A First-Order Sequent Calculus for Logical Inferentialists and Expressivists

SHUHEI SHIMAMURA[1]

Abstract: I present a sequent calculus that extends a nonmonotonic reflexive consequence relation as defined over an atomic first-order language without variables to one defined over a logically complex first-order language. The extension preserves reflexivity, is conservative (therefore nonmonotonic) and supraintuitionistic, and is conducted in a way that lets us codify, within the logically extended object language, important features of the base thus extended. In other words, the logical operators in this calculus play what Brandom (2008) calls expressive roles. Expressivist logical systems have already been proposed for propositional logics (see Hlobil, 2016, and Kaplan, 2018) but not for first-order logics. An advantage of this calculus over standard first-order calculi (e.g., those in Gentzen, 1935/1964) is that universally quantified variables behave as they should even in the presence of arbitrary nonlogical axioms. I claim that because of this robust well-behavedness of variables, this calculus also provides logical inferentialists with a way to understand the meanings of variables in terms of the roles those variables play in a wide range of inferences that is not limited to purely logical ones (e.g, mathematical inferences).

Keywords: nonmonotonic logic, first-order logic, sequent calculus, logical inferentialism, logical expressivism

1 Introduction: Two philosophical motivations

1.1 Logical inferentialism

Variables seem to play an essential role in various phases of our linguistic practices. This is most evident in mathematical practices, where we explic-

[1]This paper is the product of a joint work with the research group of Bob Brandom. The technical results reported here are mine. For valuable discussions and comments, I thank Bob Brandom, Ulf Hlobil, Dan Kaplan, Rea Golan, Stephen Mackereth, Mansooreh Kimiagari, Adrian Anhalt-Gutierrez, Yao Fan, and the audience of Logica 2018 conference and several other conferences and workshops at which earlier versions of this paper were presented.

itly employ variables for particular aims that do not seem accomplishable by other means. For example, when we want to prove that for all right-angled triangles, the square of the hypotenuse is equal to the sum of the squares of the other two sides, it seems necessary to reason with an arbitrary right-angled triangle, say, t. Similarly, when we want to specify an equation or a function, it seems inevitable for us to use variables, "x", "y", "z", etc., to talk about those arbitrary relata that are related in a particular manner within the equation or function at issue. Here, variables seem to play a distinctive role—they function as though referring to arbitrary objects.

However, it is not only in mathematics that we need to talk or think about arbitrary objects; such occasions are prevalent in our ordinary linguistic practice. Consider, for instance, how we explain the meaning of "match" to children. We say something like "if you strike a match, it lights." We are talking about neither this or that match, but rather an arbitrary match (otherwise, this explanation would be of little use). Thus, although the original sentence does not explicitly contain a variable, it seems to say something that is more explicitly said by using a variable: "For any x, if x is a match and you strike x, then x lights."

Variables seem to let us talk about objects without specifying them. This seemingly distinctive function of variables, however, perplexed Bertrand Russell—one of the first philosophers to notice the great potential use of variables for the analysis of natural-language sentences (see, e.g., Russell, 1905). This is because he also clung to the view that the meaning of an expression is specified in terms of what it represents or refers to. What then does a variable, say "x," mean? This question confronts Russell with a formidable dilemma (see, e.g., Russell, 1994, p. xxxv). Apparently, no particular object counts as the proper referent of "x." If the meaning of "x" is specified by its referent, it follows that "x" has no specifiable meaning. Or one may bite the bullet here and claim that there exist arbitrary objects along with usual particular ones, and "x" refers to one of them. This horn, however, immediately invites many tough questions, such as where and how such arbitrary objects exist, how they can ever be distinguished from each other despite their arbitrariness, and so on.[2] Thus, both horns appear difficult to grasp. Let us call this *Russell's dilemma of the meaning of variables*.

Several attempts have been made to solve Russell's dilemma by seeking an account of the meanings of variables while maintaining his representa-

[2]Frege (1979, p. 160), for instance, expresses his doubts about the notion of arbitrary objects.

tionalist assumption that meanings are explained in terms of references (e.g., Fine, 1985, 2007). However, one of the two main aims of this paper is to propose a different way out of this dilemma. It seems to me quite natural to regard variables as (parts of) logical operators. According to some, the meanings of logical operators may be explained more naturally by looking at the rules governing their proper inferential use than by looking for things that they might refer to. A common example is the conjunction. It is notoriously difficult to seek the referent of the conjunction. Nevertheless, we all seem to know what it means. What do we explain, then, when we explain the meaning of the conjunction? A natural answer seems to be the rule governing its proper use—in particular, the rule governing what a conjunctive is properly inferred from and what is properly inferred from it. This line of thought is sometimes called *logical inferentialism*.[3] What I pursue in this paper is a logical inferentialist approach to Russell's dilemma of the meanings of variables. That is, I explain the meanings of variables by looking for a set of rules governing their proper inferential use, rather than for what they might refer to.

One may wonder, though, if we already have such rules, because due to Gentzen (1935/1964) and his successors, there are suitable proof systems for various first-order logics in which such rules are conveniently isolated for the variable-involving logical operators.

Somewhat surprisingly, however, it is difficult to find a proof system in the literature that can fulfill the current aim. It seems that in order to evaluate whether given rules for the universal quantifier—one of the major variable-involving logical operators—do justice to its intuitive meaning, it is an essential criterion that those rules guarantee the following biconditional: Γ implies $\forall x A \Leftrightarrow$ for any a: Γ implies $A[a/x]$, where x does not freely occur in Γ. Let us call this *the universal principle*. As far as I know, the universal rules of most proof systems guarantee this principle only within the limits of purely logical inferences—namely, only under the condition that they are free from proper axioms (I discuss why in the next section). As the examples mentioned at the beginning of this paper illustrate, however, variables seem to play an essential role not only in purely logical inferences, but also in mathematical or even more casual inferences. Thus, logical in-

[3]See, e.g., Peregrin, 2014. The possibility of logical inferentialism, which is inspired by the famous remark of Gentzen (1935/1964, p. 295) on his natural deductions ("The introductions represent, as it were, the 'definitions' of the symbols concerned . . . "), has been investigated by many both philosophically and technically. Peregrin (ibid. pp. 3-6) offers a nice overview of this research tradition that covers most recent works.

ferentialists who wish to systematically explain the meanings of variables in these wider contexts[4] need a new proof system equipped with a set of rules that guarantees the well-behavedness of universals more robustly.

