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On the computation of probabilistic coalition structures

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Abstract
In Coalition Structure Generation (CSG), one seeks to form a partition of a given set of agents into coalitions such that the sum of the values of each coalition is maximized. This paper introduces a model for Probabilistic CSG (PCSG), which extends the standard CSG model to account for the stochastic nature of the environment, i.e., when some of the agents considered at start may be finally defective. In PCSG, the goal is to maximize the expected utility of a coalition structure. We show that the problem is $\mathsf{NP}^{\mathsf{PP}}$-hard in the general case, but remains in NP for two natural subclasses of PCSG instances, when the characteristic function that gives the utility of every coalition is represented using a marginal contribution network (MC-net). Two encoding schemes are presented for these subclasses and empirical results are reported, showing that computing a coalition structure with maximal expected utility can be done efficiently for PCSG instances of reasonable size. This is an extended and revised version of the paper entitled "Probabilistic Coalition Structure Generation" published in the proceedings of KR'18, pages 663–664 [33].

Keywords Coalition Structure Generation · Uncertainty · Computational complexity · Marginal contribution networks

1 Introduction

One of the most important challenges in multi-agent systems consists in dividing a set of agents into groups to create synergies and improve the overall performance. The Coalition Structure Generation (CSG) framework is a well-known abstraction of this problem [27]. In a nutshell, we are given a Characteristic Function Game (CFG), i.e., a finite set of

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agents where every subset, or *coalition*, is associated with a value (through a *characteristic function*) which represents some pay-off provided by the underlying task performed by the coalition. The CSG problem then consists in forming an optimal *coalition structure*, i.e., partitioning the set of agents so that the sum of coalition values is maximized.

There is a wide range of potential applications for the CSG problem, e.g., increasing the throughput of cognitive radio networks [13], optimizing the surveillance of certain areas by autonomous sensors [10], and improving communication networks [30] (see [4], Section 7.5 for an extended list of applications in real-life settings).

Notably, a CFG assigns a value to a coalition as a whole but does not dictate how this value is distributed to its members. A number of solution concepts are available in the literature for this task, including the notions of core [9], Shapley value [34], and nucleolus [32]. However, the CSG setting does not consider this additional step: instead of focusing on the performance of individual agents within their coalition, one is only interested in maximizing the overall welfare of the system. Stated otherwise, the CSG setting assumes that the agents are fully cooperative; this is the case, for instance, when the system is “owned” by an implicit additional agent (e.g., the system’s designer) and the agents involved in the CFG are ready to commit to any task/coalition with no self-interest.

The CSG problem is known to be NP-hard [31] and many algorithms have been proposed for solving it [21, 26, 29, 43]. In the case when the characteristic function is provided extensively (i.e., as a table with 2^n entries, n being the number of agents), the main algorithms are based on dynamic programming (DP) [29, 43], tree-search (IP) [26] and hybrid approaches (ODP-IP) [21]. Such a representation of a characteristic function requires listing every coalition together with its value, and considering such an input for the CSG problem is impractical for most real-world scenarios unless the number of agents is very small [20]. To deal with this issue, a number of concise representation languages for characteristic functions have been proposed, including marginal contribution networks (MC-nets) [11], synergy coalition groups (SCGs) [5], skilled-based representations [22] and agent-type representations [36, 41]. Despite the fact that NP-hardness still holds for most of these representations (at least, for those universally expressive representations), constraint optimization programming techniques can be exploited and instances of larger size can be dealt with [17, 18, 23].

In the standard CSG setting, one assumes that once a coalition structure is formed, everything goes as planned: the agents are supposed to be fully reliable, the underlying task performed by each coalition is completed as expected and the pay-off is obtained from it. However, in realistic settings, this cannot be reasonably expected to hold. Some unexpected, exogenous events often occur: agents may malfunction for various reasons, and we may be uncertain about the actual capabilities or even the attendance of each agent. Then, once an optimal coalition structure is formed and some agents are found to be defective afterwards, it is not always possible to recompute an optimal coalition structure from scratch based on the remaining agents: this is the case for instance when the agents from each coalition of the initially computed structure are committed to work together on an underlying task, or simply when some contracts forbid any coalition rearrangement. To fill the gap, one could forsee the potential absence of some agents and assume that the attendance of agents is of probabilistic nature: this assumption is reasonable when the probabilitys distribution can be obtained, e.g., from the past attendance record of the agents or the reliability of the system’s components.

One could directly model this problem as a Characteristic Function Game (CFG) where the value associated with each coalition represents some *expected utility*. Doing so, solving the CSG problem results in forming a coalition structure with a maximal expected utility.

However, this approach requires the characteristic function to be provided extensively (listing every coalition with its expected value), so this prevents one to take advantage of the existing concise representations of CFGs which are not designed to be used with expected values.

Instead, one can consider a probabilistic extension of the standard CSG framework, by assuming that the uncertain nature of the agents' attendance is provided as a probability distribution, *separately* from a standard CFG. Doing so additionally requires to enrich the definition of a CFG so as to make precise how the value of a coalition structure should evolve when some agents are missing. This is the approach adopted in this paper.

As a matter of illustration, let us consider the following scenario which will serve as a running example in the rest of this paper:

Example 1 A service company with three employees Alice, Bob and Charles has received customer orders requiring different skills:

- order #1 can be performed only by Alice, and the company gets 30€ for it;
- order #2 corresponds to Bob's skills, and pays 40€;
- order #3 pays 90€ and needs Alice and Bob;
- order #4 pays 120€ and needs Alice and Charles;
- order #5 pays 100€ and needs Bob and Charles;
- order #6 requires the presence of all three employees, for a reward of 150€.

Employees cannot be assigned to more than one task. The manager of the company wants to assign employees to task(s) so as to immediately generate the maximum profit.

At a first glance, assigning Alice and Charles to order #4 and Bob to order #2 seems to be the best plan (named hereby as plan I): doing so, the company should get a total of 160€. However, the schedule may not go as planned. Some of the workers could suddenly become unavailable due to illness and other unexpected matters. So assume in this example that the company manager has some information about the reliability of each worker. Bob is fully reliable: he will do the job he has been assigned to for sure; Alice is "somewhat" reliable: sometimes she fails to do the job; and Charles is not reliable: he cancels appointments almost all the time. With such a setting, is the "optimistic" plan I always the best choice? The answer to this question depends on what actually happens if some employees are lacking.

On the one hand, let us assume that each task requires a solid preparation, e.g., some appointments with the customers must be made ahead of time. Then the manager cannot revise the assignment if she suddenly discovers that some of the employees are missing once the plan is set up: in plan I, if Charles is missing for placing order #4 with Alice, no reward can be obtained from the remaining coalition with Alice alone. Thus it seems risky to involve Charles in any assignment, and assigning Alice and Bob to order #3 is a safer option (plan II). On the contrary, let us consider a context where it is possible to reassign the remaining workers to another available order right away. This happens when customers have a flexible schedule, and that no cost (in terms of time or budget) is involved in the transfer of a worker into another order. Then if Charles would happen to be missing in plan I, Alice could still place order #1 and the company would get 30€. So placing order #6 (plan III) may be a better option than the other two plans: since Bob will attend for sure and Alice is likely to do the job, in the lucky case where Charles is also here the company will get a reward of 150€; if Charles is absent (which is likely to be the case), then order

#3 is available for Alice and Bob, and if in addition Alice turns out to be absent, Bob can place order #2.

To model such situations, our first step is to extend the standard definition of a CFG. As mentioned earlier, a CFG is defined as a set of agents and a characteristic function associating each coalition with a value. A *Probabilistic CFG* (PCFG) considers a set of agents, and instead of a characteristic function it has two additional components. The first one is a probability distribution which provides the probability of any "scenario" to occur, i.e., a precise situation when some of the agents are actually present and the remaining agents are missing. The second component of a PCFG is a *situational* characteristic function, which associates each coalition *and each such "scenario"* with a value. This function, for instance, can be derived from a standard characteristic function by associating each coalition with the value it would obtain in the case where everything goes as planned (as specified by the standard characteristic function), and by making precise how the value of that coalition should be updated in any scenario. Based on a PCFG, the *Probabilistic CSG* (PCSG for short) problem consists in finding a coalition structure of maximal expected utility.

Some of the clear benefits of defining a PCFG as two separate components (a probabilistic function and a situational characteristic function) are in terms of modularity. Since both components are independent from each other, the situational characteristic function can be provided in any representation language without affecting (or being affected by) the definition and representation of the probability distribution. As it will be illustrated in the following sections, this allows one to naturally derive a PCFG from an existing CFG and an additional probability distribution, without making any assumption on the representation language used to describe the CFG.

In the following, we first formalize PCFGs and the PCSG problem, and point out some basic properties. We then focus on two subclasses of PCFGs which are adapted to the two application contexts illustrated in Example 1 above: the *cautious* PCFGs and the *flexible* PCFGs. In a flexible PCFG, it is assumed that when some agents are finally found to be defective, the agents from the residual coalitions can always be re-assigned to other tasks and thus produce a reward. In a cautious PCFG, it is assumed that it is never the case, and so no reward can be obtained from a residual coalition.

This paper also aims to investigate the PCSG problem from a computational perspective. More precisely, we consider two decision problems: the first one is related to the computation of the expected utility of a given coalition structure, and the second one is related to PCSG itself, i.e., the computation of a coalition structure of maximal expected utility. These problems are shown to be respectively, PP-hard and $\mathsf{NP}^{\mathsf{PP}}$-hard in the general case, even when flexible PCFGs are considered. However, we show that computing the expected utility of a coalition structure can be done in polynomial time for cautious PCFGs, and for flexible PCFGs when the situational characteristic function takes advantage of an MC-net representation. As a result, for these classes of PCFGs the decision problem related to PCSG falls in NP. Interestingly, this means that while the PCSG problem can be viewed as an "extension" of the CSG problem (this is made more precise in the next section), this does not lead to a computational shift for cautious PCFGs and MC-net based flexible PCFGs. We then point out and evaluate mixed integer linear programming (MILP) encodings for both MC-net cautious and flexible PCFGs and show that cautious (resp. flexible) PCFGs can be solved within seconds for reasonable sized instances.

The proofs of propositions are given in an "Appendix". The run-time codes of the PCFG generator and of the translator of PCFG instances into MILP instances used in our experiments are available at https://nicolas-schwind.github.io/PCSG-generator-translator.zip [33].

2 Other related work

The idea of forming coalition structures under uncertainty is not new (see [4], Section 6.2.3 for an overview). Works in non-cooperative domains [1–3, 8, 12, 37] depart from our framework since in the (probabilistic) CSG setting, the agents forming the coalitions have no self-interest and the goal is to maximize the overall (expected) welfare of the system. Some notions of uncertainty have been considered in coalition formation when the agents are not assumed to be entirely selfish [14, 28, 35]. Yet the underlying frameworks are domain-specific or more complex than the CSG setting, since additional parameters such as (sub-)tasks and available resources are explicitly represented. These approaches are mainly validated through simulations, as agents are re-assigned to different tasks reactively in face of exogenous events.

Nevertheless, there are a few recent works more closely related to our CSG setting. In [24], the authors study how to form a coalition structure CS such that, if at most k agents were to be removed from their coalition in CS, the value of the coalition structure CS' consisting of the remaining agents should be kept above a certain threshold. Forming a robust coalition structure is a useful property: it provides one with a certain guarantee of coalition structure value (given k). However, a drawback of robustness is that the focus is given on the worst case scenario, and thus all agents are considered as equally (un)reliable. Instead, in this paper we assume that the attendance of agents is of stochastic nature, which sounds reasonable when the probability distribution can easily be estimated, e.g., from the attendance record of the agents or the reliability of the system's components.

To account for the stochastic nature of the environment, Doherty et. al [7] proposed to extend the definition of a CFG to a *Contextual Coalitional Game* (CCG). In addition to a standard characteristic function, a CCG considers a set of environmental states together with a probability distribution on states, which allows for a natural formulation of expected coalitional values. The focus in [7] was given on the computation of an agent's *expected Shapley value*. This notion naturally extends the notion of Shapley value from the standard CFG setting which evaluates an agent's potential marginal contribution to every other coalition. In the stochastic setting, an agent's expected Shapley value is evaluated using expected coalitional values. This departs from our goal which is to form a coalition structure with a maximal expected utility.

Most closely related to our work is [19], in which a similar probabilistic CSG framework is considered and empirically evaluated. However, our PCSG framework and results differ from those reported in [19] on many aspects. The model used in [19] is more restrictive than the PCSG framework introduced here, and no investigation was performed from a computational complexity viewpoint (a more detailed and technical discussion about how our framework relates to the one introduced in [19] can be found at the end of Sect. 9).

3 Coalition structure generation

Let us introduce some preliminaries and formalize the CSG problem.

Table 1 Running example: the value $F(CS_i)$ and the expected utilities $U_{cau}(CS_i)$ and $U_{fle}(CS_i)$ corresponding respectively to the cautious and flexible PCSG settings.

Π_A	$F(CS_i)$	$U_{cau}(CS_i)$	$U_{fle}(CS_i)$
$CS_1 = \{\{a_1, a_3\}, \{a_2\}\}$	**160**	49.6	71.2
$CS_2 = \{\{a_1, a_2\}, \{a_3\}\}$	90	**72**	80
$CS_3 = \{\{a_1, a_2, a_3\}\}$	150	12	**86**
$CS_4 = \{\{a_2, a_3\}, \{a_1\}\}$	130	34	66.4
$CS_5 = \{\{a_1\}, \{a_2\}, \{a_3\}\}$	70	64	64

For each case, the number in a bold font refers to the optimal coalition structure

Definition 1 (*Characteristic Function Game*) A *Characteristic Function Game* (CFG for short) is a pair $\langle A, f \rangle$ where $A = \{a_1, \dots, a_n\}$ is a set of agents and $f : 2^A \to \mathbb{R}$ is a function called *characteristic function*.

Given a CFG $\langle A, f \rangle$, a *coalition* is a non-empty subset of A. A *coalition structure* CS is a set of coalitions that forms a partition of A, i.e., $CS = \{C_1, \dots, C_m\}$, where for all $C_i \in CS$, $C_i \neq \emptyset$; for all $C_i, C_j \in CS$ such that $i \neq j$, $C_i \cap C_j = \emptyset$; and $\bigcup_{C_i \in CS} C_i = A$. The set Π_A denotes the set of all coalitions structures (over A). In a CFG, each coalition C produces a reward $f(C)$ corresponding to some implicit "task" to be performed by C. $f(C)$ is called the *value* of a coalition C, and $F(CS)$ denotes the value of a coalition structure CS: it is defined as the sum of the values of all coalitions, i.e., $F(CS) = \sum_{C_i \in CS} f(C_i)$. The CSG problem is to find an *optimal* coalition structure, i.e., a coalition structure $CS \in \Pi_A$ such that for each $CS' \in \Pi_A$, $F(CS') \leq F(CS)$.

Example 2 (continued) Let us formalize the example from the introduction. We consider the CFG $\langle A, f \rangle$, where $A = \{a_1, a_2, a_3\}$, where a_1, a_2, a_3 respectively correspond to Alice, Bob and Charles; and f is defined as $f(\{a_1\}) = 30$, $f(\{a_2\}) = 40$, $f(\{a_3\}) = 0$, $f(\{a_1, a_2\}) = 90$, $f(\{a_1, a_3\}) = 120$, $f(\{a_2, a_3\}) = 100$, and $f(\{a_1, a_2, a_3\}) = 150$. There are five coalition structures in Π_A: $CS_1 = \{\{a_1, a_3\}, \{a_2\}\}$, $CS_2 = \{\{a_1, a_2\}, \{a_3\}\}$, $CS_3 = \{\{a_1, a_2, a_3\}\}$, $CS_4 = \{\{a_2, a_3\}, \{a_1\}\}$ and $CS_5 = \{\{a_1\}, \{a_2\}, \{a_3\}\}$. Plans I, II and III described in the introduction respectively correspond to CS_1, CS_2 and CS_3. We get that $F(CS_1) = f(\{a_1, a_3\}) + f(\{a_2\}) = 120 + 40 = 160$. Similarly, $F(CS_2) = 90$, $F(CS_3) = 150$, $F(CS_4) = 130$ and $F(CS_5) = 70$ (the results are reported in Table 1.) Thus CS_1 is optimal.

