Tropical method for proving plethysm

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Introduction

Today's talk is based on my papers

1. *Tropical integrable systems and Young tableaux: shape equivalence and Littlewood-Richardson correspondence*, Journal of Integrable Systems **3** (2018), no. 1, xyy011.
2. *Jeu de taquin, uniqueness of rectification and ultradiscrete KP*, Journal of Integrable Systems **4** (2019), no. 1, xyz012.

Young tableaux (Young 盤)

Young tableaux of shape λ (形 λ の Young 盤)

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline \end{array}, \qquad \lambda = (2, 2).$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 4 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}, \qquad \lambda = (4, 3, 1, 1).$$

- A Young tableau is obtained by putting a natural number in each box of a Young diagram $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$.
- Weakly increasing from left to right.
- Strongly increasing from top to bottom.

Skew Young tableaux (歪 Young 盤)

Skew Young tableaux of shape λ/μ

$$\begin{array}{|c|c|c|} \hline \ & 1 & 2 \\ \hline 3 & 3 & \\ \hline \end{array}, \qquad \lambda = (3,2),\ \mu = (1)$$

$$\begin{array}{|c|c|c|c|} \hline \ & \ & 1 & 4 \\ \hline \ & 2 & 2 & \\ \hline 1 & & & \\ \hline 3 & & & \\ \hline \end{array}, \qquad \lambda = (4,3,1,1),\ \mu = (2,1).$$

- Weakly increasing from left to right.
- Strongly increasing from top to bottom.

Tropical method

	Field	Tropical semi-field
Language	Language of fields	Language of semi-fields
Addition	$a + b$	$\min[a, b]$
Multiplication	$a \cdot b$	$a + b$
Division	a/b	$a - b$

Tropical method

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Addition	$a+b$	$\min[a,b]$
Multiplication	$a \cdot b$	$a+b$
Division	a/b	$a-b$
Polynomial	$\sum_i a_i x^i$	$\min_i[a_i + i \cdot x]$
Rational function	$\dfrac{\sum_i a_i x^i}{\sum_j b_j x^j}$	$\min_i[a_i + i \cdot x] - \min_j[b_j + j \cdot x]$

Tropicalization

Tropicalization

$$\{\text{Subtraction-free rational func.}\} \to \{\text{Tropical rational func.}\}$$

$$f = \frac{\sum_i a_i x^i}{\sum_j b_j x^j} \mapsto \widehat{f} = \min_i[a_i + i \cdot x] - \min_j[b_j + j \cdot x]$$

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1. $\widehat{f+g} = \min[\widehat{f}, \widehat{g}]$,
2. $\widehat{f \cdot g} = \widehat{f} + \widehat{g}$,
3. If f and g are subtraction-free, the composition $f \circ g$ is also subtraction-free:
$$\widehat{f \circ g} = \widehat{f} \circ \widehat{g}.$$

Combinatorics of Young tableaux

Today we consider the following two combinatorial algorithms:

1. Row insertion ($=$ Row bumping, "行挿入")
2. Jeu de taquin (A french word means "15 パズル")

Combinatorics of Young tableaux

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- The row insertion is characterized by a tropical recursion formula called Tropical tableaux (Bernstein-Kirillov [1] 2001, Noumi-Yamada [7] 2004), which is a tropical analogue of discrete Toda equation.
- Jeu de taquin is characterized by the tropical KP equation (Mikami [6] 2006, Katayama-Kakei [5] 2015), which is a tropical analogue of discrete KP equation.

Row insertion

Example

1	3	4	5	$\leftarrow 3$
2	4	6		
4				

- The procedure starts from the top row.

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- The procedure starts from the top row.
- Insert 3 to the rightmost "possible position" as far as it does not violate the rule of Young tableau.

Row insertion

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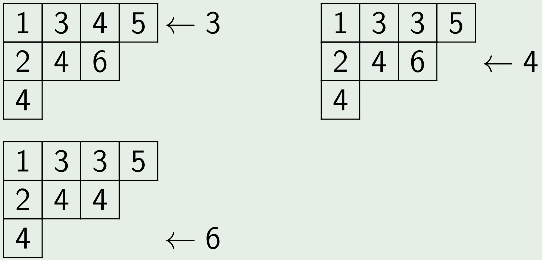

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- The “bumped” number 4 proceeds to the next row.

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- Insert 3 to the rightmost "possible position" as far as it does not violate the rule of Young tableau.
- The "bumped" number 4 proceeds to the next row.

