Non-Smooth Frobenius' Theorem

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Motivation (1)

I start this talk with an explanation of an inverse problem in economic theory called **integrability problem**. Let $n \ge 2$, $\Omega = \mathbb{R}^n_{++} \equiv \{x \in \mathbb{R}^n | x_i > 0 \text{ for all } i\}$. x_i represents the consumption plan of i-th commodity, and $p_i > 0$ denotes the corresponding price of i-th commodity. Therefore, the price system is represented by a positive vector $p = (p_1, ..., p_n)$. For a given price vector p and income level $m > 0$, let $f(p, m) \in \Omega$ be the consumer's choice of consumption vector. This function f is called the **demand function**. It is natural that $p \cdot f(p, m) \le m$ is assumed, but in consumer theory, $p \cdot f(p, m) = m$ is usually assumed for some reason. Now, consider a vector field $g : \Omega \to \mathbb{R}^n_{++}$ such that

$$f(g(x), g(x) \cdot x) = x$$

for all $x \in \Omega$. When the price vector p is $g(x)$, the consumer will purchase x if he has only enough money to buy just x under p. This vector field is called the **inverse demand function**.

Motivation (2)

If there is a function $u : \Omega \to \mathbb{R}$ such that $f(p, m)$ is the unique maximizer of u in the set $\Delta(p, m) = \{x \in \Omega | p \cdot x \leq m\}$, then we call this u a **utility function** of the consumer. The utility function is considered as a function that represents the consumer's preference. In economic theory, researchers sometimes want to determine this u. However, because u represents the consumer's preference, the information of u is hidden in their mind. Therefore, it is necessary to estimate $u(x)$ from only purchase behavior, although it is difficult. Fortunately, both $f(p, m)$ and $g(x)$ can directly be estimated from purchase data. Hence, if we can derive $u(x)$ from $g(x)$, then the estimation problem of u can be changed to the estimation problem of g, which reduces the difficulty of the problem.

Motivation (3)

If a quasi-concave function $u : \Omega \to \mathbb{R}$ is differentiable and there exists a positive real-valued function $\lambda(x)$ such that

$$\nabla u(x) = \lambda(x) g(x) \tag{1}$$

for all $x \in \Omega$, then by Lagrange's multiplier rule, u must be a utility function. Therefore, we can obtain u if we can solve the above **total differential equation** (1). If $g(x)$ is C^1, then Frobenius' theorem provides a necessary and sufficient condition for the 'local' existence of the solution (u, λ).

Frobenius' Theorem (1)

Frobenius' Theorem

Suppose that $n \ge 2$, $U \subset \mathbb{R}^n$ is open and nonempty, and $g : U \to \mathbb{R}^n \setminus \{0\}$ is continuously differentiable. Then, for any $x^* \in U$, there exists a solution to (1) around x^* if and only if, the following **Jacobi's integrability condition** holds for all $x \in U$: for each $i, j, k \in \{1, ..., n\}$ such that $i \neq j \neq k \neq i$,

$$g_i \left(\frac{\partial g_j}{\partial x_k} - \frac{\partial g_k}{\partial x_j} \right) + g_j \left(\frac{\partial g_k}{\partial x_i} - \frac{\partial g_i}{\partial x_k} \right) + g_k \left(\frac{\partial g_i}{\partial x_j} - \frac{\partial g_j}{\partial x_i} \right) = 0. \quad (2)$$

Moreover, for any such solution (u, λ), any nonempty level set $u^{-1}(c)$ is an $n - 1$ dimensional C^2 manifold.

Frobenius' Theorem (2)

Note: we call not u but a pair (u, λ) a **solution** to equation (1). Formally, consider the total differential equation (1):

$$\nabla u(x) = \lambda(x)g(x),$$

where $n \geq 2$, $U \subset \mathbb{R}^n$ is open and nonempty, and $g : U \to \mathbb{R}^n \setminus \{0\}$ is given. Then, a pair (u, λ) of real-valued functions defined on $V \subset U$ is called a **solution** to equation (1) around x^* if and only if, 1) V is an open neighborhood of x^*, 2) u is continuously differentiable, 3) λ is positive, and 4) the equation (1) holds for all $x \in V$. Frobenius' theorem is a result that provides a necessary and sufficient condition for the existence of a local solution to the above total differential equation under the differentiability of g.

Frobenius' Theorem (3)

However, in economic environment, g sometimes becomes not differentiable. Therefore, we want to relax the differentiability assumption of g from Frobenius' theorem. This is the purpose of this study.