Computing an optimal coalition structure is an NP-hard problem in the general case [31]. However, there are properties on the characteristic function f which, when satisfied, makes the CSG problem a trivial one, computationally speaking. Thus, given a set E, a mapping $v : E \to \mathbb{R}$ is said to be *subadditive*, i.e., if for every $E_1, E_2 \subseteq E$, $v(E_1) + v(E_2) \geq v(E_1 \cup E_2)$. Subadditive characteristic functions are used in domains where each agent would always do better if it were alone. This contrasts with widely cooperative environments such as the postmen problem [44], where the formation of a large coalition out of disjoint coalitions, guarantees at least the value that is obtained by the disjoint coalitions separately. In such a case, $v : E \to \mathbb{R}$ is said to be *superadditive*, i.e., if for every $E_1, E_2 \subseteq E$, $v(E_1) + v(E_2) \leq v(E_1 \cup E_2)$. We obviously have that:

Proposition 1 (folklore) *Let* $\langle A, f \rangle$ *be a* CFG.

(i) *If f is subadditive, then the coalition structure $CS = \{\{a_i\} \mid a_i \in A\}$ is an optimal one.*
(ii) *If f is superadditive, then the coalition structure $CS = \{A\}$ is an optimal one.*

4 Probabilistic coalition structure generation

We now present a formal framework for Probabilistic Coalition Structure Generation (PCSG).

4.1 General setting

The PCSG framework is based on an extension of a CFG which we simply call *Probabilistic Characteristic Function Game* (PCFG). A PCFG consists of a set of agents, a probability distribution p on a set of outcomes characterized by binary events associated with each agent, and a refined definition of a standard characteristic function, called *situational* characteristic function and denoted by g.

4.1.1 The probability distribution of a PCFG

Our goal is to deal with the events which may occur *after* forming a coalition structure: each agent may be "fully functional" as it is always assumed in the standard CSG framework, but it may also not fulfill its function as initially expected. This typically happens when an agent is found to be defective afterwards, or when an unexpected, exogenous event requires the agent to be removed from the coalition it has been assigned to. A PCFG thus considers a probability distribution p which associates every possible situation with a probability value.

Formally, let A be a finite set of agents. Given $P \subseteq A$, we denote $\overline{P} = A \setminus P$, and ω_P denotes an *outcome*, which identifies the situation where each agent from P is present and each agent from $\overline{P}$ is absent, after being assigned to a coalition. The set Ω_A denotes the set of all outcomes. An *event* is a set of outcomes. Given $Q, R \subseteq A$ such that $Q \cap R = \emptyset$, we denote by $\langle Q, R\rangle$ the event which corresponds to the set of outcomes where all agents in Q are present and all agents in R are absent. Formally, $\langle Q, R\rangle = \{\omega_P \in \Omega_A \mid Q \subseteq P, P \cap R = \emptyset\}$. The set $\mathcal{E}_A$ denotes the set of all such events $\langle Q, R\rangle$. For instance, for any $Q \subseteq A$, we have that $\langle Q, \overline{Q}\rangle = \{\omega_Q\}$, i.e., the event $\langle Q, \overline{Q}\rangle$ contains a single outcome which corresponds to the situation where each agent from Q is present and each remaining agent (i.e., each agent from $\overline{Q}$) is absent. As another example, the event $\langle Q, \emptyset\rangle$ contains $2^{|\overline{Q}|}$ outcomes, it corresponds to the situation where each agent from Q is present and nothing is known about the remaining agents from $\overline{Q}$. Now, let $p : \Omega_A \mapsto [0, 1]$ be a probability distribution over the set of outcomes. The domain of p naturally extends to events from $\mathcal{E}_A$ as follows, for each $\langle Q, R\rangle \in \mathcal{E}_A$:

$$p(\langle Q, R\rangle) = \sum_{\omega_P \in \langle Q,R\rangle} p(\omega_P).$$

Thus $p(\langle Q, R\rangle)$ represents the probability of the event $\langle Q, R\rangle$ to occur after a coalition structure is formed.

Example 1 (continued) Let us go back to the running example. For instance, consider the event $\langle\{a_1, a_2\}, \{a_3\}\rangle$ from $\mathcal{E}_A$: it contains a single outcome from Ω_A, which corresponds to the situation where a_1 and a_2 are present and a_3 is absent. On the other hand, $\langle\{a_1\}, \{a_3\}\rangle \in \mathcal{E}_A$ is an event from $\mathcal{E}_A$ which contains two outcomes from Ω_A: it corresponds to the situation where a_1 is present and a_3 is absent, but nothing is known about a_2. Let us assume that a_1 is present or functional 80% of the time, a_2 100%, and a_3 10%. So $p(\langle\{a_1\}, \emptyset\rangle) = 0.8$, $p(\langle\{a_2\}, \emptyset\rangle) = 1$, and $p(\langle\{a_3\}, \emptyset\rangle) = 0.1$. We also assume that the situation of any agent does not affect the one of any other agent: the events $\langle\{a_1\}, \emptyset\rangle$, $\langle\{a_2\}, \emptyset\rangle$ and $\langle\{a_3\}, \emptyset\rangle$ are independent.[1] Hence, the probability distribution $p : \Omega_A \to [0, 1]$ can be fully derived, e.g., $p(\langle\{a_1, a_2\}, \{a_3\}\rangle) = p(\langle\{a_1\}, \emptyset\rangle) \cdot p(\langle\{a_2\}, \emptyset\rangle) \cdot p(\langle\emptyset, \{a_3\}\rangle) = 0.8 \cdot 1 \cdot (1 - 0.1) = 0.72$.

4.1.2 The situational characteristic function of a PCFG

A PCFG also considers an extension of the standard characteristic function $f : 2^A \to \mathbb{R}$ in a CFG (cf. Definition 3), which we call *situational* characteristic function. Denoted by g, this function maps each coalition *and each outcome* to a value. That is, g makes precise how some underlying value should be updated when some agents are found missing after the formation of a coalition structure. Now notably, in a CFG the value of a given coalition depends only on its members.[2] We naturally extend this assumption to outcomes in PCFGs, i.e., the value of a coalition in a given outcome should only depend on the presence/absence of the members of that coalition. Formally, g is a mapping $g : 2^A \times \Omega_A \to \mathbb{R}$ such that for each coalition C and each outcome $\omega_P \in \Omega_A$,

$$g(C, \omega_P) = g(C, \omega_{P \cap C}). \tag{1}$$

This assumption plays a key role in most of our results in the following sections, including how the expected utility of a coalition structure can be computed coalition-wise (cf. Sect. 4.2.3).

Then, the value of a coalition structure CS in any outcome ω_P, denoted by $G(CS, \omega_P)$, is defined as the sum of the values $g(C, \omega_P)$ for each coalition $C \in CS$, i.e., $G : \Pi_A \times \Omega_A \to \mathbb{R}$ is defined for each $CS \in \Pi_A$ and each outcome $\omega_P \in \Omega_A$ as

$$G(CS, \omega_P) = \sum_{C \in CS} g(C, \omega_P). \tag{2}$$

There are a number of ways g can be defined. An interesting option is to derive g from an existing characteristic function f of a given CFG, i.e., to define a PCFG as a particular "extension" of a given CFG. In Sect. 5, we point out two such derivations of g of practical interest illustrated using our running example.

4.1.3 Probabilistic characteristic function game

We are now ready to introduce the definition of a Probabilistic Characteristic Function Game (PCFG):

[1] The independence assumption is not required in our subsequent results. It is made in the example for simplicity.

[2] Characteristic functions for which the value of a coalition C depends on the entire coalition structure to which C belongs are called *Partition Function Games* [38], but are not considered in this paper.

Definition 2 (*Probabilistic Characteristic Function Game*) A *Probabilistic Characteristic Function Game* (PCFG for short) is a tuple $\langle A, g, p\rangle$ where $A = \{a_1, \dots, a_n\}$ is a set of agents; $p : \Omega_A \to [0, 1]$ is a probability distribution; and $g : 2^A \times \Omega_A \to \mathbb{R}$ is a situational characteristic function.

In the standard CSG framework, the issue of interest is to find a coalition structure of maximal utility. In PCSG, one seeks to maximize the *expected utility* $U(CS)$ of a coalition structure CS, which, given a PCFG $\langle A, g, p\rangle$, is defined as follows:

$$U(CS) = \sum_{\omega_P \in \Omega_A} p(\omega_P) \cdot G(CS, \omega_P). \tag{3}$$

Given a PCFG, a coalition structure CS is said to be *optimal* if for each $CS' \in \Pi_A$, $U(CS') \leq U(CS)$. The PCSG problem is then to find an optimal coalition structure.

4.2 Properties

4.2.1 PCFGs extend CFGs

Let us first emphasize that PCFGs are at least as expressive as CFGs, in the sense that every CFG $\langle A,f\rangle$ can be "extended" to a PCFG $\langle A, g, p\rangle$ so that the value of any coalition structure in $\langle A,f\rangle$ coincides with its expected utility in $\langle A, g, p\rangle$. More formally, let us show how one can simply extend any CFG $\langle A,f\rangle$ into a PCFG $\langle A, g, p\rangle$ that is *equivalent* to it, i.e., when for each coalition structure CS, $F(CS) = U(CS)$. Let us formalize the notion of "CFG extension":

Definition 3 (*CFG extension*) Let $\langle A,f\rangle$ be a CFG. A *CFG extension* of $\langle A,f\rangle$ is any PCFG $\langle A, g, p\rangle$ such that for each $C \subseteq A$, $g(C, \omega_C) = f(C)$.

Intuitively, in a CFG extension the value taken by each coalition C is characterized by an underlying CFG in the situation where all of the agents from that coalition remain present, i.e., in the outcome ω_C and any outcome $\omega_P, C \subseteq P$ (cf. Equation 1). Notable CFG extensions are the most "optimistic" ones: an *optimistic* PCSG, denoted by $\langle A, g, p_\top\rangle$, is such that for every outcome $\omega_P \in \Omega_A$, $p_\top(\omega_P) = 1$ if $P = A$, and $p_\top(\omega_P) = 0$ otherwise. As expected:

Proposition 2 *Let $\langle A,f\rangle$ be a* CFG. *Then every optimistic* CFG *extension of $\langle A,f\rangle$ is equivalent to $\langle A,f\rangle$.*

4.2.2 Subadditivity and superadditivity in PCSG

Interestingly, the counterparts of the results given in Proposition 1 in the standard CSG framework also hold in our PCSG framework. That is, when the situational characteristic function g is "subadditive", the coalition structure formed of singleton coalitions is an optimal one; and when g is "superadditive", the grand coalition structure is an optimal one. This requires to make precise the notion of subadditivity and superadditivity for g, which simply extends the respective counterpart properties for f as follows:

Definition 4 (*Sub/superadditivity for g*) The function $g : 2^A \times \Omega_A \to \mathbb{R}$ is said to be *subadditive* if for all $C_1, C_2 \subseteq A$ and every $\omega_P \in \Omega_A$,

$$g(C_1, \omega_P) + g(C_2, \omega_P) \geq g(C_1 \cup C_2, \omega_P).$$

It is said to be *superadditive* if for all $C_1, C_2 \subseteq A$ and every $\omega_P \in \Omega_A$,

$$g(C_1, \omega_P) + g(C_2, \omega_P) \leq g(C_1 \cup C_2, \omega_P).$$

Proposition 3 *Let* $\langle A, g, p \rangle$ *be a* PCFG.

(i) *If g is subadditive, then the coalition structure* $CS = \{\{a_i\} \mid a_i \in A\}$ *is an optimal one.*
(ii) *If g is superadditive, then the coalition structure* $CS = \{A\}$ *is an optimal one.*

4.2.3 Coalition-wise computation of the expected utility

Now, computing $U(CS)$ for a given coalition structure CS *a priori* requires 2^n computation steps, according to Equation 3, one for each outcome. However, Proposition 4 below shows that the computation of $U(CS)$ can be characterized in a coalition-wise fashion. That is to say, computing the expected utility of a coalition structure CS boils down to computing the expected utilities of coalitions from CS and to sum them up, where the expected utility of each coalition $C \subseteq A$, denoted $u(C)$, is defined as:

$$u(C) = \sum_{\omega_P \in \Omega_A} p(\omega_P) \cdot g(C, \omega_P). \tag{4}$$

Indeed, $U(CS)$ can be characterized as follows:

Proposition 4 *Given a* PCFG $\langle A, g, p \rangle$, *for every coalition structure* $CS \subseteq \Pi_A$, *we have that*

$$U(CS) = \sum_{C \in CS} u(C).$$

Interestingly, to compute the utility of a coalition $u(C)$, one does not need to enumerate all 2^n outcomes from the set Ω_A, but only the $2^{|C|}$ events $\langle P, C \setminus P \rangle$ from $\mathcal{E}_A$, for all $P \subseteq C$. Indeed, as a consequence of Equation 1, the expected utility of C does not depend on the events involving agents outside of C:

Proposition 5 *Given a* PCFG $\langle A, g, p \rangle$, *for every coalition* $C \subseteq A$, *we have that*

$$u(C) = \sum_{P \subseteq C} p(\langle P, C \setminus P \rangle) \cdot g(C, \omega_P).$$

To summarize, from Propositions 4 and 5, we get that:

Corollary 1 *Given a* PCFG $\langle A, g, p \rangle$, *for every coalition structure* $CS \subseteq \Pi_A$, *we have that*

$$U(CS) = \sum_{C \in CS} \sum_{P \subseteq C} p(\langle P, C \setminus P \rangle) \cdot g(C, \omega_P).$$

5 Two subclasses of PCFG

Let us recall that in a CFG, the characteristic function f associates a value with each coalition $C \subseteq A$; and in contrast, in a PCFG the situational characteristic function g must associate a value with each coalition $C \subseteq A$ *and each outcome* $\omega_P \in \Omega_A$, where $P \subseteq C$ (cf. Definition 2 and Eq. 1). That is, the domain of g is exponentially larger than the one of f. However, there is a number of elegant ways g can be defined to make it as succinct as f. One of them is to start with a given CFG $\langle A, f \rangle$ (assuming it is available), and to consider one of its CFG extensions $\langle A, g, p \rangle$ (cf. Definition 3). That is, one first sets $g(C, \omega_C) = f(C)$ for each coalition $C \subseteq A$. Doing so, the value of a coalition C in the resulting PCFG corresponds to the value of C using f in the case where all agents from C remain present in their coalition. In addition, one needs a policy which makes precise how the value $g(C, \omega_P)$ is characterized in the remaining cases when $C \nsubseteq P$, i.e., how the value $f(C)$ must be "updated" in the event when some agents initially assigned to C are missing after the formation of the coalition structure.

Indeed, in both the CSG and PCSG frameworks, each coalition C is implicitly assigned to some "task" and in the case when no agent from C appears missing after the formation of the coalition structure, C should produce the reward corresponding to the task to be performed, as expected. When not all agents are present in the coalition they have been assigned to, different situations may arise depending on the context, which leads one to consider different such policies. This general scheme allows one to define a PCFG directly from a given CFG, given an additional probability distribution and a policy.

In this section we consider and formalize two natural policies corresponding to the situations illustrated in our running example: the *flexible* policy and the *cautious* policy. The flexible policy considers the scenario where any residual coalition can be assigned to another task and still produce a reward in the case where some agents appear missing. In contrast, the cautious policy considers the opposite case, i.e., that it is not possible for an updated coalition to be assigned to another task; in such a case, no reward can be obtained from it. In the following, we show that the choice of the policy may easily lead to select distinct optimal coalition structures.

