Violation of the Hamilton-Jacobi-Bellman Equation in Economic Dynamics

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HJB equation

In this talk, we treat the Hamilton-Jacobi-Bellman (HJB) equation that concerns with a variational problem in macroeconomic theory. In macroeconomic theory, models in recent years are very complicated and too difficult to solve directly. Therefore, a method using the HJB equation to analyze the problem indirectly and to obtain implications has been developed and widely used.

Problem for HJB equation in economics (1)

However, there is no mathematical basis for the application of the HJB equation to problems that frequently arise in economics. In economics, it is treated as if the 'fact' that the value function for a given variational problem is a solution to the HJB equation had already been proved, and indeed, one can find a 'proof' in several books, although this is doubtful. In fact, what is actually used in economics is that the value function is the 'unique' 'classical' solution to the HJB equation, and the above 'proof' does not guarantee such a requirement. Hence, economists use results for the HJB equation that is not rigorously verified.

Problem for HJB equation in economics (2)

By the way, in variational problems with infinite time-interval, which are commonly used in economics, there are well known examples where the value function is not even a viscosity solution, whereas there are infinitely many other solutions. See, e.g., Barles (1990). Therefore, there are two possibilities: 1) such examples do not arise in economic models. 2) the ‘proof’ used by economists is wrong, and there are many examples like the above in economics. We first verify that the latter is correct, even though there is no stochastic shock.

A counterexample (1)

We consider the following classical Ramsey-Cass-Koopmans (RCK) capital accumulation model.

$$\begin{aligned} \max \quad & \int_0^\infty e^{-\rho t} u(c(t))dt \\ \text{subject to.} \quad & k(t) \ge 0,\ c(t) \ge 0, \\ & c(t) \text{ is locally integrable}, \\ & \dot{k}(t) = f(k(t)) - c(t) \text{ a.e.}, \\ & k(0) = \bar{k} > 0, \end{aligned} \tag{1}$$

where c denotes the consumption and k denotes the capital stock. For a given $\rho > 0$, let V_ρ denote the value function of the above problem.

A counterexample (2)

The HJB equation for the problem (1) is the following differential equation with unknown function V:

$$\sup_{c\geq 0}\{(f(k)-c)V'(k)+u(c)\}-\rho V(k)=0. \tag{2}$$

A counterexample (3)

We consider that $u(c) = c$ and $f(k) = \sqrt{k}$. Then, the following facts can be proved.

1) The value function V_ρ is a finite concave function.
2) The equation (2) has infinitely many classical solutions.
3) Any viscosity solution to (2) cannot be concave.

In particular, by 1) and 3), we have that V_ρ is not a viscosity solution to (2).

Sketch of the proof of our counterexample (1)

Define $g(k) = \rho k + \frac{1}{4\rho}$. Then, $g(k) \geq f(k)$ for every k. We consider the following 'modified' problem.

$$\begin{aligned}
\max \quad & \int_0^{\infty} e^{-\rho t} u(c(t)) dt \\
\text{subject to.} \quad & k(t) \geq 0,\ c(t) \geq 0, \\
& c(t) \text{ is locally integrable,} \\
& \dot{k}(t) = g(k(t)) - c(t) \text{ a.e.,} \\
& k(0) = \bar{k} > 0.
\end{aligned} \tag{3}$$

Sketch of the proof of our counterexample (2)

We can show that if we define $k^*(t) \equiv \bar{k}$ and $c^*(t) \equiv \rho\bar{k} + \frac{1}{4\rho}$, then this pair $(k^*(t), c^*(t))$ is a solution to (3). Therefore, the value function of (3) is finite.
Moreover, we can easily show that the value function of (1) is lower than or equal to (3). Therefore, V_ρ is finite, as desired. Now we can easily show the concavity of V_ρ by the concavity of f and u.

