Understanding Negation Implicationally in the Relevant Logic **R**

Takuro Onishi*
takuro.onishi@gmail.com

Version 0.99

Abstract

A *star-free* relational semantics for relevant logic is presented together with a sound and complete sequent proof theory (display calculus). It is an extension of the *dualist approach* to negation regarded as modality, according to which de Morgan negation in relevant logic is better understood as the confusion of two negative modalities. The present work shows a way to define them in terms of implication and a new connective, co-implication, which is modeled by respective ternary relations. The defined negations are confused by a special constraint on ternary relation, called the *generalized star postulate*, which implies definability of the Routley star in the frame. The resultant logic is shown to be equivalent to the well-known relevant logic **R**. Thus it can be seen as a reconstruction of **R** in the dualist framework.

Introduction

We present a *star-free* relational semantics for the relevant logic together with a sound and complete sequent proof theory in the style of display calculus and prove that the resultant logic is equivalent to the well-known relevant logic **R**. Thus it can be seen as a star-free reconstruction of **R**. The operation called the Routley star which models de Morgan negation in Routley-Meyer semantics [18, 19, 20] for relevant logic has often been criticized for a lack of intuitive meaning [2, 22]. This paper presents a way to make sense of the Routley star in terms of ternary relation on frames, or equivalently, understand de Morgan negation implicationally.

The novelty of this work compared to preceding attempts [8, 25] lies in its *dualist* approach. Based on the conception of negation as modal operator

*The final publication is available at Springer via http://dx.doi.org/10.1007/s11225-016-9676-x.

[3], Onishi [12] showed that de Morgan negation is better understood as the *confusion of two negative modalities*, which are modeled by binary relations on the frame. The confusion is forced by the dualist reformulation of the *star postulate* studied in the literature [9, 16, 10], which implies definability of the Routley star by binary relation(s). The present paper further develops this idea and generalizes it to an implicational framework in the following way.

We employ two implication-type connectives: the relevant implication and a new, relevant *co-implication*, in terms of which the two negative modalities are respectively defined. Correspondingly, frames and models are equipped with two ternary relations: one for implication and the other for co-implication, in terms of which the two binary relations for negative modalities are respectively defined. We also present the *generalized star postulate* GS, which is a constraint on the two ternary relations and implies the dualist star postulate. Then, by applying the result of [12] to these defined negative modalities and defined binary relations, we obtain de Morgan negation in the language and the Routley star in the model.

In section 1, the background is explained. Section 2 model-theoretically introduces a logic with the generalized star postulate and proves its equivalence with the well-known relevance logic **R**. Thus, the logic, called **GS**, can be seen as a reconstruction of **R**. Section 3 is for proof theory. We provide a display calculus formulation to **GS** in which the generalized star postulate is expressed as a structural rule. In the concluding section, the significance of the dualist framework is briefly discussed.

1 Ternary relation, Routley-star, and negation as modality

In Routley-Meyer semantics, the truth clauses for implication and negation are given as follows: for any state (or set-up, situation, point, etc.) x,

$$\begin{aligned} x \models A \to B &\iff \forall y, z : Rxyz \,\&\, y \models A \Rightarrow z \models B; \\ x \models {\sim} A &\iff x^* \not\models A, \end{aligned}$$

where R is a ternary relation and $*$ is an operation on the frame.

It has become standard to interpret the ternary relation R in terms of *information*. Mares' SEP article [11] lists three types of such interpretations. According to Dunn's version, $Rxyz$ is best understood to say that "the combination of the pieces of information in x and y ... is a piece of information in z" [5]. Notions such as "information channel" or "information link" invoked by Restall and Mares also make sense [13, 9, 10] under this interpretation. The first argument x of $Rxyz$ is (or represents) an information channel that transmits information from y to z. The channel x contains constraints or information links of the form $A \to B$, which, together with a piece of information A in y, produces B in z, as the truth clause indicates.

There have been some complaints about a lack of intuitive interpretation of the star operation [2, 22]. Relevant logicians such as Dunn, Restall, and Mares [4, 15, 10] responded that the Routley star can be made sense of through the conception of negation as modality of impossibility [3]. According to it, negation as impossibility is understood in terms of a binary relation C on frames:

$$x \models {\sim} A \iff \forall y : xCy \Rightarrow y \not\models A.$$

If y supports A ($y \models A$) while x rejects A ($x \models {\sim} A$), the two states should be said to be incompatible (not xCy). Hence the binary relation C here may be understood as compatibility.

Imposing different constraints on C generates different valid formulas and inferences that involve negation. De Morgan negation modeled by the Routley star, which supports double negation laws and all four de Morgan laws, is identified as impossibility with symmetry of compatibility ($xCy \Rightarrow yCx$) and the *star postulate* found by Mares [8, p.587][1], which claims the existence of maximal compatible states:

$$\forall x \exists y : xCy \,\&\, \forall z : xCz \Rightarrow z \leq y, \tag{*}$$

where $\leq$ is an ordering in the frame along which truth is inherited. Now the star operation is definable: just set x^* to be y above.

Thus the problematic Routley star is reducible to a more intuitive binary relation. It would be nice if we could go one step further. Routley and Meyer explained $Rxyz$ as saying that x and y are "*compossible* (better, maybe, *compatible*) relative to" z [18, p.200, original italics]. For them, ternary relation is a generalization of binary compatibility relation. It must be possible, then, to incorporate the latter into the former and thereby reduce negation to implication.

