Hydrodynamic limit of stochastic ranking process

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2019.11.07 Stochastic analysis on large scale interacting systems (Osaka Univ.)

0. Plan of the talk

1. Move-to-front rules

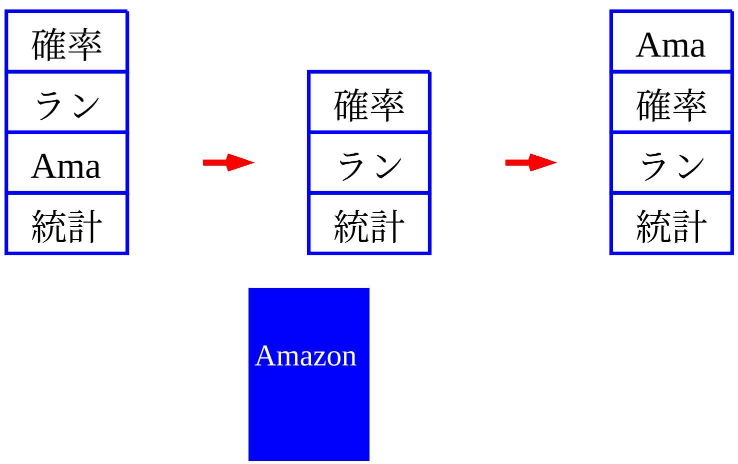
2. 'Solving the mystery of Amazon sales rank'

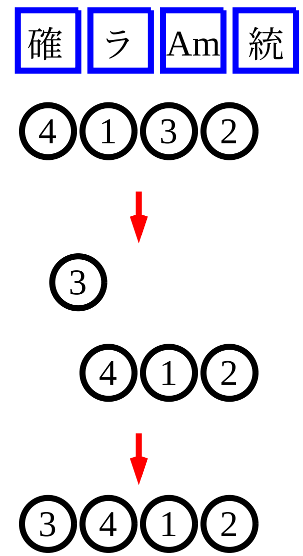
3. Hydrodynamic limit of the stochastic ranking process with position dependent intensities

4. Proof of the main theorem (Flow driven stochastic ranking process)

1. Move-to-front rules

A heap of books (pull out, and back to top after use) movie





Background materials on web

web「服部哲弥」「Tetsuya Hattori」

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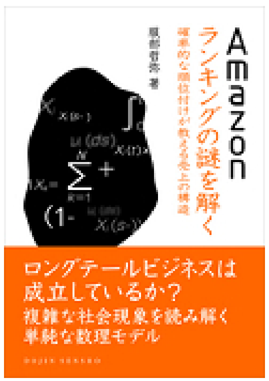


服部哲弥「Amazon ランキングの謎を解く」化学同人出版

↓

「4. 本書内容以降の研究の紹介」のすぐ下の行のリンク

Simple case (move-to-front by Poisson proc.)

N: number of particles ($i = 1, \ldots, N$) interested in $N \to \infty$
$T > 0$, $t \in [0, T]$
particle system $Y_i^{(N)}(t) \in \{\frac{i}{N} \mid i = 0, 1, \ldots, N-1\} \subset [0, 1)$,
(ranking number $N\, Y_i^{(N)}(t) + 1$)
initial value problem: $Y_i^{(N)}(0) = y_i^{(N)}$, $y_i^{(N)} \neq y_j^{(N)}$, $i \neq j$

move-to-front rules: when ptcl i moves to front other ptcls lose ranks by 1 ($Y_i^{(N)}(\tau) = 0 \;\to\; Y_j^{(N)}(\tau) = Y_j^{(N)}(\tau-) + \frac{1}{N}$)

$\tilde{\nu}_i^{(N)}$: independent Poisson proc., for the simple case of the first half of this talk, with intensities w_i depending on i.
The time the particle i moves to front $\tau =$ arrival time of $\tilde{\nu}_i^{(N)}$ ($\tilde{\nu}_i^{(N)}(\tau) - \tilde{\nu}_i^{(N)}(\tau-) > 0$)

Our model (Stochastic ranking process)

$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \notin A. \end{cases}$$ (defining function of a set A)

Nagahata (2013)

$$Y_i^{(N)}(t) = y_i^{(N)} + \frac{1}{N}\sum_{j=1}^{N}\int_0^t \mathbf{1}_{Y_j^{(N)}(s-) > Y_i^{(N)}(s-)}\,\tilde{\nu}_j^{(N)}(ds) - \int_0^t Y_i^{(N)}(s-)\tilde{\nu}_i^{(N)}(ds)$$

3rd term on r.h.s.: $\tilde{\nu}_i^{(N)}(\tau_{i,k}) - \tilde{\nu}_i^{(N)}(\tau_{i,k}-) = 1$ (arrival time) $\Rightarrow$

$$Y_i^{(N)}(\tau_{i,k}) - Y_i^{(N)}(\tau_{i,k}-) = -\int_{\tau_{i,k}-}^{\tau_{i,k}} Y_i^{(N)}(s-)\tilde{\nu}_i^{(N)}(ds)$$

$$= -Y_i^{(N)}(\tau_{i,k}-)\,(\tilde{\nu}_i^{(N)}(\tau_{i,k}) - \tilde{\nu}_i^{(N)}(\tau_{i,k}-)) = -Y_i^{(N)}(\tau_{i,k}-)$$