5.1 Flexible PCFG

A first option is to assume that if an unexpected event deprives a given coalition of some of its agents, then the residual coalition can be assigned to another task. More precisely, the residual coalition can still produce the reward it has been associated with through the characteristic function f. This assumption is reasonable in the case where one considers, for instance, a set of wireless sensor networks whose goal is to optimize some global connectivity in a utilitarian fashion. In our running example, this corresponds to the scenario when the workers can be transferred right away to another task without any further cost.

Definition 5 (*Flexible PCFG*) A *flexible* PCFG is a CFG extension $\langle A, g^f_{fle}, p \rangle$, where $g^f_{fle} : 2^A \times \Omega_A \rightarrow \mathbb{R}$ is such that for each coalition $C \subseteq A$ and each outcome $\omega_P \in \Omega_A$,

$$g^f_{fle}(C, \omega_P) = f(C \cap P).$$

The utility of a coalition $C \subseteq A$ and a coalition structure $CS \in \Pi_A$ in a flexible PCFG are respectively denoted by $u_{fle}(C)$ and $U_{fle}(CS)$.

Example 1 (continued) Let us illustrate the computation of some expected utilities from our running example. We detail only the computation of the utility of $CS_1 = \{\{a_1, a_3\}, \{a_2\}\}$. From Corollary 1, we get that:

$$\begin{aligned}
U_{fle}(CS_1) &= \textstyle\sum_{C \in CS_1} \sum_{P \subseteq C} p(\langle P, C \setminus P \rangle) \cdot g^f_{fle}(C, \omega_P) \\
&= p(\langle \{a_1, a_3\}, \emptyset \rangle) \cdot g^f_{fle}(\{a_1, a_3\}, \omega_{\{a_1, a_3\}}) \\
&\quad + p(\langle \{a_1\}, \{a_3\} \rangle) \cdot g^f_{fle}(\{a_1, a_3\}, \omega_{\{a_1\}}) \\
&\quad + p(\langle \{a_3\}, \{a_1\} \rangle) \cdot g^f_{fle}(\{a_1, a_3\}, \omega_{\{a_3\}}) \\
&\quad + p(\langle \emptyset, \{a_1, a_3\} \rangle) \cdot g^f_{fle}(\{a_1, a_3\}, \omega_{\emptyset}) \\
&\quad + p(\langle \{a_2\}, \emptyset \rangle) \cdot g^f_{fle}(\{a_2\}, \omega_{\{a_2\}}) + p(\langle \emptyset, \{a_2\} \rangle) \cdot g^f_{fle}(\{a_2\}, \omega_{\emptyset}) \\
&= 0.8 \cdot 0.1 \cdot 120 + 0.8 \cdot (1 - 0.1) \cdot 30 + 1 \cdot 40 = 71.2.
\end{aligned}$$

Similarly, it can be checked that $U_{fle}(CS_2) = 80$, $U_{fle}(CS_3) = 86$, $U_{fle}(CS_4) = 66.4$ and $U_{fle}(CS_5) = 64$ (cf. Table 1). Here, CS_3 (plan III) is the best choice among all coalition structures. Compared to CS_2, in CS_3 the agent a_3 is in the grand coalition, together with a_1 and a_2. This is harmless since in the flexible PCFG case, the absence of a_3 will not result in a breakdown of the whole coalition. And in the case when a_3 is absent, as the remaining coalition $\{a_1, a_2\}$ is relatively "reliable" and $f(\{a_1, a_2\}) > f(\{a_1\}) + f(\{a_2\})$, we get that $u_{fle}(\{a_1, a_2\}) > u_{fle}(\{a_1\}) + u_{fle}(\{a_2\})$, so that $U_{fle}(\{\{a_1, a_2\}\}) > U_{fle}(\{\{a_1\}, \{a_2\}\})$. Thus CS_3 is also preferred to CS_1.

We have shown in Proposition 3 that when g^f_{fle} is subadditive or superadditive, solving the PCSG problem is a trivial task. Obviously enough, when f is subadditive (resp. superadditive), then g^f_{fle} is subadditive (resp. superadditive). So as a consequence of Proposition 3, we get that:

Corollary 2 *Let $\langle A, g^f_{fle}, p \rangle$ be a flexible* PCFG.

(i) *If f is subadditive, then the coalition structure $CS = \{\{a_i\} \mid a_i \in A\}$ is an optimal one.*
(ii) *If f is superadditive, then the coalition structure $CS = \{A\}$ is an optimal one.*

5.2 Cautious PCFG

Another option is to assume that every coalition C in a coalition structure is implicitly assigned to a task and that the agents from C commit themselves to perform this task and no other. Thus the underlying task can be achieved only if *all* agents from C are present, and no other task can be assigned to the updated coalition in case of agent loss. That is to say, the value of the coalition C is equal to 0 in any outcome $\omega_P \in \Omega_A$ where $C \nsubseteq P$. In our running example, this option corresponds to the scenario where an appointment with the customers must be made ahead of time.

Definition 6 (*Cautious PCFG*) A *cautious* PCFG is a CFG extension $\langle A, g^f_{cau}, p \rangle$, where $g^f_{cau} : 2^A \times \Omega_A \to \mathbb{R}$ is such that for each coalition $C \subseteq A$ and each outcome $\omega_P \in \Omega_A$,

$$\text{if } C \not\subseteq P \text{ then } g^f_{cau}(C, \omega_P) = 0.$$

Note that a cautious PCSG is fully characterized given a CFG $\langle A,f \rangle$, since $g^f_{cau}(C, \omega_P) = f(C)$ when $C \subseteq P$ (cf. Definition 3 and Equation 1).

The utility of a coalition $C \subseteq A$ and a coalition structure $CS \in \Pi_A$ in a cautious PCFG are respectively denoted by $u_{cau}(C)$ and $U_{cau}(CS)$.

For any $CS \in \Pi_A$, computing $U_{cau}(CS)$ can be done coalition-wise, as shown by Proposition 4. Yet from Proposition 5, for any coalition $C \subseteq A$, computing $u_{cau}(C)$ may require summing up $2^{|C|}$ values in the general case, i.e., one for each outcome $\omega_P \in \Omega_A$ with $P \subseteq C$. However, here one only needs to consider the single event where no agent in the coalition breaks down:

Proposition 6 *Let $\langle A, g^f_{cau}, p \rangle$ be a cautious* PCFG. *For any coalition $C \subseteq A$, we have that*

$$u_{cau}(C) = p(\langle C, \emptyset \rangle) \cdot g^f_{cau}(C, \omega_C).$$

Example 1 (continued) Let us illustrate the computation of the expected utilities of the coalition structures from our running example. We detail only the computation of the utility of $CS_1 = \{\{a_1, a_3\}, \{a_2\}\}$. From Proposition 6, we get that:

$$\begin{aligned} U_{cau}(CS_1) &= \textstyle\sum_{C \in CS_1} p(\langle C, \emptyset \rangle) \cdot g^f_{cau}(C, \omega_C) \\ &= p(\langle \{a_1, a_3\}, \emptyset \rangle) \cdot g^f_{cau}(\{a_1, a_3\}, \omega_{\{a_1, a_3\}}) \\ &\quad + p(\langle \{a_2\}, \emptyset \rangle) \cdot g^f_{cau}(\{a_2\}, \omega_{\{a_2\}}) \\ &= 0.8 \cdot 0.1 \cdot 120 + 1 \cdot 40 = 49.6. \end{aligned}$$

Similarly, it can be checked that $U_{cau}(CS_2) = 72$, $U_{cau}(CS_3) = 12$, $U_{cau}(CS_4) = 34$ and $U_{cau}(CS_5) = 64$. These results are reported in Table 1. One can remark that although CS_1 is optimal for the CFG and CS_3 is optimal for its flexible CFG extension, these coalition structures are clearly not the best choice for its cautious CFG extension. Intuitively, as the probability of attendance of a_3 is quite low ($p(\langle \{a_3\}, \emptyset \rangle) = 0.1$), it is risky to assign a_3 to a coalition with some other agents: the utilities of these coalitions are low. Indeed, it can be checked that $u_{cau}(\{a_1, a_3\}) = 9.6$, $u_{cau}(\{a_2, a_3\}) = 10$ and $u_{cau}(\{a_1, a_2, a_3\}) = 12$. Instead, CS_2 is the best choice. In CS_2, a_3 is left alone in its coalition: even if its coalition produces no reward in any case ($u_{cau}(\{a_3\}) = 0$), the coalition $\{a_1, a_2\}$ is formed of reliable agents: it will produce a value of 90 with a probability of $p(\langle \{a_1\}, \emptyset \rangle) \cdot p(\langle \{a_2\}, \emptyset \rangle) = 0.8$, so that $u_{cau}(\{a_1, a_2\}) = 72$.

Now, it can easily be seen that g^f_{cau} is subadditive when the function f is subadditive. Hence, from Proposition 3 we get that:

Corollary 3 *Let $\langle A, g^f_{cau}, p \rangle$ be a cautious* PCFG. *If f is subadditive, then the coalition structure $CS = \{\{a_i\} \mid a_i \in A\}$ is an optimal one.*

However, even if f is superadditive the grand coalition $CS = \{A\}$ is not necessarily optimal: in general g^f_{cau} is not superadditive even if f is. This can be viewed using the simple following example of the cautious PCFG $\langle A, g^f_{cau}, p \rangle$, where $A = \{a_1, a_2\}$, $f(\{a_1\}) = 20$, $f(\{a_2\}) = 20$, $f(\{a_1, a_2\}) = 60$, $p(\langle \{a_1\}, \emptyset \rangle) = p(\langle \{a_2\}, \emptyset \rangle) = 0.5$: it can be easily verified that f is superadditive, $U_{cau}(\{\{a_1\}, \{a_2\}\}) = 20$ and $U_{cau}(\{\{a_1, a_2\}\}) = 15$, so $U_{cau}(\{\{a_1, a_2\}\}) < U_{cau}(\{\{a_1\}, \{a_2\}\})$.

6 The computational complexity of PCSG

In this section, we investigate the complexity of PCSG, more precisely we focus on the two following decision problems:

Definition 7 (*Decision Problem DP-CS*)

- **Input:** *A PCFG* $\langle A, g, p\rangle$ *such that g and p are computable in polynomial time,*[3] *a coalition structure CS, and a non-negative rational number k.*
- **Question:** *Does* $U(CS) \geq k$ *hold?*

Definition 8 (*Decision Problem DP-∃CS*)

- **Input:** *A PCFG* $\langle A, g, p\rangle$ *such that g and p are computable in polynomial time, and a non-negative rational number k.*
- **Question:** *Does there exist a coalition structure CS such that* $U(CS) \geq k$?

The rest of this paper will focus on such inputs, i.e., where g and p are computable in polynomial time.

We assume that the reader is familiar with the complexity class NP. The class PP is the set of problems that can be solved by a nondeterministic Turing machine in polynomial time where the acceptance condition is that a majority (more than half) of computation paths accept (see [25] for more details). Higher complexity classes are defined using oracles: $\mathsf{NP}^{\mathsf{PP}}$ corresponds to the class of decision problems that are solved in polynomial time by non-deterministic Turing machines using an oracle for PP. An indication of the high difficulty of solving PP (and thus, $\mathsf{NP}^{\mathsf{PP}}$) -hard problems is reflected by Toda's theorem, stating that a Turing machine with a PP oracle can solve in polynomial time every problem in the polynomial hierarchy [39]. It turns out that both problems **DP-CS** and **DP-∃CS** are hard for PP and $\mathsf{NP}^{\mathsf{PP}}$, respectively:

Proposition 7 **DP-CS** *is* PP*-hard and* **DP-∃CS** *is* $\mathsf{NP}^{\mathsf{PP}}$*-hard. Hardness results hold for flexible* PCFGs, *and even when the set of events* $\{\langle\{a_i\}, \emptyset\rangle \mid a_i \in A\}$ *are pairwise independent.*

When considering cautious PCFGs, a significant drop in computational complexity can be obtained. Indeed, in the previous section Proposition 6 told us that computing the expected utility of any given coalition requires to consider only a single event, i.e., the event where no agent is missing. Thus computing $u_{cau}(C)$ for any coalition is not a hard task, and as a consequence:

Proposition 8 **DP-CS** *is in* P *and* **DP-∃CS** *is* NP*-complete for cautious* PCFGs.

[3] In Definitions 7 and 8, one does not make any assumption about the way g and p are represented. However, one assumes that the corresponding mappings can be computed in polynomial time.

7 MC-net based cautious/flexible PCFGs

In this section, we identify some interesting subclasses of cautious and flexible PCFGs, i.e., those represented as marginal contribution networks (MC-nets) [11].

MC-nets are a simple, rule-based formalim, in which each rule takes the form *condition* → *number*, where "condition" is a Boolean condition over agents. To compute the value of a coalition given an MC-net, one checks for each rule whether the corresponding condition is satisfied by the coalition, and in such a case one adds the associated "number" to the value of the coalition. MC-nets are useful to represent in a succinct and natural way a characteristic function whose size would be lower bounded by 2^n if represented extensively as a table, n being the number of agents. In the general case, one can consider characteristic functions which cannot be represented through an MC-net in a succinct way, i.e., the size of an MC-net for a "realistic" characteristic function depends on the application. But MC-nets are fully expressive: they permit the computation of solution concepts such as the Shapley value [11] and weighted goals [40], and include some useful classes of representation languages such as weighted graphs [6]. They are particularly appropriate when the value associated with a coalition results in the combination of natural patterns identified in the coalition: it is indeed fair to assume that some agents are fit to work together, while others are not. For instance, consider the case when two agents contribute to a certain reward in an underlying task when they work together on it, provided that a third agent is not with them. Such a pattern can be represented thanks to an MC-net through a single rule, irrespective of the total number of agents involved. In contrast, representing the characteristic function explicitly would require to consider this pattern in the computation of the value of all $2^n - 1$ coalitions, which is not feasible in practice. Note that there exist a number of representation languages for representing a characteristic function f in a CFG, e.g., Synergy Coalition Group (SCG) representation [5], skilled-based representation [22] and agent-type representation [36, 41]. Considering all of them at once is out of the scope of this paper, and we focus on the language of MC-nets in the following.

Now, we have seen in Proposition 7 that computing the expected utility of a coalition structure in the flexible case is a PP-hard task. Interestingly, we will show in this section that when flexible PCFGs are represented by an MC-net, this task can be done in polynomial time. We actually consider MC-net based flexible *and* MC-net based cautious PCFGs. This allows us to provide encodings for each of them, and thus tackle the problem of computing an optimal coalition structure experimentally in each case, which will be done in the next section.

By definition, cautious and flexible PCFGs are CFG extensions, so they are induced by a CFG $\langle A, f\rangle$. From now on, we assume f to be represented by an MC-net [11]:

Definition 9 (*MC-net*) Given a finite set of agents A, a *marginal contribution network (MC-net)* is a finite set $\mathcal{R} = \{r_1, \dots, r_m\}$ of rules $r_i = (\gamma_i, w_i)$ ($i \in \{1, \dots, m\}$), where:

- γ_i is the condition of r_i, denoted by a consistent conjunction of literals over A (where each element of A is viewed as a Boolean variable), i.e., $\gamma_i = a_{i_1} \wedge \dots \wedge a_{i_j} \wedge \overline{a_{i_{j+1}}} \wedge \dots \wedge \overline{a_{i_k}}$, $0 \leq j \leq k$, and for all literals $a_{i_l}, \overline{a_{i_s}} \in \gamma_i$, we have that $a_{i_l} \in A$ and $a_{i_s} \in A$;
- w_i is a real number (the weight of r_i).