Sketch of the proof of our counterexample (3)

Define

$$V(k) = Ae^{2\rho\sqrt{k}-2\rho} \tag{4}$$

for some $A > 0$. Then,

$$V'(k) = \frac{\rho V(k)}{\sqrt{k}}.$$

Therefore,

$$\lim_{k\to 0} V'(k) = \lim_{k\to\infty} V'(k) = +\infty, \tag{5}$$

and thus $V'(k)$ has a positive minimum and this mimimum is linear with respect to A. Hence, if $A > 0$ is sufficiently large, then $V'(k) \geq 1$ for all $k > 0$. We can easily show that, in this case, $V(k)$ is a classical solution to the HJB equation (2).

Sketch of the proof of our counterexample (4)

Using Picard-Lindelöf's theorem, we can actually show that every locally Lipschitz viscosity solution to the HJB equation (2) can be represented by (4), where $V(1) = A$. If V is a concave viscosity solution to (2), then it is locally Lipschitz, and thus it is represented by (4). However, by (5), we conclude that V is not concave, which is a contradiction. Therefore, such a solution does not exist.

What is the problem? (1)

We introduce a ‘proof’ for the value function to be a solution to the HJB equation used in economics. (This is the deterministic version of the ‘proof’ in Malliaris and Brock (1988), in which the stochastic version is contained.) Consider that V is the value function. First, by simple evaluation, we have that

$$V(\bar{k}) = \sup_{k(s),c(s)} \left\{ \int_0^t e^{-\rho s} u(c(s))ds + e^{-\rho t} V(k(t)) \right\}.$$

This implies that

$$\sup_{k(s),c(s)} \left\{ \int_0^t e^{-\rho s} u(c(s))ds + e^{-\rho t} V(k(t)) - e^{-\rho 0} V(k(0)) \right\} = 0.$$

What is the problem? (2)

Under several assumptions, we can show that V is continuously differentiable, and thus

$$\begin{aligned} 0 &= \sup_{k(s),c(s)} \left\{ t \times \frac{d}{dT}\left[\int_0^T e^{-\rho s}u(c(s))ds + e^{-\rho T}V(k(T))\right]\Bigg|_{T=0} + o(t) \right\} \\ &= \sup_{k(s),c(s)} \{t[u(c(0)) - \rho V(k(0)) + V'(k(0))\dot{k}(0)] + o(t)\}. \end{aligned}$$

Because $k(0) = \bar{k}$ and $\dot{k}(0) = f(\bar{k}) - c(0)$, the last line depends only on $c(0)$, and thus

$$\sup_{c \geq 0}\{t[u(c) + V'(\bar{k})(f(\bar{k}) - c)] + o(t)\} - \rho V(\bar{k})t = 0.$$

Dividing this by t and letting $t \downarrow 0$, we obtain the HJB equation.

What is the problem? (3)

However, we have already found a counterexample V_ρ, and thus this 'proof' is wrong. We think that the evaluation of $o(t)$ is doubtful. In previous slide, we say that "the last line depends only on $c(0)$", but $o(t)$ actually depends on the whole trajectory of $(k(s), c(s))$. And moreover, this '$o(t)$' part becomes $o(t)$ because the functions $k(s), c(s)$ are fixed, and taking supremum may break this property. Because of the above problem, we need conditions that guarantees that the value function satisfies the HJB equation. From now on, we consider this problem in a much more generalized model than the RCK model, although there is no stochastic shock.

The model

We consider the following problem.

$$\begin{aligned} \max \quad & \int_0^\infty e^{-\rho t} u(c(t), k(t)) dt \\ \text{subject to.} \quad & c(t) \ge 0, \ k(t) \ge 0, \\ & c(t) \text{ is locally integrable}, \\ & \int_0^\infty e^{-\rho t} u(c(t), k(t)) dt \text{ can be defined}, \\ & \dot{k}(t) = F(k(t), c(t)) \text{ a.e.}, \\ & k(0) = \bar{k} > 0. \end{aligned} \tag{6}$$

In the usual RCK model, $u(c, k) = u(c)$ and $F(k, c) = f(k) - dk - c$ (where f is the production function and $d \ge 0$ denotes capital depressiation rate).