$\Rightarrow Y_i^{(N)}(\tau_{i,k}) = 0$ (move-to-front)

2nd term: i loses rank by 1 when j at lower rank moves to front

$$Y_i^{(N)}(\tau_{j,k}) - Y_i^{(N)}(\tau_{j,k}-) = \frac{1}{N}\int_{\tau_{j,k}-}^{\tau_{j,k}} \mathbf{1}_{Y_j^{(N)}(s-) > Y_i^{(N)}(s-)}\,\tilde{\nu}_j^{(N)}(ds) = \frac{1}{N}$$

non-symmetric Markov process (particle system of N ptcls)

2. 'Solving the mystery of Amazon rank'

Amazon.co.jp sales rank

amazon.co.jp

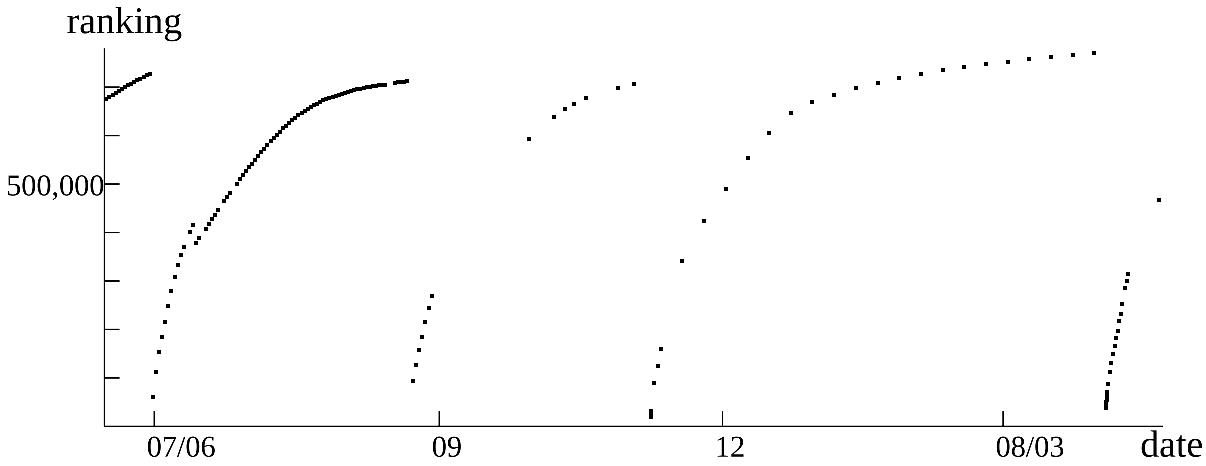
- details hidden
- explained to reflect popularity

What if use the stochastic ranking process as simplest model of popularity rank?

Amazon sales rank moves-to-front!

Ad hoc correspondence between Amazon.co.jp sales rank and stochastic ranking process: Assume an internet order (by an Amazon user) time for book i is the arrival time of $\tilde{\nu}_i^{(N)}$ (move-to-front time for ptcl i).

First chek point of this assumption is: Will an internet-ordered book move-to-front (gain rank 1)?



Next step: Is the observed concave deterministic looking curve a result of LLN? Furthermore, could it be derived from our Poisson process based model?

Characteristic curve

Recall: $Y_i^{(N)}(t) = y_i^{(N)}$

$$+\frac{1}{N}\sum_{j=1}^{N}\int_0^t \mathbf{1}_{Y_j^{(N)}(s-)>Y_i^{(N)}(s-)}\,\tilde{\nu}_j^{(N)}(ds) - \int_0^t Y_i^{(N)}(s-)\tilde{\nu}_i^{(N)}(ds)$$

Dynamics of dummy (without move-to-front，代本板)

$$Y_C^{(N)}((y_0,t_0),t) = y_0 + \frac{1}{N}\sum_{j=1}^{N}\int_{t_0}^t \mathbf{1}_{Y_j^{(N)}(s-)\geqq Y_C^{(N)}((y_0,t_0),s-)}\,\tilde{\nu}_j^{(N)}(ds)$$

If $y_0 = 0$, $Y_C^{(N)}((0,t_0),t) = \frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{\tilde{\nu}_j^{(N)}(t)>\tilde{\nu}_j^{(N)}(t_0)}$ is an average of i.i.d. r.v.s, hence strong LLN

(complete LLN: $*^{(N)}$)

- Refinement to spacial distribution fcn. possible.