Rademacher's Theorem

Recall that $n \geq 2$, $U \subset \mathbb{R}^n$ is open and nonempty, and $g : U \to \mathbb{R}^n \setminus \{0\}$ is given. Instead of differentiability, we assume that g is locally Lipschitz. The following result is the basis of this talk.

Rademacher's Theorem

Suppose that $U \subset \mathbb{R}^n$ is open and nonempty, and $g : U \to \mathbb{R}^m$ is locally Lipschitz. Then, g is differentiable almost everywhere.

Using this theorem, we can consider Jacobi's integrability condition a.e. for locally Lipschitz g.

Weak Solution to Total Differential Equations

Recall the total differential equation (1):

$$\nabla u(x) = \lambda(x) g(x).$$

We extend the notion of solutions. Let $x^* \in U$. We call a pair (u, λ) of real-valued functions defined on $V \subset U$ a **weak solution** to (1) around x^* if 1) V is an open neighborhood of x^*, 2) u is locally Lipschitz, 3) λ is positive a.e., 4) any nonempty level set $u^{-1}(c)$ is an $n-1$ dimensional C^1 manifold, and 5) equation (1) holds for almost every $x \in V$. Note that, every usual solution (u, λ) is a weak solution. Indeed, because u is C^1 and $\nabla u(x) \neq 0$ for all $x \in V$, by the preimage theorem, $u^{-1}(c)$ is either the empty set or a $n-1$ dimensional C^1 manifold.

Main Result

The main result of this talk is as follows.

Theorem

Suppose that $n \geq 2$, $U \subset \mathbb{R}^n$ is open and nonempty, and $g : U \to \mathbb{R}^n \setminus \{0\}$ is locally Lipschitz. Then, g satisfies Jacobi's integrability condition (2) for almost every $x \in U$ if and only if, for every $x^* \in U$, there exists a weak solution (u, λ) to (1) around x^*.

Idea: Nikliborc's Theorem (1)

Hereafter, I try to explain the idea of the proof. First, Frobenius' theorem is sometimes said to be 'equivalent' to the following theorem.

Theorem (Nikliborc, 1929)

Let $n \geq 2$, $U \subset \mathbb{R}^{n+1}$ be open and nonempty, and $f : U \to \mathbb{R}^n$ be C^1. Choose $(x^*, y^*) \in U$ and consider the following differential equation:

$$\nabla E(x) = f(x, E(x)),\ E(x^*) = y^*. \tag{3}$$

Define

$$s_{ij}(x, y) = \frac{\partial f_i}{\partial x_j}(x, y) + \frac{\partial f_i}{\partial y}(x, y) f_j(x, y). \tag{4}$$

Then, $s_{ij} \equiv s_{ji}$ for every $i, j \in \{1, ..., n\}$ if and only if for any $(x^*, y^*) \in U$, there exists a C^2 solution $E : V \to \mathbb{R}$ to (3), where $V \subset \mathbb{R}^n$ is an open neighborhood of x^*. Moreover, if f is C^k, then E is C^{k+1}.

Idea: Nikliborc's Theorem (2)

If we assume that Frobenius' theorem is correct, we can easily derive Nikliborc's theorem. Actually, define $g(x,y) = (f(x,y), -1)$. We can easily check that $s_{ij} = s_{ji}$ for every $i, j \in \{1, ..., n\}$ if and only if g satisfies Jacobi's integrability condition, and thus, there exists a solution (u, λ) around (x^*, y^*) such that

$$\nabla u(x,y) = \lambda(x,y) g(x,y).$$

By implicit function theorem, the level set $u^{-1}(u(x^*, y^*))$ can locally be identified with the graph of a C^2 function E defined on an open neighborhood of x^*, and we can easily show that E is a solution to (3). Therefore, Nikliborc's theorem holds.

Idea: Nikliborc's Theorem (3)

The converse is also true: that is, if we assume that Nikliborc's theorem is correct, then we can relatively easily derive Frobenius' theorem. To show this, because $g(x) \neq 0$, we assume without loss of generality that $g_n(x) \neq 0$. Let $\tilde{x} = (x_1, ..., x_{n-1})$ when $x = (x_1, ..., x_n)$. Define $f_i(x) = g_i(x)/g_n(x)$. Then, $s_{ij} = s_{ji}$ for $i, j \in \{1, ..., n-1\}$ if and only if g satisfies Jacobi's integrability condition, and thus for each $x^* \in U$, there exists a C^2 solution $E(\tilde{x})$ to the following partial differential equation:

$$\nabla E(\tilde{x}) = f(\tilde{x}, E(\tilde{x})),\ E(\tilde{x}^*) = x_n^*.$$

Let u be a continuously differentiable function such that the level set $u^{-1}(u(x^*))$ coincides with the graph of the solution E to the above differential equation, and let $\lambda(x) = \frac{\partial u}{\partial x_n}(x)/g_n(x)$. Then, (u, λ) is a solution to (1), as desired.

