Point process with last-arrival-time dependent intensity and 1-dimensional incompressible fluid system with evaporation.

Tetsuya Hattori *

Laboratory of Mathematics, Faculty of Economics, Keio University,

Hiyoshi Campus, 4–1–1 Hiyoshi, Yokohama 223-8521, Japan

URL: http://web.econ.keio.ac.jp/staff/hattori/research.htm

email: hattori@econ.keio.ac.jp

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Abstract

We consider an infinite system of quasilinear first-order partial differential equations, generalized to contain spacial integration, which describes an incompressible fluid mixture of infinite components in a line segment whose motion is driven by unbounded and space-time dependent evaporation rates. We prove unique existence of the solution to the initial-boundary value problem, with conservation-of-fluid condition at the boundary. The proof uses a map on the space of collection of characteristics, and a representation based on a non-Markovian point process with last-arrival-time dependent intensity.

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1 Introduction.

Consider an incompressible fluid mixture in a line segment, say $[0,1]$, which flow in order preserving manner and in one direction, with $y=0$ being the upper stream boundary, and no leaking occurs at $y=1$. Each fluid component, say α, evaporates with rate w_α which may vary among different components and may depend on time. Flow of the fluid is driven by filling the evaporated portion of the fluid toward the down stream. To formulate a system of partial differential equations which explains the dynamics of this fluid up to time $T>0$,

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let $U_\alpha(y,t)$ be the total volume (length) of fluid component α at time t in the interval $[y,1)$. Then we have

$$\begin{aligned}(1)\qquad &\frac{\partial U_\alpha}{\partial t}(y,t) + \sum_\beta w_\beta(t) U_\beta(y,t)\,\frac{\partial U_\alpha}{\partial y}(y,t) = -w_\alpha(t)U_\alpha(y,t),\\ &(y,t)\in[0,1]\times[0,T].\end{aligned}$$

We will preserve the total volume of each component by supplying the evaporated portion from upper stream boundary through the boundary condition

$$(2)\qquad U_\alpha(0,t) = r_\alpha \quad\text{and}\quad U_\alpha(1,t)=0, \quad t \geqq 0,$$

for non-negative constants r_α satisfying $\sum_\alpha r_\alpha = 1$. The incompressibility condition is formulated as

$$(3)\qquad \sum_\alpha U_\alpha(y,t) = 1-y, \quad t\geqq 0.$$

The number of fluid components may be finite or infinite. (For the latter case we regard the summations in α as series.) If the system is infinite, we should impose an additional condition

$$(4)\qquad \sum_\alpha r_\alpha \sup_t w_\alpha(t) < \infty,$$

to keep the velocity of the flow finite, namely, to keep coefficient of the y-derivative term in (1) well-defined at $y=0$. With appropriate initial conditions, these define an initial/boundary value problem of a one dimensional first order quasilinear partial differential equations, and we will look for non-negative solutions non-increasing in y (corresponding to non-negativity of density of fluid components). The motion of the fluid mixture described by the equations is driven by evaporation, a kind of situation where poluted water in a drain or groove slowly flows into an inland lake, where the water escapes mostly by evaporation.

In this paper we consider a generalization of (1) to allow for spacial dependence for the evaporation rates w_α, as well as time dependence. Such generalization seems practically natural, because if the fluid container has spacial non-uniformity in temperature, the evaporation rates would also have spacial dependence. $U_\alpha(y,t)$ is the volume of type α fluid component in the interval $[y,1)$, and we need to consider its density to consider spacially varying evaporation rates, hence a natural generalization of (1) is

$$\begin{aligned}(5)\qquad &\frac{\partial U_\alpha}{\partial t}(y,t) - \sum_\beta \int_y^1 w_\beta(z,t)\,\frac{\partial U_\beta}{\partial z}(z,t)\,dz\,\frac{\partial U_\alpha}{\partial y}(y,t)\\ &= \int_y^1 w_\alpha(z,t)\,\frac{\partial U_\alpha}{\partial z}(z,t)\,dz,\\ &\alpha = 1,2,\ldots,\ \ (y,t)\in[0,1]\times[0,T].\end{aligned}$$

Note that the equation is now non-local and contains integration. If w_α are independent of y, then, with (2), (5) reduces to (1).

