K-theoretic symmetric polynomials and noncommutative Schur functions

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Background

Grothendieck polynomials are a "K-theoretic variant" of Schubert polynomials, which represent (a class of) a Schubert variety in the K-theory of the Flag variety (cf. [11, 12], 1980's).

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For the Grassmannian case, Schubert varieties are represented by a symmetric polynomial. Such symmetric Grothendieck polynomial is seen as a K-analog of the Schur polynomial.

(When referring to the symmetric polynomials, we often cut the word "symmetric" and call them "Grothendieck polynomials" simply.)

Background

	Cohomology	K-theory
Flag variety	Schubert	Grothendieck
Grassmannian	Schur	(symmetric) Grothendieck

Grothendieck polynomials (Preceding Research)

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- Combinatorial aspects:
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  - Various generalizations that are “combinatorially reasonable” [17].

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- Combinatorial aspects:
  - Grothendieck polynomial is a "generating function" of set-valued tableaux [1, 17].
  - Various generalizations that are "combinatorially reasonable" [17].
- Representation theory:
  - Determinantial formulas (Weyl-type [4, 10, 17], Jacobi-Trudi-type [10, 13]).
  - K-theoretic crystal ("Krystal") [16].

Grothendieck polynomials (Preceding Research)

- K-theoretic symmetric polynomials are realized as a solution of various integrable systems:
    - Grothendieck polynomials are realized as an "inverse scattering solution" for solvable lattice models (TASEP, melting crystal) [14, 15].
    - Dual stable Grothendieck polynomials form a family of special solutions for a discrete Toda equation [9].

Motivation

Realize K-theoretic symmetric polynomials in a simpler way.

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Realize K-theoretic symmetric polynomials in a simpler way.
$\rightarrow$ Realize them as a vacuum expectation value of a fermionic operator.

Example (Combinatorial presentation)

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For $T \in ST^n(\lambda)$, put

$$x^T := \prod_{i=1}^{\infty} x_i^{\sharp(i'\mathrm{s\ in}\ T)}.$$

Schur polynomial

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in ST^n(\lambda)} x^T.$$

Example (Combinatorial presentation)

$\lambda = (2,1)$

$ST^3(\lambda) =$

$$\left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right\}.$$

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$s_{(2,1)}(x_1, x_2, x_3) =$

$$x_1^2x_2 + x_1x_2^2 + x_1x_2x_3 + x_1^2x_3 + x_1x_2x_3 + x_1x_3^3 + x_2^2x_3 + x_2x_3^2.$$

Basic facts

1. Schur polynomials are symmetric.
2. (elementary symmetric polynomial).

$$s_{(1^i)}(x_1, \ldots, x_n) = e_i(x_1, \ldots, x_n)$$

3. (complete symmetric polynomial).

$$s_{(i)}(x_1, \ldots, x_n) = h_i(x_1, \ldots, x_n)$$

Set-valued tableaux

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$\lambda = (2, 1)$

$SST^3(\lambda) =$

$$\left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,,\; \begin{array}{|c|c|}\hline 1 & 12 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,,\; \begin{array}{|c|c|}\hline 1 & 13 \\ \hline 23 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,,\; \begin{array}{|c|c|}\hline 12 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,,\; \begin{array}{|c|c|}\hline 1 & 123 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \ldots \right\}.$$

(27 elements).

Grothendieck polynomial

Definition

Define the (symmetric) Grothendieck polynomial $G_\lambda(x_1,\dots,x_n)$ as

$$G_\lambda(x_1,\dots,x_n) = \sum_{T\in SST^n(\lambda)} \beta^{ex(T)} x^T.$$

Here, β is a parameter and

$$ex(T) = \text{(number of numbers)} - \text{(number of boxes)}: \text{ exceed.}$$

Grothendieck polynomial

Example

$$
\begin{aligned}
&G_{(2,1)}(x_1, x_2, x_3) \\
&= (x_1^2 x_2 + x_1^2 x_3 + \cdots) + 2x_1x_2x_3 \\
&\qquad + \beta(x_1^2 x_2^2 + \cdots) + 3\beta(x_1^2 x_2 x_3 + \cdots) \\
&\qquad\qquad + 2\beta^2(x_1^2 x_2^2 x_3 + \cdots) + \beta^3 x_1^2 x_2^2 x_3^2
\end{aligned}
$$

(27 terms).