Tropical tableaux

Tropical tableaux (Cf. [1],[7])

The row insertion to a one row tableau is characterized by the tropical recursion formula

$$\begin{cases} y_i + b_i = a_i + x_i \\ b_{i+1} = a_{i+1} + X_{i+1} - X_i \\ X_i := \min_{r=1}^{i}[\sum_{k=1}^{r-1}(x_{i-k+1} - a_{i-k})] \end{cases}$$

where x_i (*resp.* y_i) is the number i's in the tableau before (*resp.* after) the insertion, and

$$a_i = \begin{cases} 1 & \text{(if } i \text{ is inserted)} \\ 0 & \text{(otherwise)} \end{cases} \qquad b_i = \begin{cases} 1 & \text{(if } i \text{ is bumped)} \\ 0 & \text{(otherwise)} \end{cases}$$

Tropical tableaux

Example

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 7 \\ \hline 2 & 4 & 8 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array} \leftarrow 3 \quad \overset{\text{row insertion}}{\Longrightarrow} \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 3 & 7 \\ \hline 2 & 4 & 8 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array} \;\; \leftarrow 6$$

Tropical tableaux

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$$(x_i)_i = (1, 0, 1, 0, 0, 1, 1, 0, \dots), \qquad (y_i)_i = (1, 0, 2, 0, 0, 0, 1, 0, \dots),$$

$$(a_i)_i = (0, 0, 1, 0, 0, 0, 0, 0, \dots), \qquad (b_i)_i = (0, 0, 0, 0, 0, 1, 0, 0, \dots),$$

$$\begin{cases} a_i + x_i = y_i + b_i, & (\text{“}\sharp\{i\text{'s}\} \text{ remains unchanged”}) \\ b_{i+1} = a_i + X_{i+1} - X_i, & \\ X_i = \min_{r=1}^{i}[\sum_{k=1}^{r-1}(x_{i-k+1} - a_{i-k})]. & \end{cases}$$

Tropical tableaux

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$X_1 = \min[0] = 0,$

Tropical tableaux

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$X_1 = \min[0] = 0,\ X_2 = \min[0, x_2 - a_1] = 0,$

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$X_1 = \min[0] = 0,\ X_2 = \min[0, x_2 - a_1] = 0,$
$X_3 = \min[0, x_3 - a_2, x_3 + x_2 - a_2 - a_1] = 0,$

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$$(X_i)_i = (0,0,0,-1,-1,0,0,0,\dots)$$

The equation $b_{i+1} = a_i + X_{i+1} - X_i$ matches with the combinatorial description of the row insertion.

Ultradiscrete Toda equation

Noumi-Yamada [7] showed that the row insertion algorithm is obtained from the discrete Toda equation; $(x_i, a_i) \mapsto (y_i, b_i)$

$$\begin{cases} y_i b_i = a_i x_i \\ y_i + b_{i+1} = a_i + x_{i+1} \quad (i \geq 2) \\ b_2 = a_1 + x_2 \end{cases}$$

by ultradiscretization (= tropicalization).

Ultradiscrete Toda equation

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$$b_2 = a_1 + x_2 = a_1 \cdot (1 + \frac{x_2}{a_1})$$

$$b_3 = a_2 + x_3 - y_2 = a_2 + x_3 - \frac{a_2 x_2}{a_1 + x_2} = a_2 \cdot \frac{1 + \frac{x_3}{a_2} + \frac{x_3 x_2}{a_2 a_1}}{1 + \frac{x_2}{a_1}}$$

$$b_4 = a_3 + x_4 - y_2 = \cdots = a_3 \cdot \frac{1 + \frac{x_4}{a_3} + \frac{x_4 x_3}{a_3 a_2} + \frac{x_4 x_3 x_2}{a_3 a_2 a_1}}{1 + \frac{x_3}{a_2} + \frac{x_3 x_2}{a_2 a_1}}$$

Ultradiscrete Toda equation

Discrete Toda equation is rewritten as

$$\begin{cases} y_i b_i = a_i x_i \\ b_{i+1} = a_i \cdot \dfrac{\sum_{r=1}^{i+1} \prod_{k=1}^{r-1} \frac{x_{i-k+2}}{a_{i-k+1}}}{\sum_{r=1}^{i} \prod_{k=1}^{r-1} \frac{x_{i-k+1}}{a_{i-k}}} \end{cases}$$

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Tropicalizing this, we have

$$\begin{cases} a_i + x_i = y_i + b_i, \\ b_{i+1} = a_i + X_{i+1} - X_i, \\ X_i = \min_{r=1}^{i}[\sum_{k=1}^{r-1}(x_{i-k+1} - a_{i-k})]. \end{cases}$$