Assumptions (1)

We use the following notations.

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x_i \geq 0 \text{ for all } i\},$$

$$\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n | x_i > 0 \text{ for all } i\}.$$

If $n = 1$, then n is omitted.

Assumption 1

$\rho > 0$.

Assumptions (2)

Assumption 2

$u : \mathbb{R}^2_+ \to \mathbb{R} \cup \{-\infty\}$ is continuous and concave. Moreover, $u(c,k)$ is nondecreasing in k and increasing in c, and C^1 on $\mathbb{R}^2_{++}$. Furthermore, there exists $c > 0$ such that $u(c,0) > -\infty$.

Because u can be independent of k, we admit $u(c,k) = u(c)$.

Assumption 3

$F : \mathbb{R}^2_+ \to \mathbb{R}$ is a continuous and concave function such that $F(0,0) = 0$, and $F(k,c)$ is decreasing in c. Moreover, there exist $d_1, d_2 \geq 0$ such that if $k > 0$ and $c \geq 0$, then $F(k,c) > -d_1 k - d_2 c$. Furthermore, for every c, there exists k such that $F(k,c) > F(0,c)$.

If $F(k,c) = f(k) - dk - c$ and $f'(k) < d$ for some k, then F is not increasing even if f is increasing. Thus, we cannot assume that F is increasing in k.

Assumptions (3)

Assumption 4

The function $\frac{\partial u}{\partial c}(c,k)$ is decreasing in c, and if $k>0$, then $\lim_{c\to 0}\frac{\partial u}{\partial c}(c,k)=+\infty$ and $\lim_{c\to\infty}\frac{\partial u}{\partial c}(c,k)=0$. Moreover, for every $k>0$ and $M>0$, $c\mapsto \frac{\partial u}{\partial k}(c,k)$ is bounded on $]0,M]$.

The utility function $u(c)=c$ that appears in our example violates this assumption.

Assumption 5

$\frac{\partial F}{\partial c}(k,c)$ is defined on $\mathbb{R}^2_{++}$ and continuous.

If $F(k,c)=f(k)-dk-c$, then this condition is automatically satisfied.

Assumptions (4)

Definition 1 (CRRA function)

For $\theta > 0$, define

$$u_\theta(x) = \begin{cases} \frac{x^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1, \\ \log x & \text{if } \theta = 1. \end{cases}$$

This function u_θ is called the CRRA function.

Note: this function is the solution to the following equation:

$$-\frac{cu''(c)}{u'(c)} = \theta,\ u'(1) = 1,\ u(1) = 0.$$

In economics, $-\frac{cu''(c)}{u'(c)}$ is sometimes called the **relative risk aversion**. Because u_θ has the constant relative risk aversion, it is called the CRRA function.

Assumptions (5)

Assumption 6

There exist $k^*, k^+ > 0, c^* \geq 0, \gamma, \delta, \theta > 0, a > 0, b \geq 0, C \in \mathbb{R}$ such that

$$F(k^*, c^*) > 0, \tag{7}$$
$$(\gamma, -\delta) \in \partial F(k^*, c^*),\ 0 < D_{k,+}F(k^+, 0) \leq \gamma, \tag{8}$$
$$\rho - (1-\theta)\gamma > 0, \tag{9}$$
$$u(c,k) \leq au_\theta(c) + bu_\theta(k) + C \text{ on } \mathbb{R}^2_{++}. \tag{10}$$

This assumption is not so strong. For example, if $F(k,c) = f(k) - dk - c$ and f satisfies the Inada condition $f'(\mathbb{R}_{++}) = \mathbb{R}_{++}$, then restrictions except (10) vanishes. And (10) simply states that $u(c,k)$ is bounded by the positive affine transform of the CRRA function. Note also that we admit $b = 0$.

Assumptions (6)

Assumption 7

$\frac{\partial F}{\partial c}$ and $\frac{\partial u}{\partial c}$ are continuously differentiable in k. Moreover, there exists $k > 0$ such that $\inf_{c \geq 0} D_{k,+}F(k,c) > \rho$.