Basic examples

$\lambda = (1^3)$, $n = 4$

$$G_{(1^3)}(x_1, x_2, x_3, x_4)$$
$$= x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + 3\beta x_1x_2x_3x_4$$

If $\beta = 0$, it reduces to $s_{(1^3)}(x_1, x_2, x_3, x_4)$.

Facts

1. Grothendieck polynomials are symmetric.
2. When $\beta = 0$, they reduce to Schur polynomials.

Idea

Fermionic presentation of the Schur functions:

*Cf.) T. Miwa, M. Jimbo and E. Date, "***ソリトンの数理***(Solitons, Differential equations, symmetries and infinite dimensional algebras (En)) ," Iwanami Shoten, Tokyo, 1993 (Jp), Cambridge University Press, 2000 (En).*

Idea: Generalize this result to obtain a fermionic presentation of the Grothendieck polynomials.

Free fermions

Let $[A, B]_+ = AB + BA$.

Let $\mathcal{A}$ be the $\mathbb{C}$-algebra of free-fermions generated by $\{\psi_m, \psi_n^*\}_{m,n\in\mathbb{Z}}$:

$$[\psi_m, \psi_n]_+ = [\psi_m^*, \psi_n^*]_+ = 0, \qquad [\psi_m, \psi_n^*]_+ = \delta_{m,n}.$$

Schur polynomial as expectation value

vacuum vector $|0\rangle$, $\langle 0|$

$$\psi_m|0\rangle = \psi_n^*|0\rangle = 0, \quad \langle 0|\psi_n = \langle 0|\psi_m^* = 0, \qquad m < 0,\ n \geq 0.$$

For $m < 0,\ n \geq 0$,

$\psi_m,\ \psi_n^*$	annihilation operators
$\psi_n,\ \psi_m^*$	creation operators

Schur polynomial as expectation value

$\mathcal{F}$: Fock space, the left $\mathcal{A}$-module generated by

$$\psi_{n_1}\psi_{n_2}\cdots\psi_{n_r}\psi^*_{m_1}\psi^*_{m_2}\cdots\psi^*_{m_s}|0\rangle$$

for all $r, s \geq 0$, $n_1 > \cdots > n_r \geq 0 > m_s > \cdots > m_1$.

Schur polynomial as expectation value

$\mathcal{F}$: Fock space, the left $\mathcal{A}$-module generated by

$$\psi_{n_1}\psi_{n_2}\cdots\psi_{n_r}\psi^*_{m_1}\psi^*_{m_2}\cdots\psi^*_{m_s}|0\rangle$$

for all $r, s \geq 0$, $n_1 > \cdots > n_r \geq 0 > m_s > \cdots > m_1$.

$\mathcal{F}^\dagger$: Dual Fock space, the right $\mathcal{A}$-module generated by

$$\langle 0|\psi_{m_s}\cdots\psi_{m_2}\psi_{m_1}\psi^*_{n_r}\cdots\psi^*_{n_2}\psi^*_{n_1}$$

for all $r, s \geq 0$, $n_1 > \cdots > n_r \geq 0 > m_s > \cdots > m_1$.

Schur polynomial as expectation value

Fact/Definition

There exists a unique bilinear form

$$\mathcal{F}^\dagger \otimes_{\mathbb{C}} \mathcal{F} \to \mathbb{C}; \qquad \langle w| \otimes |v\rangle \mapsto \langle w|v\rangle$$

satisfying

1. $\langle 0|0\rangle = 1$,
2. $(\langle w|\psi_n)|v\rangle = \langle w|(\psi_n|v\rangle)$,
3. $(\langle w|\psi_n^*)|v\rangle = \langle w|(\psi_n^*|v\rangle)$.