1.2 Logical expressivism

The second important aim of this paper is to submit a proof system that can also help advance the enterprise of *logical expressivism*, proposed and developed by Brandom (1994, 2000, 2008). Once we widen our focus from purely logical inferences to more casual ones, we may also be able to widen the scope of inferentialism: we may be able to explain not only the meanings of logical vocabulary but also those of nonlogical vocabulary in terms of the roles they play in the widened range of inferences.[5] If the meanings of logical and non-logical vocabulary are explained alike in terms of their inferential roles, it may be wondered how these two types of vocabulary can be distinguished in the first place. Logical expressivism is an inferentialist answer to this demarcation problem of logic.

Brandom (2009, p. 11) aspires to characterize logical vocabulary as "the organ of semantic self-consciousness"—the organ that is potentially available to anyone who can talk and that, once actualized, lets one talk about what one means when one talks. In Brandom (2008, pp. 52–54), this slogan is cashed out as two conditions for a piece of vocabulary to count as logical. First, the inferential roles of sentences involving that would-be logical operator can be mechanically determined on the basis of the inferential roles of sentences without it. Thus, logical vocabulary comes for free, as it were, for anyone who master nonlogical vocabulary. This condition, which Brandom calls *algorithmic elaboration*, is shown to be met by providing a proof system for a language containing the would-be logical operator that systematically determines the inferential roles of all the sentences involving that operator on the basis of those of atomic sentences. Second, the would-be logical operator must let us codify, without disturbing it, some aspect of the inferential role of the nonlogical vocabulary. A typical instance of this role, which Brandom calls *explicitation* or *expression*, can be played by the conditional. In a proof system in which the so-called deduction theorem and its converse hold (i.e., Γ implies $A \to B \Leftrightarrow \Gamma \cup \{A\}$ implies B), the conditional lets us codify, within the object language, the information that

[4] It may be worth stressing here that Gentzen (1935/1964) himself explicitly states his intention to use his proof systems for the analysis of inferences in mathematics (p. 288, p. 291).

[5] Strictly speaking, this is what Brandom (2000, p. 28) calls "weak inferentialism."

one thing (i.e., A) implies another (i.e., B) in a given context (i.e., Γ). In this way, the conditional lets us talk about implications, which, according to inferentialism, (partly) constitute the meanings of the sentences invovled. The nondisturbance proviso is also met if the proof system is conservative.

The ambition of logical expressivists as sketched above imposes several requirements on a proof system. First, since logical expressivists want to talk about the meanings of nonlogical (as well as logical) vocabulary (such as "match"), it needs to deal with nonlogical (as well as logical) inferences in terms of which those meanings are supposed to be understood. For instance, to understand the meaning of "match," it seems that one needs to understand, say, that "a is a match" and "a is struck" jointly implies, other things being equal, "a lights." As this example illustrates, inferences of this type, which are sometimes called "material inferences,"[6] often seem to be defeasible, and therefore nonmonotonic. Thus, a logical expressivist's proof system must be able to accommodate such a nonmonotonic material consequence relation as proper axioms, and to conservatively extend it to a logically complex one. Let us call this constraint *Nonmonotonicity*. Second, as illustrated above, the deduction theorem and its converse are unnegotiable for logical expressivists, because otherwise the conditional cannot play its expressive role to codify an implication. Similar biconditional constraints are imposed on the negation and other logical operators that are supposed to express different aspects of the underlying material consequence relation. Let us call this constraint *Expressivity*.[7]

There are already several nonmonotonic proof systems equipped with logical operators playing different expressive roles, such as expressing implication, incoherence, local monotonicity (see, e.g., the original work by Hlobil, 2016 for a supraintuitionistic system and its extension—with several improvements—to a supraclassical system proposed by Kaplan, 2018). One limitation of these systems, however, is that they are all propositional logics. From the logical expressivist viewpoint, this means that we can only talk about the meanings of entire sentences (e.g., "a is a match"), but we cannot yet talk about the meanings of their component expressions (e.g., "match"). For if we want to talk specifically about the meaning of, say, "match," it seems that we must say something like this: "For any x, if x is a match and x is struck, then x lights." Given the universal principle, the universally quantified variable "x" here codifies, roughly, that the pattern

[6]See, e.g., Sellars (1953) and Brandom (1994, 2000, 2008).

[7]For a more rigrous and generalized characterization of *Expressivity*, see Kaplan (2018, sec. 2).

of inference stated above holds irrespective of the individual term involved. This operator thus seems to let us talk purely about the inferential role of "is a match" (in connection with the other predicates such as "strike" and "lights") while bracketing that of "a".

So far, we have seen how the two philosophical ideas, logical inferentialism and logical expressivism, motivate us to pursue a proof system in which the universal principle robustly holds even in the presence of arbitrary proper axioms. In the next section, I explain why most of the currently available proof systems fail to satisfy this demand. In section 3, I introduce my alternative nonmonotonic proof system. Finally, in section 4, I prove that this system has several desirable properties, such as preserving reflexivity, being conservative, and guaranteeing the universal principle even in the presence of arbitrary proper axioms, along with other such biconditional properties demanded by *Expressivity*. In that section, I also show that the system is supraintuitionistic.