Law of large numbers (LLN)

Recall (dummy in the sequence): $Y_C^{(N)}((0,t_0),t) = \frac{1}{N}\sum_{j=1}^{N} \mathbf{1}_{\tilde{\nu}_j^{(N)}(t)>\tilde{\nu}_j^{(N)}(t_0)}$

Recall (Poisson): $\mathrm{P}[\,\nu(t) > \nu(s)\,] = 1-\mathrm{P}[\,\nu(t)-\nu(s) = 0\,] = 1-e^{-w(t-s)}$

Complete LLN (finite variance case: no relation necessary among rvs with different N (martingale structure not used in my proof)

- Assume $\lambda^{(N)} := \frac{1}{N}\sum_{i=1}^{N} \delta_{w_i} \to \lambda$, then

$$y_C((0,t_0),t) := \lim_{N\to\infty} Y_C^{(N)}((0,t_0),t) = \lim_{N\to\infty} \mathrm{E}[\,Y_C^{(N)}((0,t_0),t)\,]$$

$$= \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \mathrm{P}[\,\tilde{\nu}_j^{(N)}(t) > \tilde{\nu}_j^{(N)}(t_0)\,] = 1 - \int_W e^{-w(t-t_0)}\,\lambda(dw)$$

- Random time developement of ranks is well approximated by deterministic one. movie2
- The latter has explicit expression in terms of exponentials (probability of Poisson process) (cf. Second half of this talk isn't that simple)

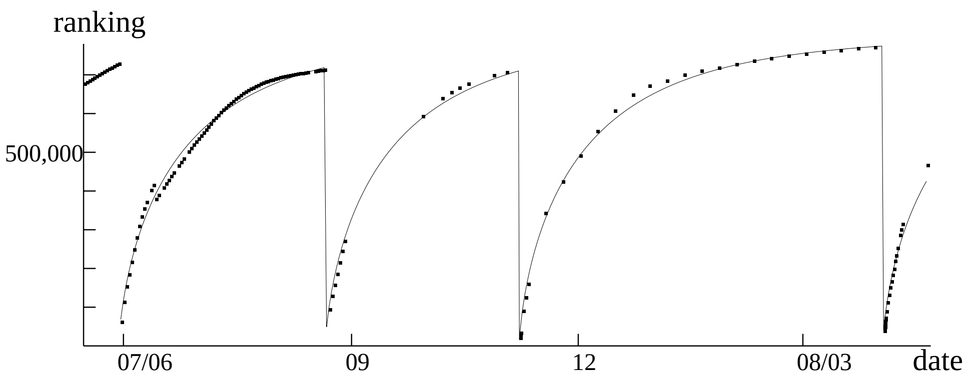
Amazon webstore isn't a long-tail business

Recall: i moves-to-front = arriaval of $\tilde{\nu}_i^{(N)}$ =order of book i at Amazon.

Recall: $y_C((0,t_0),t) = 1 - \int_W e^{-w(t-t_0)}\lambda(dw)$; λ is distribution of rate (intensity) of order w.

$X_i^{(N)}(t) = N\,Y_i^{(N)}(t)+1 \doteqdot N\,y_C((O,\tau),t)$ is the rank of book i at time $t > \tau$ after the latest order at τ.

Statistical fit of observed data: Assume undisclosed λ to be Pareto (power law) distribution.



- (Solid line) seems quantatively a good fit.
- The fit implies traditional big hit business model rather than expected (?) long-tail business model.

Joint empirical distribution

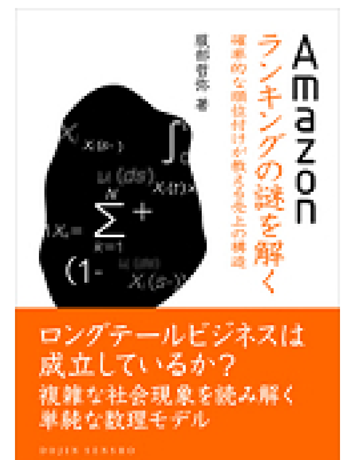
Some more quantities for use in second half of the talk:

- Joint empirical distribution position and intensity $\mu_t^{(N)}$, and its spacial distribution function $\varphi^{(N)}$, is a refinement of 'characteristic curve' $Y_C^{(N)}$.

$$\mu_t^{(N)} = \frac{1}{N} \sum_{i=1}^{N} \delta_{(w_i, Y_i^{(N)}(t))}$$

$$\varphi^{(N)}(dw, \gamma, t) = \mu_t^{(N)}(dw \times [Y_C^{(N)}(\gamma, t), 1]),\ \ \gamma = (y_0, t_0)$$

As for $Y_C^{(N)}(\gamma, t) = \varphi^{(N)}(W, \gamma, t)$, complete LLN of independent r.v.s



- t dependence of intensity w:

$$w\,(t - t_0) \ \mapsto \ \int_{t_0}^{t} w(s)\,ds.\ \ W :\ \mathbb{R}_+ \mapsto C([0, T])$$

(eg., day-night activity cycle)

3. SRP with position dependent intensities

Recall: $Y_i^{(N)}(t) = y_i^{(N)}$

$$+ \frac{1}{N}\sum_{j=1}^{N}\int_0^t \mathbf{1}_{Y_j^{(N)}(s-)>Y_i^{(N)}(s-)}\,\tilde{\nu}_j^{(N)}(ds) - \int_0^t Y_i^{(N)}(s-)\tilde{\nu}_i^{(N)}(ds)$$

Recall: time dependence of intensity w_i also possible by independent increment (non-uniform Poisson) process $\tilde{\nu}_j^{(N)}$.

