

Proposition 2 is easily generalized as follows.

Proposition 3 *For any value of fundamental shock ε in period t, if $\phi = \infty$ under $i_t = \phi E_t\pi_{t+j}$ for $j \geq 0$, the equilibrium solution has j-dimensional indeterminacy.*

The proof is similar to that of Proposition 2.

Batini and Haldane (1999) argue that an inflation forecast rule with a long forecast horizon risks macroeconomic instability. However, theirs is a reduced-form model, and thus they do not provide any theoretical reason for the instability. Proposition 3 suggests that the instability is amplified by the increase in the dimension of indeterminacy.

4 Simulation

We simulate stable sunspot equilibrium dynamics to show that fluctuations in the current variables are amplified to the extent that the central bank respond to the expected inflation actively. The values of our parameters are taken from Gali (2008, p.51): $\sigma = 1$, $\kappa = 0.1275$, $\beta = 0.99$, and $\bar{\phi} \approx 32.2157$. In response to a sunspot shock in period 0 that generates a 0.1% rise in the expected inflation rate in period $t + 1$, we simulate sunspot dynamics under $\phi = \{32.5, 40, 50\}$. This sunspot shock abstracts a situation in which, for example, households expect expansions of the future economy. The methodology applied to calculate sunspot equilibria is taken from Sims (2002) and Lubik and Schorfheide (2003).[4]

Figure 1 shows the sunspot impulse responses. By assumption, the sunspot shock always raises the inflation rate in period $t + 1$ by 0.1%. Sunspot equilibria under $\phi > \bar{\phi}$ are oscillatory and converging to the steady state. The oscillatory convergence under $\phi > \bar{\phi}$ originates from the fact that in response to an increase in expected inflation above the steady state, the bank raises the current nominal rate so actively that the current variables drop below the steady state. Thus, variables in the current and next periods continue to have signs opposite to each other around the steady state.[5][6]

The magnitude of ϕ has a positive effect on the future economy and a negative one on the current economy. As ϕ becomes large, future variables stabilize while current variables fluctuate. This is in line with the implication of Proposition 1 that the central bank succeeds in stabilizing the future economy on the one hand and leaves the current economy to fluctuate arbitrarily on the other.[7]