2 The problem

Most of the currently available proof systems fail to assure the universal principle in the presence of some proper axioms. Although logical inferentialism has usually been pursued within the framework of natural deductions, I focus on sequent calculi because they can make the problem at issue more straightforwardly visible. As long as the different frameworks of proof systems capture the same logic, however, problems shown to occur in one must also occur in the other. Furthermore, as will be seen below, the problem arises easily from a few common features shared by many different logics, including classical, intuitionistic, relevant, and modal logics. Thus, I claim, the scope of the problem is quite wide.

In sequent calculi of various logics, the inferential contributions from the universal quantifier are usually specified by the following pair of left and right rules[8]:

$$\frac{\Gamma, A[\tau/\xi] \mid\!\sim \Theta}{\Gamma, \forall\xi A \mid\!\sim \Theta}\ \text{L}\forall \qquad \frac{\Gamma \mid\!\sim A[\zeta/\xi], \Theta}{\Gamma \mid\!\sim \forall\xi A, \Theta}\ \text{R}\forall$$

where no ζ freely occurs in Γ, $\forall\xi A$, or Θ

[8]Note that below I use the snake turnstile (i.e., $\mid\!\sim$) instead of the regular turnstile (i.e., $\vdash$) to indicate that the consequence relation at issue can be nonmonotonic.

The additional clause of the right universal rule is the so-called "eigenvariable condition," which is supposed to secure the "arbitrariness" of the variable substituted (i.e., "ζ") and thereby justify its substitution by the universally quantified variable (i.e., "ξ"). Notice that in the standard setting it is only via free occurrences of eigenvariables that we can introduce universal quantifiers on the right. How, then, can free eigenvariables be introduced in the first place? In the standard systems, there is no inference rule for introducing a free variable. Thus, they must come from axioms. And this is where the problem arises.

To see how, suppose that we are given a sequent calculus with a set of inferential rules including the ones mentioned above and a set of logical axioms. Also suppose that our language contains two predicates, say P and Q, standing for "is a bachelor" and "is unmarried," respectively. Given this translation, it must hold that for each individual constant a: $Pa \mathrel{|\!\sim} Qa$. So let us add these implications as proper atomic axioms (i.e., these are *material* inferences in which the meanings of "P" and "Q" partly consist). Now, given the standard right rule of the conditional, it follows that for each individual constant a: $\mathrel{|\!\sim} Pa \rightarrow Qa$. However, because in the standard setting any open sequents in which free eigenvariables occur must come from logical axioms (i.e., those implications that are formally valid irrespective of the expressions involved in them, such as tautologies [i.e., $A \mathrel{|\!\sim} A$] or the explosion [i.e., $\bot \mathrel{|\!\sim}$]), there is no way to derive the sequent in which the corresponding free eigenvariable occurs: $\mathrel{|\!\sim} Py \rightarrow Qy$. And because it is only via such an open sequent that we can derive the target sequent with the corresponding universal on the right—i.e., $\mathrel{|\!\sim} \forall x(Px \rightarrow Qx)$—, the left-to-right direction of the universal principle fails.

There are two possible routes to blocking this underproduction problem: adding more open sequents or modifying the right universal rule. One immediate response from the first route would simply be to add the required open sequent as an extra proper axiom: $Py \mathrel{|\!\sim} Qy$. After all, one may think, given that for any a, $Pa \mathrel{|\!\sim} Qa$ is materially good, $Py \mathrel{|\!\sim} Qy$ must also be materially good, and therefore count as a proper axiom. This option is, however, not available to us as logical inferentialists with respect to variables. Our aim is to *explain* the meaning of a variable in terms of the rules governing its inferential use. Yet, if we were simply to stipulate a crucial aspect of variable use as above in the form of a proper axiom, we would rather *exploit* our presupposed understanding of its meaning than *explain* that meaning. The aspect in question should be derived from some fundamental rule(s) governing the inferential use of the variable in question (i.e.,

"y") instead of being simply stipulated.

Before turning to my own solution, I should also mention the response from the second route. One may think that the underproduction problem can also be straightforwardly solved by letting the right rule for the universal "look directly at" the material inferences—the inferences in which the meanings of individual constants and predicates such as "a," "P," and "Q" (partly) consist. After all, if $\mid\!\sim Pa \to Qa$ for an arbitrary a, it ipso facto seems plausible to derive $\mid\!\sim \forall x(Px \to Qx)$. Thus, the following modified right universal rule may suggest itself:

$$\frac{\Gamma \mid\!\sim A[\alpha/\xi], \Theta}{\Gamma \mid\!\sim \forall \xi A, \Theta}\ \text{R}\forall'$$

where no α occurs in Γ or Θ,

where the modified eigenvariable condition at the bottom is supposed to ensure the "arbitrariness" of the relevant constant α. A serious problem with this solution, however, is that in the presence of proper axioms, the modified condition no longer ensures the arbitrary substitutability of α at all. For that matter, the eigenvariable condition and any of their variants can play its intended role only under the assumption that each term has exactly the same inferential potential (i.e., $\Gamma[\tau_1/\xi] \mid\!\sim \Theta[\tau_1/\xi] \Leftrightarrow \Gamma[\tau_2/\xi] \mid\!\sim \Theta[\tau_2/\xi]$). This holds as long as we limit our focus to logical inferences in which the inferential potential that is characteristic of individual constants is generally ignored; but that assumption may no longer hold once we take such potential into account by adding the relevant proper axioms. Suppose, for instance, that b, c, and R stand for "Tokyo," "Japan," and "is hit by a typhoon," respectively. Given this translation, the following implication seems to count as a proper axiom: $Rb \mid\!\sim Rc$. However, in the presence of this axiom, the modified right universal rule presented above lets us derive an obviously invalid implication: $Rb \mid\!\sim \forall x Rx$ (note that the modified eigenvariable condition is satisfied here as no c occurs in Rb).

In my view, the real culprit of the problem of underproduction is not the right universal rule, but the fact that the standard systems are devoid of any inferential rules that govern the use of variables in such a way as to make them mean what they mean—any inferential rules that let us derive the set of new open sequents that are not formally valid but materially good given the inferential potential of the other expressions (encapsulated in the set of proper axioms). In the next two sections, I demonstrate how such rules can be built into a supraintuitionistic sequent calculus that meets *Nonmonotonicity* and several conditions of *Expressivity*.

