Non-Smooth Integrability Theory

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Motivation: Integrability Theory (1)

I start this talk by explaining the classical consumer theory. Let $\mathbb{R}^N_+ \equiv \{x \in \mathbb{R}^N | x_i \ge 0 \text{ for all } i\}$, and $\mathbb{R}^N_{++} \equiv \{x \in \mathbb{R}^N | x_i > 0 \text{ for all } i\}$. Let $n \ge 2$ be given. For each $i \in \{1, ..., n\}$, $x_i \ge 0$ represents the consumption plan of i-th commodity, and $p_i > 0$ denotes the corresponding price of i-th commodity. Therefore, the consumption plan and the price system are represented by a non-negative vector $x = (x_1, ..., x_n)$ and a positive vector $p = (p_1, ..., p_n)$. For a given price vector p and an income level $m > 0$, let $f(p, m) \in \mathbb{R}^n_+$ be the consumer's choice of consumption vector. It is natural that $p \cdot f(p, m) \le m$.
Define $\Omega = \mathbb{R}^n_+$. We consider that Ω is the set of all possible consumption vectors, and call this set the **consumption set**. Hence, the consumer's choice function $f(p, m)$ satisfies $f(p, m) \in \Omega$ and $p \cdot f(p, m) \le m$.

Motivation: Integrability Theory (2)

Consider that a function $u : \Omega \to \mathbb{R}$ is given, and suppose that $f^u(p, m)$ is the unique maximizer of the following problem.

$$\begin{aligned} \max \quad & u(x) \\ \text{subject to.} \quad & x \in \Omega, \\ & p \cdot x \le m. \end{aligned}$$

Then, we call this u a **utility function** of the consumer, and f^u the **demand function** corresponding to u. The utility function is considered as a function that represents the consumer's preference, and the demand function represents the consumer's choice. Of course, any demand function satisfies $f^u(p, m) \in \Omega$ and $p \cdot f^u(p, m) \le m$, and if u is increasing, then $p \cdot f^u(p, m) = m$. The last equality is called **Walras' law**.

Motivation: Integrability Theory (3)

In economic theory, researchers sometimes want to determine this u. However, because u represents the consumer's preference, the information of u is hidden in their mind. Therefore, it is necessary to estimate u from only purchase behavior, although it is difficult. Fortunately, f can directly be estimated from purchase data. Hence, if we can derive u from f, then the estimation problem of u can be changed to the estimation problem of f, which reduces the difficulty of the problem. The research area that studies this inverse problem is called the **integrability theory**.

Motivation: Integrability Theory (4)

Hosoya (2017) studied this theory, and gave a condition for a C^1 function f to have a corresponding utility function and a procedure for computing its utility function u_f. The paper also discussed the continuity of the mapping $f \mapsto u_f$, and showed that this mapping is continuous for local C^1 topology in the space of f and local uniform topology in the space of u_f. This continuity is related to the evaluation of estimation errors, and is used to show that if the estimate of f is sufficiently close to the true value, then u_f is also close to the true value.
The problem is the topologies used in the continuity result. Local C^1 topology is too strong to be applicable for actual estimation of f. Therefore, we should challenge to derive these results under a weaker topology.

Motivation: Integrability Theory (5)

So, let us consider weakening the topology of the space of f to a local uniform topology. However, if we do so, the space of C^1 functions becomes incomplete. In other words, we cannot assure the smoothness for the true value f, even if the estimate of f is smooth. Because our previous results require C^1 for f to calculate u_f, this implies that the true u_f cannot be calculated. This problem makes the application of many statistical methods difficult.

Thus, we aim to remove C^1 assumption of f. However, it is known that f can correspond to two u, v that represent different orders if only continuity is assumed (Mas-Colell, 1977), and in this case, we cannot determine which order is the true consumer's preference. Hence, we start the discussion by assuming that f is locally Lipschitz. As is well-known, a locally Lipschitz function is Fréchet differentiable almost everywhere (Rademacher's theorem). This property plays an important role in the present study.

Candidate of Demand

Recall that $\Omega = \mathbb{R}^n_+$. A function $f : \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to \Omega$ is called a **candidate of demand** (CoD) if it satisfies the budget inequality $p \cdot f(p, m) \le m$. If this inequality holds with equality for any (p, m), then we say that f satisfies **Walras' law**. For a CoD f, let $R(f)$ be the range of f.
Suppose that $U \subset \mathbb{R}^n \times \mathbb{R}$ and $f : U \to \mathbb{R}^n$ is differentiable at (p, m). Define

$$S_f(p, m) = D_p f(p, m) + D_m f(p, m) f^T(p, m).$$

This matrix-valued function S_f is called the **Slutsky matrix**. Note that, if f is a locally Lipschitz CoD, then $S_f(p, m)$ can be defined almost everywhere.