Row insertion as a tropical map

	Field	Tropical
x_i	input at a discrete time t	$\sharp$(i's in a row)
a_i	"	$\sharp$(i's to be inserted)
y_i	output at a discrete time t	$\sharp$(i's in a row after insertion)
b_i	"	$\sharp$(i's proceed to the next row)
	$\begin{cases} a_i x_i = y_i b_i \\ a_i + x_{i+1} = y_i + b_{i+1} \\ b_2 = a_1 + x_2 \end{cases}$ (discrete Toda equation)	$\begin{cases} a_i + x_i = y_i + b_i \\ b_{i+1} = a_i + X_{i+1} - X_i \end{cases}$ (Row insertion)

(cf. Noumi-Yamada [7])

Jeu de taquin

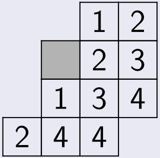
Example

		1	2
		2	3
	1	3	4
2	4	4	

- Choose an "inside corner."

Jeu de taquin

Example



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Example

		1	2
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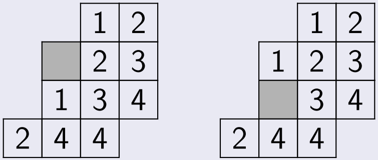
- Choose an "inside corner."
- Compare numbers on the right and below, and slide the smaller one. (If they are equal, move the below one).

Jeu de taquin

Example



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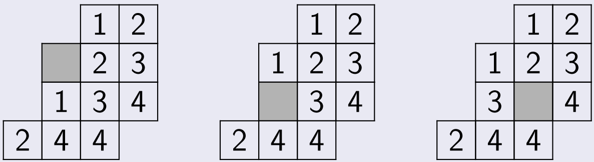

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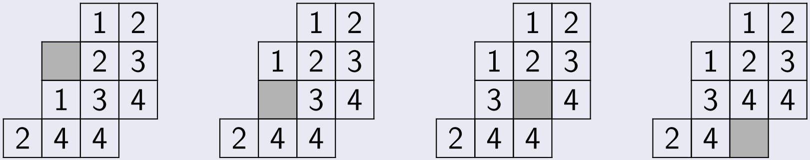

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Jeu de taquin as a tropical map

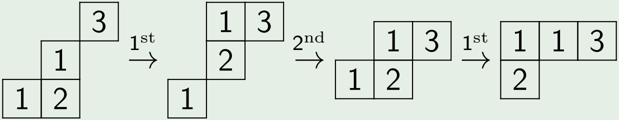
	Field	Tropical
$f_{i,j}^t$	input of the discrete KP eq.	$\sharp\begin{pmatrix} 0,1,2,\ldots,j\text{'s contained} \\ \text{in the } 1^{\text{st}},2^{\text{nd}},\ldots,i^{\text{th}} \text{ rows} \\ \text{after the } t^{\text{th}} \text{ move}\end{pmatrix}$
	$f_{i,j}^t f_{i,j+1}^{t+1} = f_{i+1,j+1}^t f_{i-1,j}^{t+1} + f_{i,j+1}^t f_{i,j}^{t+1}$ (discrete KP equation)	$F_{i,j}^t + F_{i,j+1}^{t+1} = \max[F_{i+1,j+1}^t + F_{i-1,j}^{t+1}, F_{i,j+1}^t + F_{i,j}^{t+1}]$ (Jew de taquin)

(cf. Katayama-Kakei [5])

Rectification

A rectification of a skew tableau is a (non-skew) Young tableau that is obtained by a sequence of Jeu de taquin.

Example



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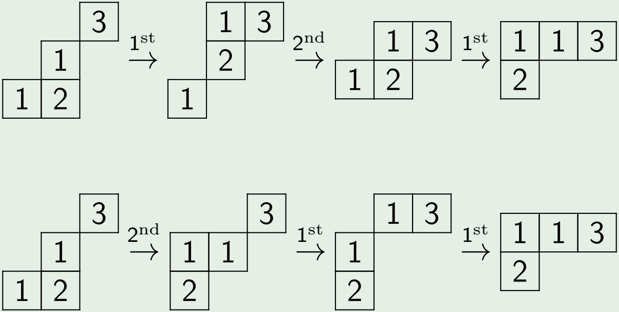

Rectification

Theorem (Uniqueness of rectification, cf.[2])

A rectification of a skew tableau does not depend on the choice of a sequence of jeu de taquin.

Combinatorial proofs are well-known (cf. Fulton's textbook "Young Tableau [2]")

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A rectification of a skew tableau does not depend on the choice of a sequence of jeu de taquin.

Combinatorial proofs are well-known (cf. Fulton's textbook "Young Tableau [2]")

Today I give (an outline of) an alternative proof of this theorem by using tropical integrable systems.