3 The system

Let $\mathcal{L}_0$ be the atomic language with the bottom ("$\bot$"), Var the set of all the variables, and Con the set of all the individual constants, where I assume that Con is finite. Let $\mathcal{L}_{0-} = \mathcal{L}_0 - \{\bot\}$, and $\mathcal{L}^c_{0[-]}$ is the largest closed subset of $\mathcal{L}_{0[-]}$ (i.e., $\mathcal{L}^c_{0[-]} = \{p \mid p \in \mathcal{L}_{0[-]}$, and p is a closed sentence$\}$. Note that the square brackets are used here to indicate that the bracketed element (i.e., "$-$") is optional (thus, here I set $\mathcal{L}^c_0$ and $\mathcal{L}^c_{0-}$ at a stroke). For notational convenience, I shall often use such square brackets.

Next, let $\mid\!\sim_0$ be a material consequence relation over $\mathcal{L}^c_0$. Because $\mid\!\sim_0$ is *material*, $\mid\!\sim_0$ varies depending on what particular vocabulary $\mathcal{L}^c_0$ contains and how such vocabulary should be used in our discursive practice. Although it is an important task to think about how to identify $\mid\!\sim_0$ within a given discursive practice, that is beyond and orthogonal to my purposes here. Thus, I simply take $\mid\!\sim_0$ as given. I assume, though, that $\mid\!\sim_0$ meets at least a few structual constraints, as follows:

Definition 1 $\mid\!\sim_0 \subseteq \mathcal{P}(\mathcal{L}^c_{0-}) \times \mathcal{L}^c_0$, *where (i)* $\mathcal{L}^c_{0-} \mid\!\sim_0 \bot$*; (ii)* $\emptyset \not\mid\!\sim_0 \bot$*; (iii)* $\mid\!\sim_0$ *is reflexive; (iv) if for any* $\Delta_0 \subseteq \mathcal{L}^c_{0-} : \Delta_0, \Gamma_0 \mid\!\sim_0 \bot$*, then for any* $p \in \mathcal{L}^c_0 : \Gamma_0 \mid\!\sim_0 p$*.*

Note that (iv) is a weak version of the explosion principle, which I call, after Hlobil (2016), "Ex Falso *Fixo* Quodlibet" (ExFF). Because I treat the premises as a set, contraction and permutation are also granted. However, I do not impose weakening on this base consequence relation in order to make room for it to be defeasible (and therefore nonmonotonic). Transitivity is not imposed either, for the same reason.[9]

Now, let us turn to the extension of a material consequence relation thus defined over $\mathcal{L}^c_0$ (i.e., $\mid\!\sim_0$) to the (indexed) consequence relation over the logically complex language $\mathcal{L}$ (i.e., $\mid\!\sim^{[\uparrow X]}_{[S]}$). $\mathcal{L} = \mathcal{L}_- \cup \{\bot\}$, where the syntax of $\mathcal{L}_-$ is given as follows:

Syntax of $\mathcal{L}_-$: $\varphi ::= p \mid \varphi \rightarrow \varphi \mid \neg\varphi \mid \varphi \& \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \forall x_i \varphi \mid \exists x_i \varphi$.

For technical convenience, I assume that variables in $\mathcal{L}$ consist of "x" with varying indices (i.e., $Var = \{x_1, x_2, \ldots x_i, \ldots\}$).

The extension of $\mid\!\sim_0$ over $\mathcal{L}^c_0$ in this logically complex language is conducted via a sequent calculus, which I call the first-order nonmonotonic

[9]Transitivity forces monotonicity in the presence of the conditional satisfying the deduction theorem and its converse. See, ibid. sec. 4.3.

sequent calculus (FNM). Note that FNM not only specifies the snake turnstile ($\mid\!\sim$) that represents the extended consequence relation, but also introduces those with indices ($\mid\!\sim^{\uparrow X}_{S}$) in order to keep track of important features of $\mid\!\sim$ concerning subjunctive robustness and universalizability of inferential pattern. The indexed upward-arrow is supposed to track a range of subjunctive robustness (i.e., $\Gamma \mid\!\sim^{\uparrow X} A \Leftrightarrow$ for any $\Delta \in X \subseteq \mathcal{P}(\mathcal{L}^{c}_{0-})$: $\Gamma, \Delta \mid\!\sim A$), whereas the substitution memory S is supposed to register a range of variable substitutability *salva consquentia* (i.e., $\Gamma \mid\!\sim_{S} A \Leftrightarrow$ for any $\alpha \in N \subseteq Con$ s.t. $< N, i > \in S$: $\Gamma[\alpha/x_i] \mid\!\sim A[\alpha/x_i]$). Although FNM thus deals with countably many turnstiles, the main turnstile is the plain snake, within which several important features of the extended consequence relation (including those tracked by the two indices above) can eventually be expressed with the help of logical operators such as $\rightarrow$, $\neg$, $\Box$, and $\forall$ (see section 4.3).

First, a given material consequence relation is taken as the axioms of FNM.

Axiom: If $\Gamma \mid\!\sim_0 p$, then $\Gamma \mid\!\sim^{\uparrow\emptyset}_{\emptyset} p$ is an axiom.
Convention 1: $\mid\!\sim^{\uparrow\emptyset}_{S}$ can be abbreviated as $\mid\!\sim_{S}$.
Convention 2: $\mid\!\sim^{[\uparrow X]}_{\emptyset}$ can be abbreviated as $\mid\!\sim^{[\uparrow X]}$.
Convention 3: $\mid\!\sim^{\uparrow\mathcal{P}(\mathcal{L}^{c}_{0-})}_{[S]}$ can be abbreviated as $\mid\!\sim^{\uparrow}_{[S]}$.