For $X \in \mathcal{A}$,

$$\langle X\rangle := \langle 0|X|0\rangle \qquad \text{(vacuum expectation value of } X).$$

Vacuum expectation value

Example

$$\langle 0|\psi_0^*\psi_1^*\psi_1\psi_0|0\rangle$$

Vacuum expectation value

Example

$$
\begin{aligned}
& \langle 0|\psi_0^*\psi_1^*\psi_1\psi_0|0\rangle \\
= \; & \langle 0|\psi_0^*(1-\psi_1\psi_1^*)\psi_0|0\rangle
\end{aligned}
$$

Vacuum expectation value

Example

$$
\begin{aligned}
& \langle 0|\psi_0^*\psi_1^*\psi_1\psi_0|0\rangle \\
= \; & \langle 0|\psi_0^*(1-\psi_1\psi_1^*)\psi_0|0\rangle \\
= \; & \langle 0|\psi_0^*\psi_0|0\rangle - \langle 0|\psi_0^*\psi_1\psi_1^*\psi_0|0\rangle
\end{aligned}
$$

Vacuum expectation value

Example

$$\begin{aligned} & \langle 0|\psi_0^*\psi_1^*\psi_1\psi_0|0\rangle \\ = & \langle 0|\psi_0^*(1-\psi_1\psi_1^*)\psi_0|0\rangle \\ = & \langle 0|\psi_0^*\psi_0|0\rangle - \langle 0|\psi_0^*\psi_1\psi_1^*\psi_0|0\rangle \\ = & \langle 0|\psi_0^*\psi_0|0\rangle \end{aligned}$$

Vacuum expectation value

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= & \langle 0|\psi_0^*\psi_0|0\rangle - \langle 0|\psi_0^*\psi_1\psi_1^*\psi_0|0\rangle \\
= & \langle 0|\psi_0^*\psi_0|0\rangle \\
= & \langle 0|(1-\psi_0\psi_0^*)|0\rangle
\end{aligned}
$$

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= \; & \langle 0|(1-\psi_0\psi_0^*)|0\rangle \\
= \; & \langle 0|0\rangle - \langle 0|\psi_0\psi_0^*|0\rangle \\
= \; & 1.
\end{aligned}
$$

Vacuum expectation value

Example

$$\langle 0|\psi_0^*\psi_1^*\psi_2\psi_0|0\rangle$$

Vacuum expectation value

Example

$$
\begin{aligned}
& \langle 0|\psi_0^*\psi_1^*\psi_2\psi_0|0\rangle \\
= \; & -\langle 0|\psi_0^*\psi_2\psi_1^*\psi_0|0\rangle
\end{aligned}
$$

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= & \langle 0|\psi_2\psi_0^*\psi_1^*\psi_0|0\rangle
\end{aligned}
$$

Vacuum expectation value

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= & \langle 0|\psi_2\psi_0^*\psi_1^*\psi_0|0\rangle \\
= & \ 0.
\end{aligned}
$$

Vacuum expectation value

Wick's theorem

$$\langle \psi_{m_1} \cdots \psi_{m_r} \psi^*_{n_r} \cdots \psi^*_{n_1} \rangle = \det(\langle \psi_{m_i} \psi^*_{n_j} \rangle)_{1 \leq i,j \leq r}.$$

Schur polynomial as expectation value

Shifted vacuum vector

$$|m\rangle = \begin{cases} \psi_{m-1}\psi_{m-2}\cdots\psi_0|0\rangle, & m \geq 0, \\ \psi^*_m \cdots \psi^*_{-2}\psi^*_{-1}|0\rangle, & m < 0, \end{cases}$$

$$\langle m| = \begin{cases} \langle 0|\psi^*_0\psi^*_1 \dots \psi^*_{m-1}, & m \geq 0, \\ \langle 0|\psi_{-1}\psi_{-2} \dots \psi_m, & m < 0. \end{cases}$$

Schur polynomial as expectation value

$: \cdot :$ normal ordering (which moves annihilator operators to the right),
$p_n(x) = x_1^n + x_2^n + \cdots$ (n-th power sum)

Operators a_m, $H(x)$, $e^{H(x)}$: $\mathcal{F} \to \mathcal{F}$

- $a_m = \sum_{k \in \mathbb{Z}} : \psi_k \psi^*_{m+k} :$
- $H(x) = \sum_{n>0} \frac{p_n(x)}{n} a_n$
- $e^{H(x)} = 1 + H(x) + \frac{H(x)^2}{2!} + \frac{H(x)^3}{3!} + \cdots .$

Schur polynomial as expectation value

Fact

$$s_\lambda(x) = \langle 0|e^{H(x)}\psi_{\lambda_1-1}\psi_{\lambda_2-2}\cdots\psi_{\lambda_r-r}|-r\rangle.$$