Idea: Nikliborc's Theorem (4)

Hence, we find that Frobenius' theorem implies Nikliborc's theorem, and vice versa. By the way, recently I obtained an extension of Nikliborc's theorem.

Theorem (Hosoya, 2021)

Suppose that $n \geq 2$, $U \subset \mathbb{R}^{n+1}$ is open and nonempty, and $f : U \to \mathbb{R}^n$ is locally Lipschitz. Let $(x^*, y^*) \in U$ and consider the partial differential equation (3). Define $s_{ij}(x)$ as the equation (4) if it can be defined. Then, $s_{ij} = s_{ji}$ almost everywhere if and only if for every $(x^*, y^*) \in U$, there exists a local C^1 solution $E : V \to \mathbb{R}$ to (3), where $V \subset \mathbb{R}^n$ is an open neighborhood of x^*.

If we can use this theorem instead of Nikliborc's theorem, then we expect to obtain the corresponding extension of Frobenius' theorem. Actually, our main result is such an extension.

A Lemma (1)

To derive the main result, we need the following lemma.

Lemma (Hosoya, 2021)

Suppose that $n \geq 2$, $U \subset \mathbb{R}^{n+1}$ is open and nonempty, $f : U \to \mathbb{R}^n$ is locally Lipschitz, and if we define s_{ij} as (4), $s_{ij}(x, y) = s_{ji}(x, y)$ for all $i, j \in \{1, ..., n\}$ and almost all $(x, y) \in U$. Choose any $(x^*, y^*) \in U$ and consider the following ordinary differential equation:

$$\dot{c}(t) = f((1-t)x^* + tx, c(t)) \cdot (x - x^*),\ c(0) = y. \tag{5}$$

Let $c(t; x, y)$ be the solution function of (5): that is, $t \mapsto c(t; x, y)$ is a nonextendable solution to (5) with parameter (x, y). Let $V \subset \mathbb{R}^n$ be an open and convex neighborhood of x^*. Then, there exists a classical solution $E : V \to \mathbb{R}$ to (3) if and only if the domain of $c(t; x, y)$ includes $[0, 1] \times V \times \{y^*\}$. Moreover, if such a solution E exists, then $E(x) = c(1; x, y^*)$.

A Lemma (2)

Recall that the domain of the solution function $c(t; x, y)$ to (5) is open. Hence, we have obtained the following corollary:

Corollary

Suppose that $n \geq 2$, $U \subset \mathbb{R}^{n+1}$ is open and nonempty, $f : U \to \mathbb{R}^n$ is locally Lipschitz, and if we define s_{ij} as (4), $s_{ij}(x, y) = s_{ji}(x, y)$ for all $i, j \in \{1, ..., n\}$ and almost all $(x, y) \in U$. Choose any $(x^*, y^*) \in U$, and for an open and convex neighborhood $V \subset \mathbb{R}^n$ of x^*, suppose that there exists a C^1 solution $E^{y^*} : V \to \mathbb{R}$ to (3). Then, there exists $\varepsilon > 0$ such that if $|y - y^*| < \varepsilon$, then the following partial differential equation

$$\nabla E(x) = f(x, E(x)),\ E(x^*) = y$$

also has a solution $E^y : V \to \mathbb{R}$.

Sketch of the Proof (1)

Now, suppose that $g : U \to \mathbb{R}^n \setminus \{0\}$ is locally Lipschitz and $x^* \in U$. Because $g(x) \neq 0$ for all $x \in U$, we assume without loss of generality that $g_n(x^*) = 0$. In addition to this, to simplify our arguments, we consider only the case in which $g_n(x) > 0$ for all $x \in U$. Then, we can define $f_i(x) = g_i(x)/g_n(x)$. Then, g satisfies (2) at x if and only if $s_{ij}(x) = s_{ji}(x)$. Hereafter, we write $\tilde{x} = (x_1, ..., x_{n-1})$ if $x = (x_1, ..., x_n)$.