Then, FNM extends this base consequence relation by closing it under the following sequent rules.

$$\frac{\Gamma \mid\!\sim^{\uparrow}_{S} A \qquad \Gamma, B \mid\!\sim^{\uparrow X}_{S} C}{\Gamma, A \rightarrow B \mid\!\sim^{\uparrow X}_{S} C}\ \text{LC} \qquad \frac{\Gamma, A \mid\!\sim^{\uparrow X}_{S} B}{\Gamma \mid\!\sim^{\uparrow X}_{S} A \rightarrow B}\ \text{RC}$$

$$\frac{\Gamma \mid\!\sim^{\uparrow X}_{S} A}{\Gamma, \neg A \mid\!\sim^{\uparrow X}_{S} \bot}\ \text{LN} \qquad \frac{\Gamma, A \mid\!\sim^{\uparrow X}_{S} \bot}{\Gamma \mid\!\sim^{\uparrow X}_{S} \neg A}\ \text{RN}$$

$$\frac{\Gamma, A, B \mid\!\sim^{\uparrow X}_{S} C}{\Gamma, A\&B \mid\!\sim^{\uparrow X}_{S} C}\ \text{L\&} \qquad \frac{\Gamma \mid\!\sim^{\uparrow X}_{S} A \qquad \Gamma \mid\!\sim^{\uparrow X}_{S} B}{\Gamma \mid\!\sim^{\uparrow X}_{S} A\&B}\ \text{R\&}$$

$$\frac{\Gamma, A \mid\!\sim_S^{\uparrow X} C \qquad \Gamma, B \mid\!\sim_S^{\uparrow X} C}{\Gamma, A \vee B \mid\!\sim_S^{\uparrow X} C}\ \mathrm{L}\vee \qquad \frac{\Gamma \mid\!\sim_S^{\uparrow X} A_i}{\Gamma \mid\!\sim_S^{\uparrow X} A_1 \vee A_2}\ \mathrm{R}\vee$$

where i = 1 or 2

$$\frac{\Gamma, p_1, \ldots, p_n \mid\!\sim_S A \qquad \Gamma \mid\!\sim_S^{\uparrow X} A}{\Gamma \mid\!\sim_S^{\uparrow\{\{p_1,\ldots,p_n\}\}\cup X} A}\ \mathrm{PushUpUN}$$

where $p_1, \ldots, p_n \in \mathcal{L}^c_{0-}$

$$\frac{\Gamma, A \mid\!\sim_S^{\uparrow X} B}{\Gamma, \Box A \mid\!\sim_S^{\uparrow X} B}\ \mathrm{LB} \qquad \frac{\Gamma \mid\!\sim_S^{\uparrow} A}{\Gamma \mid\!\sim_S^{[\uparrow]} \Box A}\ \mathrm{RB}$$

$$\frac{\Gamma[\alpha/x_i] \mid\!\sim_S^{\uparrow X} A[\alpha/x_i]}{\Gamma \mid\!\sim_{S\cup\{<\{\alpha\},i>\}}^{\uparrow X} A}\ \mathrm{AB1} \qquad \frac{\Gamma[\alpha/x_i] \mid\!\sim_S^{\uparrow X} A[\alpha/x_i]}{\Gamma \mid\!\sim_S^{\uparrow X} A}\ \mathrm{AB2}$$

where there is no $< N, i >\in S$ (AB1); where there is no $< N, i >\in S$ (AB2)

$$\frac{\Gamma \mid\!\sim_{S\cup\{<N,i>\}}^{\uparrow X} A \qquad \Gamma \mid\!\sim_{S\cup\{<N',i>\}}^{\uparrow X} A}{\Gamma \mid\!\sim_{S\cup\{<N\cup N',i>\}}^{\uparrow X} A}\ \mathrm{vUN}$$

$$\frac{\Gamma, A \mid\!\sim_{S\cup\{<N,i>\}}^{\uparrow} B}{\Gamma, \forall x_i A \mid\!\sim_{S-\{<N,i>\}}^{[\uparrow]} B}\ \mathrm{pLA} \qquad \frac{\Gamma \mid\!\sim_{S\cup\{<Con,i>\}}^{\uparrow X} A}{\Gamma \mid\!\sim_{S-\{<Con,i>\}}^{\uparrow X} \forall x_i A}\ \mathrm{RA}$$

$$\frac{\Gamma, A \mid\!\sim_{S\cup\{<Con,i>\}}^{\uparrow X} B}{\Gamma, \exists x_i A \mid\!\sim_{S-\{<Con,i>\}}^{\uparrow X} B}\ \mathrm{LE} \qquad \frac{\Gamma \mid\!\sim_{S\cup\{<N,i>\}}^{\uparrow X} A}{\Gamma \mid\!\sim_{S-\{<N,i>\}}^{\uparrow X} \exists x_i A}\ \mathrm{RE}$$

$$\frac{\Gamma \mid\!\sim_S^{\uparrow} A}{\Gamma, B \mid\!\sim_S^{[\uparrow]} A}\ \mathrm{pW} \qquad \frac{\Gamma \mid\!\sim_S^{\uparrow} \bot}{\Gamma \mid\!\sim_S^{[\uparrow]} A}\ \mathrm{ExFF}$$

Most of these rules are adopted from the Non-Monotonic Modal sequent calculus in Hlobil (2016) (with obvious adjustments and a few minor modifications), for which I omit justifications. Among others, AB1 (for "abstraction") and vUN (for "variable unification") are crucial for the substitution memory to do its intended job, and given them RA is the key for the universal principle to hold. The top sequent of pLA must be indefeasible, since otherwise replacement of A by $\forall x_i A$ on the left might defeat the implication or incoherence at issue. In this way, FNM maps $\mid\!\sim_0 \subseteq \mathcal{P}(\mathcal{L}^c_{0-}) \times \mathcal{L}^c_0$ to $\mid\!\sim_S^{\uparrow X} \subseteq \mathcal{P}(\mathcal{L}_-) \times \mathcal{L}$, where $X \subseteq \mathcal{P}(\mathcal{L}^c_{0-})$ and $S \subset \mathcal{P}(Con) \times \mathbb{N}$.