• Hydrodynamic limit of the stochastic ranking process with position dependent intensities, to formulate position dependenc of intensity (e.g., advertising effect). $w_i:\ [0,1]\times[0,T]\to\mathbb{R}_+$

$\nu_i^{(N)}$: Poisson random measure on ${\mathbb{R}_+}^2$ of unit intensity.

(i.e., $U(A)=\nu_i^{(N)}(A)$ is Poisson r.v. with mean $=$ area of $A\subset{\mathbb{R}_+}^2$
$A\cap B=\emptyset\Rightarrow U(A)\perp U(B)$)

$$\tilde{\nu}_i^{(N)}(ds) = \int_{\xi=0}^{\xi=w_i(Y_i^{(N)}(s-),s)} \nu_i^{(N)}(d\xi\times ds)\text{: Kusuoka (2012)}$$

• dependence through w, implying $N\to\infty$ limit expressed by dependent increments

The latter half of this talk

First half of this talk:

- Revolutionary new phenomena
  - Large, real-time popularity ranking made visible for the first time in history
- Natural mathematics made ready for the new phenomena

Second half of the talk:

- A new formulation by Nagahata and Kusuoka leads to new natural mathematical problems

— position dependence of intensity =dependent stochastic variables leading to a natural class of process with dependent increments

Hydrodynamic limit

Assume: $W \subset C^{1,0}$; $\sup_{w\in W} \|w'\| < \infty$, $\mu_0^{(N)} \to \mu_0$ weakly', $\int \|w\|\, d\lambda < \infty$

(Note, $\lambda^{(N)} = \mu_t^{(N)}(\cdot \times [0,1]) \to \lambda$ with unbdd w allowed)

Main Theorem. $\exists \mu_t$; with prob. 1, uniform in t, $\mu_t^{(N)} \to \mu_t$ weakly. Moreover, For $L \in \mathbb{N}$ and $y_1, \dots, y_L$, with $\nu_i^{(N)} = \nu_i$, $N \in \mathbb{N}$, and $\lim_{N\to\infty} y_i^{(N)} = y_i$,with prob. 1, $Y_i^{(N)}(t)$ converges uniformly in t (propagation of chaos);

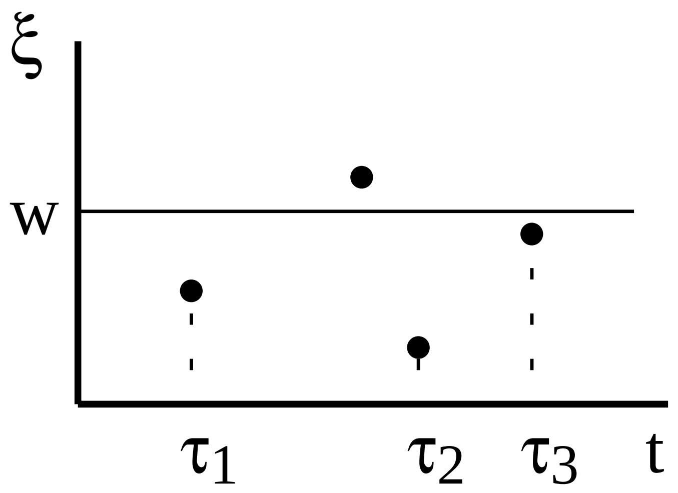
$$Y_i(t) = y_i + \int_{s\in(0,t]} \int_{(w,z)\in W\times[Y_i(s-).1]} w(z,s)\, \mu_s(dw \times dz)\, ds$$
$$- \int_{s\in(0,t]} \int_{\xi\in\mathbb{R}_+} Y_i(s-)\, \mathbf{1}_{\xi\in[0,w_i(Y_i(s-),s))}\, \nu_i(d\xi ds) \qquad \diamond$$

Weak conv. topology on set of distributions, ufm in time, for almost all Poisson sample. (Weak conv.: LLN with fluctuation cancellation among processes with distinct w_is)