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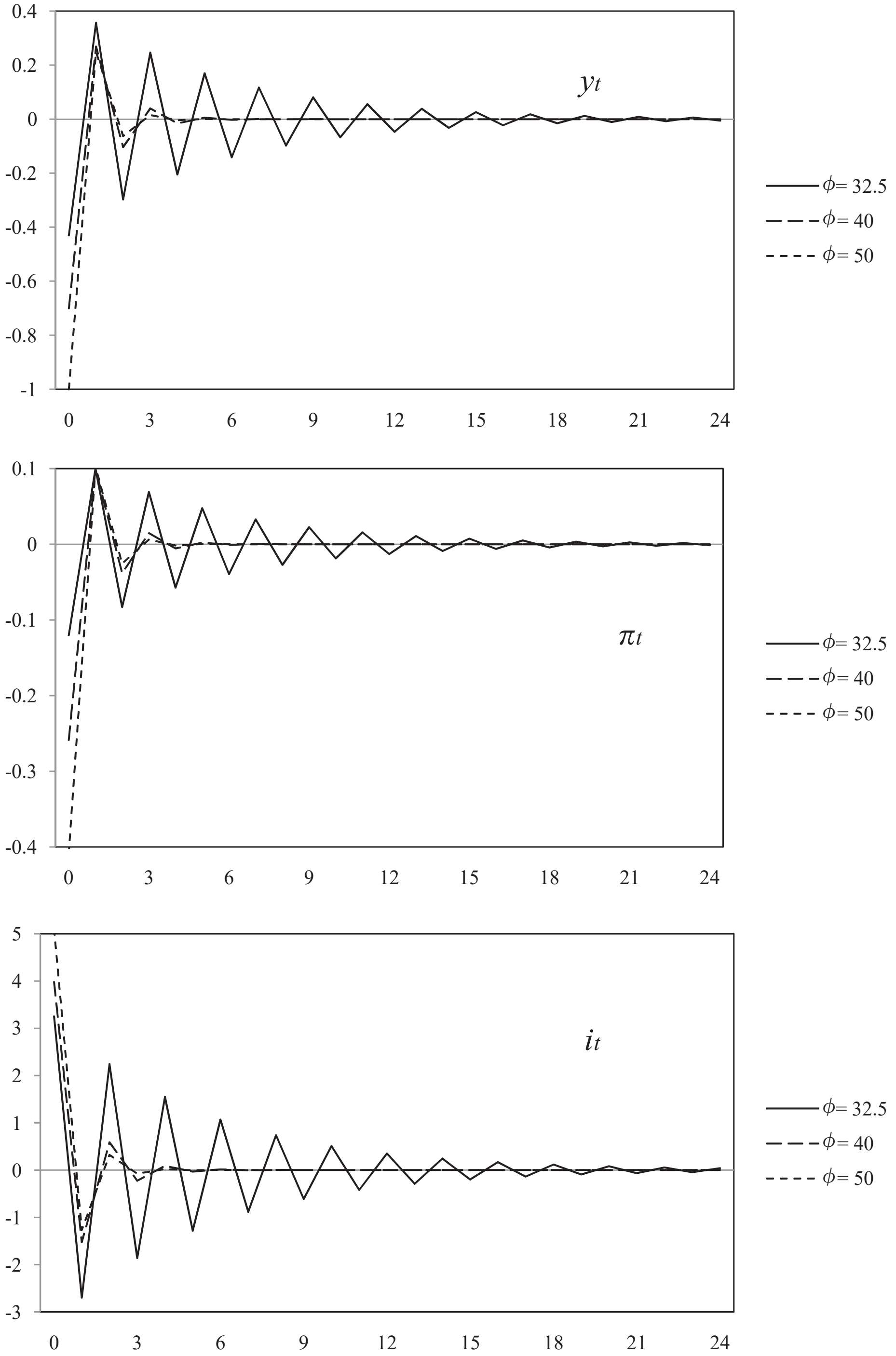
4) Sims (2002) generalizes the methodology of Blanchard and Kahn (1980) to obtain a solution of the rational expectations equilibrium. Lubik and Schorfheide (2003) show a methodology to obtain the impulse responses of a sunspot equilibrium using Sims' methodology. The details are in Appendix D.

5) This mechanism is inherently equivalent to that of the sunspot equilibria under $\phi < 1$ that smoothly converge to the steady state, as simulated by Lubik and Schorfheide (2003). The smooth convergence under $\phi < 1$ is because in response to a rise in the expected inflation, the central bank raises the current nominal rate less than onefor-one. Then, the positive sunspot shock in the expected inflation reduces the real interest rate, and the economy is modestly adjusted toward the steady state.

6) If ϕ reaches unity or $\bar{\phi}$, sunspot responses stop converging. The equilibrium solutions are shown in Appendices E and F, respectively. If ϕ exceeds these values, the sunspot responses explode, leaving only fundamental responses in the neighborhood of the steady state.

7) Note that this oscillatory dynamic is different from the *limit cycle* found by Benhabib, Schmitt-Grohe, and Uribe (2003). They show the possibility of *global indeterminacy* under a backward-looking rule.

Figure 1 Impulse Responses to a Sunspot Shock



5 Conclusion

In this paper, we have investigated the reason why the equilibrium is indeterminate when a forward-looking rule is too active. We also consider the reason for the economic instability when the forecast horizon of the rule becomes long.

The reason for the former is that if an active forward-looking rule is adopted, the central bank primarily focuses on stabilizing the expectations of future variables on the one hand and allows the current variables to fluctuate arbitrarily due to a sunspot shock on the other. The reason for the latter is that indeterminacy expands in the dimension as the central bank focuses on stabilizing the expectations of inflation in the distant future.