Define ${\sim} A := A \to f$ with some false constant f. Zhou [25] treated f (he used the symbol $\perp$) as the universally false proposition ($x \not\models f$ for any x). Then the compatibility relation is defined as

$$xCy \iff \exists z : Rxyz.$$

In this case, xCy represents the compatibility of x and y in *some* respect z. Mares [8] defined f and C in a slightly different manner. His definition was, in effect,

$$x \models f \iff x \not\leq 0; \qquad\qquad xCy \iff \exists z \leq 0 : Rxyz,$$

where 0 is the so-called *regular* (or *normal*) state where all the logical truths hold. Mares' compatibility is more specific than Zhou's, but it is harder to

[1] Restall found this postulate independently later [16]. The name "star postulate" is from Mares' book [10].

figure out what it amounts to. We will later present a modified definition of C that makes clear how and in what sense two states are compatible.

Aside from this, the unification of negation into implication is not completed because the star postulate has not been generalized in the above works. In the present setting where C is defined in terms of the ternary relation R, the star postulate is a constraint on a restricted part of R. If we are to understand negation implicationally, it is desirable to have it generalized to an intuitive constraint on unrestricted ternary relation, which implies the star postulate as a special case. Therefore the task of this paper is to find a generalization of the star postulate.

For this purpose, we introduce a different perspective: another negative modality of *unnecessity* and a binary relation of *exhaustiveness*, which have been studied alongside impossibility and compatibility [25, 6, 21]. Let the formula $\blacktriangleright A$ read "A is unnecessary" and xEy "the states x and y are jointly exhaustive" where E is a binary relation of the frame. Then set

$$x \models \blacktriangleright A \iff \exists y : xEy \,\&\, y \not\models A.$$

In keeping with this, we shall write $\triangleright A$ for negation as impossibility instead of $\sim A$. $\triangleright$ and $\blacktriangleright$ are dual to each other. For example, whereas the symmetry of compatibility C corresponds to double negation introduction for $\triangleright$ ($A \vdash \triangleright\triangleright A$), the symmetry of E corresponds to double negation elimination for $\blacktriangleright$ ($\blacktriangleright\blacktriangleright A \vdash A$).

Onishi [12] showed that, by identifying these two negations, i.e., by imposing the axioms

$$\triangleright A \vdash \blacktriangleright A \quad \text{and} \quad \blacktriangleright A \vdash \triangleright A,$$

we obtain *self-dual negations* such as de Morgan negation in a decent way. This method is decent because, while the axioms characterizing the star postulate do not correspond to the postulate (i.e., although they are sound and complete with respect to the frames with (*), there exists a frame that validates them but fails to satisfy (*)), the conjunction $\triangleright A \dashv\vdash \blacktriangleright A$ of the identifying axioms corresponds to the *dualist* version of (*):

$$\forall x \exists y : xCy \,\&\, xEy \,\&\, (\forall z : xCz \Rightarrow z \leq y) \,\&\, (\forall w : xEw \Rightarrow y \leq w). \qquad (**)$$

This asserts the existence of x's mate state which is not only maximal among those compatible with x but also minimal among those jointly exhaustive with x. Moreover, this dualist conception enables us to express double negation laws by structural rules of sequent calculus (display calculus), which seems impossible in a single-negation framework.

We generalize the dualist approach to implication and ternary relation. An additional ternary relation S that generalizes exhaustiveness is introduced.[2] Unnecessity is defined using a new connective of *co-implication* which is modeled by S. Compatibility and exhaustiveness are obtained by restricting R and S in a natural way. The *generalized star postulate* is presented as a constraint on the relationship between R and S.

2 A logic with the generalized star postulate

In this section, we define **GS**, a logic with the generalized star postulate and prove that it is equivalent to the well-known relevant logic **R**.

2.1 Languages

Here, we deal with only the intensional part of relevant logic to save space. Extensional conjunction and disjunction can be added in the usual way without any trouble.

Definition 1 (Language)**.** Let $\mathcal{L}$ be the propositional language over a set PV of propositional variables, defined by the following grammar:

$$A ::= t \mid f \mid p \in PV \mid A \circ B \mid A \to B \mid A \bullet B \mid A \mathrel{\triangleright\!\!-} B.$$

We define, in $\mathcal{L}$, two negative modalities as

$$\triangleright A = A \to f \qquad\qquad \blacktriangleright A = t \mathrel{\triangleright\!\!-} A.$$

Let $\mathcal{L}_R$ (for **R**) be the language consisting of t, $\circ$, $\to$, and negation $\sim$.

The pair of $\bullet$ (fission, intensional disjunction) and $\mathrel{\triangleright\!\!-}$ (co-implication) is the dual of the familiar residual pair of fusion $\circ$ (intensional conjunction) and relevant implication $\to$. Thus, we will have

$$A \circ B \vdash C \iff A \vdash B \to C \quad \text{and} \quad A \vdash B \bullet C \iff A \mathrel{\triangleright\!\!-} B \vdash C.$$

The co-implication $A \mathrel{\triangleright\!\!-} B$ may be read as "it is possible to assert A while rejecting B."

2.2 Models for GS

We define the notion of GS-model for the logic **GS** and show that the Routley star is definable in GS-models.

[2] As to the use of additional ternary relation, Routley's attempt should be mentioned [17]. He used it to give the *falsity condition of implication* in a four-valued framework ("American Plan") and to eliminate the star operation. Unfortunately the semantics was very complicated. The idea behind the present paper is that the ternary relation S would be for co-implication rather than for falsity of implication.