Given a condition $\gamma_i = a_{i_1} \wedge \ldots \wedge a_{i_j} \wedge \overline{a_{i_{j+1}}} \wedge \ldots \wedge \overline{a_{i_k}}$, we note $\gamma_i^+ = \{a_{i_l} \mid l \leq j\}$ the positive part of γ_i and $\gamma_i^- = \{a_{i_l} \mid l > j\}$ the negative part of γ_i. We say that a rule $r_i = (\gamma_i, w_i)$ is *partially activated* by a coalition $C \subseteq A$ when $\gamma_i^+ \subseteq C$; and r_i is *activated* by C when it is partially activated by C and moreover $\gamma_i^- \cap C = \emptyset$. Given an MC-net $\mathcal{R}$, f is characterized for each $C \subseteq A$ as $f(C) = \sum_{r_i \in \mathcal{R}_C^*} w_i$, where $\mathcal{R}_C^*$ is the set of rules from $\mathcal{R}$ that are activated by C.

Example 1 (continued) The characteristic function f used in the running example can be encoded (for instance) as $\mathcal{R} = \{r_1, r_2, r_3, r_4, r_5\}$, where:

$$r_1 = (\{a_1\}, 30) \qquad r_2 = (\{a_2\}, 40) \qquad r_3 = (\{a_1, a_2\}, 20)$$
$$r_4 = (\{a_2, a_3\}, 60) \qquad r_5 = (\{a_1, \overline{a}_2, a_3\}, 90).$$

For instance, the coalition $\{a_1, a_2, a_3\}$ partially activates all five rules, and activates precisely the first four rules.

As another example, let us detail the computation of $F(CS_1)$, where $CS_1 = \{\{a_1, a_3\}, \{a_2\}\}$ (cf. Table 1). The coalition $\{a_1, a_3\}$ activates precisely the rules r_1 and r_5, so $f(\{a_1, a_3\}) = w_1 + w_5 = 30 + 90 = 120$. And the coalition $\{a_2\}$ activates precisely the rule r_2, so $f(\{a_2\}) = w_2 = 40$. Accordingly, we get that $f(\{a_1, a_3\}, \{a_2\}) = 120 + 40 = 160$, thus $F(CS_1) = 160$, which corresponds to the value of $F(CS_1)$ given in Table 1.

Obviously enough, f is computable in polynomial time when it is characterized by an MC-net, and thus both g^f_{cau} and g^f_{fle} are also computed in polynomial time.

Now without loss of generality, one assumes that each γ_i contains at least one positive literal. This assumption does not lead to an exponential blow-up: for instance, if a rule has the form $(\{\overline{a_1}\}, w)$, then it can equivalently be replaced by the set of rules $\{(\{\overline{a_1}, a_2\}, w), (\{\overline{a_1}, \overline{a_2}, a_3\}, w), \ldots, (\{\overline{a_1}, \ldots, \overline{a_{n-1}}, \ a_n\}, w)\}$ (see [17, 18]). Under this assumption, each r_i of $\mathcal{R}$ can be partially activated by at most one coalition, denoted by $CS(i)$ when it exists, of the coalition structure CS under consideration: indeed, the agent corresponding to this positive literal cannot belong to several coalitions.

Interestingly, Propositions 9 and 10 below show that the expected utility of any given coalition structure CS can be computed on a rule-by-rule basis, for both cautious and flexible PCFGs:

Proposition 9 *For any $CS \in \Pi_A$,*

$$U_{cau}(CS) = \sum_{r_i \in \mathcal{R}^*} w_i \cdot p(\langle CS(i), \emptyset \rangle),$$

where $\mathcal{R}^*$ is the set of rules r_i activated by a coalition $CS(i) \in CS$.

Example 1 (continued) Let us detail how $U_{cau}(CS_1)$ can be computed on a rule-by-rule basis according to Proposition 9, where $CS_1 = \{\{a_1, a_3\}, \{a_2\}\}$ (cf. Table 1). Recall that in our example, we have that $p(\langle \{a_1\}, \emptyset \rangle) = 0.8$, $p(\langle \{a_2\}, \emptyset \rangle) = 1$, and $p(\langle \{a_3\}, \emptyset \rangle) = 0.1$, that the coalition $\{a_1, a_3\}$ activates precisely the rules r_1 and r_5, and that the coalition $\{a_2\}$ activates precisely the rule r_2. Thus:

$$\begin{aligned} U_{cau}(CS_1) &= \textstyle\sum_{r_i \in \mathcal{R}^*} w_i \cdot p(\langle CS(i), \emptyset \rangle) \\ &= w_1 \cdot p(\langle \{a_1, a_3\}, \emptyset \rangle) + w_5 \cdot p(\langle \{a_1, a_3\}, \emptyset \rangle) + w_2 \cdot p(\langle \{a_2\}, \emptyset \rangle) \\ &= w_1 \cdot (0.8 \cdot 0.1) + w_5 \cdot (0.8 \cdot 0.1) + w_2 \cdot 1 \\ &= 40 \cdot 0.08 + 90 \cdot 0.08 + 40 \cdot 1 \\ &= 2.4 + 7.2 + 40 = 49.6, \end{aligned}$$

which corresponds to the value of $U_{cau}(CS_1)$ given in Table 1.

Proposition 10 *For any* $CS \in \Pi_A$,

$$U_{fle}(CS) = \sum_{r_i \in \mathcal{R}^+} w_i \cdot p(\langle \gamma_i^+, CS(i) \cap \gamma_i^- \rangle),$$

where $\mathcal{R}^+$ *is the set of rules* r_i *partially activated by a coalition* $CS(i) \in CS$.

Example 1 (continued) Similarly, we detail the computation of $U_{fle}(CS_1)$ on the rule-by-rule basis as given by Proposition 10. Since the coalition $\{a_1, a_3\}$ partially activates precisely the rules r_1 and r_5, and that the coalition $\{a_2\}$ partially activates precisely the rule r_2, we get that:

$$\begin{aligned} U_{fle}(CS_1) &= \textstyle\sum_{r_i \in \mathcal{R}^+} w_i \cdot p(\langle \gamma_i^+, CS(i) \cap \gamma_i^- \rangle) \\ &= w_1 \cdot p(\langle \{a_1\}, \emptyset \rangle) + w_5 \cdot p(\langle \{a_1, a_3\}, \emptyset \rangle) + w_2 \cdot p(\langle \{a_2\}, \emptyset \rangle) \\ &= w_1 \cdot 0.8 + w_5 \cdot (0.8 \cdot 0.1) + w_2 \cdot 1 \\ &= 40 \cdot 0.8 + 90 \cdot 0.08 + 40 \cdot 1 \\ &= 24 + 7.2 + 40 = 71.2, \end{aligned}$$

which corresponds to the value of $U_{fle}(CS_1)$ given in Table 1.

We already know from Proposition 8 that the expected utility of a given coalition structure can be computed in polynomial time for any cautious PCFG. Furthermore, we showed in Proposition 7 that the problem remains PP-hard for flexible PCFGs. Interestingly, Proposition 10 tells us that for MC-net based flexible PCFGs, computing the expected utility of a coalition structure can be done in polynomial time. As a consequence:

Proposition 11 **DP-CS** *is in* P *and* **DP-∃CS** *is* NP-*complete for MC-net based flexible* PCFGs.

8 Computing optimal coalition structures for MC-net based PCFGs

We now tackle the problem of practically computing a coalition structure of maximal expected utility, for both MC-net based cautious and flexible PCFGs. In both cases, we reduce the corresponding problem to a mixed integer linear programming (MILP) problem. Then we can take advantage of the IBM ILOG CPLEX Optimizer[4] for generating an optimal solution of the latter, which can easily be interpreted as an optimal solution of the former, i.e., a coalition structure of maximal expected utility. For simplicity, the

[4] https://www-01.ibm.com/software/commerce/optimization/cplex-optimizer/.

encodings provided in this section are designed to the situation where the set of events $\{\langle\{a_i\}, \emptyset\rangle \mid a_i \in A\}$ are pairwise independent. Nevertheless, Propositions 9 and 10 tell us that the general case can be dealt with, as far as for any event $\langle Q, R\rangle \in \mathcal{E}_A$, $p(\langle Q, R\rangle)$ is computed in polynomial time.

Obviously enough, coalitions structures can be represented by means of equivalence relations. So we consider $n \cdot (n-1)/2$ binary variables $l_{i,j}, 1 \leq i < j \leq n$, where $l_{i,j}$ is true (1) if and only if agents a_i and a_j belong to the same coalition within the coalition structure ("l" stands for "linked"). We set a cubic (in n) number of constraints of the form $l_{i,j} + l_{j,k} - l_{i,k} \leq 1$ for establishing the transitivity of the "linked" relation. The assignments of the $l_{i,j}$ variables satisfying the constraints correspond precisely to the feasible coalition structures from Π_A.

We now need to consider variables and constraints which will be used to compute the expected utility of each coalition structure. Yet the conditions under which each rule $r_i \in R$ contributes to the global expected utility of a coalition structure differ between a cautious and a flexible PCFG (see Propositions 9 and 10). As a consequence, we need to introduce an encoding for each case.

8.1 A MILP encoding scheme for cautious PCFGs

First, a binary variable r_i is introduced for each rule $r_i = (\gamma_i, w_i)$ of R. Then for a rule r_i of $\mathcal{R}$, let us consider a positive literal a_k occurring in γ_i. We generate a set of inequalities equivalent to the constraint

$$r_i \Leftrightarrow \left(\bigwedge_{a_j \in \gamma_i, j \neq k} l_{k,j} \wedge \bigwedge_{\overline{a}_j \in \gamma_i} \neg l_{k,j} \right).$$

The two indexes k, j of each $l_{k,j}$ are switched if $k > j$. This set contains as many inequalities as literals in γ_i, leading to a total number of inequalities which is upper bounded by the number m of rules of $\mathcal{R}$ multiplied by the size d of the largest rule r_i (i.e., the number of literals in γ_i) plus one. On our running example, the rule $r_5 = (\{a_1, \overline{a}_2, a_3\}, 90)$ gives rise to the three inequalities $l_{1,3} - l_{1,2} - r_5 \leq 0$, $-l_{1,3} + r_5 \leq 0$, and $l_{1,2} + r_5 \leq 1$. This means that r_5 must be set to 1 precisely when the rule r_5 is activated by a coalition $CS(5)$ where $a_1, a_3 \in CS(5)$ and $a_3 \notin CS(5)$.

We need some further notations at this step. Let L_k be the list with a head equal to k and a tail which consists of all the integers from $\{1, \ldots, n\} \setminus \{k\}$ ordered in ascending way. For every j in the tail of L_k, $pred(j)_k$ denotes the index of the agent which is the predecessor of j in L_k. And $last(k)$ denotes the last element of L_k. Then we iterate the following process. Initially no rule of $\mathcal{R}$ is marked. Let $rank(j)$ ($j \in \{1, \ldots, n\}$) be the number of rules r_i of $\mathcal{R}$ which are not marked and are such that γ_i contains a_j as a positive literal. We consider one of the agents a_k such that a_k maximizes $rank(.)$ and we mark all the rules r_i such that γ_i contains a_k as a positive literal. We introduce a variable $p_{k,k}$ and we set the constraint $p_{k,k} = p(a_k)$. Then, for each agent a_j except a_k, we consider a variable $p_{k,j}$ and set two conditional rules:

$$l_{k,j} = 1 \rightarrow p_{k,j} = p_{k,pred(j)_k} \cdot p(\langle\{a_j\}, \emptyset\rangle)$$

and

$$l_{k,j} = 0 \rightarrow p_{k,j} = p_{k,pred(j)_k}.$$

By construction, $a_k \in CS(i)$ in the current coalition structure CS and $p_{k,last(k)}$ is equal to the probability $p(\langle CS(i), \emptyset\rangle)$. For each rule r_i which has been marked at this step, it remains to introduce a variable u_i and the following two conditional rules: $r_i = 1 \rightarrow u_i = w_i \cdot p_{k,last(k)}$, and $r_i = 0 \rightarrow u_i = 0$. This means that the rule r_i is activated precisely when its contribution to the expected utility of CS is equal to $w_i \cdot p(\langle CS(i), \emptyset\rangle)$, as required (cf. Proposition 9). We resume at the step where the ranks of the agents are computed, until all rules are marked. At the end of the process, for every rule r_i of $\mathcal{R}$, there is at least one $p_{k,last(k)}$ which has been computed such that a_k is a positive literal of r_i. The objective function $\sum_{i=1}^{m} u_i$ is to be maximized. For our running example, the following program has been obtained:

`maximize` $u_1 + u_2 + u_3 + u_4 + u_5$
`subject to`
$l_{1,2} + l_{2,3} - l_{1,3} \leq 1, l_{1,3} + l_{2,3} - l_{1,2} \leq 1,$
$l_{1,2} + l_{1,3} - l_{2,3} \leq 1, l_{2,3} + l_{1,3} - l_{1,2} \leq 1,$
$l_{1,3} + l_{1,2} - l_{2,3} \leq 1, l_{2,3} + l_{1,2} - l_{1,3} \leq 1,$
$r_1 = 1, r_2 = 1, l_{1,2} - r_3 \leq 0, -l_{1,2} + r_3 \leq 0,$
$l_{2,3} - r_4 \leq 0, -l_{2,3} + r_4 \leq 0,$
$l_{1,3} - l_{1,2} - r_5 \leq 0, -l_{1,3} + r_5 \leq 0, l_{1,2} + r_5 \leq 1,$
$p_{1,1} = 0.8, l_{1,2} = 1 \rightarrow p_{1,2} = p_{1,1} \cdot 1, l_{1,2} = 0 \rightarrow p_{1,2} = p_{1,1},$
$l_{1,3} = 1 \rightarrow p_{1,3} = p_{1,2} \cdot 0.1, l_{1,3} = 0 \rightarrow p_{1,3} = p_{1,2},$
$r_1 = 1 \rightarrow u_1 = 30 \cdot p_{1,3}, r_1 = 0 \rightarrow u_1 = 0,$
$r_3 = 1 \rightarrow u_3 = 20 \cdot p_{1,3}, r_3 = 0 \rightarrow u_3 = 0,$
$r_5 = 1 \rightarrow u_5 = 90 \cdot p_{1,3}, r_5 = 0 \rightarrow u_5 = 0,$
$p_{2,2} = 1,$
$l_{1,2} = 1 \rightarrow p_{2,1} = p_{2,2} \cdot 0.8, l_{1,2} = 0 \rightarrow p_{2,1} = p_{2,2},$
$l_{2,3} = 1 \rightarrow p_{2,3} = p_{2,1} \cdot 0.1, l_{2,3} = 0 \rightarrow p_{2,3} = p_{2,1},$
$r_2 = 1 \rightarrow u_2 = 40 \cdot p_{2,3}, r_2 = 0 \rightarrow u_2 = 0,$
$r_4 = 1 \rightarrow u_4 = 60 \cdot p_{2,3}, r_4 = 0 \rightarrow u_4 = 0,$
$0 \leq p_{1,1}, p_{1,2}, p_{1,3}, p_{2,2}, p_{2,1}, p_{2,3} \leq 1,$
Binary $l_{1,2}, l_{1,3}, l_{2,3}, r_1, r_2, r_3, r_4, r_5$

This encoding scheme when applied to a PCFG instance for n agents and m rules bearing on at most d agents generates a MILP instance containing a number of constraints upper bounded by $n^3 + 4 \cdot n^2 + m \cdot (d + 3)$, over $\frac{3}{2} \cdot n^2 - \frac{n}{2} + 2 \cdot m$ variables (including $\frac{n^2}{2} - \frac{n}{2} + m$ binary variables), and a linear objective function over m variables.