A new fact

Let $\Theta : \mathcal{F} \to \mathcal{F}$ を

$$\Theta = \beta a_{-1} - \frac{\beta^2}{2} a_{-2} + \frac{\beta^3}{3} a_{-3} - \cdots .$$

Then, we have

Theorem (I [5] (2020))

$$G_\lambda(x) = \langle 0 | e^{H(x)} \psi_{\lambda_1 - 1} e^{\Theta} \psi_{\lambda_2 - 2} e^{\Theta} \cdots \psi_{\lambda_r - r} e^{\Theta} | -r \rangle .$$

A new fact

$$s_\lambda(x) = \langle 0| e^{H(x)} \psi_{\lambda_1-1}\psi_{\lambda_2-2}\cdots\psi_{\lambda_r-r}|-r\rangle$$
$$G_\lambda(x) = \langle 0| e^{H(x)} \psi_{\lambda_1-1} e^{\Theta} \psi_{\lambda_2-2} e^{\Theta} \cdots \psi_{\lambda_r-r} e^{\Theta} |-r\rangle.$$

Outline of the proof

Once you have an answer, the proof is easy.

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Calculate the vacuum expectation value by using Wick's theorem

$$\langle \psi_{m_1} \cdots \psi_{m_r} \psi^*_{n_r} \cdots \psi^*_{n_1} \rangle = \det(\langle \psi_{m_i} \psi^*_{n_j} \rangle)_{1 \le i,j \le r}$$

and compare it to the known determinantal formula [10, 13]:

$$G_\lambda(x) = \det \left(\sum_{m=0}^{\infty} \binom{i-1}{m} \beta^m h_{\lambda_i - i + j + m}(x) \right)_{1 \le i,j \le r} .$$

(An easy exercise of linear algebra.)

Outline of the proof

Let $[A, B] = AB - BA$.

If $[A, B]$ commutes with both A and B, then $e^A e^B = e^{[A,B]} e^B e^A$.

$$[\Theta, \psi_n] = \beta\psi_{n+1} - \frac{\beta^2}{2}\psi_{n+2} + \frac{\beta^3}{3}\psi_{n+3} - \cdots .$$

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If $[A, B]$ commutes with both A and B, then $e^A e^B = e^{[A,B]} e^B e^A$.

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We have

$$e^{k\Theta} \psi_n e^{-k\Theta} = \sum_{i=0}^{k} \binom{k}{i} \beta^i \psi_{n+i}. \qquad \text{(A)}$$

Outline of the proof

Applying (A) and Wick's theorem to

$$
\begin{aligned}
&\langle 0|e^{H(x)}\psi_{\lambda_1-1}e^{\Theta}\psi_{\lambda_2-2}e^{\Theta}\cdots\psi_{\lambda_r-r}e^{\Theta}|-r\rangle \\
&=\langle 0|e^{H(x)}\psi_{\lambda_1-1}(e^{\Theta}\psi_{\lambda_2-2}e^{-\Theta})(e^{2\Theta}\psi_{\lambda_3-3}e^{-2\Theta})\cdots \\
&\qquad\qquad (e^{r\Theta}\psi_{\lambda_r-r}e^{-r\Theta})e^{(r+1)\Theta}|-r\rangle,
\end{aligned}
$$

we have

$$
\det\left(\sum_{m=0}^{\infty}\binom{i-1}{m}\beta^m h_{\lambda_i-i+j+m}(x)\right)_{1\le i,j\le r}.
$$

Variants

We can obtain a fermionic presentation of

- $g_\lambda(x)$: dual stable Grothendieck polynomials [7, 5]
- $GQ_\lambda(x)$: K-theoretic Q-functions [8]
- $GP_\lambda(x)$: K-theoretic P-functions
- $gp_\lambda(x), gq_\lambda(x)$: dual K-theoretic P- and Q-functions
- Multi-Schur functions [6], etc...