Definition 2 (Notation)**.** Let Q_1 and Q_2 be some ternary relations. We introduce the following abbreviation for their composition:

$$\begin{aligned} Q_1Q_2(x_1x_2)x_3x_4 &\iff \exists v : Q_1x_1x_2v \,\&\, Q_2vx_3x_4; \\ Q_1Q_2x_1(x_2x_3)x_4 &\iff \exists v : Q_1x_1vx_4 \,\&\, Q_2x_2x_3v; \\ Q_1Q_2x_1)x_2x_3(x_4 &\iff \exists v : Q_1x_1vx_4 \,\&\, Q_2vx_2x_3. \end{aligned}$$

When $Q_1 = Q_2$, say it is R, we write $R^2(x_1x_2)x_3x_4$.

Definition 3 (GS-models)**.** A *GS-frame* is a structure $\langle 0, 0^*, K, R, S\rangle$, where K is a set of states, $0, 0^* \in K$, and $R, S \subseteq K^3$ such that

$$\begin{aligned} &R0xy \iff Sx0^*y \quad \text{(we then write } x \leq y\text{)} \\ &x \leq x && \text{(identity)} \\ &Rxyz \,\&\, x' \leq x \Rightarrow Rx'yz; \quad Sxyz \,\&\, x \leq x' \Rightarrow Sxy'z && \text{(monotony)} \\ &Rxxx \;\&\; Sxxx && \text{(idempotence)} \\ &Rxyz \Rightarrow Ryxz; \quad Sxyz \Rightarrow Sxzy && \text{(commutativity)} \\ &SRx_1)x_2x_3(x_4 \iff RS(x_1x_2)x_3x_4. && \text{(GS, Generalized Star postulate)} \end{aligned}$$

A *GS-model* is a GS-frame equipped with a valuation $\models \,\subseteq K \times PV$ which satisfies the hereditary condition that for any $p \in PV$ and any $x, y \in K$,

$$x \leq y \,\&\, x \models p \Rightarrow y \models p.$$

The valuation $\models$ extends to arbitrary formulas in $\mathcal{L}$ as follows:

$$\begin{aligned} x \models t &\iff 0 \leq x \\ x \models f &\iff x \not\leq 0^* \\ x \models A \circ B &\iff \exists y, z : Ryzx \,\&\, y \models A \,\&\, z \models B \\ x \models A \to B &\iff \forall y, z : Rxyz \,\&\, y \models A \Rightarrow z \models B \\ x \models A \bullet B &\iff \forall y, z : Sxyz \,\&\, z \not\models A \Rightarrow y \models B \\ x \models A \rhd\!\!- B &\iff \exists y, z : Syxz \,\&\, y \models A \,\&\, z \not\models B \end{aligned}$$

Sometimes we write like $M, x \models A$ to indicate which model is considered.

We say that A *entails* B *in a GS-model* M and write $A \vdash_M B$ if for any $x \in K$, $x \models B$ holds whenever $x \models A$. A formula A is said to be *verified in a GS-model* if $0 \models A$. A is called *GS-valid* if it is verified in every GS-model. The logic **GS** is the set of all GS-valid formulas.

The newly introduced ternary relation S can be motivated by dualizing the information-theoretic interpretation of R. $Sxyz$ may be understood as saying that any combination of pieces of *negative* information taken from y and z is a piece of *negative* information in x. Fission (with negative polarity) is the connective that represents the combination of negative information (consider the contraposition of its truth clause). If $A \rhd\!\!- B$ is negative (not true) at y with $Sxyz$, it gives negative information A in x when put together with negative B in z. Thus co-implication represents a negative information link.

We look at a few basic properties of GS-models. By virtue of monotony, the hereditary condition extends to complex formulas:

Fact 4. In any GS-model M, for any $A \in \mathcal{L}$ and $x, y \in K$,

$$x \models A \,\&\, x \leq y \Rightarrow y \models A.$$

Proof. By induction on the construction of A. □

The binary relation $\leq$ is reflexive. We further assume the anti-symmetry of $\leq$ and additional monotony conditions:

$$Rxyz \,\&\, z \leq z' \Rightarrow Rxyz' \qquad\qquad Sxyz \,\&\, x' \leq x \Rightarrow Sx'yz,$$

which imply the transitivity of $\leq$. Thus we assume $\leq$ is a partial order. This does not change the set of valid formulas because in the canonical models used in the completeness proof, $\leq$ represents the subset relation between sets of formulas.

The regular state 0 codifies the entailments in the model in the form of implication, and the constant t can be seen as the conjunction of all such logical truths of the model. The state 0^*, which we call the *coregular state*, behaves dually with f and $\rhd\!\!-$:

Fact 5. In any GS-model M, we have for any $A, B \in \mathcal{L}$,

$$\begin{aligned} A \vdash_M B \iff 0 \models A \to B \iff t \vdash_M A \to B \\ \iff 0^* \not\models A \rhd\!\!- B \iff A \rhd\!\!- B \vdash_M f. \end{aligned}$$

Proof. Immediate. □

If A entails B in a model, then accepting A while rejecting B (i.e., accepting $A \rhd\!\!- B$) would be excluded as violating the logic of the model. This fact shows that 0^* excludes such co-implications. Thus, the coregular state rejects all the logical falsities. One can also see that since any logically false co-implication entails f, it is their disjunction.

Now we look at how the Routley star is defined on GS-models. Fix a GS-model $M = \langle 0, 0^*, K, R, S, \models \rangle$.