8.2 A MILP encoding scheme for flexible PCFGs

As to the case of cautious PCFGs, a binary variable r_i is introduced for each rule $r_i = (\gamma_i, w_i)$ of $\mathcal{R}$. Then for each rule r_i of $\mathcal{R}$, let us consider the index $plus(i)$ of a positive literal occurring in γ_i. We generate a set of inequalities equivalent to the constraint

$$r_i \Leftrightarrow \left(\bigwedge_{a_j \in \gamma_i, j \neq plus(i)} l_{plus(i),j} \right).$$

Again, the two indexes $plus(i), j$ of each $l_{plus(i),j}$ are switched if $plus(i) > j$. This set contains as many inequalities as positive literals in γ_i, leading to a total number of inequalities upper bounded by the number m of rules of $\mathcal{R}$ multiplied by the size d^+ of the rule r_i with a largest "positive part" (i.e., the largest number of positive literals in γ_i) plus one. On the running example, the rule $r_5 = (\{a_1, \overline{a}_2, a_3\}, 90)$ gives rise to the two inequalities $l_{1,3} - r_5 \leq 0$ and $-l_{1,3} + r_5 \leq 0$. This means that r_5 must be set to 1 precisely when the rule r_5 is partially activated by a coalition $CS(5)$ where $a_1, a_3 \in CS(5)$.

This time again, we need some further notations at this step. For every rule r_i of $\mathcal{R}$, let L_i^- be the list with head $plus(i)$ and tail the indexes of the negative literals of γ_i ordered in ascending way. For every j in the tail of L_k^-, $pred^-(j)_k$ denotes the index of the agent which is the predecessor of j in L_k^-. And $last^-(k)$ denotes the last element of L_k^-. For every rule r_i of $\mathcal{R}$, we introduce a variable $p_{i,plus(i)}$ and set $p_{i,plus(i)} = \Pi_{a_k \in pos(r_i)} p(\langle \{a_k\}, \emptyset \rangle)$. For each index j in the tail of L_i^-, we introduce a variable $p_{i,j}$ and generate the following conditional rules:

$$l_{plus(i),j} = 1 \rightarrow p_{i,j} = p_{i,pred^-(j)_i} \times p(\langle \emptyset, \{a_j\} \rangle)$$

and

$$l_{plus(i),j} = 0 \rightarrow p_{i,j} = p_{i,pred^-(j)_i}.$$

By construction, $a_{plus(i)} \in CS(i)$ in the current coalition structure CS and $p_{i,last^-(i)}$ is equal to the probability $p(\langle \gamma_i^+, CS(i) \cap \gamma_i^- \rangle)$. As in the cautious case, it remains to introduce one variable u_i and two conditional constraints per rule r_i of $\mathcal{R}$:

$$r_i = 1 \rightarrow u_i = w_i \cdot p_{i,last^-(i)} \text{ and } r_i = 0 \rightarrow u_i = 0.$$

This means that the rule r_i is partially activated precisely when its contribution to the expected utility of CS is equal to $w_i \cdot p(\langle \gamma_i^+, CS(i) \cap \gamma_i^- \rangle)$, as required (cf. Proposition 10). The objective function $\sum_{i=1}^m u_i$ is to be maximized. For our running example, the following program has been obtained:

`maximize` $u_1 + u_2 + u_3 + u_4 + u_5$
`subject to`
$l_{1,2} + l_{2,3} - l_{1,3} \leq 1, l_{1,3} + l_{2,3} - l_{1,2} \leq 1,$
$l_{1,2} + l_{1,3} - l_{2,3} \leq 1, l_{2,3} + l_{1,3} - l_{1,2} \leq 1,$
$l_{1,3} + l_{1,2} - l_{2,3} \leq 1, l_{2,3} + l_{1,2} - l_{1,3} \leq 1,$
$r_1 = 1, r_2 = 1, l_{1,2} - r_3 \leq 0, -l_{1,2} + r_3 \leq 0,$
$l_{2,3} - r_4 \leq 0, -l_{2,3} + r_4 \leq 0,$
$l_{1,3} - r_5 \leq 0, -l_{1,3} + r_5 \leq 0,$
$p_{1,1} = 0.8, r_1 = 1 \rightarrow u_1 = p_{1,1} \cdot 30, r_1 = 0 \rightarrow u_1 = 0,$
$p_{2,2} = 1, r_2 = 1 \rightarrow u_2 = p_{2,2} \cdot 40, r_2 = 0 \rightarrow u_2 = 0,$
$p_{3,1} = 0.8 \cdot 1, r_3 = 1 \rightarrow u_3 = p_{3,1} \cdot 20, r_3 = 0 \rightarrow u_3 = 0,$
$p_{4,2} = 1 \cdot 0.1, r_4 = 1 \rightarrow u_4 = p_{4,2} \cdot 60, r_4 = 0 \rightarrow u_4 = 0,$
$p_{5,1} = 0.8 \cdot 0.1, l_{1,2} = 1 \rightarrow p_{5,2} = p_{5,1} \cdot (1 - 1), l_{1,2} = 0 \rightarrow p_{5,2} = p_{5,1},$
$r_5 = 1 \rightarrow u_5 = p_{5,2} \cdot 90, r_5 = 0 \rightarrow u_5 = 0,$
$0 \leq p_{1,1}, p_{2,2}, p_{3,1}, p_{4,2}, p_{5,1}, p_{5,2} \leq 1,$
Binary $l_{1,2}, l_{1,3}, l_{2,3}, r_1, r_2, r_3, r_4, r_5$

This encoding scheme when applied to a PCFG instance for n agents and m rules bearing on at most d agents generates a MILP instance containing a number of constraints upper bounded by $n^3 + 3 \cdot m \cdot d + 5 \cdot m$, over at most $\frac{n^2}{2} - \frac{n}{2} + m \cdot (d+3)$ variables (including $\frac{n^2}{2} - \frac{n}{2} + m$ binary variables), and a linear objective function over m variables.

9 Experiments

9.1 Empirical setting

In order to evaluate our approach empirically, we generated benchmarks following a protocol similar to the one considered in [16–18, 42]. Namely, in order to generate a rule r_i, one first generates a set γ_i containing one agent picked up at random under a uniform distribution, then we repeatedly add a new agent picked up with probability α to the set γ_i, until an agent is not added or γ_i includes half of the total number of agents. Then we proceed in the same way with the remaining agents a but add $\overline{a}$ to γ_i with probability β until an agent is not added or γ_i contains all the agents. When γ_i^+ contains more than one agent, one switches an agent of γ_i^+ (picked up uniformly at random) from positive to negative, with some probability σ. The weight w_i of r_i is then chosen between 1 and the number of agents in the rule, uniformly at random. Lastly, this weight w_i is turned into its opposite $-w_i$ with probability δ. We took $\alpha = 0.55$, $\beta = 0.15$, $\sigma = 0.2$, $\delta = 0.2$, and as many rules as agents ranging from 10 to 120. In each case, 100 instances have been generated. Finally, each $p(\langle\{a_j\}, \emptyset\rangle)$ is a random value between 0 and 1 with a step of 0.01. The version of CPLEX used was IBM(R) ILOG(R) CPLEX(R) Interactive Optimizer 12.6.0.0 with the option set parallel 1. Our experiments were conducted on Intel Xeon E5-2643 (3.30 GHz) processors with 32 GiB RAM on Linux CentOS. A time-out of 900s and a memory-out of 7.6 GiB have been considered for each instance.

9.2 Results

Table 2 gives the empirical results that have been obtained for the MC-net cautious and flexible PCFG benchmarks we considered. The leftmost column indicates the numbers m of rules (equal to the number n of agents[5]), the second and third (resp. fourth and fifth) columns give the numbers of instances solved in the time limit and the average computation times for the instances solved when the cautious (resp. flexible) policy is considered.

The table shows that when the cautious policy (resp. the flexible policy) is considered, all the PCFG instances with up to 20 agents (resp. 60 agents) can be easily solved in a few seconds (less than a minute in the first case and approximately two minutes in the second case). The discrepancy between the two cases can be explained by the fact that, in the flexible case, for each rule $r_i \in R$ the computation of the probability of C_i (if it exists) is based only on the negative literals occurring in r_i and there are only few negative literals in the rules considered in the experiments ($\sigma = 0.2$).

To test this assumption, we performed additional experiments in the flexible case. For $\sigma = 0$, all the PCFG instances with up to 70 agents have been easily solved in

[5] As in the protocol used in [16–18, 42], here the number of rules generated is equal to the number of agents for simplicity, but this does not impact scalability.

Table 2 Results with $\alpha = 0.55$, $\beta = 0.15$, $\sigma = 0.2$, $\delta = 0.2$, on an average of 100 instances per number of rules

Instance	Cautious PCFG		Flexible PCFG	
#rules	#solved	Avg. time (in s)	#solved	Avg. time (in s)
5	100	0.6352	100	0.5583
10	100	0.5719	100	0.5901
15	100	1.9259	100	0.6861
20	99	43.3949	100	0.6913
25	79	91.6741	100	0.6587
30	21	321.564	100	1.1141
35	12	148.655	100	2.3204
40	4	268.385	100	6.2978
45	1	432.92	100	14.9835
50	2	169.8	99	44.5536
55	0	–	99	68.2371
60	0	–	100	123.965
65	0	–	95	159.822
70	0	–	84	241.943
75	0	–	64	211.608
80	0	–	49	201.161
85	0	–	35	247.294
90	0	–	31	272.593
95	0	–	26	265.485
100	0	–	19	313.278

around two minutes, while when $\sigma = 0.8$, the limit where some instances are unsolved was reached for 35 agents.

Obviously enough, we cannot compare our approach with previous ones for solving PCFG instances since such approaches do not exist. So the best we can do is to make some comparisons with existing works for solving MC-net based CFG instances. In [23] the MILP encoding relies on a graph where each vertex is a rule from the MC-net, and each edge is a relationship between rules. Then in [17, 18], the authors exploit the same idea and present a specific Boolean encoding of CFG instances into MAX-SAT instances. They take advantage of these encodings and of state-of-the-art MAX-SAT solvers for solving those CFG instances. Both SAT-based MAX-SAT solvers and MAX-SAT solvers based on a branch and bound technique are considered, and the former class of solvers prove to be much more efficient than the latter one (solvers from the first class succeeded in solving instances based on 300 agents/rules in less than 10 s, while solvers from the second class succeeded in solving instances based on at most 10 agents/rules). Unfortunately, one cannot take advantage of SAT-based MAX-SAT solvers for the PCSG problem because of the computation of the probabilities (roughly, computing products). Indeed, in MAX-SAT solvers based on a branch and bound technique, weights are considered as such: they do not need to be encoded into the constraints but participate in the control of the search. Incorporating weights into SAT-based MAX-SAT solvers would require instead to encode them using Boolean variables and to encode using Boolean circuits all the products required by the computation. Clearly enough, the resulting encoding would be very heavy and inefficient.

Since cautious and flexible PCFG instances are CFG extensions (cf. Proposition 2), it is not surprising that solving them proves to be harder in practice; especially, drawing any sound conclusion about the performance of our approach compared to the one reported in [17, 18, 23] would not really make sense. That mentioned, our results show that the branch and bound (actually, branch and cut) approach used by CPLEX Optimizer to solve PCFG instances performs well compared to the branch and bound approach used by some MAX-SAT solvers for solving CFG instances.

Before concluding this section, let us mention another closely related work [19]. In their paper, a similar probabilistic CSG framework is considered and empirically evaluated. Our PCSG framework and results differs from those reported in [19] on many aspects. The model used in [19] is more restrictive than our PCSG model, and no investigation was performed from a computational complexity viewpoint. Indeed, the model from [19] requires all the events associated with agents to be mutually independent. In comparison, all of our results hold without making this assumption, including the computational complexity results from Sect. 6, and the ability to compute the expected utility of any coalition structure in polynomial time on a rule-by-rule basis for MC-net based flexible and cautious PCFGs (cf. Sect. 7). The other way around, the authors of [19] considered an additional parameter k in their model corresponding to the number of agents which, when found defective and removed from a coalition, still allowing the residual coalition to produce a certain reward. So $k = 0$ corresponds to our cautious policy and $k = |A|$ corresponds to our flexible policy. The authors of [19] implemented some approximation algorithms and considered benchmarks where the value of each coalition through the characteristic function is generated randomly according to some probability distributions commonly used in the literature. They tested their algorithms on instances up to 14 agents and for a value for k varying between 0 and 5. They showed that a suboptimal solution for instances based on 14 agents can be found in about 10 s in the cautious case (i.e., $k = 0$), and in about 1000 s when $k = 5$. In comparison, we showed that (i) an optimal solution on MC-net based cautious PCFG instances (i.e., $k = 0$) can be found within 2 s for instances based on 15 agents, and (ii) an optimal solution on MC-net based flexible PCFG instances (i.e., $k = n$) can be found within 15 s for instances based on 45 agents. This clearly shows the benefits of MC-net based representations in the PCSG framework, and indicates that our encodings are quite efficient.

10 Conclusion

We have introduced the Probabilistic Coalition Structure Generation (PCSG) problem, which can be viewed as a natural stochastic extension of the CSG problem. In CSG, the system is characterized by a Characteristic Function Game (CFG), i.e., a characteristic function which associates with each coalition a value representing the pay-off obtained by the coalition after performing an underlying task; the CSG problem consists in forming a coalition structure such that the sum of all coalitional values is maximized. In PCSG, one considers an extension of a CFG, simply called Probabilistic CFG (PCFG). A PCFG considers a *situational* characteristic function, i.e., an extension of the standard characteristic function which associates with each coalition *and each possible outcome* a value, where an outcome identifies the situation where each agent is either present or absent from the coalition it has been assigned. The uncertain nature of the agents' functionality is represented as an additional probability distribution on the set of all outcomes. While in CSG one seeks to form a coalition structure of maximal value, the PCSG problem consists in forming a coalition structure of maximal expected utility.

We have shown that the PCSG problem is $\mathsf{NP}^{\mathsf{PP}}$-hard in the general case. Then, we have considered two specific PCFG classes, the cautious PCFGs and the flexible PCFGs, corresponding to two policies of interest, and where the situational characteristic function can naturally be derived from the one of a standard CFG. We have shown that for both cautious and flexible PCFGs, a significant drop in computational complexity can be obtained when the utility of every coalition is represented by an MC-net; more precisely, we have shown that for these two classes the PCSG problem is NP-complete, and thus it is theoretically not harder than the standard CSG problem.

We have also provided an encoding scheme for computing a coalition structure of maximal expected utility by associating with each PCFG a MILP (Mixed Integer Linear Programming) instance. We have empirically evaluated the efficiency of these encodings, and showed that cautious (resp. flexible) PCFGs with up to 20 (resp. 40) agents could be solved within a few seconds.

A first perspective for further research consists in studying how to design encodings and algorithms which would be more scalable. In this respect, we plan to investigate how one could take advantage of the SMT-based optimization algorithm reported in [15]. Another perspective consists in generalizing the PCSG model to deal with more complex scenarios than those considered in the paper, considering different representation languages than MC-nets for the characteristic function [5, 22, 36, 41], and to evaluate the corresponding expressiveness/efficiency trade-off.

Appendix: Proofs of propositions

Proposition 2 *Let $\langle A,f\rangle$ be a* CFG. *Then every optimistic* CFG *extension of $\langle A,f\rangle$ is equivalent to $\langle A,f\rangle$.*

Proof Let $\langle A,f\rangle$ be a CFG and let $\langle A,g,p_\top\rangle$ be any optimistic CFG extension of $\langle A,f\rangle$. By definition of a CFG extension (cf. Definition 3), we have that $g(C,\omega_A)=f(C)$ for each coalition $C\subseteq A$. So by Eq. 2, for each coalition structure $CS\in\Pi_A$, we get that $G(CS,\omega_A)=\sum_{C\in CS} g(C,\omega_A)=\sum_{C\in CS} f(C)=F(CS)$. Hence, by definition of $p_\top$ and $U(CS)$, we get that $U(CS)=\sum_{\omega_P\in\Omega_A} p(\omega_P)\cdot G(CS,\omega_P)=G(CS,\omega_A)=F(CS)$. This concludes the proof. □

Proposition 3 *Let $\langle A,g,p\rangle$ be a* PCFG.