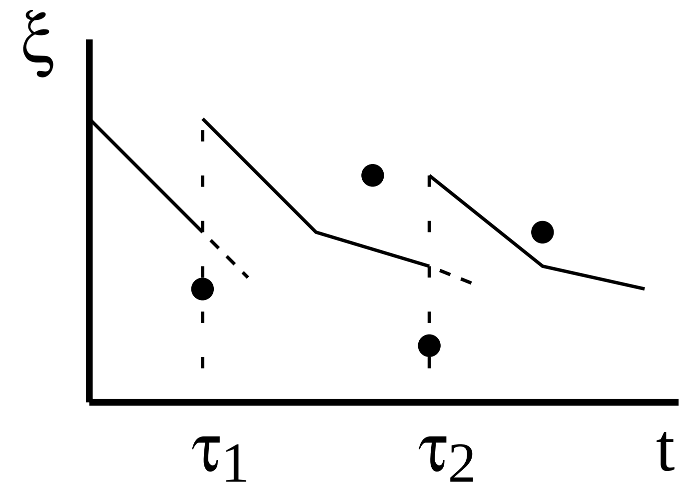
Process with dependent increments

Best expected results with reasonable assumptions, proof by construction of limit 1,2

1. Expression of limit requires processes with dependent increments.
$\tilde{\nu}(t) = k$, $\tau_k \leqq t < \tau_{k+1}$ defined with unit Poisson r.m. ν on ${\mathbb{R}_+}^2$ by
$\tau_k = \inf\{t \geqq \tau_{k-1} \mid \nu(\{(\xi, s) \in {\mathbb{R}_+}^2 \mid \xi \leqq \underline{w}(\tau_{k-1}, s),\ \tau_{k-1} < s \leqq t\}) > 0\}$
Note. $\underline{w}$ is fcn of 2 time vars. cf. SRP use $w(y, s)$ with $y = Y_i^{(N)}(s)$.



$\tilde{\nu}$: Poisson proc. with intensity w



Point proc. with last-arrival-time dependent intensity

Proc. with last-arrival-time dep intensity

- A class of non-trivial but natural extension of Poisson process

$$\mathrm{P}[\, t < \tau_k \mid \mathcal{F}_{\tau_{k-1}} \,] = \exp\Big(-\int_{\tau_{k-1}}^{t} \underline{w}(\tau_{k-1}, u)\, du\Big),\ t \geqq \tau_{k-1}$$

$$\Omega(t_0, t) = \int_{t_0}^{t} \underline{w}(t_0, u)\, du, \quad \mathrm{P}[\, \tilde{\nu}(t) = \tilde{\nu}(s) \,] = \sum_{k \geqq 0} \int_{0 =: u_k < u_{k-1} < \cdots < u_0 \leqq s}$$

$$\times\, e^{-\sum_{i=0}^{k-1} \Omega(u_{i+1}, u_i) - \Omega(u_0, t)} \Big(\prod_{i=0}^{k-1} \underline{w}(u_{i+1}, u_i)\, du_i \Big)$$

$\mathrm{P}[\, \tilde{\nu}(t) = \tilde{\nu}(s) \,] = e^{-\Omega(s,t)}$ reproduced, if $\underline{w}$ indep of 1st variable, by

$$\int_{0 \leqq u_1 \leqq u_2 \leqq \cdots \leqq u_k \leqq s} \prod_{i=1}^{k} f(u_i) du_1\, du_2 \ldots du_k = \frac{1}{k!} \Big(\int_0^s f(v) dv \Big)^k$$

System of PDE with non-local terms

Construction of limit 2. generalized PDE system solvable by characteristic curves

Recall: $\varphi(dw,(y_0,t_0),t) = \mu_t(dw \times [y_C((y_0,t_0),t),1])$;
$y_C((y_0,t_0),t) = 1 - \varphi_{y_C}(W,(y_0,t_0),t)$

Theorem 1. $\exists!$ Lipschitz conti. (y_C,μ_t);
$y_C((y_0,t_0),t_0) = y_0\,,\ \ (y_0,t_0) \in$ init-bdry values$\Gamma_T\,,$
$\mu_t(dw \times [0,1)) = \lambda(dw),\quad t \in [0,T]$ (conservation),
$\mu_t(W \times [y,1)) = 1 - y,\quad (y,t) \in [0,1] \times [0,T]\quad (y_C = 1 - \varphi(W)),$

$$\mu_t(dw \times [y_C(\gamma,t),1)) = \mu_{t_0}(dw \times [y_0,1)) - \int_{s=t_0}^{t} \int_{z=y_C(\gamma,s)}^{1} w(z,s)\mu_s(dw \times dz)\,ds,\ \gamma = (y_0,t_0),\ (\gamma,t) \in \Delta_T = \{\text{pair of } (t,\Gamma_t)\}\ \diamond$$

Example of finite fluid components α. $U_\alpha(y,t) = \mu_t(\{w_\alpha\}\times[y,1])$ satisfy

$$\frac{\partial U_\alpha}{\partial t}(y,t) - \sum_\beta \int_y^1 w_\beta(z,t)\frac{\partial U_\beta}{\partial z}(z,t)dz\frac{\partial U_\alpha}{\partial y}(y,t) = \int_y^1 w_\alpha(z,t)\frac{\partial U_\alpha}{\partial z}(z,t)dz$$

Solution by stochastic process

Solution of gen. PDE system expressed in terms of processes: On Γ_T: $(0,T) \succeq O = (0,0) \succeq (1,0)$