Appendix

A Proof of Proposition 1

Suppose $\phi = +\infty$. For any value of fundamental shock ε in period t, Eq. (3) leads to $E_t\pi_{t+1} = 0$. This is applied to the expectations in period $t + 1$ onward. Then,

$$E_t\pi_{t+j} = 0 \text{ for } j \geq 1. \tag{6}$$

Eqs. (2) and (6) derive $\pi_t = \kappa\tilde{y}_t$. This is applied to the expectations in period t + 1 onward. Then, from Eq. (6),

$$E_t\tilde{y}_{t+j} = \begin{cases} \pi_t/\kappa & \text{for } j = 0 \\ 0 & \text{for } j \geq 1 \end{cases}. \tag{7}$$

In addition, Eqs. (1), (6), and (7) provide

$$E_t i_{t+j} = \begin{cases} -\sigma\tilde{y}_t & \text{for } j = 0 \\ 0 & \text{for } j \geq 1 \end{cases}.$$

Here, i_t has an arbitrary value due to a sunspot shock in period t that is irrespective of fundamental shock ε.

To summarize, for any value of fundamental shock ε in period t, if $\phi = \infty$ under a forward-looking rule $i_t = \phi E_t\pi_{t+1}$, the equilibrium is

$$\begin{cases} \pi_t = \kappa\tilde{y}_t = -\frac{\kappa}{\sigma} i_t, \\ E_t\pi_{t+j} = E_t\tilde{y}_{t+j} = E_t i_{t+j} = 0 \text{ for } j \geq 1. \end{cases}$$

B Equilibrium under $i_t = \infty \cdot \pi_t$

Suppose $\phi = +\infty$. For any value of ε in period t, Eq. (3) leads to

$$E_t\pi_{t+j} = 0 \text{ for } \forall j. \tag{8}$$

Next, Eqs. (2) and (8) provide

$$E_t y_{t+j} = 0 \text{ for } \forall j. \tag{9}$$

However, if the economy satisfies Eqs. (8) and (9), Eq. (1) indicates that the nominal interest rate must satisfy

$$E_t i_{t+j} = 0 \text{ for } \forall j.$$

In summary, for any value of fundamental shock ε in period t, if $\phi = \infty$ under a current-looking rule $i_t = \phi\pi_t$, the equilibrium is

$$E_t\pi_{t+j} = E_t\tilde{y}_{t+j} = E_t i_{t+j} = 0 \text{ for } \forall j.$$

C Proof of Proposition 2

Suppose $\phi = +\infty$. For any value of fundamental shock ε in period t, Eq. (5) leads to $E_t\pi_{t+2} = 0$. This is applied to the expectations in period $t + 2$ onward. Then,

$$E_t\pi_{t+j} = 0 \text{ for } j \geq 2. \tag{10}$$

Eqs. (2) and (10) give $E_t\pi_{t+1} = \kappa E_t\tilde{y}_{t+1}$. This is applied to the expectations in period $t + 2$ onward. Then, from Eq. (10),

$$E_t\tilde{y}_{t+j} = \begin{cases} E_t\pi_{t+1}/\kappa & \text{for } j = 1 \\ 0 & \text{for } j \geq 2 \end{cases}. \tag{11}$$

In addition, Eqs. (1), (10), and (11) provide

$$E_t i_{t+j} = \begin{cases} -\sigma E_t\tilde{y}_{t+1} & \text{for } j = 1 \\ 0 & \text{for } j \geq 2 \end{cases}.$$

Summarizing the solutions from period $t + 1$ onward, for any value of fundamental shock ε in period t, if $\phi = \infty$ under $i_t = \phi E_t\pi_{t+2}$, the equilibrium is

$$\begin{cases} E_t\pi_{t+1} = \kappa E_t\tilde{y}_{t+1} = -\frac{\kappa}{\sigma}E_t i_{t+1}, \\ E_t\pi_{t+j} = E_t\tilde{y}_{t+j} = E_t i_{t+j} = 0 \text{ for } j \geq 2. \end{cases}$$

$E_t i_{t+1}$ has an arbitrary value due to a sunspot shock in period t that is irrespective of fundamental shock ε. This means that the equilibrium solution in period $t + 1$ has one-dimensional indeterminacy.