(i) *If g is subadditive, then the coalition structure $CS=\{\{a_i\}\mid a_i\in A\}$ is an optimal one.*

(ii) *If g is superadditive, then the coalition structure $CS=\{A\}$ is an optimal one.*

Proof We provide the proof only for (i), as (ii) can be proved in a similar manner. Let $\langle A,g,p\rangle$ be a PCFG, and assume that g is subadditive. Let $CS=\{\{a_i\}\mid a_i\in A\}$ and $CS'\in\Pi_A$, and let us prove that $U(CS')\leq U(CS)$. By definition of $U(CS)$ and $U(CS')$ (cf. Equation 3), we have that

$$U(CS') = \sum_{\omega_P \in \Omega_A} p(\omega_P) \cdot \sum_{C \in CS} g(C, \omega_P)$$

and

$$U(CS) = \sum_{\omega_P \in \Omega_A} p(\omega_P) \cdot \sum_{a \in A} g(\{a\}, \omega_P).$$

So we just need to show that $\sum_{C \in CS} g(C, \omega_P) \leq \sum_{a \in A} g(\{a\}, \omega_P)$. Yet g is subadditive, so

$$\sum_{C \in CS} g(C, \omega_P) \leq \sum_{a \in A} g(\{a\}, \omega_P),$$

from which we can conclude that

$$U(CS') \leq U(CS).$$

□

Proposition 4 *Given a* PCFG $\langle A, g, p \rangle$, *for every coalition structure* $CS \subseteq \Pi_A$, *we have that*

$$U(CS) = \sum_{C \in CS} u(C).$$

Proof We have that:

$$\begin{aligned} U(CS) &= \textstyle\sum_{P \subseteq A} p(\omega_P) \cdot G(CS, \omega_P) \\ &= \textstyle\sum_{P \subseteq A} p(\omega_P) \cdot \sum_{C \in CS} g(C, \omega_P) \\ &= \textstyle\sum_{C \in CS} \sum_{P \subseteq A} p(\omega_P) \cdot g(C, \omega_P) \\ &= \textstyle\sum_{C \in CS} u(C). \end{aligned}$$

This concludes the proof. □

Proposition 5 *Given a* PCFG $\langle A, g, p \rangle$, *for every coalition* $C \subseteq A$, *we have that*

$$u(C) = \sum_{P \subseteq C} p(\langle P, C \setminus P \rangle) \cdot g(C, \omega_P).$$

The proof uses the following lemma:

Lemma 1 *Let* $\langle A, g, p \rangle$ *be a* PCFG, *let* $\langle Q, R \rangle$ *be an event from* $\mathcal{E}_A$ *and* $C \subseteq A$ *such that* $C \cap (Q \cup R) = \emptyset$. *We have that* $\langle Q, R \rangle = \bigcup_{P \subseteq C} \langle Q \cup P, R \cup (C \setminus P) \rangle$.

Proof We prove it by induction on the size of C. The result trivially holds if $|C| = 0$, i.e., if $C = \emptyset$. Assume the result holds for $|C| = k$, for some k such that $0 \leq k \leq n - 1 - |Q \cup R|$. So:

$$\langle Q, R \rangle = \bigcup_{P \subseteq C} \langle Q \cup P, R \cup (C \setminus P) \rangle.$$

Let $a_i \in A \setminus (Q \cup R)$. We know that $\langle \{a_i\}, \emptyset \rangle \cup \langle \emptyset, \{a_i\} \rangle = \Omega_A$. Thus:

$$
\begin{aligned}
\langle Q, R\rangle &= \bigcup_{P\subseteq C}\langle Q\cup P, R\cup(C\setminus P)\rangle \\
&= \bigcup_{P\subseteq C}\langle Q\cup P, R\cup(C\setminus P)\rangle \cap (\langle\{a_i\},\emptyset\rangle \cup \langle\emptyset,\{a_i\}\rangle) \\
&= (\bigcup_{P\subseteq C}\langle Q\cup P, R\cup(C\setminus P)\rangle \cap \langle\{a_i\},\emptyset\rangle)\cup \\
&\quad (\bigcup_{P\subseteq C}\langle Q\cup P, R\cup(C\setminus P)\rangle \cap \langle\emptyset,\{a_i\}\rangle) \\
&= (\bigcup_{P\subseteq C}\langle Q\cup P\cup\{a_i\}, R\cup(C\setminus P)\rangle)\cup \\
&\quad (\bigcup_{P\subseteq C}\langle Q\cup P, R\cup(C\setminus P)\cup\{a_i\}\rangle) \\
&= \bigcup_{P\subseteq C\cup\{a_i\}}\langle Q\cup P, R\cup(C\setminus P)\rangle.
\end{aligned}
$$

We have just proved that the result holds for any C, $|C| = k+1$, which concludes the proof. □

We can now prove Proposition 5:

Proof of Proposition 5 By definition (cf. Eq. 4), we have that:

$$u(C) = \sum_{P\subseteq A} p(\omega_P)\cdot g(C,\omega_P).$$

For any subset C of A, every subset P of A is by construction the union of a subset P_C of C with a subset $P_{\overline{C}}$ of $\overline{C}$, such that $P_C \cap P_{\overline{C}} = \emptyset$. Accordingly,

$$u(C) = \sum_{P_C\subseteq C, P_{\overline{C}}\subseteq\overline{C}} p(\langle P_C\cup P_{\overline{C}}, \overline{P_C\cup P_{\overline{C}}}\rangle)\cdot g(C,\omega_{P_C\cup P_{\overline{C}}}).$$

Yet from Equation 1, we know that $g(C,\omega_{P_C\cup P_{\overline{C}}}) = g(C,\omega_{P_C})$. So we can rewrite $u(C)$ as:

$$u(C) = \sum_{P_C\subseteq C} g(C,\omega_{P_C})\cdot \sum_{P_{\overline{C}}\subseteq\overline{C}} p(\langle P_C\cup P_{\overline{C}}, \overline{P_C\cup P_{\overline{C}}}\rangle).$$

Yet for any $P_C \subseteq C$ and any $P_{\overline{C}} \subseteq \overline{C}$, we have that $\overline{P_C\cup P_{\overline{C}}} = (C\setminus P_C)\cup(\overline{C}\setminus P_{\overline{C}})$, so the event $\langle P_C\cup P_{\overline{C}}, \overline{P_C\cup P_{\overline{C}}}\rangle$ corresponds to the event $\langle P_C\cup P_{\overline{C}}, (C\setminus P_C)\cup(\overline{C}\setminus P_{\overline{C}})\rangle$. And by Lemma 1, for any $P_C\subseteq C$, we have that $\bigcup_{P_{\overline{C}}\subseteq\overline{C}}\langle P_C\cup P_{\overline{C}}, (C\setminus P_C)\cup(\overline{C}\setminus P_{\overline{C}})\rangle = \langle P_C, C\setminus P_C\rangle$. That is to say, for any $P_C\subseteq C$, we get that $\bigcup_{P_{\overline{C}}\subseteq\overline{C}}\langle P_C\cup P_{\overline{C}}, \overline{P_C\cup P_{\overline{C}}}\rangle = \langle P_C, C\setminus P_C\rangle$. Hence, $\sum_{P_{\overline{C}}\subseteq\overline{C}} p(\langle P_C\cup P_{\overline{C}}, \overline{P_C\cup P_{\overline{C}}}\rangle) = p(\langle P_C, C\setminus P_C\rangle)$. Therefore, we can rewrite $u(C)$ as follows:

$$u(C) = \sum_{P_C\subseteq C} p(\langle P_C, C\setminus P_C\rangle)\cdot g(C,\omega_{P_C}),$$

which, using Equation 1 again, can be written equivalently as:

$$u(C) = \sum_{P\subseteq C} p(\langle P, C\setminus P\rangle)\cdot g(C,\omega_P).$$

This concludes the proof. □

Proposition 6 *Let* $\langle A, g^f_{cau}, p\rangle$ *be a cautious* PCFG. *For any coalition* $C\subseteq A$, *we have that*

$$u_{cau}(C) = p(\langle C,\emptyset\rangle)\cdot g^f_{cau}(C,\omega_C).$$

Proof According to Corollary 1, for any coalition structure $CS \in \Pi_A$, we have that:

$$U_{cau}(CS) = \sum_{C \in CS} \sum_{P \subseteq C} p(\langle P, C \setminus P \rangle) \cdot g^f_{cau}(C, \omega_P).$$

Yet by definition of g^f_{cau}, for any given coalition $C \subseteq A$ and any $P \subseteq C$, we have that $g^f_{cau}(C, \omega_P) = 0$ whenever $C \nsubseteq P$. Hence, in the equation, for any coalition C one can restrict ourselves to events $\langle P, C \setminus P \rangle$ where $P = C$, that corresponds to the single event $\langle C, \emptyset \rangle$. Therefore, $U_{cau}(CS)$ is simplified as $U_{cau}(CS) = \sum_{C \in CS} p(\langle C, \emptyset \rangle) \cdot g^f_{cau}(C, \omega_C)$. This concludes the proof. □

Proposition 7 **DP-CS** *is* PP*-hard and* **DP-∃CS** *is* $\mathsf{NP}^{\mathsf{PP}}$*-hard. Hardness results hold for flexible* PCFGs, *and even when the set of events* $\{\langle \{a_i\}, \emptyset \rangle \mid a_i \in A\}$ *are pairwise independent.*

Proof **1.** We prove that **DP-CS** is PP-hard for flexible PCFGs and when all events from $\{\langle \{a_i\}, \emptyset \rangle \mid a_i \in A\}$ are pairwise independent by considering a reduction in polynomial time from the PP-complete problem *MAJSAT*: given a propositional formula φ in Conjunctive Normal Form (CNF), does the majority of assignments satisfy φ? Consider such a formula φ defined over the set of propositional variables $X = \{x_1, \dots, x_n\}$. Since φ is in CNF, it can be viewed as a set of clauses $\{cl_i \mid cl_i \in \varphi\}$ interpreted conjunctively, where each clause is a disjunction of literals over X. Now, let us associate with φ the flexible PCFG $\langle A, g^f_{fle}, p \rangle$ and the coalition structure CS, where $A = \{a_1, \dots, a_n\}$, $CS_* = \{A\}$, all events from $\{\langle \{a_i\}, \emptyset \rangle \mid a_i \in A\}$ are pairwise independent, $p(\langle \{a_i\}, \emptyset \rangle) = 0.5$ for each $a_i \in A$, and g^f_{fle} be characterized by a function $f : 2^A \mapsto \mathbb{N}$ defined for each coalition $C \subseteq A$ as

$f(C) = 1$ if for every clause $cl_i \in \varphi$, there is a literal $l_j \in cl_i$ such that ($a_j \in C$ if and only if l_j is a positive literal); and $f(C) = 0$ in the remaining cases.

Note that each value $f(C)$ is not explicitely represented for each coalition $C \subseteq A$ and each outcome $\omega_P \in \Omega_A$ in a table of exponential size, since each such value is completely characterized in polynomial time by φ given C. Then we recall that g^f_{fle} is characterized for each coalition $C \subseteq A$ and each outcome $\omega_P \in \Omega_A$ as $g^f_{fle}(C, \omega_P) = f(C \cap P)$.

Let us show that the majority of propositional assignments satisfies φ if and only if $U(CS_*) \geq 0.5$ (i.e., $k = 0.5$). Consider the one-to-one correspondence between the set of all outcomes Ω_A and the set of all propositional assignments associating each outcome $\omega_P \in \Omega_A$ with the propositional assignment I_P defined for every variable $x_i \in X$ as $I_P(x_i) = 1$ if and only if $a_i \in P$. Then it can be seen by definition of f that for each outcome $\omega_P \in \Omega_A$, $f(P) = 1$ if I_P satisfies φ (denoted by $I_P \vDash \varphi$), otherwise $f(P) = 0$. Now, by definition of p, for each outcome $\omega_P \in \Omega_A$, we have that $p(\omega_P) = 0.5^n$. Hence,

$$\begin{aligned} U_{fle}(CS_*) &= \textstyle\sum_{\omega_P \in \Omega_A} p(\omega_P) \cdot G_{fle}(CS_*, \omega_P) \\ &= \textstyle\sum_{\omega_P \in \Omega_A} p(\omega_P) \cdot G_{fle}(\{A\}, \omega_P) \\ &= \textstyle\sum_{\omega_P \in \Omega_A} p(\omega_P) \cdot g^f_{fle}(A, \omega_P) \\ &= \textstyle\sum_{\omega_P \in \Omega_A} 0.5^n \cdot f(P) \\ &= 0.5^n \cdot \textstyle\sum_{\omega_P \in \Omega_A} f(P) \\ &= 0.5^n \cdot |\{I_P \mid I_P \vDash \varphi\}|. \end{aligned}$$

The majority of propositional assignments satisfies φ if and only if $|\{I_P \mid I_P \vDash \varphi\}| \geq 2^{n-1}$ if and only if $U_{fle}(CS_*) \geq 0.5^n \cdot 2^{n-1} = 0.5$. Hence, **DP-CS** is PP-hard.

2. We prove that **DP-∃CS** is $\mathsf{NP}^{\mathsf{PP}}$-hard for flexible PCFGs and when all events from $\{\langle\{a_i\},\emptyset\rangle \mid a_i \in A\}$ are pairwise independent by considering a reduction in polynomial time from the $\mathsf{NP}^{\mathsf{PP}}$-complete problem *E-MAJSAT*: given a propositional formula φ in Conjunctive Normal Form (CNF), defined over $X \cup Y$ where X and Y are two disjoint sets of propositional variables, does there exist an assignment γ_X over X such that the majority of assignments over Y satisfies $\varphi_{|\gamma_X}$ (i.e., φ conditioned on γ_X)? Consider such a formula φ defined over the set of propositional variables $X \cup Y$, where $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$. Without loss of generality, we assume that $n \geq 2$, for technical reasons that will be of use in the latter part of the proof. Since φ is in CNF, it can be viewed as a set of clauses $\{cl_i \mid cl_i \in \varphi\}$ interpreted conjunctively, where each clause is a disjunction of literals over $X \cup Y$.

Now, let us associate with φ the flexible PCFG $\langle A, g^f_{fle}, p\rangle$, where $A = \{new, a_1, a'_1, b_1, b'_1, \dots, a_n, a'_n, b_n, b'_n\}$, all events from $\{\langle\{new\}, \emptyset\rangle\} \cup \bigcup\{\langle\{a_i\},\emptyset\rangle, \langle\{a'_i\},\emptyset\rangle, \langle\{b_i\},\emptyset\rangle, \langle\{b'_i\},\emptyset\rangle \mid i \in \{1,\dots,n\}\}$ are pairwise independent, $p(\langle\{new\},\emptyset\rangle) = 1$, and $p(\langle\{a_i\},\emptyset\rangle) = p(\langle\{a'_i\},\emptyset\rangle) = p(\langle\{b_i\},\emptyset\rangle) = p(\langle\{b'_i\},\emptyset\rangle) = 0.5$ for each $i \in \{1, \dots, n\}$. The function g^f_{fle} is characterized by a function $f : 2^A \mapsto \mathbb{N}$; yet before describing how f is defined, let us introduce some preliminary notions on coalitions from A that will be useful in the proof.

A set $C \subseteq A$ is said to be *canonical* if the following conditions are jointly satisfied:

1. $new \in C$;
2. for each $i \in \{1, \dots, n\}$, $\{a_i, a'_i\} \subseteq C$ or $\{b_i, b'_i\} \subseteq C$;
3. for each $i \in \{1, \dots, n\}$, $\{a_i, a'_i\} \cap C = \emptyset$ or $\{b_i, b'_i\} \cap C = \emptyset$.

For instance, for $n = 4$, $C_1 = \{new, a_1, a'_1, b_2, b'_2, b_3, b'_3, a_4, a'_4\}$ is a canonical coalition. Note that a canonical coalition always contains $2n + 1$ elements, and that there are exactly 2^n canonical coalitions.

A set $C \subseteq A$ is said to be *sub-canonical* if the following conditions are jointly satisfied:

1. C is a proper subset of a canonical coalition;
2. $new \in C$;
3. for each $i \in \{1, \dots, n\}$, $a_i \in C$ if and only if $a'_i \notin C$;
4. for each $i \in \{1, \dots, n\}$, $b_i \in C$ if and only if $b'_i \notin C$.