Outline

Language of fields

$p : A \to B \qquad$ (Subtraction-free rational map)

(Geometrization) $\Uparrow \quad \Downarrow$ (Tropicalization)

Language of semi-fields

$$\widehat{p} : \widehat{A} \to \widehat{B}$$

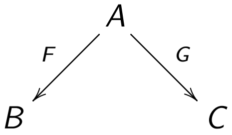
(A piecewise linear map obtained from p by $\begin{cases} + \mapsto \min \\ \times \mapsto + \end{cases}$)

Main trick

Language of fields

⇑ ⇓

Language of semi-fields (=combinatorial objects)



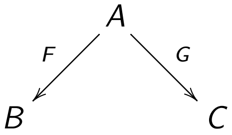
F, G: Maps that are defined combinatorially (e.g. row insertion, jeu de taquin)

Main trick

Language of fields

(Lift) $\Uparrow$ $\Downarrow$

Language of semi-fields (=combinatorial objects)



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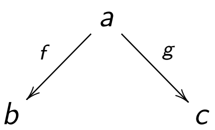
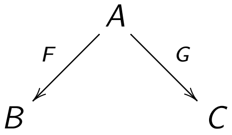


$\widehat{f} = F, \ \widehat{g} = G$

(Lift) $\Uparrow$ $\quad\Downarrow$

Language of semi-fields (=combinatorial objects)



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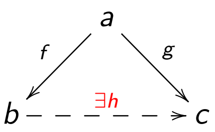

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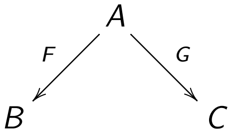
h: Subtraction-free

(Lift) $\Uparrow$ $\Downarrow$

Language of semi-fields (=combinatorial objects)



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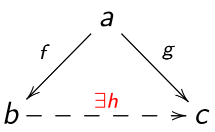

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(Lift) $\Uparrow$ $\Downarrow$ (Tropicalize)

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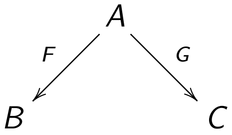

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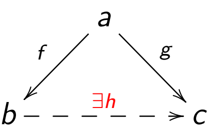

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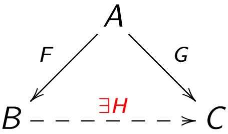

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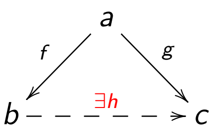

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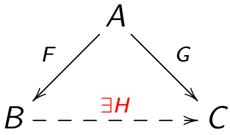
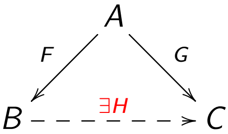


F, G: Maps that are defined combinatorially (e.g. row insertion, jeu de taquin)

H: Well-defined map

Main trick

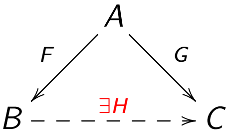


F, G: Maps that are <u>well-known</u>

H: A map <u>just defined</u>

I am happy enough <u>because</u>:

Main trick



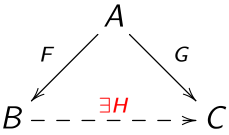
F, G: Maps that are well-known

H: A map just defined

I am happy enough because:

"There exists a map H that makes the diagram commutative."

Main trick



F, G: Maps that are <u>well-known</u>

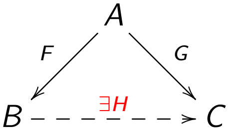
H: A map <u>just defined</u>

I am happy enough <u>because</u>:

"There exists a map H that makes the diagram commutative."

$=$ "The procedure of taking an inverse image of F and sending it by G does not depend on the choice of the inverse image. "

Main trick



F, G: Maps that are <u>well-known</u>

H: A map <u>just defined</u>

I am happy enough <u>because</u>:

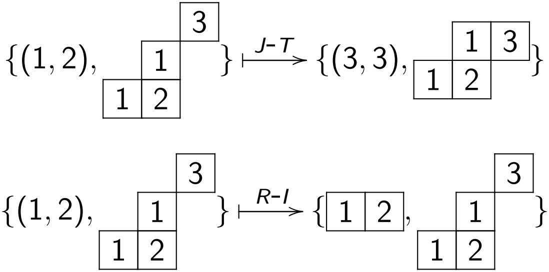
"There exists a map H that makes the diagram commutative."