Set of flow θ $\Theta_T := \{\theta : \ \Delta_T \to [0,1] \mid \theta(z,s),s) = z, \ (z,s) \in \Gamma_T,$
conti., non-increasing in γ, surj., non-decreasing in $t\}$

For intensity $w(y,s)$: $[0,1]\times[0,T] \to \mathbb{R}_+$ of SRP (THE MODEL) and a flow $\theta \in \Theta_T$, define intensity $\underline{w}_{\theta,w,z}$: $[0,T]^2 \to \mathbb{R}_+$ of point proc. $\tilde{\nu}_{\theta,w,z}$ with last-arrival-time dep intensity by

$$\underline{w}_{\theta,w,z}(s,t) = \begin{cases} w(\theta((z,0),t),t), & \text{if} \quad s=0, \\ w(\theta((0,s),t),t), & \text{if} \quad s>0. \end{cases} \quad \text{and put}$$

$$\varphi_\theta(dw,(y_0,t_0),t) := \int_{z\in[y_0,1]} \mathrm{P}[\, \tilde{\nu}_{\theta,w,z}(t) = \tilde{\nu}_{\theta,w,z}(t_0)\,]\,\mu_0(dw\times dz)$$

$\mathcal{G}$: $\Theta_T \to \Theta_T$ by $\mathcal{G}(\theta)((y_0,t_0),t) := 1 - \varphi_\theta(W,(y_0,t_0),t)$

Theorem 1'. $\exists!\theta = y_C \in \Theta_T$ satisfying $\theta = \mathcal{G}(\theta)$. ◇

$$\begin{aligned}\varphi_{y_C}(dw,(y_0,t_0),t) &= \int_{z\in[y_0,1]} \mathrm{P}[\, \tilde{\nu}_{y_C,w,z}(t) = \tilde{\nu}_{y_C,w,z}(t_0)\,]\,\mu_0(dw\times dz) \\ &= \mu_t(dw \times [y_C((y_0,t_0),t),1])\end{aligned}$$

4. Proof of the main theorem

— Define an intermediate model, the flow driven stochastic ranking process
— Strong law of indep. fcn. valued stochastic variables (disappearance of fluctuation of intermediate model)
— hierarchical Gronwall type estimates (extinction of difference between THE MODEL and intermediate model)

Intermediate model attains same limit as THE MODEL by LLN of independent stochastic variables

Pt proc $\tilde{\nu}_i^{(N,\theta)}$ with last-arrival-time dep intensity $\underline{w}_{\theta,w_i,y_i^{(N)}}$

$$Y_i^{(N,\theta)}(t) = y_i^{(N)} + \frac{1}{N}\sum_{j=1}^{N}\int_{s\in(0,t]} \mathbf{1}_{Y_j^{(N,\theta)}(s-)>Y_i^{(N,\theta)}(s-)}\, \tilde{\nu}_j^{(N,\theta)}(ds) - \int_{s\in(0,t]} Y_i^{(N,\theta)}(s-)\tilde{\nu}_i^{(N,\theta)}(ds)$$

arbitray θ, $\tilde{\nu}_j^{(N,\theta)}$ indep but proc with dep increments, otherwise same as the original model.

Intermediate model and common limit

The only difference in SDE is choice of $\tilde{\nu}$ (both are move-to-front rule). $\tilde{\nu}$:

Poisson, for original, dependence through $w_i(Y_i, t)$
$\downarrow$ ae conv as $N \to \infty$ by coupling of Poisson random measure

indep but dep increment proc for intermediate model $(\forall \theta)$
$\downarrow$ $N \to \infty$ by complete LLN+conv. of expectation for $\theta = y_C$

limit Soln of fctnl eq expressed by pt proc with dependent increment

Recall the intermediate model and the limit (propagation of chaos result)

$$Y_i^{(N,\theta)}(t) = y_i^{(N)} + \int_{s\in(0,t]} \frac{1}{N}\sum_{j=1}^{N} \mathbf{1}_{Y_j^{(N,\theta)}(s-)>Y_i^{(N,\theta)}(s-)}\, \tilde{\nu}_j^{(N,\theta)}(ds) - \int_{s\in(0,t]} Y_i^{(N,\theta)}(s-)\,\tilde{\nu}_i^{(N,\theta)}(ds)$$

$$Y_i(t) = y_i + \int_{s\in(0,t]} \int_{(w,z)\in W\times[Y_i(s-).1]} w(z,s)\,\mu_s(dw\times dz)\,ds - \int_{s\in(0,t]} Y_i(s-)\,\tilde{\nu}_i(ds)$$

Uniform complete LLN

- details of the point process $\to$ E[$\varphi^{(N,\theta)}$] $\to \varphi_\theta$
- LLN (general theory) $\varphi^{(N,\theta)}$ – E[$\varphi^{(N,\theta)}$] $\to 0$ with

moment estimate uniform in initial-final times (to prepare for Gronwall type estimate for difference with original model