4 Main properties

4.1 Conservativeness

To begin with, it is straightforward to show that FNM is a conservative extension of the underlying material consequence relation, as there is no simplifying rule (such as *Cut*) in FNM.

Proposition 1 *For any* $\Gamma_0 \subseteq \mathcal{L}^c_{0-}$ *and for any* $p \in \mathcal{L}^c_0$*,* $\Gamma_0 \mid\!\sim p \Leftrightarrow \Gamma_0 \mid\!\sim_0 p$*.*

Proof. ($\Rightarrow$) is straightforward. ($\Leftarrow$) is also straightward from the fact that no rule of FNM reduces the complexity of a formula on either side of $\mid\!\sim$. ■

As we stipulated that $\mid\!\sim_0$ can be nonmonotonic, so can $\mid\!\sim$, which is a conservative extention of $\mid\!\sim_0$. Thus, we have shown that FNM satisfies *Nonmonotonicity*.

4.2 Preservation of reflexivity

Next, reflexivity as stipulated at the base is preserved by FNM. With some preparations, the preservation of reflexivity is first shown with respect to the atomic *open* sentences.

Definition 2 *S is unique* $\Leftrightarrow$ *for any* N *such that* $< N, i >\in S$*, (i)* $N \neq \emptyset$*, and (ii) there is no N' such that* $N' \neq N$ *and* $< N', i >\in S$*.*

Lemma 1 *For any* $\Gamma \subseteq \mathcal{L}_-$*, for any* $p \in \mathcal{L}_{0-}$*, for any* $X \subseteq \mathcal{P}(\mathcal{L}^c_{0-})$*, and for any finite unique* $S \subset \mathcal{P}(Con) \times \mathbb{N}$*,* $\Gamma, p \mid\!\sim^{\uparrow X}_S p$*.*

Proof. By double induction on the number of distinct variables occuring in p and the cardinality of S. ■

We are now in a position to show reflexivity across the board.

Proposition 2 *For any* $\Gamma \subseteq \mathcal{L}_-$*, for any* $A \in \mathcal{L}_-$*, and for any finite unique* $S \subset \mathcal{P}(Con) \times \mathbb{N}$*,* $\Gamma, A \mid\!\sim^{[\uparrow]}_S A$*.*

Proof. By induction on the complexity of A. The base case is straightforward from Lemma 1, and the only tricky cases in the induction step are those in which A is the universal or the existential. Let A be $\forall x_i B$. By induction hypothesis, for any $a \in Con$, $B[a/x_i] \mid\!\sim^{\uparrow}_S B[a/x_i]$, from which leaves $\forall x_i B \mid\!\sim^{[\uparrow]}_S \forall x_i B$ is derivable via AB1, pLA, AB1, vUN ($|Con|-1$ times), and RA. Note, however, that if for some $N, < N, i > \in S$, then the leaves must instead be $B[x_j/x_i][a/x_j] \mid\!\sim_S^{[\uparrow]} B[x_j/x_i][a/x_j]$, where such j is chosen that $x_j$does not freely occur in B, and there is no $< N, j > \in S$ (otherwise, AB1 could not be applied). These leaves lead to $\forall x_j(B[x_j/x_i]) \mid\!\sim_S^{[\uparrow]} \forall x_k(B[x_j/x_i])$, where $\forall x_j(B[x_j/x_i])$ is, by definition, syntactically equivalent to (or an "alphabetical variant" of) $\forall x_i B$. Finally, if needed, Γ can be added to the left by pW (|Γ| times). The case in which A is the existential can be treated in the parallel manner. ■

4.3 Expressivity

4.3.1 Implication and incoherence

According to logical expressivism, logical operators codify, within the object language, some features of the underlying material consequence relation. For instance, the conditional lets us talk about implications, whereas the negation lets us talk about incoherences. To justify such expressivist readings of logical operators in FNM, we first need to show the following lemma, which is repeatedly used in the proofs below.

Lemma 2 *If* $\Gamma \mid\!\sim_S^{\uparrow X} A$*, then* S *is unique.*

Proof. By induction on proof height. ■

Now, the following proposition justifies a reading of the conditional and negation on the right as codifying implications and incoherences, respectively.

Proposition 3 ***Conditional:*** $\Gamma \mid\!\sim_S^{\uparrow X} A \to B \Leftrightarrow \Gamma, A \mid\!\sim_S^{\uparrow X} B$. ***Negation:*** $\Gamma \mid\!\sim_S^{\uparrow X} \neg A \Leftrightarrow \Gamma, A \mid\!\sim_S^{\uparrow X} \bot$. ***Conjunction:*** $\Gamma \mid\!\sim_S^{\uparrow X} A\&B \Leftrightarrow \Gamma \mid\!\sim_S^{\uparrow X} A$ *and* $\Gamma \mid\!\sim_S^{\uparrow X} B$.

Proof. (⇒) is by induction on proof height. (⇐) is straightforward from RC, RN, and R&. ■

4.3.2 Local monotonicity

Another important feature of the underlying material consequnece relation that logical expressivists wish to codify is its local monotonicity. Although material inferences and incoherences often seem defeasible (e.g., "a is a match," "a is struck" $\mid\!\sim$ "a lights"), some may be *in*defeasible (e.g., "a is a bachelor" $\mid\!\sim$ "a is unmarried"). In FNM, such local monotonicity is kept track of by the upward-arrowed snake turnstile ($\mid\!\sim^{\uparrow}$) and eventually expressed by the monotonicity box ($\Box$).[10] This claim is underwritten by the following lemmas and proposition.

Lemma 3 $\Gamma \mid\!\sim_S^{\uparrow X} A \Leftrightarrow$ *for any* $\Delta \in X$, $\Gamma, \Delta \mid\!\sim_S A$.