Further, given the value of $E_t\tilde{y}_{t+1}$, the solution in period t is expressed as functions of i_t as

$$\begin{cases} \pi_t = \kappa(1+\beta+\frac{\kappa}{\sigma})\, E_t\tilde{y}_{t+1} - \frac{\kappa}{\sigma} i_t, \\ \tilde{y}_t = (1+\frac{\kappa}{\sigma})\, E_t\tilde{y}_{t+1} - \frac{1}{\sigma} i_t. \end{cases}$$

Here, i_t also has an arbitrary value due to another sunspot shock in period t that is irrespective of fundamental shock ε. That is, the solution in period t has greater one-dimensional indeterminacy even if $E_t\tilde{y}_{t+1}$ is given.

Therefore, for any value of fundamental shock ε in period t, if $\phi = \infty$ under $i_t = \phi E_t\pi_{t+2}$, the equilibrium has *two-dimensional* indeterminacy.

D Derivation of a Sunspot Equilibrium

Our system is described in the manner employed by Sims (2002) as follows:

$$\Gamma_0 Y_t = \Gamma_1 Y_{t-1} + \Psi z_t + \Pi \eta_t$$

$$\text{where } Y_t = \begin{bmatrix} \tilde{y}_t \\ \pi_t \\ E_t\tilde{y}_{t+1} \\ E_t\pi_{t+1} \end{bmatrix},\ z_t = \varepsilon_t,\ \eta_t = \begin{bmatrix} \eta_t^y \\ \eta_t^\pi \end{bmatrix},$$

$$\Gamma_0 = \begin{bmatrix} 1 & 0 & -1 & (\phi-1)/\sigma \\ -\kappa & 1 & 0 & -\beta \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},\ \Gamma_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Psi = \begin{bmatrix} -1/\sigma \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \Pi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where η^x represents the forecast error of variable x (i.e., $x_t = E_{t-1}x_t + \eta_t^x$).

A solution to the fundamental equilibrium under determinacy is given by equation (44) of Sims (2002), and this solution corresponds to the solution of the fundamental equilibrium under indeterminacy that Lubik and Schorfheide (2003) define under the assumption of *orthogonality*. A solution to the sunspot equilibrium under indeterminacy is the sum of the above fundamental solution and the forecast error component $H \begin{bmatrix} Q_{1\cdot} - \Phi Q_{2\cdot} \\ 0 \end{bmatrix} \Pi\eta_t$, the notations of which are taken from Sims (2002). These solutions are computed using a MATLAB program ”gensys.m,” which is provided on Professor Sims’s homepage.

E Equilibrium under $i_t = E_t\pi_{t+1}$

Suppose that a sunspot shock causes households to believe that the inflation rate is non-zero and permanently constant; for example, $E_t\pi_{t+j} = \pi \neq 0$ for $j \geq 0$. Then, if

$\phi = 1$, the following smooth and non-converging sunspot equilibrium satisfies our equations:

$$E_t \pi_{t+j} = \pi,$$

$$E_t \tilde{y}_{t+j} = \frac{1-\beta}{\kappa}\pi,$$

$$E_t i_{t+j} = \pi \qquad \text{for } j \geq 0.$$

F Equilibrium under $i_t = \bar{\phi} E_t \pi_{t+1}$

Suppose that a sunspot shock causes households to believe that the absolutes of variables are non-zero and constant and that the signs of variables change period by period; for example, $E_t \pi_{t+j} = (-1)^j \pi \neq 0$ for $j \geq 0$. Then, if $\phi = \bar{\phi} = 1 + \frac{2\sigma(1+\beta)}{\kappa}$, the following cyclical and non-converging sunspot equilibrium satisfies our equations:

$$\pi_{t+j} = (-1)^j \pi,$$

$$\tilde{y}_{t+j} = (-1)^j \frac{1+\beta}{\kappa}\pi,$$

$$i_{t+j} = (-1)^j \left(1 + \frac{2\sigma(1+\beta)}{\kappa}\right)\pi \qquad \text{for } j \geq 0.$$

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