Applications (“Ginata-reading” (ぎなた読み))

If we change the “punctuation”

$$\begin{aligned}&\langle 0|e^{H(x)}\psi_{\lambda_1-1}e^{\Theta}\psi_{\lambda_2-2}e^{\Theta}\cdots\psi_{\lambda_r-r}e^{\Theta}|-r\rangle\\&=\langle 0|e^{H(x)}\psi_{\lambda_1-1}(e^{\Theta}\psi_{\lambda_2-2}e^{-\Theta})(e^{2\Theta}\psi_{\lambda_3-3}e^{-2\Theta})\cdots\\&\qquad\qquad(e^{r\Theta}\psi_{\lambda_r-r}e^{-r\Theta})e^{(r+1)\Theta}|-r\rangle\end{aligned}$$

as

Applications (“Ginata-reading” (ぎなた読み))

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as

$$\begin{aligned}&\langle 0|e^{H(x)}e^{r\Theta}(e^{-r\Theta}\psi_{\lambda_1-1}e^{r\Theta})(e^{-(r-1)\Theta}\psi_{\lambda_2-2}e^{(r-1)\Theta})\cdots\\&\qquad(e^{-\Theta}\psi_{\lambda_r-r}e^{\Theta})|-r\rangle,\end{aligned}$$

Applications ("Ginata-reading" (ぎなた読み))

(...), we obtain the Hudson-Ikeda-Matsumura-Naruse-type formula [3]

$$G_\lambda(x) = \det\left(\sum_{s=0}^{\infty}\binom{i-j}{s}(-\beta)^s G_{\lambda_i-i+j+s}(x)\right)_{1\le i,j\le r}.$$

Application (non-commutative Schur function)

β-twisted Schur operator $u_1, u_2, \ldots$ acting on the space of Young diagrams as:

$$u_1 \cdot \ydiagram{3,1,1} = \ydiagram{4,1,1},$$

$$u_2 \cdot \ydiagram{3,1,1} = \ydiagram{3,2,1},$$

$$u_3 \cdot \ydiagram{3,1,1} = (-\beta) \cdot \ydiagram{3,2,2}.$$

Application (non-commutative Schur function)

Fact

The noncommutative family $\{u_1, u_2, \dots\}$ satisfies the Knuth relation

$$u_j u_i u_k = u_j u_k u_i \qquad (i \geq j > k),$$
$$u_i u_k u_j = u_k u_i u_j \qquad (i > j \geq k),$$

which is a sufficient condition to ensure that Fomin-Greene's theorem of noncommutative Schur functions [2] is applicable here.

Application (non-commutative Schur function)

Definition (Non-commutative Schur function [2])

$\{u_1, u_2, \ldots\}$: non-commutative elements satisfying the Knuth relation. Let

$$s_\lambda(u_1, u_2, \ldots) := \sum_{T \in ST(\lambda)} c(T),$$

where $c(T)$ is the column-reading word of T.

Application (non-commutative Schur function)

Definition (Non-commutative Schur function [2])

$\{u_1, u_2, \ldots\}$: non-commutative elements satisfying the Knuth relation. Let

$$s_\lambda(u_1, u_2, \ldots) := \sum_{T \in ST(\lambda)} c(T),$$

where $c(T)$ is the column-reading word of T.

Theorem (Fomin-Greene [2] (1998))

Noncommutative Schur polynomials commute with each other.

Application (non-commutative Schur function)

Define

$$G : \bigoplus_{\lambda:\text{partition}} \mathbb{Z}\cdot\lambda \to \Lambda; \qquad \sum_{\lambda} c_\lambda \cdot \lambda \mapsto \sum_{\lambda} c_\lambda G_\lambda(x).$$

Then we have

Proposition

1. $e_i(x)\cdot G_\lambda(x) = G(e_i(u_1,u_2,\dots)\cdot\lambda)$,
2. $s_\lambda(x)\cdot G_\mu(x) = G(s_\lambda(u_1,u_2,\dots)\cdot\mu)$.

(Proof)

1. Direct calculation using the fermionic presentation.
2. Use Fomin-Greene's theory.

Application (non-commutative Schur function)

Example

$$e_2(u_1, u_2, \dots) = u_2u_1 + u_3u_1 + u_3u_2 + u_4u_1 + u_4u_2 + u_4u_3 + \cdots .$$

$$\begin{aligned} e_2(x) &= e_2(x) \cdot G_\emptyset(x) \\ &= G_{1,1}(x) - 2\beta G_{1,1,1}(x) + 3\beta^2 G_{1,1,1,1}(x) - \cdots . \end{aligned}$$

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