Demand Function

For a weak order $\succsim$ on Ω, define

$$f^{\succsim}(p,m) = \{x \in \Omega | p \cdot x \le m \text{ and } \forall y \in \Omega,\ p \cdot y \le m \Rightarrow x \succsim y\}.$$

If $u : \Omega \to \mathbb{R}$ satisfies

$$u(x) \ge u(y) \Leftrightarrow x \succsim y,$$

then $f^{\succsim}$ is also written as f^u. For a CoD f, If $f = f^{\succsim}$, then f is called the **demand function** corresponding to $\succsim$. If $f = f^u$, then it is also said that f corresponds to u. We say that f is a demand function if $f = f^{\succsim}$ for some weak order $\succsim$.

The First Main Result (1)

Theorem 1

Suppose that f is a locally Lipschitz CoD that satisfies Walras' law, and the Slutsky matrix is symmetric and negative semi-definite almost everywhere. Fix $\bar{p} \gg 0$, and define a function $u_{f,\bar{p}}$ as follows. FIrst, if $x \notin R(f)$, then define $u_{f,\bar{p}}(x) = 0$. Second, if $x = f(p, m)$, then solve the following ODE:

$$\dot{c}(t) = f((1-t)p + t\bar{p}, c(t)) \cdot (\bar{p} - p), \; c(0) = m, \tag{1}$$

and define $u_{f,\bar{p}}(x) = c(1)$. Then, the following hold. (next page)

The First Main Result (2)

Theorem 1 (cont.)

1. The value $c(1)$ can be defined and is independent of the choice of $(p, m) \in f^{-1}(x)$, and thus $u_{f,\bar{p}}$ is well-defined.
2. $f = f^{u_{f,\bar{p}}}$.
3. The restriction of $u_{f,\bar{p}}$ to $R(f)$ is upper semi-continuous.
4. If $f = f^{\succsim}$ and $\succsim$ is upper semi-continuous on $R(f)$, then for every $x, y \in R(f)$,

$$u_{f,\bar{p}}(x) \ge u_{f,\bar{p}}(y) \Leftrightarrow x \succsim y.$$

NOTE: a weak order $\succsim$ on Ω is said to **upper semi-continuous** on $A \subset \Omega$ if for any $x \in A$, $\{y \in A | y \succsim x\}$ is relatively closed in A.

The First Main Result (3)

As a corollary, we can obtain the following result.

Corollary 1

Suppose that f is a locally Lipschitz CoD that satisfies Walras' law. Then, the following statements are equivalent.

1) $f = f^{\succsim}$ for a weak order $\succsim$.
2) $f = f^{u_{f,\bar{p}}}$, where $u_{f,\bar{p}}$ is defined in Theorem 1.
3) S_f is symmetric and negative semi-definite almost everywhere.
4) For every $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$, there exists a C^1 concave solution $E : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ to the following PDE:

$$\nabla E(q) = f(q, E(q)), \ E(p) = m. \tag{2}$$

The Idea of the Proof of Theorem 1 (1)

We separate the idea of the proof of Theorem 1 into three steps. In the first step, suppose that we already know that $f = f^u$ for some continuous function $u : \Omega \to \mathbb{R}$. Define

$$E^x(q) = \inf\{q \cdot y | u(y) \ge u(x)\}. \tag{3}$$

It is known that

$$u(x) \ge u(y) \Leftrightarrow E^x(\bar{p}) \ge E^y(\bar{p}),$$

and thus, if we define $u_{f,\bar{p}}(x) = E^x(\bar{p})$, then $u_{f,\bar{p}}$ has the same information of the consumer's preference as u.

The Idea of the Proof of Theorem 1 (2)

In the second step, we suppose that we already know that there exists u such that $f = f^u$, but the shape of u is unknown. If we can calculate $E^x(\bar{p})$, then we can define $u_{f,\bar{p}}(x)$ as in the first step, and this $u_{f,\bar{p}}$ has the same information of the consumer's preference as u. Unfortunately, because the right-hand side of (3) includes information on u, we cannot calculate $E^x(\bar{p})$ directly. However, it is well-known that E^x satisfies the following equation:

$$\nabla E^x(q) = f(q, E^x(q))$$

for all q (called Shephard's lemma). Moreover, E^x is concave and C^1, and if $x = f(p, m)$, then $E^x(p) = m$. In conclusion, E^x is a concave solution to the PDE (2).