Note that when C is sub-canonical, there is exactly one canonical coalition C' such that $C \subseteq C$, and we also say that C is a sub-canonical set *w.r.t.* C'. For instance, for $n = 4$, $C_2 = \{new, a_1, b'_2, b_3, a_4\}$ is a sub-canonical coalition of C_1. Additionally, one can remark that a sub-canonical coalition C always contains $n + 1$ elements, that for each $i \in \{1, \dots, n\}$, exactly one element from $\{a_i, a'_i, b_i, b'_i\}$ belongs to a sub-canonical coalition C, and that for any given canonical coalition C', there are exactly 2^n sub-canonical coalitions w.r.t. C'.

Lastly, let us define the function α associating any literal l_i over $X \cup Y$ with a pair of agents from A, defined as:

$$\alpha(l_i) = \begin{cases} \{a_i, a'_i\} & \text{if } l_i \text{ is a positive literal over } X, \\ \{b_i, b'_i\} & \text{if } l_i \text{ is a negative literal over } X, \\ \{a_i, b_i\} & \text{if } l_i \text{ is a positive literal over } Y, \\ \{a'_i, b'_i\} & \text{if } l_i \text{ is a negative literal over } Y. \end{cases}$$

We are now ready to define f, for each coalition $C \subseteq A$, as:

$$f(C) = \begin{cases} 2^{5n} & \text{if } C \text{ is canonical,} \\ 2^{n+1} & \text{if } C \text{ is sub-canonical and for each clause } cl_i \in \varphi, \\ & \text{there is a literal } l_j \in cl_i \text{ such that } \alpha(l_j) \cap C \neq \emptyset, \\ 0 & \text{in the remaining cases.} \end{cases}$$

Then we recall that g^f_{fle} is characterized for each coalition $C \subseteq A$ and each outcome $\omega_P \in \Omega_A$ as $g^f_{fle}(C, \omega_P) = f(C \cap P)$.

We intend now to prove that there is an assignment γ_X over X such that the majority of assignments over Y satisfies $\varphi_{|\gamma_X}$ if and only if there exists a coalition structure CS such that $U_{fle}(CS) \geq 2^{3n} + 1$ according to the flexible PCFG $\langle A, g^f_{fle}, p\rangle$.

(Only if part) Assume that there is an assignment γ_X over X such that the majority of assignments over Y satisfies $\varphi_{|\gamma_X}$. We associate with γ_X the coalition structure $CS_{\gamma_X} = \{C_{\gamma_X}, A \setminus C_{\gamma_X}\}$, where C_{γ_X} is the canonical coalition defined as:

$$\begin{aligned} C_{\gamma_X} = & \bigcup \{\{a_i, a'_i\} \mid \gamma_X(x_i) = 1, i \in \{1, \dots, n\}\} \\ & \cup \bigcup \{\{b_i, b'_i\} \mid \gamma_X(x_i) = 0, i \in \{1, \dots, n\}\} \\ & \cup \{new\}. \end{aligned}$$

Let us show that $U_{fle}(CS_{\gamma_X}) \geq 2^{3n} + 1$. From Proposition 4 we know that

$$U_{fle}(CS_{\gamma_X}) = u_{fle}(C_{\gamma_X}) + u_{fle}(A \setminus C_{\gamma_X}).$$

So it is enough to prove that $u_{fle}(C_{\gamma_X}) \geq 2^{3n} + 1$. Let us first recall that according to Proposition 5, $u_{fle}(C_{\gamma_X})$ is defined as:

$$u_{fle}(C_{\gamma_X}) = \sum_{P \subseteq C_{\gamma_X}} p(\langle P, C_{\gamma_X} \setminus P\rangle) \cdot g^f_{fle}(C_{\gamma_X}, \omega_P).$$

Equivalently,

$$u_{fle}(C_{\gamma_X}) = \sum_{P \subseteq C_{\gamma_X}} p(\langle P, C_{\gamma_X} \setminus P\rangle) \cdot f(P).$$

Yet we know by definition of f that $f(P) = 2^{5n}$ when P is canonical, that is – under the condition that $P \subseteq C_{\gamma_X}$ – exactly the case when $P = C_{\gamma_X}$. On the other hand, since $p(\langle \{new\}, \emptyset\rangle) = 1$, since $p(\langle \{a_i\}, \emptyset\rangle) = p(\langle \{a'_i\}, \emptyset\rangle) = p(\langle \{b_i\}, \emptyset\rangle) = p(\langle \{b'_i\}, \emptyset\rangle) = 0.5$ for each $i \in \{1, \dots, n\}$, since all these events are pairwise independent and since C_{γ_X} contains $2n + 1$ elements, we have that $p(\langle C_{\gamma_X}, \emptyset\rangle) = 0.5^{2n}$. Thus $p(\langle C_{\gamma_X}, \emptyset\rangle) \cdot f(C_{\gamma_X}) = 0.5^{2n} \cdot 2^{5n} = 2^{3n}$. Hence,

$$u_{fle}(C_{\gamma_X}) = 2^{3n} + \delta,$$

where $\delta = \sum_{P \subsetneq C_{\gamma_X}} p(\langle P, C_{\gamma_X} \setminus P\rangle) \cdot f(P)$. Since we need to show that $u_{fle}(C_{\gamma_X}) \geq 2^{3n} + 1$, what remains to be shown is that $\delta \geq 1$.

Now, let us consider the set $\mathcal{E}^{C_{\gamma_X}}_A \subseteq \mathcal{E}_A$ defined as the set of events $\langle Q, R\rangle$ satisfying the following set of conditions:

(i) Q is a sub-canonical set w.r.t. C_{γ_X};
(ii) $R = C_{\gamma_X} \setminus Q$.

Note that for each $\langle Q, R\rangle \in \mathcal{E}_A^{C_{\gamma_X}}$, $Q \cup R = C_{\gamma_X}$, $|Q| = n+1$ and $|R| = n$, and that $|\mathcal{E}_A^{C_{\gamma_X}}| = 2^n$.

By construction of $\mathcal{E}_A^{C_{\gamma_X}}$, we have that $\mathcal{E}_A^{C_{\gamma_X}} \subseteq \{\langle Q, C_{\gamma_X} \setminus Q\rangle \mid Q \subsetneq C_{\gamma_X}\}$. Thus:

$$\delta \geq \sum_{\langle Q,R\rangle \in \mathcal{E}_A^{C_{\gamma_X}}} p(\langle Q, R\rangle) \cdot f(Q).$$

Now, since $p(\langle\{new\}, \emptyset\rangle) = 1$, since $p(\langle\{a_i\}, \emptyset\rangle) = p(\langle\{a_i'\}, \emptyset\rangle) = p(\langle\{b_i\}, \emptyset\rangle) = p(\langle\{b_i'\}, \emptyset\rangle) = 0.5$ for each $i \in \{1, \dots, n\}$, and since all these events are pairwise independent, we get for each $\langle Q, R\rangle \in \mathcal{E}_A^{C_{\gamma_X}}$ that:

$$\begin{aligned} p(\langle Q, R\rangle) &= \textstyle\prod_{a \in Q} p(\langle\{a\}, \emptyset\rangle) \cdot \prod_{a \in R} p(\langle \emptyset, \{a\}\rangle) \\ &= 1 \cdot 0.5^n \cdot (1-0.5)^n \\ &= 0.5^{2n}. \end{aligned}$$

So we get that

$$\delta \geq \sum_{\langle Q,R\rangle \in \mathcal{E}_A^{C_{\gamma_X}}} 0.5^{2n} \cdot f(Q).$$

Equivalently,

$$\delta \geq 0.5^{2n} \cdot \sum_{\langle Q,R\rangle \in \mathcal{E}_A^{C_{\gamma_X}}} f(Q).$$

Now, let us build a one-to-one correspondence β between the set $\mathcal{E}_A^{C_{\gamma_X}}$ and the set of propositional assignments over Y as follows. For each event $\langle Q, R\rangle \in \mathcal{E}_A^{C_{\gamma_X}}$, we set $\beta(\langle Q, R\rangle)$ to be the propositional assignment γ_Y over Y defined for each $y_i \in Y$ as $\gamma_Y(y_i) = 1$ if and only if $\{a_i, b_i\} \cap Q \neq \emptyset$ (i.e., $\gamma_Y(y_i) = 0$ when $\{a_i', b_i'\} \cap Q \neq \emptyset$).

It can be easily verified by construction of C_{γ_X} and $\mathcal{E}_A^{C_{\gamma_X}}$ that for every propositional assignment γ_Y over Y that satisfies $\varphi_{|\gamma_X}$, the event $\langle Q, R\rangle \in \mathcal{E}_A^{C_{\gamma_X}}$ defined as $\langle Q, R\rangle = \beta^{-1}(\gamma_Y)$ meets the conditions given in the definition of f to satisfy $f(Q) = 2^{n+1}$. Since a majority of propositional assignments over Y satisfies $\varphi_{|\gamma_X}$, this means that $f(Q) = 2^{n+1}$ for a majority of events $\langle Q, R\rangle$ from $\mathcal{E}_A^{C_{\gamma_X}}$. Yet we know that $|\mathcal{E}_A^{C_{\gamma_X}}| = 2^n$, so this means that $f(Q) = 2^{n+1}$ for at least 2^{n-1} events $\langle Q, R\rangle$ from $\mathcal{E}_A^{C_{\gamma_X}}$. Hence,

$$\sum_{\langle Q,R\rangle \in \mathcal{E}_A^{C_{\gamma_X}}} f(Q) \geq 2^{n-1} \cdot 2^{n+1} = 2^{2n}.$$

We got that $\delta \geq 0.5^{2n} \cdot 2^{2n}$, i.e., $\delta \geq 1$, thus $u_{fle}(C_{\gamma_X}) \geq 2^{3n} + 1$, so $U_{fle}(CS_{\gamma_X}) \geq 2^{3n} + 1$. This concludes the (only if) part of the proof.

(If part) Assume that there exists a coalition structure CS such that $U_{fle}(CS) \geq 2^{3n} + 1$ according to the flexible PCFG $\langle A, g^f_{fle}, p\rangle$. From Proposition 4, we know that $U_{fle}(CS)$ is defined as:

$$U_{fle}(CS) = \sum_{C \in CS} u_{fle}(C),$$

where

$$u_{\mathit{fle}}(C) = \sum_{\omega_P \in \Omega_A} p(\omega_P) \cdot g^f_{\mathit{fle}}(C, \omega_P),$$

or equivalently, where

$$u_{\mathit{fle}}(C) = \sum_{\omega_P \in \Omega_A} p(\omega_P) \cdot f(C \cap P),$$

Yet for every coalition C such that $new \notin C$, we know from the definition of f that $f(C \cap P) = 0$ for every $\omega_P \in \Omega_A$; and thus for each such coalition C, $u_{\mathit{fle}}(C) = 0$. This means that CS contains a coalition C^* such that $u_{\mathit{fle}}(C^*) \geq 2^{3n} + 1$.

Let us show that such coalition C^* is canonical. Let us recall from Proposition 5 that $u_{\mathit{fle}}(C^*)$ is characterized as follows:

$$u_{\mathit{fle}}(C^*) = \sum_{P \subseteq C^*} p(\langle P, C^* \setminus P \rangle) \cdot g(C^*, \omega_P).$$

Equivalently,

$$u_{\mathit{fle}}(C^*) = \sum_{P \subseteq C^*} p(\langle P, C^* \setminus P \rangle) \cdot f(P).$$

Toward a contradiction, assume that C^* is not canonical. We fall into two cases:

- Case 1: C^* is a strict superset of a canonical coalition C. Then $u_{\mathit{fle}}(C^*)$ can be written as $u_{\mathit{fle}}(C^*) = \delta_1 + \delta_2$, where

$$\delta_1 = p(\langle C, C^* \setminus C \rangle) \cdot f(C),$$

and

$$\delta_2 = \sum_{P \subseteq C^*, P \neq C} p(\langle P, C^* \setminus P \rangle) \cdot f(P).$$

We first compute δ_1. On the one hand, $p(\langle \{new\}, \emptyset \rangle) = 1$, $p(\langle \{a_i\}, \emptyset \rangle) = p(\langle \{a'_i\}, \emptyset \rangle) = p(\langle \{b_i\}, \emptyset \rangle) = p(\langle \{b'_i\}, \emptyset \rangle) = 0.5$ for each $i \in \{1, \ldots, n\}$, and all these events are pairwise independent, so we have that $p(\langle C, C^* \setminus C \rangle) = 0.5^{|C^*|-1}$. On the other hand, according to the definition of f, we have that $f(C) = 2^{5n}$ since C is canonical. Hence, $\delta_1 = 0.5^{|C^*|-1} \cdot 2^{5n}$, that is, $\delta_1 = 2^{5n-|C^*|+1}$.

Let us now provide an upper bound for δ_2. On the one hand, we know that for each $P \subseteq C^*$ such that $P \neq C$, $p(\langle P, C^* \setminus P \rangle) = 0.5^{|C^*|-1}$; and $P = C^* \cap P$ is not canonical, so according to the definition of f, $f(P) \leq 2^{n+1}$. Thus for each $P \subseteq C^*$ such that $P \neq C$,

$$p(\langle P, C^* \setminus P \rangle) \cdot f(P) \leq 0.5^{|C^*|-1} \cdot 2^{n+1}.$$

Hence,

$$\delta_2 \leq \sum_{P \subseteq C^*, P \neq C} 0.5^{|C^*|-1} \cdot 2^{n+1},$$

so

$$\delta_2 < 2^{|C^*|} \cdot 0.5^{|C^*|-1} \cdot 2^{n+1},$$

that is, $\delta_2 < 2^{n+2}$.

We got that $\delta_1 + \delta_2 < 2^{5n-|C^*|+1} + 2^{n+2}$. Yet since C is canonical, $|C| = 2n + 1$. Since C^* is a strict superset of C, we get that $|C^*| \geq 2n + 2$. So $\delta_1 + \delta_2 < 2^{3n-1} + 2^{n+2}$. As we initially assumed at the beginning of this proof and without loss of generality that $n \geq 2$, we get that $2^{n+2} < 2^{3n-1}$, so $\delta_1 + \delta_2 < 2^{3n-1} \cdot 2^{3n-1}$, thus $\delta_1 + \delta_2 < 2^{3n}$, and so $u_{fle}(C^*) < 2^{3n}$. This contradicts $u_{fle}(C^*) \geq 2^{3n} + 1$.

- Case 2: C^* is not a strict superset of a canonical coalition C. Since C^* is assumed not to be canonical, we know that no subset of C^* is canonical. So for each $P \subseteq C^*$, according to the definition of f, $f(P) \leq 2^{n+1}$; and we know that $p(\langle P, C^* \setminus P\rangle) = 0.5^{|C^*|-1}$. Thus

$$u_{fle}(C^*) \leq \sum_{P \subseteq C^*} 0.5^{|C^*|-1} \cdot 2^{n+1},$$

that is,

$$u_{fle}(C^*) \leq 2^{|C^*|} \cdot 0.5^{|C^*|-1} \cdot 2^{n+1},$$

or equivalently,

$$u_{fle}(C^*) \leq 2^n.$$

This contradicts $u_{fle}(C^*) \geq 2^{3n} + 1$.

Both cases lead to a contradiction, so we know that C^* is canonical.

Now, since C^* is canonical, we have that $f(C^*) = 2^{5n}$ by definition of f. And since $|C^*| = 2n + 1$, we have that $p(\langle C^*, \emptyset\rangle) = 0.5^{2n}$. Thus

$$p(\langle C^*, \emptyset\rangle) \cdot f(C^*) = 0.5^{2n} \cdot 2^{5n} = 2^{3n}.$$

This means that $u_{fle}(C^*)$ can be written as

$$u_{fle}(C^*) = 2^{3n} + \delta_3,$$

where

$$\delta_3 = \sum_{P \subseteq C^*, P \neq C^*} p(\langle P, C^* \setminus P\rangle) \cdot f(P).$$

Since we know that $u_{fle}(C^*) \geq 2^{3n} + 1$, we get that $\delta_3 \geq 1$.