Definition 6. Define two binary relations C (compatibility) and E (exhaustiveness) on K as follows: for any $x, y \in K$,

$$xCy \iff Rxy0^* \quad \text{and} \quad xEy \iff S0xy.$$

It should be clear that $\triangleright$ is impossibility and $\blacktriangleright$ is unnecessity under our definition. That is, for any $x \in K$,

$$\begin{aligned} x \models \triangleright A\,(= A \to f) &\iff \forall y : xCy \Rightarrow y \not\models A \\ x \models \blacktriangleright A\,(= t \mathrel{\triangleright\!\!-} A) &\iff \exists y : xEy \,\&\, y \not\models A. \end{aligned}$$

The definitions of compatibility and exhaustiveness make good sense. Suppose that xCy, i.e., $Rxy0^*$ holds. According to the information theoretic interpretation, $Rxy0^*$ means the combination of pieces of (positive) information taken from x and y *never* leads to a logical clash (recall that the coregular state 0^* rejects all logical falsity). In this *logical* sense, the information states x and y are compatible.

Exhaustiveness is a little trickier. Suppose that xEy, i.e. $S0xy$. Suppose further that $y \not\models A$ and $x \not\models B$. Then, $0 \not\models A \bullet B$. Thus, jointly rejecting A and B does *not* violate the logical truth of the model, and the combination of A and B does not exhaust all logical possibilities. Contrapositively, if it *does*, that is, $0 \models A \bullet B$, then we have $y \models A$ or $x \models B$. Thus, at least one option of the logically exhaustive pair A and B is realized in either x or y. In this sense, x and y are jointly exhaustive when xEy holds.

Next, from the generalized star postulate (GS) we derive the dualist star postulate (**) in the following form:

Fact 7.

$$\forall x, y, z : xCy \,\&\, xEz \Longrightarrow y \leq z \tag{S1}$$
$$\forall x \exists y : xCy \,\&\, xEy, \tag{S2}$$

the conjunction of which is equivalent to the dualist star postulate (**).

Proof. (S1) Suppose xCy ($Rxy0^*$) and xEz ($S0xz$). By the left-to-right direction of (GS), there is a state v such that $R0yv$ and $Sv0^*z$. Thus, $y \leq v$ and $v \leq z$, hence $y \leq z$.

(S2) By (identity), we have $R0xx$ and $Sx0^*x$. Then by the right-to-left direction of (GS), we can take some state y such that $S0yx$ (yEx) and $Ryx0^*$ (yCx). Note that (commutativity) implies the symmetry of C and E:

$$xCy \Rightarrow yCx \quad \text{and} \quad xEy \Rightarrow yEx. \tag{sym}$$

Therefore, we can conclude that xEy and xCy. $\square$

Now that we have the dualist star postulate, the Routley star is definable in GS-models, and the defined impossibility and unnecessity are identified or confused into de Morgan negation whose truth clause is represented in terms of the defined the Routley star:

Proposition 8. We can define a (unique) operation $\star$ on K such that

$$x \models \triangleright A \iff x^\star \not\models A \iff x \models \blacktriangleright A \tag{1}$$
$$x \leq y \Longrightarrow y^\star \leq x^\star \tag{2}$$
$$x = x^{\star\star}. \tag{3}$$

Proof. As noted earlier, for each $x \in K$, $x^\star$ is defined as the y in (S2) above. Then these properties follow from (S1), (S2), and (sym). See [12, Proposition 5.7] and related remarks for proof.

As to uniqueness, suppose there is another operation $*$ that satisfies (I). Then $*$-states and $\star$-states are equivalent in the sense that $x^* \models A$ if and only if $x^\star \models A$. We can assume that in any model such equivalent states are the same because each state of canonical models is constructed as the set of formulas it supports. Hence $\star = *$, the defined star operation is unique. [3] □

2.3 Equivalence between **GS** and **R**

We defined a logic, **GS**, within which de Morgan negation is definable. In this subsection we show that **GS** is equivalent to **R**. We follow [5] on the definition of models for **R**.

Definition 9 (R-models)**.** An *R-frame* is a structure $\langle 0, K, R, * \rangle$, where $0 \in K$, $R \subseteq K^3$, and $* : K \to K$, satisfying

$$x \leq x \quad (\text{where } x \leq y \iff R0xy) \tag{identity}$$
$$Rxyz \,\&\, x' \leq x \Rightarrow Rx'yz \tag{monotony}$$
$$Rxxx \tag{idempotence}$$
$$Rxyz \Rightarrow Ryxz \tag{commutativity}$$
$$R^2(x_1x_2)x_3x_4 \Rightarrow R^2x_1(x_2x_3)x_4 \tag{associativity}$$
$$Rxyz \Rightarrow Rxz^*y^* \tag{*1}$$
$$x = x^{**} \tag{*2}$$

We assume $\leq$ is a partial order as in GS-models.

An *R-model* is an R-frame with a hereditary valuation $\models \subseteq K \times PV$. The truth clause for ~ is: $x \models \sim A \iff x^* \not\models A$. The notions of verification and entailment in the model are defined in an obvious way. **R** is the logic on the language $\mathcal{L}_R$ defined by the class of R-models.

We observe that there is a bijective correspondence between GS-models and R-models.

[3] Notice that we use $\star$ for the *defined* star operation. We also note here that 0^*, the designated coregular state, is indeed the value of the defined star function of 0, thus $0^\star = 0^*$.

Lemma 10. Given a GS-model $M = \langle 0, 0^*, K, R, S, \models\rangle$, the structure $\Phi(M) = \langle 0, K, R, \star, \models\rangle$, where $\star$ is the operation uniquely defined in Proposition 8, is an R-model.