The Idea of the Proof of Theorem 1 (3)

Hence, choose any $x \in \Omega$ and $(p, m) \in f^{-1}(x)$, and define

$$c(t) = E^x((1-t)p + t\bar{p}).$$

Then, $c(1) = E^x(\bar{p}) = u_{f,\bar{p}}(x)$. Moreover, $c(t)$ is a solution to the following ODE:

$$\dot{c}(t) = f((1-t)p + t\bar{p}, c(t)) \cdot (\bar{p} - p),\ c(0) = m,$$

which is the same as equation (1) appears in Theorem 1. Therefore, we can obtain $u_{f,\bar{p}}(x)$ for the same method as in Theorem 1, and thus $u_{f,\bar{p}}$ represents the same order as u. Thus, we can calculate the information of u by the above method.

The Idea of the Proof of Theorem 1 (4)

In the third step, a CoD f is given, and any other information is absent. If $f = f^{\succsim}$ for some weak order $\succsim$, then by almost the same arguments as in the second step, we can calculate $u_{f,\bar{p}}$, and prove that $f = f^{u_{f,\bar{p}}}$. Theorem 1 states that the necessary and sufficient condition for the existence of such $\succsim$ is the symmetry and negative semi-definiteness of S_f almost everywhere. This fact can relatively easily be shown if f is C^1, but is valid even when f is not differentiable but only locally Lipschitz.

The Second Main Result (1)

The existence of a solution to the PDE (2) is robust under limit manipulation. Hence, using condition 4) of Corollary 1, we can show the following theorem.

Theorem 2

Suppose that (f^k) is a sequence of locally Lipschitz demand functions that satisfy Walras' law that uniformly converges to f on any compact set. If f is also locally Lipschitz, then f is a demand function.

Hence, the space of demand function is closed in some sense.

The Second Main Result (2)

Choose any sequence $L = (L_\nu)$ of positive real numbers, and define $\mathscr{F}_L$ as the set of all demand functions f that has a Lipschitz constant L_ν on the set $\Delta_\nu = [\nu^{-1}, \nu]^{n+1}$. Define a metric ρ as

$$\rho(f, f') = \sum_{\nu=1}^{\infty} 2^{-\nu} \arctan\left(\sup_{(p,m)\in\Delta_\nu} \|f(p, m) - f'(p, m)\| \right).$$

It is easy to show that (f^k) converges to f on any compact set if and only if $\rho(f^k, f)$ converges to 0.
By Theorem 2 and Ascoli-Arzelà's theorem, we can show the following corollary.

Corollary 2

$\mathscr{F}_L$ is a compact metric space under the metric ρ.

A Problem on Calculation to Utility Function

We can define $u_{f,\bar{p}}$ appropriately only on $R(f)$. Hence, we want that $R(f)$ is so wide that $\mathbb{R}^n_{++} \subset R(f)$. However, this property may be broken by limit manipulation. That is, the exists a sequence (f^k) on $\mathscr{F}_L$ such that $R(f^k) = \mathbb{R}^n_{++}$ for any k and $f^k \to f$, where $R(f)$ is very narrower than $\mathbb{R}^n_{++}$. For example, let

$$u^k(x_1, x_2) = \begin{cases} (x_1^{-k} + x_2^{-k})^{-\frac{1}{k}} & \text{if } x_1 \neq 0 \neq x_2, \\ 0 & \text{otherwise,} \end{cases}$$

and $f^k = f^{u^k}$. We can show that for $L_\nu = \nu^5$, $f^k \in \mathscr{F}_L$ and $R(f^k) = \mathbb{R}^2_{++}$. However, f^k converges to f uniformly on any compact set for $f = f^u$ with $u(x_1, x_2) = \min\{x_1, x_2\}$, and $R(f) = \{x \in \mathbb{R}^2_{++} | x_1 = x_2\}$. This example indicates that the wideness of $R(f)$ must be exogenously assumed.

The C Axiom

We hereafter assume that $R(f)$ includes $\mathbb{R}^n_{++}$. Define an **inverse demand correspondence** $G^f : \mathbb{R}^n_{++} \twoheadrightarrow \mathbb{R}^n_{++}$ of a CoD f as

$$G^f(x) = \left\{ p \in \mathbb{R}^n_{++} \middle| \sum_{i=1}^n p_i = 1,\ f(p, p \cdot x) = x \right\}.$$

We say that f satisfies the **C axiom** if and only if G^f is compact- and convex-valued, and upper hemi-continuous. The following result is shown in Hosoya (2020).

Lemma 1

$u_{f,\bar{p}}$ is continuous on $\mathbb{R}^n_{++}$ if and only if f satisfies the C axiom.

Continuity of Calculation (1)

Using Lemma 1, we obtain the following theorem.