$\Delta = \{(t_1, t_2) \in \mathbb{R}^2 \mid 0 \leqq t_1 \leqq t_2 \leqq T\}$
$D_\uparrow$: set of fcns in 2 vars with monotonicity as difference of 2 cadlag non-decresing fcns on Δ

Theorem. $r > 0$, $q > 2$, For each N, $\{Z_i^{(N)}\}_{i=1}^N$ are indep $D_\uparrow$ valued s.vs.
E[$|Z_i^{(N)}(0,T)|^q$]$^{1/q} \leqq M$, $\quad$ |E[$Z_i^{(N)}(s,t)$]| $\leqq Mw|t-s|^r$
imply $\exists C_q, N_0 > 0$; $(\forall N \geqq N_0)$

$$\mathrm{E}\Big[\sup_{(t_1,t_2)\in\Delta} \left| \frac{1}{N} \sum_{i=1}^{N} \Big((Z_i^{(N)}(t_1,t_2)) - \mathrm{E}[\, Z_i^{(N)}(t_1,t_2) \,] \Big) \right|^q \Big]$$

$$\leqq \frac{M^q 2^{q-1}}{N^{q^2 r/(2qr+2r+2)}} \left(C_q^q \,(2T\, w^{1/r} + 1) + 2^{2q}\right) \qquad \diamond$$

Note. $\dfrac{\text{fluctuation}}{\text{mean}} = O(N^{-1/2+\epsilon})$ (ϵsmall for large q but positive)

Hattori–Kusuoka (2012)

Hattori–Kusuoka, ALEA **9 (2)** (2012) 571–607.

• Result same with today's talk (convergence of joint empirical distribution and propagation of chaos for tagged ptcls)

• Method: martingale convergence (cf. Today used intermediate model and details of limit)

• Assumption on $t = 0$ ($\mu_0^{(N)}$):

1. bdd intensity (lack of expression at that time using process with dependent increment)

2. $\mu_0^{(N)} \to \mu_0$ in total variation norm (cf. today assumed weak convergence only) we couldn't consider continuous distribution as Pareto (power law)

Proof based on LLN cancelling fluctuations among ptcls with the same w. i.e., infinite ptcls with same w required

'Amazon ranking': Zipf law for ptcls (typically, all ptcl have distinct intensities), limit is generalized Pareto, continuous unbdd distribution. (I gave up martingale arguments and generalized to weak convergence.)

Nagahata's talk this afternoon

The limit: each (prob 1) sample orbit $Y_1^{(N)}(\omega)$ of a tagged particle converges. In particular, arrival times converge samplewise.

Nagahata's Functional CLT: conditioning on arrival times of tagged particles, CLT for fluctuations of the orbit, IF $\sharp W < \infty$ (finite varieties of intensity w)

cf. Hattori–Kusuoka. Again cancellation among the same intensities??

• Cancellation among different ws mathematically natural and practically important.

• Cancellation among different ws hold for position indepe-nent case.

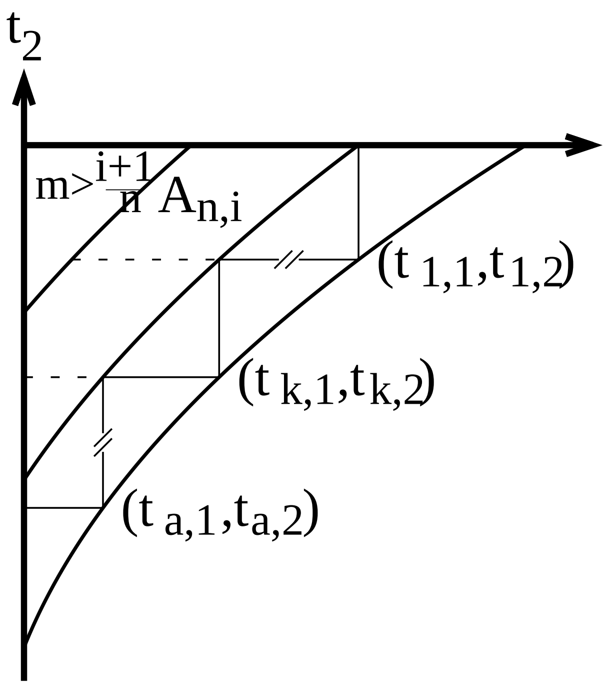
• However, LLN (and Gronwall) for today's talk no better:

$$\frac{\text{fluctuation}}{\text{mean}} = O(N^{-1/2+\epsilon})$$

Puzzling...