Proof. It suffices to show (associativity) and (*1) hold in $\Phi(M)$. First notice that the equivalence

$$Sxyz \iff Rxy^\star z \tag{SR}$$

follows from (GS). $Sxyz$, together with $Ryy^\star 0$ ($yCy^\star$), implies by (GS) that there is a state v such that $Rxy^\star v$ and $Sv0^\star z$ ($v \leq z$), and hence $Rxy^\star z$ by monotony. Conversely, applying the right-to-left direction of (GS) to $Rxy^\star z$ and $Sz0^\star z$ ($z \leq z$), we obtain some state v such that $Sxvz$ and $Rvy^\star 0^\star$ ($vCy^\star$). Recall that the star state is maximal among compatible states. Thus $y^\star \leq v^\star$, hence $v \leq y$. Then we can conclude that $Sxyz$ using (commutativity).

Now (*1) is immediate (Com for commutativity):

$$Rxyz \underset{\text{(SR)}}{\iff} Sxy^\star z \underset{\text{(Com)}}{\iff} Sxzy^\star \underset{\text{(SR)}}{\iff} Rxz^\star y^\star.$$

(Associativity) is derived as follows:

$$\begin{aligned} Rx_1x_2v \,\&\, Rvx_3x_4 &\underset{\text{(SR)}}{\iff} Rx_1x_2v \,\&\, Svx_3^\star x_4 \underset{\text{(GS)}}{\iff} Sx_1vx_4 \,\&\, Rvx_2x_3^\star \\ &\underset{\text{(SR)}}{\iff} Rx_1v^\star x_4 \,\&\, Rvx_2x_3^\star \underset{\text{(Com,*1)}}{\iff} Rx_1v^\star x_4 \,\&\, Rx_2x_3v^\star. \end{aligned}$$

(Existential quantification on v is suppressed.) □

Lemma 11. Given an R-model $N = \langle 0, K, R, *, \models\rangle$, the structure $\Psi(N) = \langle 0, 0^*, K, R, S, \models\rangle$, where S is defined as (SR) $Sxyz \Leftrightarrow Rxy^*z$, is a GS-model.

Proof. It is tedious but easy to check all the constraints for GS-models. For example, suppressing existential quantification again (Asso represents associativity),

$$\begin{aligned} Rx_1x_2v \,\&\, Svx_3x_4 &\underset{\text{(SR)}}{\iff} Rx_1x_2v \,\&\, Rvx_3^*x_4 \underset{\text{(Asso)}}{\Longrightarrow} Rx_1vx_4 \,\&\, Rx_2x_3^*v \\ &\underset{\text{(SR)}}{\iff} Sx_1v^*x_4 \,\&\, Rx_2x_3^*v \underset{\text{(Com,*1)}}{\iff} Sx_1v^*x_4 \,\&\, Rv^*x_2x_3. \end{aligned}$$

This is the right-to-left direction of (GS). □

It should be clear from (SR) and the uniqueness of the defined star operation that Φ and Ψ are bijective: $\Psi \circ \Phi(M) = M$ and $\Phi \circ \Psi(N) = N$.

Now **GS** and **R** are equivalent in the following sense. First, **R** can be embedded into **GS**:

Theorem 12. Define inductively a translation $\varphi : \mathcal{L}_R \to \mathcal{L}$ as almost homomorphic except for the case $\varphi({\sim} A) = \varphi(A) \to f$. Then for any $A \in \mathcal{L}_R$,

$$A \text{ is } R\text{-valid} \iff \varphi(A) \text{ is } GS\text{-valid}.$$

Proof. ($\Longrightarrow$) Given a GS-model $\mathcal{M} = \langle 0, 0^*, K, R, S, \models \rangle$, by Lemma 10 we obtain an R-model $\Phi(M) = \langle 0, K, R, \star, \models \rangle$. It is clear that

$$\Phi(M), x \models A \iff M, x \models \varphi(A) \tag{4}$$

for any $x \in K$ and $A \in \mathcal{L}_R$. The crucial case is when A is of the form $\sim B$, but this has already been completed in Proposition 8. This establishes that if $\varphi(A)$ is falsified in some GS-model, then there is an R-model that falsifies A.

($\Longleftarrow$) Consider an R-model $N = \langle 0, K, R, *, \models \rangle$. Then, since the GS-model $\Psi(N)$ satisfies (4) above, we have

$$N, x \models A \iff \Psi(N), x \models \varphi(A)$$

for any $x \in K$ and $A \in \mathcal{L}_R$ (recall that $N = \Phi(\Psi(N))$). The right-to-left direction of the theorem follows immediately. $\square$

This theorem shows that $\varphi[\mathbf{R}] \subseteq \mathbf{GS}$ (considering $\mathbf{R}$ and $\mathbf{GS}$ as sets of valid formulas). The next theorem shows that the converse direction *essentially* holds:

Theorem 13. We say two formulas A and $B \in \mathcal{L}$ are *equivalent* (in GS-models) if A and B entail each other ($A \vdash_M B$ and $B \vdash_M A$) in any GS-model M. Then, for any $A \in \mathcal{L}$, there is a formula $A' \in \mathcal{L}_R$ such that $\varphi(A')$ is equivalent to A.

Proof. By induction on the construction of $A \in \mathcal{L}$. Fix a GS-model $M = \langle 0, 0^*, K, R, S, \models \rangle$. The cases for atomic formulas, t, $\circ$ and $\rightarrow$ are trivial.