Theorem 3

Suppose that (f^k) is a sequence of locally Lipschitz demand functions that satisfy Walras' law and the C axiom. Suppose also that (f^k) converges to f with respect to ρ, and f is a locally Lipschitz CoD that satisfies Walras' law and the C axiom. Then, $u_{f^k,\bar{p}}$ uniformly converges to $u_{f,\bar{p}}$ on any compact set $C \subset \mathbb{R}^n_{++}$.

Continuity of Calculation (2)

Choose a sequence $M = (M_\nu)$ of positive real numbers, and let $\mathscr{F}_{L,M}$ be the set of all $f \in \mathscr{F}_L$ such that for any $x \in]\nu^{-1}, \nu[^n$, $p \in G^f(x)$ implies that $p \in [M_\nu, 1]^n$. Then, we can show the following Corollary.

Corollary 3

$\mathscr{F}_{L,M}$ is also compact under the metric ρ, and if (f^k) is a sequence of $\mathscr{F}_{L,M}$ that converges to f, then $u_{f^k,\bar{p}}$ uniformly converges to $u_{f,\bar{p}}$ on any compact set $C \subset \mathbb{R}^n_{++}$.

Continuity of Calculation (3)

Using the compactness of $\mathscr{F}_L$, we can easily extend the results of Corollaries 2-3 to the case of pointwise convergence.

Theorem 4

Suppose that (f^k) is a sequence of $\mathscr{F}_L$ that pointwise converges to f. Then, $f \in \mathscr{F}_L$. If, in addition, $f^k \in \mathscr{F}_{L,M}$ for all k, then $f \in \mathscr{F}_{L,M}$, and $u_{f^k,\bar{p}}$ converges to $u_{f,\bar{p}}$ uniformly on any compact set $C \subset \mathbb{R}^n_{++}$.

A Future Task (1)

There are two approaches to the integrability theory, direct and indirect. The present talk is related to the direct approach. In contrast, the indirect approach treats a function $g : \Omega \to \mathbb{R}^n_{++}$ such that $f(g(x), g(x) \cdot x) = x$ for all $x \in \Omega$. If $f = f^u$ for some differentiable function u, then by Lagrange's first-order condition,

$$\nabla u(x) = \lambda(x) g(x) \tag{4}$$

for all $x \in \Omega$. The indirect approach treats the above equation as a differential equation, and calculates u and λ from g.
The existence result for a solution to (4) is called **Frobenius' theorem**, which is deeply related to the following PDE with an initial value condition:

$$\nabla u(x) = f(x, u(x)),\ u(x^*) = u^*. \tag{5}$$

Note that, the PDE (2) in our research is of this form.

A Future Task (2)

Frobenius' Theorem

Suppose that $U \subset \mathbb{R}^n$ is open, and $g : U \to \mathbb{R}^n \setminus \{0\}$ is C^1. Then, the following **total differential equation**

$$\nabla u(x) = \lambda(x) g(x)$$

has a local solution (u, λ) around x^* for any $x^* \in U$ if and only if the following Jacobi's integrability condition holds:

$$g_i \left(\frac{\partial g_j}{\partial x_k} - \frac{\partial g_k}{\partial x_j} \right) + g_j \left(\frac{\partial g_k}{\partial x_i} - \frac{\partial g_i}{\partial x_k} \right) + g_k \left(\frac{\partial g_i}{\partial x_j} - \frac{\partial g_j}{\partial x_i} \right) = 0. \quad (6)$$

A Future task (3)

Suppose that f is C^1. Define

$$g(p, m) = (f(p, m), -1).$$

Then, it is easy to show that Jacobi's condition (6) is equivalent to the symmetry of S_f. Applying Frobenius' theorem, we can easily obtain that the existence of a local solution to (5) is equivalent to the symmetry of S_f. This is called Nikliborc's theorem.

A Future Task (4)

Conversely, suppose that we have obtained Nikliborc's theorem, and consider a C^1 function $g : U \to \mathbb{R}^n \setminus \{0\}$. Without loss of generality, we assume that $g_n(x^*) \neq 0$, and define

$$f_i(x) = -\frac{g_i(x)}{g_n(x)}.$$

Then, g satisfies Jacobi's condition (6) if and only if S_f is symmetric, and using Nikliborc's theorem, we can easily obtain Frobenius' theorem. In conclusion, Nikliborc's theorem is equivalent to Frobenius' theorem in some sense.

A Future Task (5)

Actually, in proving Theorem 1, we extend Nikliborc's theorem to non-differentiable f. Using this result, we can obtain an extended result of Frobenius' theorem for non-differentiable g. Hence, using this result, we may be able to extend results in the indirect approach of the integrability theory such as Hosoya (2013). However, this is a future task.

Thank you for your attention.