Proof. By induction on proof height. ■

Lemma 4 $\Gamma \mid\!\sim_S^{\uparrow X} \Box A \Leftrightarrow \Gamma \mid\!\sim_S^{\uparrow} A$.

Proof. By induction on proof height. ■

Proposition 4 $\Gamma \mid\!\sim_S \Box A \Leftrightarrow$ *for any* $\Delta \subseteq \mathcal{L}_-$, $\Gamma, \Delta \mid\!\sim_S A$.

Proof. Immediate from Lemma 3 and Lemma 4. ■

4.3.3 Universalizability

Finally, let us turn to our key proposition, that is, the universal principle. We first need to do some preparation.

Sublemma 1 *If* $\tau_1 \neq \zeta, \tau_2 \neq \xi$, *and* $\xi \neq \zeta$, $A[\tau_1/\xi][\tau_2/\zeta] = A[\tau_2/\zeta][\tau_1/\xi]$.

Proof. By induction on the complexity of A. ■

This lets us prove the following key lemma, which ensures that the substitution memory can store information about the range of variable substitutability *salva consequentia*.

Lemma 5 *(i) If* $\Gamma \mid\!\sim^{\uparrow X}_{S \cup \{<N,i>\}} A$ *is proved at height* n*, then for any* $a \in N, \Gamma[a/x_i] \mid\!\sim^{\uparrow X}_{S - \{<N,i>\}} A[a/x_i]$ *is provable at a height* $\leqq n$*. (ii) If for any* $a \in N, \Gamma[a/x_i] \mid\!\sim^{\uparrow X}_{S - \{<N,i>\}} A[a/x_i]$*, then* $\Gamma \mid\!\sim^{\uparrow X}_{S \cup \{<N,i>\}} A$.

Proof. (i) is by induction on proof height, where Sublemma 1 is used for the cases of the induction step in which the sequent at issue comes by AB1 or AB2. (ii) is straightforward from AB1 and vUN. ■

The next lemma says that the information about universal substitutability (i.e., a special case of the above) stored in the memory is expressible by the universal on the right.

[10] This technical apparatus was originally devised by Hlobil (2016).

Lemma 6 *$\Gamma \mathrel{|\!\!\sim}^{\uparrow X}_{S-\{<Con,i>\}} \forall x_i A \Leftrightarrow \Gamma \mathrel{|\!\!\sim}^{\uparrow X}_{S\cup\{<Con,i>\}} A$, where x_i does not freely occur in Γ, and there is no $< N, i >\in S$.*

Proof. ($\Rightarrow$) is by induction on proof height, where Lemma 5 (i) is used for the cases of the induction step in which the sequent at issue comes by pLA or LE. ($\Leftarrow$) is straightforward from RA. ■

We are now in a position to prove the universal principle.

Proposition 5 *$\Gamma \mathrel{|\!\!\sim} \forall x_i A \Leftrightarrow$ for any $a \in Con, \Gamma \mathrel{|\!\!\sim} A[a/x_i]$, where x_i does not freely occur in Γ.*

Proof. Straightforward from Lemmas 5 and 6. ■

4.4 Logical strength

Before closing, let us see where FNM is located within the familiar terrain of the other standard logics. FNM is *supraintuitionistic*. To demonstrate this, I first specify a special region of $\mathrel{|\!\!\sim}^{\uparrow X}_{S}$. Then, I embed intuitionistic logic there. Finally, I show that the special region is a local region of the material consequence relation represented by $\mathrel{|\!\!\sim}$.

Let us start with a few definitions.

Definition 3 *S is adequate for $\Gamma \mathrel{|\!\!\sim}^{\uparrow X}_{S} A \Leftrightarrow$ for any x_i freely occurring in Γ or A, there is some N such that $< N, i >\in S$. S is minimally adequate for $\Gamma \mathrel{|\!\!\sim}^{\uparrow X}_{S} A \Leftrightarrow S$ is adequate, and if $< N, i >\in S$, then x_i freely occurs in Γ or A.*

Definition 4 *S is full $\Leftrightarrow$ for any $< N, i >\in S$, $N = Con$.*

We are now in a position to specify a special region of $\mathrel{|\!\!\sim}^{\uparrow X}_{S}$ in which intuitionistic logic is going to be embedded.

Definition 5 *$\Gamma \mathrel{|\!\!\sim}_F A \Leftrightarrow \Gamma \mathrel{|\!\!\sim}^{\uparrow}_{S} A$, where S is unique, minimally adequate, and full.*

Next, let us show how to embed intuitionistic logic in $\mathrel{|\!\!\sim}_F$. This is done by showing that all the rules of LJ—the sequent calculus for first-order intuitionistic logic proposed by Gentzen (1935/1964)—can be added without disturbing $\mathrel{|\!\!\sim}_F$—that is, they are *admissible* within $\mathrel{|\!\!\sim}_F$. First, we need to define the notion of admissibility within $\mathrel{|\!\!\sim}_F$.

Definition 6 *A rule R of the form:* $\dfrac{\Gamma_1 \mid\!\sim A_1 \qquad \ldots \qquad \Gamma_n \mid\!\sim A_n}{\Gamma \mid\!\sim A}$ *is* *admissible within* $\mid\!\sim_F$ $\Leftrightarrow$ *if* $\Gamma_1 \mid\!\sim_F A_1, \ldots, \Gamma_n \mid\!\sim_F A_n$*, then* $\Gamma \mid\!\sim_F A$*.*

Now, let us prove the admissibility of the rules of LJ within $\mid\!\sim_F$. It is convenient to disucuss *Cut* and the rest of the rules separately. The admissibility of the latter can first be shown with the help of the following two sublemmas.