[$A = f$] Take $\sim t \in \mathcal{L}_R$. Then, for any $x \in K$,

$$\begin{aligned}
x \models \varphi(\sim t) &\iff x \models t \rightarrow f \\
&\iff \forall y, z : Rxyz \,\&\, y \models t \Rightarrow z \models f \\
&\iff \forall y, z : Rxyz \,\&\, 0 \leq y \Rightarrow z \not\leq 0^* \\
&\iff \forall z : x \leq z \Rightarrow z \not\leq 0^* \\
&\iff x \not\leq 0^* \iff x \models f.
\end{aligned}$$

The fourth equivalence is due to (commutativity) and (monotony).

[$A = B \vartriangleright\!\!- C$] By IH, we obtain B' and $C' \in \mathcal{L}_R$ such that $\varphi(B')$ and $\varphi(C')$ are equivalent to B and C respectively. Take $\sim(B' \rightarrow C') \in \mathcal{L}_R$. Then,

$$\begin{aligned}
x \models \varphi(\sim(B' \rightarrow C')) &\iff x \models \vartriangleright(\varphi(B') \rightarrow \varphi(C')) \\
&\iff x^\star \not\models \varphi(B') \rightarrow \varphi(C') && \text{(Proposition 8)} \\
&\iff \exists y, z : Rx^\star yz \,\&\, y \models \varphi(B') \,\&\, z \not\models \varphi(C') \\
&\iff \exists y, z : Ryx^\star z \,\&\, y \models B \,\&\, z \not\models C && \text{(Com, IH)} \\
&\iff \exists y, z : Syxz \,\&\, y \models B \,\&\, z \not\models C && \text{(SR)} \\
&\iff x \models B \vartriangleright\!\!- C.
\end{aligned}$$

$[A = B \bullet C]$ Obtain $B', C' \in \mathcal{L}_R$ for B and C as in the previous case. Take $\sim C' \to B' \in \mathcal{L}_R$. Then,

$$\begin{aligned}
x \models \varphi(\sim C' \to B') &\iff x \models \triangleright\varphi(C') \to \varphi(B') \\
&\iff \forall y, z : Rxyz \,\&\, y \models \triangleright\varphi(C') \Rightarrow z \models \varphi(B') \\
&\iff \forall y, z : Rxyz \,\&\, y^\star \not\models \varphi(C') \Rightarrow z \models \varphi(B') \\
&\iff \forall y, z : Sxy^\star z \,\&\, y^\star \not\models C \Rightarrow z \models B \qquad \text{(IH, SR)} \\
&\iff \forall w, z : Sxwz \,\&\, w \not\models C \Rightarrow z \models B \\
&\iff x \models B \bullet C.
\end{aligned}$$

This concludes the proof. □

Thus, for any GS-valid formula A, we can find an R-valid formula A' such that A and $\varphi(A')$ are equivalent. In this sense, $\varphi[\mathbf{R}]$ covers **GS**, and hence **R** and **GS** are equivalent.

Can this correspondence between those models with GS and those with the Routley star be transferred to weaker relevant logics? Among the constraints for GS-models, identity and monotony are essential for basic properties of relevant implication and coimplication. As to idempotence, which corresponds to Contraction ($A \to A \otimes A$), one may have noticed that it does not play any substantial role in the above proofs of definability of the Routley star in GS-models and the equivalence of **GS** and **R**. Thus by dropping idempotence, one immediately obtains a weaker logic which is capable of defining de Morgan negation implicationally and equivalent to **R** minus Contraction. On the other hand, dropping commutativity will cause a mess because double negation laws depend on it. Thus, our method of "implicationalizing" negation cannot be applied straightforwardly to weaker relevant logics such as **E** and **T**.

3 Proof theory: Display calculus

In this section a sequent proof theory for **GS** is presented. [4] We work with *display calculus*, an extension of Gentzen type sequent calculus. The idea of display calculus (or display *logic*) was originally developed by Belnap, and his first formulation comprised relevant logic as a fragment [1]. Ours may look slightly different since it is based on Goré's display calculus for bi-intuitionistic logic [7] (see also [23, 24]). The only novelty is the structural rule (rGS) for the generalized star postulate. Assuming that the reader is more or less familiar with display calculus, we present essential definitions and state theorems without proof.

[4] We continue to work with the intensional part only. For a treatment of the extensional part, we refer to the discussion in [14, sec. 2].

Definition 14 (Structures)**.** The set $\mathcal{S}$ of *structures*, expressions built from formulas using *structure connectives*, is defined by the following grammar:

$$X ::= A \in \mathcal{L} \mid \mathbf{I} \mid X \,;\, Y \mid X > Y.$$

A *sequent* is an expression of the form $X \vdash Y$ where $X, Y \in \mathcal{S}$.

$\mathbf{I}$ is an empty structure, the binary ; is the meta-level counterpart of fusion *or* fission, and $>$ represents implication *or* co-implication according to its position in a sequent. Presumably the following rules are sufficiently self-explanatory (note that the display rules incorporate the commutativity of the semicolon implicitly.).

Definition 15 (δ**GS**)**.** The display calculus δ**GS** consists of the following rules (a double line indicates that the rule is bi-directional):

[*Display rules*]

$$\begin{array}{c} X \,;\, Y \vdash Z \\ \hline\hline X \vdash Y > Z \\ \hline\hline Y \,;\, X \vdash Z \end{array} \qquad \begin{array}{c} X \vdash Y \,;\, Z \\ \hline\hline X > Y \vdash Z \\ \hline\hline X \vdash Z \,;\, Y \end{array}$$