Choose any $x^* \in U$. If there exists a weak solution (u, λ) to (1) around x^*, then, for any h such that $|h|$ is sufficiently small, there exists a C^1 function $E^h : V \to \mathbb{R}$ such that $u^{-1}(u(\tilde{x}^*, x_n^* + h))$ locally coincides with the graph of E^h. It is easy to show that E^h is a C^1 solution to (3), and thus $s_{ij} = s_{ji}$ a.e.. Hence, if a weak solution exists, then g must satisfy (2) almost everywhere.

Sketch of the Proof (2)

Conversely, suppose that g satisfy (2) almost everywhere. Then, $s_{ij}(x) = s_{ji}(x)$ for all $i, j \in \{1, ..., n-1\}$ and almost all $x \in U$. Consider the following differential equation.

$$\nabla E(\tilde{x}) = f(\tilde{x}, E(\tilde{x})),\ E(\tilde{x}^*) = x_n^* + h. \tag{6}$$

By our Corollary, there exists $\varepsilon > 0$ such that if $|h| < 2\varepsilon$, then for $V' = \prod_{i=1}^{n-1}]x_i^* - 2\varepsilon, x_i^* + 2\varepsilon[$, there exists a C^1 solution $E^h : V' \to \mathbb{R}$ to (6).

Sketch of the Proof (3)

Note that, if $c(t;\tilde{x},x_n)$ be the solution function of the following differential equation

$$\dot{c}(t)=f((1-t)\tilde{x}^*+t\tilde{x},c(t))\cdot(\tilde{x}-\tilde{x}^*),\ c(0)=x_n,$$

then, by our Lemma, $E^h(\tilde{x})=c(1;\tilde{x},x_n^*+h)$. We can easily check that $c(t;\tilde{x},x_n)$ is increasing in x_n, and thus $E^h(\tilde{x})$ is increasing in h. Because $c(t;\tilde{x},x_n)$ is continuous, if $\delta>0$ is sufficiently small, then for all $\tilde{x}\in V'$ such that $|\tilde{x}_i-\tilde{x}_i^*|<\delta$ for all $i\in\{1,...,n\}$,

$$E^{-\varepsilon}(\tilde{x})<x_n^*-\delta<x_n^*+\delta<E^{\varepsilon}(\tilde{x}).$$

By the intermediate value theorem, for all $x_n\in]x_n^*-\delta,x_n^*+\delta[$, there uniquely exists $h^*\in]-\varepsilon,\varepsilon[$ such that $E^{h^*}(\tilde{x})=x_n$. Define $V=\prod_{i=1}^n]x_i^*-\delta,x_i^*+\delta[$, and $u(x)$ as such h^*.

Sketch of the Proof (4)

By definition, if $x \in V$ and $|h| < \varepsilon$, then

$$h = u(x) \Leftrightarrow c(1; \tilde{x}, x_n^* + h) = x_n. \tag{7}$$

Because $c(1; \tilde{x}, x_n)$ is increasing in x_n, we have that $u(x)$ is increasing in x_n. Note that, because f is locally Lipschitz, using Gronwall's inequality, we can show that $c(t; \tilde{x}, x_n)$ is locally Lipschitz. Therefore, we can assume that there exists $L > 0$ such that the function

$$w \mapsto f((1-t)\tilde{x}^* + t\tilde{x}, c(t; \tilde{x}, w))$$

has a Lipschitz constant L on $[0,1] \times \prod_{i=1}^{n-1}[x_i^* - \delta, x_i^* + \delta] \times [x_n^* - \varepsilon, x_n^* + \varepsilon]$. Replacing $\delta > 0$ so small if needed, we can assume that $2L\delta < 1$.

Sketch of the Proof (5)

Then,

$$\begin{aligned} &c(1;\tilde{x},w+h)-c(1;\tilde{x},w)\\ =\ &h+\int_0^1[f((1-t)\tilde{x}^*+t\tilde{x},c(t;\tilde{x},w+h))\\ &-f((1-t)\tilde{x}^*+t\tilde{x},c(t;\tilde{x},w))]\cdot(\tilde{x}-\tilde{x}^*)dt\\ \geq\ &h(1-L\|\tilde{x}-\tilde{x}^*\|)>2^{-1}h. \end{aligned}$$

Therefore, by equation (7), if $x_n < x_n'$, then $u(\tilde{x},x_n')-u(\tilde{x},x_n)<2(x_n'-x_n)$. In other words, the function $u(x)$ has the Lipschitz constant 2 in x_n.