If $F(k,c) = f(k) - dk - c$, then $\frac{\partial F}{\partial c} \equiv -1$, and thus the first requirement of the above assumption automatically holds.

Solution (1)

Definition 2 (Admissible set)

The set $A_{\bar{k}}$ denotes the set of all pairs $(k(t), c(t))$ of functions that satisfies all requirements in (6), where the initial capital stock $\bar{k}$ is given.

Definition 3 (Value function)

The following function

$$\bar{V}(\bar{k}) = \sup\left\{ \int_0^\infty e^{-\rho t} u(c(t), k(t)) dt \,\middle|\, (k(t), c(t)) \in A_{\bar{k}} \right\}$$

is called the **value function**.

Solution (2)

Definition 4 (Solution to the problem)

$(k^*(t), c^*(t)) \in A_{\bar{k}}$ is said to be a **solution** to (6) if

$$\int_0^\infty e^{-\rho t} u(c^*(t), k^*(t))dt = \bar{V}(\bar{k}) \in \mathbb{R}.$$

HJB equation

The HJB equation of (6) is the following equation with the unknown function $V(k)$.

$$\sup_{c \geq 0}\{F(k,c)V'(k) + u(c,k)\} - \rho V(k) = 0. \tag{11}$$

The extended real-valued function V defined on $\mathbb{R}_+$ is said to be a **classical solution** to (11) if V is C^1 on $\mathbb{R}_{++}$ and (11) holds for every $k > 0$. The definition of the **viscosity solution** is omitted in this talk.

Pure accumulation path

Lemma 1

Suppose that Assumption 3 holds. Then, for every $\bar{k} > 0$,

$$\dot{k}(t) = F(k(t), 0), \ k(0) = \bar{k}$$

has a solution $k^+(t, \bar{k})$ defined on $\mathbb{R}_+$. Moreover, $\inf_{t \ge 0} k^+(t, \bar{k}) > 0$ for every $\bar{k} > 0$.

This function $k^+(t, \bar{k})$ is called the **pure accumulation path**.

Growth condition

Definition 5 (Growth condition)

A function $V : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ is said to satisfy the **growth condition** if

$$\lim_{T\to\infty} e^{-\rho T} V(k^+(T, \bar{k})) = 0 \tag{12}$$

for all $\bar{k} > 0$.

Let $\mathscr{V}$ denote the set of all increasing and concave functions that satisfies the growth condition.

Proposition 1

Proposition 1

Suppose that Assumptions 1-4 hold. Then, there exists a positive continuous function $c^*(p,k)$ defined on $\mathbb{R}^2_{++}$ such that

$$F(k, c^*(p,k))p + u(c^*(p,k), k) = \sup_{c \geq 0}\{F(k,c)p + u(c,k)\}.$$

Main results

Theorem 1

Suppose that Assumptions 1-6 hold. Then, the value function $\bar{V}$ is in $\mathscr{V}$, and is a classical solution to the HJB equation.

Theorem 2

Suppose that Assumptions 1-7 hold. Then, the value function $\bar{V}$ is the unique classical solution to the HJB equation on $\mathscr{V}$.

Deriving solution to the original problem (1)

Corollary 1

Suppose that Assumptions 1-7 hold. Then, the following differential equation

$$\dot{k}(t) = F(k(t), c^*(\bar{V}'(k(t)), k(t))), \; k(0) = \bar{k} \tag{13}$$

has a solution $k^*(t)$ defined on $\mathbb{R}_+$ such that $\inf_{t \ge 0} k^*(t) > 0$. Suppose also that at least one of the following three requirements holds: 1) $u(c, k)$ is bounded from either above or below. 2) $k^*(t)$ is bounded. 3) $\liminf_{k \to \infty} c^*(p, k) > 0$. Define $c^*(t) = c^*(\bar{V}'(k^*(t)), k^*(t))$. Then, $(k^*(t), c^*(t))$ is a solution to the original problem.