関数値独立確率変数列の２重に一様な大数の強法則

- $Z_i^{(N)}$ は単調だが期待値を引く→「仕切り」を入れ期待値の変化 $< \epsilon$
- ２重に一様→等高線で網→個数制御はHölder連続性の仮定に帰着



$$Y^{(N)}(t_1,t_2) = \frac{1}{N}\sum_{i=1}^{N} Z_i^{(N)}(t_1,t_2)$$

の減る方向（右下）に $\mathsf{E}[\,Y^{(N)}\,]$ の減少が $2/n$ 以内の見張りを立てる．

$m(t_1,t_2) = \mathsf{E}[\,Y^{(N)}(t_1,t_2)\,]$

$A_{n,i} = \{(t_1,t_2)\in A \mid \frac{i}{n} < m(t_1,t_2) \leqq \frac{i+1}{n}\}$

$m(t_{k,1},t_{k,2}) = \frac{i-1}{n}$

$(\forall (t_1,t_2)\in A_{n,i})\ \exists k;$

$m(t_1,t_2) \leqq m(t_{k,1},t_{k,2}) + \frac{2}{n}$

$(t_1,t_2)\in A_{n,0}$ または $m(t_1,t_2)=0$ だと見張りが無いが，$m(t_1,t_2)\leqq\frac{1}{n}$ が成立して有効

- 時間方向は単調性しか使わないので，独立増分性等は不要で，直前の到着時刻に依存する強度を持つ点過程でも何も気にせず使える

多変数階層的Gronwall型評価

- **主定理証明の完成：**$\theta = y_C$ **なる中間模型と元の模型の極限の一致**
- LLN**で稼いだ**$N^{-1/2+\epsilon}$ vs **従属項を**Hölder**ではがす**$N^{\epsilon'}$ **の競争**

$a, c, a_q, b_q, c_q \geqq 0$, $t \in [0, T]$

命題（Gronwall不等式）． $x(t) \leqq a + c \int_0^t x(s)ds \Rightarrow \ x(t) \leqq a e^{ct}$ ◇

定理（q変数）． $\vec{t} = (t_1, \ldots, t_q)$,

$$x(\vec{t}) \leqq a e^{c(t_1+\cdots+t_q)} \frac{1}{q} \sum_{i=1}^{q} e^{-ct_i} + \frac{c}{q} \sum_{i=1}^{q} \int_0^{t_i} (x(\vec{t})|_{t_i=u})\, du$$

ならば $x(\vec{t}) \leqq a\, e^{c\,(t_1+\cdots+t_q)}$ ◇

定理（q再帰＋非線型項$0 \leqq d \leqq 1$）． $x_q \geqq 0$, $q = 1, 2, \ldots,$ **が** $x_0 = 1$ **と**

$$x_q(\vec{t}) \leqq a_q \sum_{i=1}^{q} x_{q-1}(t_1, \ldots, \not{t_i}, \ldots, t_q)^d$$

$$+ b_q \sum_{i=1}^{q} x_{q-1}(t_1, \ldots, \not{t_i}, \ldots, t_q) + c_q \sum_{i=1}^{q} \int_0^{t_i} (x_q(\vec{t})|_{t_i=s})\, ds$$ **を満たせば，**

$x_q(\vec{t}) \leqq g_q\, e^{\tilde{c}_q\,(t_1+\cdots+t_q)}$. **ここで，** $\tilde{c}_q = \max_{1 \leqq k \leqq q} k c_k$, $\tilde{c}_q = \max_{1 \leqq k \leqq q} k c_k$ ◇

多変数斉次Gronwall型不等式

- 斉次な場合のq変数拡張が恐らく一番の要点

定理． $c \geqq 0$．$x:\ [0,T]^q \to \mathbb{R}$ 可積分

$$x(\vec{t}) \leqq c \sum_{i=1}^{q} \int_0^{t_i} (x(\vec{t})|_{t_i=s}) ds,\ \vec{t} \in [0,T]^q,\ \Rightarrow\ x(\vec{t}) \leqq 0. \qquad \diamond$$

証明． $(A_{i,k}y)(\vec{t}) = \dfrac{1}{(k-1)!} \displaystyle\int_0^{t_i} (t_i - s)^{k-1} (y(\vec{t}))|_{t_i=s}\, ds$

$A_{i,k}A_{j,\ell} = A_{j,\ell}A_{i,k}$，　$A_{i,k}A_{i,\ell} = A_{i,1}^{k+\ell} = A_{i,k+\ell}$　を経て

$$x(\vec{t}) \leqq c^N \sum_{\substack{(k_1,\dots,k_q)\in\mathbb{Z}_+^q;\\ k_1+\cdots+k_q=N}} (A_{q,k_q} A_{q-1,k_{q-1}} \cdots A_{1,k_1} x)(\vec{t}).$$

他方，$(A_{i,k}y)(\vec{t}) \leqq \dfrac{t_i^k}{k!} \displaystyle\sup_{\vec{t}\in[0,T]^q} y(\vec{t})$ □

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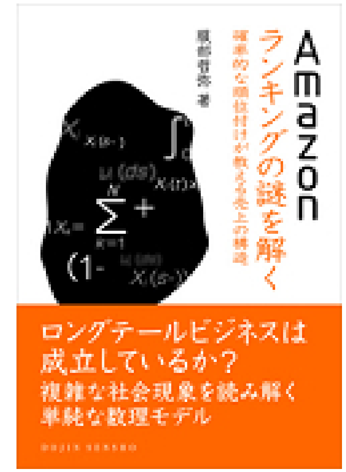

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