Sublemma 2 *All the connective rules concerning the propositional part of FNM (i.e., LC, RC, LN, RN, L&, R&, L*$\vee$*, and R*$\vee$*) are admissible within* $\mid\!\sim_F$*.*

Proof. RC, LN, RN, and L& are straightforward, whereas LC, R&, L$\vee$, and R$\vee$ are handled with the help of AB1 and vUN. ■

Sublemma 3 *If* $\Gamma \mid\!\sim_F A$*, then for any* $\Delta \subseteq \mathcal{L}_-$*,* $\Gamma, \Delta \mid\!\sim_F A$*.*

Proof. Straightforward from Lemma 3. ■

Lemma 7 *All the rules of LJ except for Cut are admissible within* $\mid\!\sim_F$*.*

Proof. All the rules except for those involving the quantifiers are straightforward from Sublemmas 2 and 3. The right universal and left existential rules are also straightforward from RA and LE respectively. Finally, the left universal and right existential rules are taken care of by pLA and RE with the help of Lemma 5. ■

As to the admissibility of *Cut*, let us first prove the so-called "substitution lemma" with respect to $\mid\!\sim_F$. This is shown via the following sublemma.

Sublemma 4 *(i) If* ζ *does not freely occur in* A*,* $A[\zeta/\xi][\tau/\zeta] = A[\tau/\xi]$*. (ii)* $A[\alpha/\xi][\alpha/\zeta] = A[\zeta/\xi][\alpha/\zeta]$*.*

Proof. By induction on the complexity of A, where (i) is used in the inductive proof of (ii). ■

Lemma 8 *If* $\Gamma \mid\!\sim_F A$*, then* $\Gamma[\tau/x_i] \mid\!\sim_F A[\tau/x_i]$*.*

Proof. By proof by cases, where cases are divided depending first on whether x_i freely occurs in Γ or A, then on whether τ is a variable, and, if so, furhter on whether τ freely occurs in Γ or A. In some of these cases, Lemma 5 and Sublemma 4 are appealed to. ■

At this stage, we can appropriate Gentzen's (ibid.) well-known result of the eliminability of *Cut* in LJ: given the substitution lemma, any sequents derivable in LJ via *Cut* are derivable without *Cut*.

Lemma 9 *Cut is also admissible within* $\mid\!\sim_F$.

Proof. Straightforward from Lemma 7, Lemma 8, and the eliminability of *Cut* in LJ. ■

Given these, we are now in a position to prove that $\mid\!\sim_F$ is supraintuitionistic.

Lemma 10 $\mid\!\sim_F$ *is at least as strong as intuitionistic logic.*

Proof. Straightforward from Proposition 2, Lemma 7, and Lemma 9. ■

Finally, let us show that $\mid\!\sim_F \subseteq \mid\!\sim$, which immediately follows from the lemma below.

Lemma 11 *(i) If* $\Gamma \mid\!\sim_S^{\uparrow X} A$*, then* $\Gamma \mid\!\sim^{\uparrow X} A$*. (ii) If* $\Gamma \mid\!\sim^{\uparrow X} A$*, then* $\Gamma \mid\!\sim A$*.*

Proof. By induction on proof height. ■

Corollary 1 *If* $\Gamma \mid\!\sim_F A$*, then* $\Gamma \mid\!\sim A$*.*

Thus, we are eventually in a position to show supraintuitionisity of $\mid\!\sim$.

Proposition 6 $\mid\!\sim$ *is at least as strong as intuitionistic logic.*

Proof. Straightforward from Lemma 10 and Corollary 1. ■

5 Conclusion

This paper was motivated by two philosophical aims. One is to explain the meanings of variables in terms of their inferential roles; the other is to implement a logical system that can codify (i.e., talk about), within the object language, the inferential roles (i.e., meanings) of predicates. I argued that it is essential for both of these aims to have a proof system in which what I call the universal principle holds even in the presence of arbitrary nonlogical axioms. To the best of my knowledge, however, no currently available system meets this demand. Thus, I propose an alternative system in which the use of variables is explicitly controlled by an extra set of inferential rules in such a way that the universal principle holds robustly. This system is nonmonotonic and conservative, preserves reflexivity, and satisfies several biconditionals essential for its logical operators to codify important features of the underlying consequence relation, such as implications, incoherences, local monotonicity, and the universalizability of inferential patterns. The system is also supraintuitionistic.

References

Brandom, R. (1994). *Making It Explicit: Reasoning, Representing, and Discursive Commitment*. MA: Harvard University Press.

Brandom, R. (2000). *Articulating Reasons: An Introduction to Inferentialism*. MA: Harvard University Press.

Brandom, R. (2008). *Between Saying and Doing: Towards an Analytic Pragmatism*. Oxford: Oxford University Press.

Brandom, R. (2009). *Reason in Philosophy: Animating Ideas*. Cambridge, MA: Harvard University Press.

Fine, K. (1985). *Reasoning with Arbitrary Objects*. Oxford: Basil Blackwell.

Fine, K. (2007). *Semantic Relationism*. MA: Blackwell Publishing.

Frege, G. (1979). *Posthumous Writings* (P. Long & R. White, Trans.). Oxford: Basil Blackwell.

Gentzen, G. (1935/1964). Investigation into logical deduction (M. E. Szabo, Trans.). *American Philosophical Quarterly*, *1*(4), 288–306.

Hlobil, U. (2016). A nonmonotonic sequent calculus for inferentialist expressivists. In P. Arazim & M. Dančák (Eds.), *The Logica Yearbook 2015* (pp. 87–105). College Publications.

Kaplan, D. (2018). A multi-succedent sequent calculus for logical expressivists. In P. Arazim & M. Dančák (Eds.), *The Logica Yearbook 2017*. College Publications.

Peregrin, J. (2014). *Inferentialism: Why Rules Mater*. New York: Palgrave Macmillan.

Russell, B. (1905). On denoting. *Mind*, *14*(56), 479–493.

Russell, B. (1994). *Foundations of Logic 1903-05* (A. Urquhart, Ed.). London: Routledge.

Sellars, W. (1953). Inference and meaning. *Mind*, *62*(247), 313–338.

Shuhei Shimamura
Nihon University, College of Science and Technology

Japan
E-mail: shuhei.shimamura0803@gmail.com