Deriving solution to the original problem (2)

We can derive that

$$\lim_{T\to\infty}\int_0^T e^{-\rho t}u(c^*(t),k^*(t))dt=\bar{V}(\bar{k})$$

without the aid of requirements 1)-3). However, it is possible that $e^{-\rho t}u(c^*(t),k^*(t))$ is not integrable in the sense of Lebesgue integral. Hence, 1)-3) are needed.

We note that 1)-3) in Corollary 1 are all mild requirements. For example, if $u=u_\theta(c)$ for some $\theta\neq 1$, then it is bounded from either above or below, and 1) holds. If $F(k,c)=f(k)-dk-c$ and there exists $k>0$ such that $f'(k)<d$, then $k^+(t,\bar{k})$ is bounded, and thus $k^*(t)$ is also bounded, which implies that 2) holds. If $F(k,c)=f(k)-dk-c$ and $u(c,k)=u(c)$, then we can easily show that $c^*(p,k)$ is independent of k, and 3) holds.

Deriving solution to the original problem (3)

The converse of the Corollary 1 holds unconditionally.

Corollary 2

Suppose that Assumptions 1-7 hold. If there exists a solution $(k^*(t), c*(t))$ to (6) such that $\inf_{t\geq 0} k^*(t) > 0$, then $k^*(t)$ solves (13), and $c^*(t)$ coincides with $c^*(\bar{V}(k^*(t)), k^*(t))$ almost everywhere.

Calculation (1)

Suppose that $u(c,k) = \log c$ and $F(k,c) = \gamma k - c$, where $\gamma > \rho$. It is easy to show that Assumptions 1-7 hold. Moreover, $c^*(p,k)$ is independent of k, and thus 3) of Corollary 1 holds. We derive the value function by the "guess and verify" method. Suppose that

$$\bar{V}(k) = A \log k + B.$$

Then,

$$c^*(p,k) = \frac{1}{p}$$

and

$$\bar{V}'(k) = \frac{A}{k}.$$

Calculation (2)

Therefore,

$$\sup_{c\geq 0}\{(\gamma k - c)\bar{V}'(k) + u(c)\} = \log k + A\gamma - 1 - \log A,$$

and thus $\bar{V}$ solves the HJB equation if and only if

$$\log k + A\gamma - 1 - \log A = \rho A \log k + \rho B,$$

which is equivalent to

$$A = \frac{1}{\rho},\ B = \frac{1}{\rho}\left[\log \rho + \frac{\gamma}{\rho} - 1\right].$$

Hence, the value function is as follows.

$$\bar{V}(k) = \frac{1}{\rho}\left[\log k - \log \rho + \frac{\gamma}{\rho} - 1\right].$$

Calculation (3)

In this case, the equation (13) is as follows.

$$\dot{k}(t) = (\gamma - \rho)k(t),\ k(0) = \bar{k}.$$

This can be easily solved, and thus, by Corollary 1,

$$k^*(t) = \bar{k}e^{(\gamma-\rho)t},\ c^*(t) = \rho\bar{k}e^{(\gamma-\rho)t}$$

is a solution to (6).
Actually, this solution is in Barro and Sala-i-Martin (2003), where the Euler equation and the transversality condition are used for derivation. We present another method for derivation in a solution.

The rest task

We want to extend these results to models with probability shock. Introducing probability shock is difficult because the HJB equation becomes the second-order differential equation. Fortunately, by Alexandrov's theorem, C^1 concave function is twice differentiable a.e., and thus we think that this problem is solvable.
Next, we want to solve the HJB equation approximately. More concretely, we want to derive that if V is sufficiently near to $\bar{V}$, then the solution $k^*(t)$ to (13) using V instead of $\bar{V}$ is also sufficiently near to the true solution. If this is valid, then we can obtain an approximate solution to the original problem by an approximate solution to the HJB equation.

Thank you for your attention.