[*Structural rules*]

$$A \vdash A \ (Id) \quad \frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\ (\text{Cut}) \quad \frac{X \vdash Y}{\mathbf{I} \,;\, X \vdash Y}\ (\mathbf{I}_L) \quad \frac{X \vdash Y}{X \vdash Y \,;\, \mathbf{I}}\ (\mathbf{I}_R)$$

$$\frac{X \,;\, X \vdash Y}{X \vdash Y}\ (W_L) \quad \frac{X \vdash Y \,;\, Y}{X \vdash Y}\ (W_R) \quad \frac{X \,;\, Y \vdash W \,;\, Z}{X > Z \vdash Y > W}\ (rGS)$$

[*Logical rules*]

$$\frac{\mathbf{I} \vdash Y}{t \vdash Y}\ (t_L) \quad \mathbf{I} \vdash t\ (t_R) \quad f \vdash \mathbf{I}\ (f_L) \quad \frac{X \vdash \mathbf{I}}{X \vdash f}\ (f_R)$$

$$\frac{A \,;\, B \vdash Y}{A \circ B \vdash Y}\ (\circ_L) \quad \frac{X \vdash A \qquad Y \vdash B}{X \,;\, Y \vdash A \circ B}\ (\circ_R)$$

$$\frac{X \vdash A \qquad B \vdash Y}{A \to B \vdash X \,;\, Y}\ (\to_L) \quad \frac{X \vdash A \,;\, B}{X \vdash A \to B}\ (\to_R)$$

$$\frac{A \vdash X \qquad B \vdash Y}{A \bullet B \vdash X \,;\, Y}\ (\bullet_L) \quad \frac{X \vdash A \,;\, B}{X \vdash A \bullet B}\ (\bullet_R)$$

$$\frac{A \,;\, B \vdash Y}{A \rhd\!\!- B \vdash Y}\ (\rhd\!\!-_L) \quad \frac{X \vdash A \qquad B \vdash Y}{X \,;\, Y \vdash A \rhd\!\!- B}\ (\rhd\!\!-_R)$$

The defining feature of display calculus is the *display property* that every substructure of a sequent can be "displayed" through display rules. Cut elimination theorem can be proved in a generic way based on this property.

Definition 16 (Antecedent/succedent parts)**.** In $X \vdash Y$, we say X is an *antecedent part* (AP) and Y is a *succedent part* (SP). We define:

- If $W\,;Z$ is an AP (SP) in $X \vdash Y$, so are W and Z, and
- In any substructure $W > Z$ of $X \vdash Y$, W (Z) is an AP (SP) in $X \vdash Y$.

Definition 17. We say two sequents are *display equivalent* if they are derivable from each other using only display rules.

Theorem 18 (Display property)**.** Let Z be a substructure in $X \vdash Y$. Then,

- If Z is an AP in $X \vdash Y$, there exists a sequent $Z \vdash W$ which is display equivalent to $X \vdash Y$.
- If Z is an SP in $X \vdash Y$, there exists a sequent $W \vdash Z$ which is display equivalent to $X \vdash Y$.

Proof. We refer the reader to an elegant proof by Restall [14]. □

Theorem 19 (Cut elimination)**.** If $X \vdash Y$ is derivable in δBiR, it is derivable without Cut.

Proof. See [1] for the general Cut elimination theorem. □

Definition 20. Define a translation τ from sequents to formulas as follows:

$$\tau(X \vdash Y) = \tau_1(X) \to \tau_2(Y),$$

where τ_1, τ_2 are defined inductively as

$$\begin{aligned} \tau_1(A) &= A & \tau_2(A) &= A \\ \tau_1(\mathbf{I}) &= t & \tau_2(\mathbf{I}) &= f \\ \tau_1(X\,;Y) &= \tau_1(X) \circ \tau_1(Y) & \tau_2(X\,;Y) &= \tau_2(X) \bullet \tau_2(Y) \\ \tau_1(X > Y) &= \tau_1(X) \mathrel{\triangleright\!\!-} \tau_2(Y) & \tau_2(X > Y) &= \tau_1(X) \to \tau_1(Y). \end{aligned}$$

Theorem 21 (Soundness and completeness)**.** A sequent $X \vdash Y$ is provable in δ**GS** if and only if $\tau(X \vdash Y)$ is GS-valid.

Proof. The *only-if* part (soundness) is routine. For the *if* part (completeness), the familiar method of using canonical model applies. In a canonical model, the ternary relation S^c is defined as

$$S^c xyz \iff \forall A, B \in \mathcal{L} : A \notin z \;\&\; B \notin y \Rightarrow A \bullet B \notin x,$$

where x, y and z are prime theories. □

Let us check the correspondence of the generalized star postulate (GS) and the structural rule (rGS). We write $A \vdash_F B$ if A entails B in a *frame* F, that is, $A \vdash_M B$ for any model $M = \langle F, \models \rangle$ on F. We also define a GS^--frame (model) to be a GS-frame (model) without (GS).

Theorem 22. The generalized star postulate (GS) corresponds to the structural rule (rGS) over GS^--frames. That is, for any GS^--frame F and any $A, B, C, D \in \mathcal{L}$, the following (GSi) and (rGSi) are equivalent ($i \in \{1, 2\}$):

- (GS1) $SRx_1)x_2x_3(x_4 \Rightarrow RS(x_1x_2)x_3x_4$ holds in F;

  ($rGS1$) $A \circ B \vdash_F C \bullet D \Rightarrow A \rhd\!\!- C \vdash_F B \rhd\!\!- D$,

- (GS2) $RS(x_1x_2)x_3x_4 \Rightarrow SRx_1)x_2x_3(x_4$ holds in F;

  ($rGS2$) $A \rhd\!\!- C \vdash_F B \rhd\!\!- D \Rightarrow A \circ B \vdash_F C \bullet D$.

Note that ($rGS1$) and ($rGS2$) say the rule (rGS) preserves entailment.