$=$ "The procedure of taking an inverse image of F and sending it by G does not depend on the choice of the inverse image. "

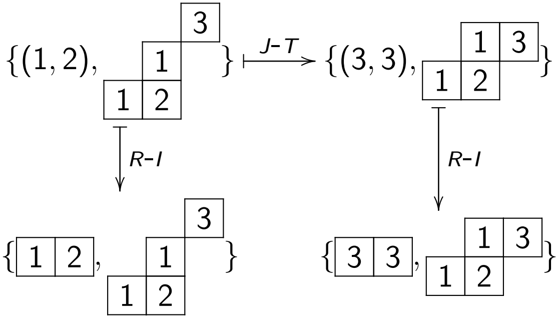
"Is the procedure X indepentent of the choice of Y ?" is a FAQ for those who study the combinatorics of Young tableaux.

Outline of proof

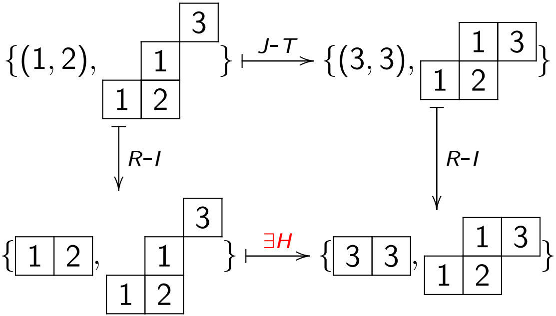
Previous researches has shown that the two maps
J-T (Noumi-Yamada 2004) and *R-I* (Katayama-Kakei 2015)
are given by a tropical rational map.

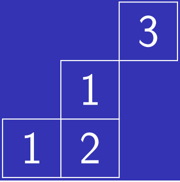


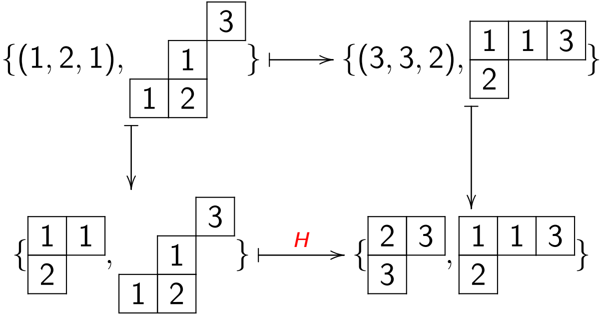
Outline of proof

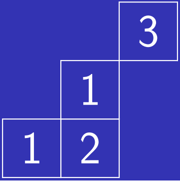


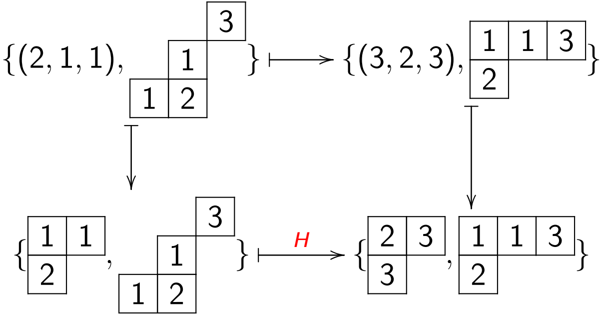
Outline of proof



Two rectifications of  (I).



Two rectifications of  (II).



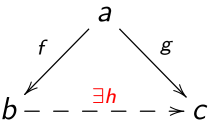
Uniqueness

In each case, we have

$$H\left(\begin{ytableau} 1 & 1 \\ 2 \end{ytableau}\,,\ \begin{ytableau} \none & \none & 3 \\ \none & 1 \\ 1 & 2 \end{ytableau}\right) = \left(\begin{ytableau} 2 & 3 \\ 3 \end{ytableau}\,,\ \begin{ytableau} 1 & 1 & 3 \\ 2 \end{ytableau}\right).$$

This is an essential reason why a rectification is unique.

Note



The subtraction-free map h is constructed by using the factorization of totally positive matrices, which is a popular technique in the theory of discrete integrable system (cf. Iwao [3, 4])

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[3] Shinsuke Iwao, *Tropical integrable systems and Young tableaux: shape equivalence and Littlewood-Richardson correspondence*, Journal of Integrable Systems **3** (2018), no. 1, xyy011.

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[4] ———, *Jeu de taquin, uniqueness of rectification and ultradiscrete KP*, Journal of Integrable Systems **4** (2019), no. 1, xyz012.

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[7] Masatoshi Noumi and Yasuhiko Yamada, *Tropical Robinson-Schensted-Knuth correspondence and birational Weyl group actions*, Representation theory of algebraic groups and quantum groups (T. Shoji, M. Kashiwara, N. Kawanaka, G. Lusztig, and K. Shinoda, eds.), vol. 40, Soc. Japan, Tokyo, 2004, pp. 371–442.