Sketch of the Proof (6)

Choose $x \in V$ and let $w = u(x)$. Recall that $\nabla E^w(\tilde{x}) = f(x)$. Therefore, there exists $\delta' > 0$ such that if $|h| < \delta'$, then $(x_1, ..., x_i + h, ..., x_n) \in V$, and $y_n(h) = E^w(x_1, ..., x_i + h, ..., x_{n-1}) \in [x_n - (|f_i(x)| + 1)|h|, x_n + (|f_i(x)| + 1)|h|]$. Define $y(h) = (x_1, ..., x_i + h, ..., x_{n-1}, y_n(h))$. Then, $u(y(h)) = u(x) = w$. Because u is Lipschitz in x_n, there exists $L' > 0$ independent of x, h such that

$$
\begin{aligned}
&|u(x_1, ..., x_i + h, ..., x_n) - u(x)| \\
= &|u(x_1, ..., x_i + h, ..., x_n) - u(y(h))| \\
\le &L'(\max_{y \in \bar{V}} \|f(y)\| + 1)|h|,
\end{aligned}
$$

which implies that u is locally Lipschitz on V in x_i.

Sketch of the Proof (7)

Therefore, u is locally Lipschitz, and by Rademacher's theorem, u is differentiable almost everywhere. If u is differentiable at $x \in V$, then for $h = u(x)$, $E_h(\tilde{x}) = x_n$ and the graph of E_h coincides with $u^{-1}(h)$, which implies that $u^{-1}(h)$ is an $n-1$ dimensional C^1 manifold. By the chain rule,

$$\begin{aligned} &\frac{\partial u}{\partial x_i}(x) + \frac{\partial u}{\partial x_n}(x) f_i(x) \\ = &\frac{\partial u}{\partial x_i}(x) + \frac{\partial u}{\partial x_n}(x) \frac{g_i(x)}{g_n(x)} = 0, \end{aligned}$$

which implies that either $\frac{\partial u}{\partial x_i}(x) = g_i(x) = 0$ or

$$\frac{\frac{\partial u}{\partial x_i}(x)}{g_i(x)} = \frac{\frac{\partial u}{\partial x_n}(x)}{g_n(x)}$$

for every $i \in \{1, ..., n-1\}$.

Sketch of the Proof (8)

Hence, if we define

$$\lambda(x) = \frac{\frac{\partial u}{\partial x_n}(x)}{g_n(x)},$$

then

$$\nabla u(x) = \lambda(x) g(x)$$

almost everywhere. We can easily show that $\lambda(x) > 0$ almost everywhere using equation (7) and local Lipschitz property, and thus this completes the proof of our main theorem.

Notes on Economic Application (1)

Unfortunately, this weak solution u is not necessarily quasi-concave, and thus Lagrange's rule may not work well. If g is C^1, the quasi-concavity of u is equivalent to the following condition:

$$v^T g(x) = 0 \Rightarrow v^T \nabla g(x) v \leq 0. \tag{8}$$

This fact is proved in Hosoya (2022, COT). I expect that this result holds even when g is only locally Lipschitz: that is, I expect that when g is locally Lipschitz, any weak solution u is quasi-concave if and only if (8) holds almost everywhere.

Notes on Economic Application (2)

Another problem is as follows: the above weak solution u is not entirely defined, but we want to obtain the globally defined utility function $\tilde{u} : \Omega \to \mathbb{R}$. This can be solved by using the following differential equation:

$$\dot{y}(t) = (g(y(t)) \cdot x)v - (g(y(t)) \cdot v)x, \; y(0) = x.$$

Choose $x, v \in \Omega$. Define $w = ((v \cdot x)v - (v \cdot v)x$, $t^*(x, v) = \inf\{t \geq 0 | y(t) \cdot w \geq 0\}$, and

$$u_v(x) = \frac{\|y(t^*(x, v))\|}{\|v\|}.$$

If g is C^1, then using the local solution u, we can derive that u_v is a utility function of this consumer. I expect that it is still true even when g is not differentiable but only locally Lipschitz.

Extension

Actually, a modern form of Frobenius' theorem is on a theorem for not a single vector field but a finite tuple of linearly independent vector fields $(g_1, ..., g_k)$. This tuple of vector fields is said to **locally integrable** around x^* if and only if there exists $u_1, ..., u_k$ defined on a neighborhood of x^* such that

$$\nabla u_i(x) = \sum_{j=1}^{k} \lambda_{ij}(x) g_j(x)$$

for all $i \in \{1, ..., k\}$, and $(\nabla u_1(x), ..., \nabla u_k)$ is also linearly independent. I want to extend my result to this problem. However, this is a future task.

Thank you for your attention.