Proof. That (GS1,2) implies ($rGS1$,2) is a part of the soundness of δGS. We show the converse. Fix a GS^- frame $F = \langle 0, 0^* K, R, S\rangle$.

First, we show that if (GS1) *fails*, so does ($rGS1$). Suppose $SRx_1)x_2x_3(x_4$ holds in F. There is a state $z \in K$ such that Sx_1zx_4 and Rzx_2x_3. Suppose also that $RS(x_1x_2)x_3x_4$ fails, i.e., there is no state $w \in K$ such that Rx_1x_2w and Swx_3x_4. Now we define a valuation $\models$ on F by

$$u \models p \iff x_2 \leq u \qquad u \models q \iff Rx_1x_2u \qquad u \models r \iff u \not\leq x_4.$$

By (monotony), $\models$ satisfies the heredity condition. Then, we obtain

$$z \models (p \to q) \rhd\!\!- r \text{ and } z \not\models p \to (q \rhd\!\!- r).$$

Indeed, we have Sx_1zx_4, $x_4 \not\models r$, and

- $x_1 \models p \to q$: Suppose Rx_1uv and $u \models p$. Then, by $x_2 \leq u$ and (monotony), it follows that Rx_1x_2v. This means $v \models q$.

This gives $z \models (p \to q) \rhd\!\!- r$. We also have Rzx_2x_3, $x_2 \models p$, and

- $x_3 \not\models q \rhd\!\!- r$: Suppose Swx_3v and $v \not\models r$. Then by (monotony) Swx_3x_4. Hence, it is impossible to have Rx_1x_2w, which means $w \not\models q$.

Therefore $z \not\models p \to (q \rhd\!\!- r)$. This shows that $(p \to q) \rhd\!\!- r \nvdash_F p \to (q \rhd\!\!- r)$. On the other hand, $(p \to q) \circ p \vdash_F r \bullet (q \rhd\!\!- r)$ (provable in δGS without (rGS)). Thus, ($rGS1$) fails to preserve entailment in F.

Next, suppose (GS2) fails. Then, there is $z \in K$ such that Rx_1x_2z and Szx_3x_4 and there is no $w \in K$ such that Sx_1wx_4 and Rwx_3x_4. We define a valuation $\models$ by

$$u \models p \iff x_2 \leq u \qquad u \models q \iff u \not\leq x_3 \qquad u \models r \iff x_1 \leq u.$$

Then, while $r \rhd\!\!- (r \rhd\!\!- (p \to q)) \vdash_F p \to q$ holds, we have

$$z \models r \circ p \text{ and } z \not\models (r \rhd\!\!- (p \to q)) \bullet q,$$

Thus $r \circ p \nvdash_F (r \rhd\!\!- (p \to q)) \bullet q$, which implies the failure of ($rGS2$). $\square$

4 Conclusion

In this paper, we introduced *GS*-models equipped with two ternary relations. They are governed by the generalized star postulate, which enables us to define the Routley star in the model. The logic **GS** defined by *GS*-models was proved to be equivalent to the well-known relevant logic **R** through the translation φ. Thus, (the intensional fragment of) **R**, a logic of implication and (primitive) negation, has been reconstructed as a logic of implication and co-implication. A proof theory for **GS** was also presented in the style of display calculus.

It might appear that we have simply replaced the mysterious Routley star with a different mysterious thing S and that good sense was not made of the relevant negation. Such an impression is mistaken.

As noted above, the ternary relation S can be well understood by adapting or dualizing the information-theoretic interpretation of R. We also note that ternary relations can be understood in terms of the structure of sequent. R corresponds to a multiple-premise (or two-premise) sequent, i.e., $A\ ;\ B \vdash C$ is *not* valid if $Rxyz$, $x \models A$, $y \models B$, and $z \not\models C$ for some x, y and z. Then it is clear that S represents the structure of the multiple-conclusion sequent. If $Sxyz$, $x \models A$, $z \not\models B$, and $y \not\models C$, then $A \vdash B\ ;\ C$ is not valid. Thus, this is something with which we are more or less familiar.

This extended setting of dual relations or multiple-conclusion sequent is not only familiar but also illuminating. It has allowed us to formulate a generalization of the star postulate in such a way that it has a simple, corresponding structural rule. Notice here that the structural rule (rGS) makes the logic appear *classical*. It is the rule that characterizes classical logic over bi-intuitionistic logic [7], that is, by adding it, bi-intuitionistic logic collapses into classicality.[5] Recall that Copeland's criticism on relevant negation was that it deviates badly from the classical conception of negation. Of course it is not entirely classical since it allows incomplete and/or inconsistent states in a frame. However, the structural rule (rGS) suggests that there *is* a sense in which relevant negation preserves the classical conception. Making this notion of classicality and its relation to the star postulate precise is a potential topic of future work.

Acknowledgement The author would like to thank Kengo Okamoto, Yuichiro Hosokawa, Sakiko Yamasaki, Ben Blumson, Tang Weng Hong, Koji Tanaka, and the anonymous reviewers for their helpful comments. The paper was

[5] Or it is just a structural version of the right introduction rule for implication:

$$\frac{X, A \vdash B, Y}{X \vdash A \to B, Y}\ (\to R)$$

which differentiate classical logic from intuitionistic logic formulated as a multiple-conclusion sequent calculus.

written while the author stayed at National University of Singapore as a fellow under the JSPS program "Japan-ASEAN Collaboration Research Program on Innovative Humanosphere in Southeast Asia: In Search of Wisdom toward Compatibility Growth and Community in the World."

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