Now, obviously enough, δ_3 can be written as $\delta_3 = \delta_3^1 + \delta_3^2$, where

$$\begin{aligned}\delta_3^1 &= \sum_{\substack{P \subseteq C^*, P \neq C^*, \\ \langle P, C^* \setminus P\rangle \in \mathcal{E}_A^{C^*}}} p(\langle P, C^* \setminus P\rangle) \cdot f(P) \\ &= \sum\nolimits_{\langle P, C^* \setminus P\rangle \in \mathcal{E}_A^{C^*}} p(\langle P, C^* \setminus P\rangle) \cdot f(P),\end{aligned}$$

and

$$\delta_3^2 = \sum_{\substack{P \subseteq C^*, P \neq C^*, \\ \langle P, C^* \setminus P \rangle \notin \mathcal{E}_A^{C^*}}} p(\langle P, C^* \setminus P \rangle) \cdot f(P),$$

where the set $\mathcal{E}_A^{C^*} \subseteq \mathcal{E}_A$ is defined as the set of events $\langle Q, R \rangle$ satisfying the following set of conditions:

(i) Q is a sub-canonical set w.r.t. C^*;
(ii) $R = C^* \setminus P$.

Yet one can see that for every $P \subseteq C^*$ such that $P \neq C^*$, if $\langle P, C^* \setminus P \rangle \notin \mathcal{E}_A^{C^*}$ then the set $P = C^* \cap P$ is neither canonical nor sub-canonical, which means that $f(P) = 0$ according to the definition of f. Hence, $\delta_3^2 = 0$. Since we know that $\delta_3 \geq 1$ and $\delta_3 = \delta_3^1 + \delta_3^2$, we get that $\delta_3^1 \geq 1$. Yet for each event $\langle P, C^* \setminus P \rangle \in \mathcal{E}_A^{C^*}$, $p(\langle P, C^* \setminus P \rangle) = 0.5^{2n}$ (C^* is canonical, so $|C^*| = 2n + 1$). Thus

$$\delta_3^1 = \sum_{\langle P, C^* \setminus P \rangle \in \mathcal{E}_A^{C^*}} 0.5^{2n} \cdot f(P).$$

Let us prove that there is a majority of events $\langle P, C^* \setminus P \rangle$ from $\mathcal{E}_A^{C^*}$ such that $f(P) = 2^{n+1}$.

We know that for each $\langle P, C^* \setminus P \rangle \in \mathcal{E}_A^{C^*}$, $P = C^* \cap P$ is not canonical, so by definition of f, we know that $f(P) \leq 2^{n+1}$. So what we want to show is that there is a majority of events $\langle P, C^* \setminus P \rangle$ from $\mathcal{E}_A^{C^*}$ that satisfy $f(P) \geq 2^{n+1}$. Assume toward a contradiction that this is not the case.

According to the definition of f, this means that there is a strict majority of events $\langle P, C^* \setminus P \rangle$ from $\mathcal{E}_A^{C^*}$ that satisfy $f(P) = 0$. Since $|\mathcal{E}_A^{C^*}| = 2^n$ (C^* is canonical and thus it admits 2^n sub-canonical subsets), this means that:

$$\delta_3^1 < 0.5 \cdot 2^n \cdot 0.5^{2n} \cdot 2^{n+1}.$$

We get that $\delta_3^1 < 1$. This contradicts $\delta_3^1 \geq 1$. Therefore, there is a majority of events $\langle P, C^* \setminus P \rangle$ from $\mathcal{E}_A^{C^*}$ such that $f(P) = 2^{n+1}$.

Now, let us associate with C^* the propositional assignment $\gamma_X^{C^*}$ over X defined for every $x_i \in X$ as $\gamma_X^{C^*}(x_i) = 1$ if and only if $\{a_i, a'_i\} \subseteq C^*$ (recall that since C^* is canonical, we set $\gamma_X^{C^*}(x_i) = 0$ when $\{a_i, a'_i\} \cap C^* = \emptyset$ and $\{b_i, b'_i\} \subseteq C^*$). And let us build a one-to-one correspondence β between the set of events $\mathcal{E}_A^{C^*}$ and the set of propositional assignments over Y as follows. For each event $\langle P, C^* \setminus P \rangle \in \mathcal{E}_A^{C^*}$, we set $\beta(\langle P, C^* \setminus P \rangle)$ to be the propositional assignment γ_Y over Y defined for each $y_i \in Y$ as $\gamma_Y(y_i) = 1$ if and only if $\{a_i, b_i\} \cap P \neq \emptyset$ (i.e., $\gamma_Y(y_i) = 0$ when $\{a'_i, b'_i\} \cap P \neq \emptyset$).

Now, consider any event $\langle P, C^* \setminus P \rangle$ from $\mathcal{E}_A^{C^*}$ such that $f(P) = 2^{n+1}$. By definition of f, this means that for each clause $cl_i \in \varphi$, there is a literal $l_j \in cl_i$ such that $\alpha(l_j) \cap P \neq \emptyset$. Then it can be directly verified by construction of $\gamma_X^{C^*}$ and $\beta(\langle P, C^* \setminus P \rangle)$ that the interpretation $\gamma_X^{C^*} \cup \beta(\langle P, C^* \setminus P \rangle)$ satisfies each clause $cl_i \in \varphi$. Since we have proved that there is a majority of events $\langle P, C^* \setminus P \rangle$ from $\mathcal{E}_A^{C^*}$ such that $f(P) = 2^{n+1}$, this means that there is a majority of assignments over Y that satisfies $\varphi_{|\gamma_X^{C^*}}$. So we have shown that there is an assignment γ_X over X such that the majority of assignments over Y satisfies $\varphi_{|\gamma_X^{C^*}}$. This concludes the (if) part of the proof.

We have proved that there is an assignment γ_X over X such that the majority of assignments over Y satisfies $\varphi_{|\gamma_X}$ if and only if there exists a coalition structure CS such that $U_{fle}(CS) \geq 2^{3n} + 1$ according to the PCFG $\langle A, g^f_{fle}, p\rangle$ Therefore, **DP-∃CS** is $\mathsf{NP}^{\mathsf{PP}}$-hard.

□

Proposition 8 **DP-CS** *is in* P *and* **DP-∃CS** *is* NP-*complete for cautious* PCFGs.

Proof **1.** Let us first prove that **DP-CS** is in P for cautious PCFGs. From Proposition 6, we have for any coalition $C \subseteq A$ that

$$u_{cau}(C) = p(\langle C, \emptyset\rangle) \cdot g^f_{cau}(C, \omega_C),$$

yet $p(\langle C, \emptyset\rangle)$ and $g^f_{cau}(C, \omega_P)$ are computed in polynomial time, so $u_{cau}(C)$ is computed in polynomial time. Yet from Proposition 4, for every coalition structure $CS \in \Pi_A$,

$$U_{cau}(CS) = \sum_{C \in CS} u_{cau}(C),$$

which is then be computed in polynomial time. Therefore, **DP-CS** is in P for cautious PCFGs.

2. Let us now prove that **DP-∃CS** is NP-complete for cautious PCFGs. **DP-∃CS** is in NP for cautious PCFGs, since the fact that a (polynomial-size) coalition structure $CS \in \Pi_A$ can be guessed in polynomial time and checked in P according to point 1 of this proof.

To show that **DP-∃CS** is NP-hard for cautious PCFGs, it is enough to remark that the standard CSG problem is NP-hard, and that from Proposition 2, any CFG $\langle A, f\rangle$ is equivalent to its optimistic cautious CFG extension $\langle A, g^f_{cau}, p_\top\rangle$, i.e., where for each outcome $\omega_P \in \Omega_A$, $p_\top(\omega_P) = 1$ if $P = A$, and $p_\top(\omega_P) = 0$ otherwise.

Therefore, **DP-∃CS** is NP-complete for cautious PCFGs. □

Proposition 9 *For any* $CS \in \Pi_A$,

$$U_{cau}(CS) = \sum_{r_i \in \mathcal{R}^*} w_i \cdot p(\langle CS(i), \emptyset\rangle),$$

where $\mathcal{R}^*$ *is the set of rules* r_i *activated by a coalition* $CS(i) \in CS$.

Proof We know from Proposition 4 that for each $CS \in \Pi_A$,

$$U_{cau}(CS) = \sum_{C \in CS} u_{cau}(C),$$

and from Proposition 6 that for each $C \in CS$,

$$u_{cau}(C) = p(\langle C, \emptyset\rangle) \cdot g^f_{cau}(C, \omega_C).$$

Yet g^f_{cau} is characterized by f, which itself is represented by an MC-net R. So by definition of f and g^f_{cau}, we get that

$$g^f_{cau}(C, \langle C, \emptyset\rangle) = f(C) = \sum_{r_i \in \mathcal{R}^*_C} w_i,$$

where R^*_C is the set of rules from R that are activated by C.

Hence,

$$u_{cau}(C) = p(\langle C, \emptyset \rangle) \cdot \sum_{r_i \in \mathcal{R}^*_C} w_i.$$

And since $p(\langle C, \emptyset \rangle)$ is independent of r_i in the above equation, we get that

$$u_{cau}(C) = \sum_{r_i \in \mathcal{R}^*_C} w_i \cdot p(\langle C, \emptyset \rangle).$$

Hence, for any $CS \in \Pi_A$,

$$U_{cau}(CS) = \sum_{C \in CS} \sum_{r_i \in \mathcal{R}^*_C} w_i \cdot p(\langle C, \emptyset \rangle).$$

Yet we know that each rule r_i from $\mathcal{R}$ is activated by at most one coalition from CS. This means that for all coalitions $C, C', C \neq C'$, we have that $\mathcal{R}^*_C \cap \mathcal{R}^*_{C'} = \emptyset$. Therefore,

$$U_{cau}(CS) = \sum_{r_i \in \mathcal{R}^*} w_i \cdot p(\langle CS(i), \emptyset \rangle),$$

where $\mathcal{R}^*$ is the set of rules r_i activated by a coalition $CS(i) \in CS$.

This concludes the proof. □

Proposition 10 *For any $CS \in \Pi_A$,*

$$U_{fle}(CS) = \sum_{r_i \in \mathcal{R}^+} w_i \cdot p(\langle \gamma_i^+, CS(i) \cap \gamma_i^- \rangle),$$

where $\mathcal{R}^+$ is the set of rules r_i partially activated by a coalition $CS(i) \in CS$.

Proof We know from Proposition 4 that for each $CS \in \Pi_A$,

$$U_{fle}(CS) = \sum_{C \in CS} u_{fle}(C),$$

and that for each $C \in CS$,

$$u_{fle}(C) = \sum_{P \subseteq C} p(\langle P, C \setminus P \rangle) \cdot g^f_{fle}(C, \omega_P).$$

Yet g^f_{fle} is characterized by f, which itself is represented by an MC-net R. So by definition of f and g^f_{fle}, we get for each $P \subseteq C$ that

$$g^f_{fle}(C, \langle P, C \setminus P \rangle) = f(C \cap P) = f(P) = \sum_{r_i \in \mathcal{R}^*_P} w_i,$$

where $\mathcal{R}^*_P$ is the set of rules from $\mathcal{R}$ that are activated by P. Hence,

$$u_{fle}(C) = \sum_{P \subseteq C} p(\langle P, C \setminus P \rangle) \cdot \sum_{r_i \in \mathcal{R}^*_P} w_i.$$

And since $p(\langle P, C \setminus P \rangle)$ is independent of r_i in the above equation, we get that

$$u_{fle}(C) = \sum_{P \subseteq C} \sum_{r_i \in \mathcal{R}_P^*} w_i \cdot p(\langle P, C \setminus P \rangle).$$

Equivalently, we can write

$$u_{fle}(C) = \sum_{r_i \in \mathcal{R}} \sum_{P \subseteq C} w_i(P) \cdot p(\langle P, C \setminus P \rangle),$$

where $w_i(P) = w_i$ if $r_i \in \mathcal{R}_P^*$, otherwise $w_i(P) = 0$.

Yet for all $P \subseteq C$, we have that $r_i \in \mathcal{R}_P^*$ precisely for those events $\langle P, C \setminus P \rangle$ where $\gamma_i^+ \subseteq P$ and $\gamma_i^- \cap C \subseteq C \setminus P$, i.e., $w_i(P) = w_i$ when $\gamma_i^+ \subseteq P$ and $\gamma_i^- \cap C \subseteq C \setminus P$, otherwise $w_i(P) = 0$. Hence,

$$u_{fle}(C) = \sum_{r_i \in \mathcal{R}, \gamma_i^+ \subseteq C} w_i \cdot p(\langle \gamma_i^+, \gamma_i^- \cap C \rangle).$$

Equivalently,

$$u_{fle}(C) = \sum_{r_i \in \mathcal{R}_C^+} w_i \cdot p(\langle \gamma_i^+, \gamma_i^- \cap C \rangle),$$

where $\mathcal{R}_C^+$ is the set of rules r_i partially activated by C. Hence, for any $CS \in \Pi_A$,

$$U_{fle}(CS) = \sum_{C \in CS} \sum_{r_i \in \mathcal{R}_C^+} w_i \cdot p(\langle \gamma_i^+, \gamma_i^- \cap C \rangle).$$

Yet we know that each rule r_i from $\mathcal{R}$ is partially activated by at most one coalition from CS. This means that for all coalitions $C, C', C \neq C'$, we have that $\mathcal{R}_C^+ \cap \mathcal{R}_{C'}^+ = \emptyset$. Therefore, for any $CS \in \Pi_A$,

$$U_{fle}(CS) = \sum_{r_i \in \mathcal{R}^+} w_i \cdot p(\langle \gamma_i^+, CS(i) \cap \gamma_i^- \rangle),$$

where $\mathcal{R}^+$ is the set of rules r_i partially activated by a coalition $CS(i) \in CS$.

This concludes the proof. □

Proposition 11 **DP-CS** *is in* P *and* **DP-∃CS** *is* NP-*complete for MC-net based flexible* PCFGs.

Proof ***1.*** From Proposition 6, we have for any $CS \in \Pi_A$ that

$$U_{fle}(CS) = \sum_{r_i \in R^+} w_i \cdot p(\langle \gamma_i^+, CS(i) \cap \gamma_i^- \rangle),$$

where $\mathcal{R}^+$ is the set of rules r_i partially activated by a coalition $CS(i) \in CS$.

Yet $p(\langle Q, R \rangle)$ is computed in polynomial time for any event $\langle Q, R \rangle$, so $\langle \gamma_i^+, CS(i) \cap \gamma_i^- \rangle$ is computed in polynomial time for any rule $r_i \in \mathcal{R}$ and the coalition $CS(i)$ that partially activated r_i in any $CS \in \Pi_A$. Hence, for any $CS \in \Pi_A$, $U_{fle}(CS)$ is computed in polynomial time in the size of $|\mathcal{R}|$. Therefore, **DP-CS** is in P for MC-net based flexible PCFGs.

2. **DP-∃CS** is in NP, since the fact that a (polynomial-size) coalition structure $CS \in \Pi_A$ can be guessed in polynomial time and checked in P according to point 1 of this proof.

To show that **DP-∃CS** is NP-hard for cautious PCFGs, it is enough to remark that the standard CSG problem is NP-hard even when f is represented as an MC-net [23], and that from Proposition 2, any MC-net based CFG $\langle A, f \rangle$ is equivalent to its optimistic flexible MC-net based CFG extension $\langle A, g^f_{fle}, p_\top \rangle$, i.e., where for each outcome $\omega_P \in \Omega_A$, $p_\top(\omega_P) = 1$ if $P = A$, and $p_\top(\omega_P) = 0$ otherwise.

Therefore, **DP-∃CS** is NP-complete for MC-net based flexible PCFGs. □

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