The equation of the form (1) is known to be solved by considering characteristic curves [2], a curve $y = y_C(t)$ whose derivative is equal to the velocity of fluid;

$$\frac{d\, y_C}{dt}(t) = \sum_\beta w_\beta(t)\, \varphi_\beta(t) \tag{6}$$

where $\varphi_\alpha(t) = U_\alpha(y_C(t), t)$. Then (1) implies an ordinary differential equation for $\varphi_\alpha(t)$, which can be solved explicitly, and (6) then implies

$$y_C(t) = 1 - \sum_\beta U_\beta(y_0, t_0)\, \exp(-\int_{t_0}^t w_\beta(u)\, du). \tag{7}$$

A natural generalization of (6) for (5) is

$$\frac{d\, y_C}{dt}(t) = -\sum_\beta \int_{y_C(t)}^1 w_\beta(z,t)\, \frac{\partial\, U_\beta}{\partial z}(z,t)\, dz. \tag{8}$$

As we will see in the present paper, this is no longer solved in such simple form as (7). Introduction of spacial dependence for w_α complicates the solution when combined with the boundary condition (2) which conserves component volumes. We will show later that y_C is determined as a fixed point to the map G defined by (77) and (78), a result which apparently deviates largely from (7). To be specific, our proof in § 5 of Theorem 10 proves an expression $y_C = \lim_{n\to\infty} G^n(\theta_0)$ for the characteristic curves, where θ_0 is a constant flow.

In the preceding work [11], the problem (5) for w_α with spacial dependence was considered under the condition

$$\sup_\alpha \sup_{(y,t)} w_\alpha(y,t) < \infty, \quad \text{and} \quad \sup_\alpha \sup_{(y,t)} \frac{\partial\, w_\alpha}{\partial y}(y,t) < \infty.$$

In view of (4) for $\{w_\alpha\}$ with spacially independent case, a natural restriction for w_α is expected to be a milder one,

$$\sum_\alpha r_\alpha \sup_{(y,t)} w_\alpha(y,t) < \infty, \quad \text{and} \quad \sup_\alpha \sup_{(y,t)} \frac{\partial\, w_\alpha}{\partial y}(y,t) < \infty, \tag{9}$$

allowing, in particular, fluid mixture with unbounded evaporation rates. The unique existence of the solution was proved for the case of bounded evaporation rates in [11], but explicit formula such as (78) were absent, and a rather strong restriction on w_α was posed. In the present paper we solve the equation under a natural assumption (9).

Besides mathematical naturalness of the assumption (9), removal of boundedness condition on $\{w_\alpha\}$ also has a practical meaning in analyses of behaviors of web ranking data for on-line retail businesses, using the stochastic ranking processes[4, 5, 3, 6, 7, 8, 12, 13, 11, 10]. For example, the time evolution of the sales ranks in an on-line bookstore Amazon.co.jp has been found to be approximately explained by those solutions in the above references, which correspond to the earlier version of the main result of the present paper. By regarding the rank as the position y (popular books corresponding to upstream small y and professional books mainly in the downstream around $y = 1$), and by regarding each book as fluid particle (molecule) α, and evaporation rate w_α as the popularity of the book α, and by taking hydrodynamic limit of book titles $N \to \infty$, we can consider the set of sales ranks for the bookstore as a fluid system with infinitely many fluid components. Distinct book corresponds to distinct fluid comoponent, which is well-approximated to be infinitely many, and compared to the sales (popularity) of professional books which we normalize to be $w_\alpha = O(1)$, the sales of top-sales books are usually considered to be of order positive power of N in the framework of studies in economy, hence the set $\{w_\alpha\}$ becomes unbounded in the hydrodynamic limit. The dependence of w_α on position in this application corresponds to the possibility that winning high sales ranks attract people to enhance the sales further, a possibility which would be of commercial interest. In fact, we consider in [10] a stochastic ranking process with space-time dependent intensities, for which we have the solution considered in the present paper as a hydrodynamic limit.

As we will show in this paper, the solution U_α turns out to have a concise expression using the stochastic processes $N_{\theta,w,z}$ which we introduce in § 3,

$$U_\alpha(y,t) = \int_{z\in[y_0,1)} \mathrm{P}[\, N_{y_C,w_\alpha,z}(t) = N_{y_C,w_\alpha,z}(t_0)\,]\, \mu_{0,\alpha}(dz), \tag{10}$$

where $\mu_{0,\alpha}$ denotes initial spacial distribution of the fluid component α, and (y_0, t_0) is a initial/boundary point such that the characteristic curve starting from the point satisfies $y = y_C(t)$. (See (88) with (28).) The map G of (77) and (78) also has a corresponding expression (76). The processes $N_{\theta,w,z}$ may be regarded as generalizations of the Poisson process, but, in contrast to the Poisson process, lacks independent increment properties, resulting in the complexity of the solution. In the case of spacially independent evaporation rates, this underlying process reduces to the Poisson process, whose independent increment property implies simple explicit formula such as (7).

We mentioned earlier that the characteristic curve, which is the key quantity for a solution to a one dimensional first order quasilinear partial differential equation, will no longer be obtained by ordinary differential equation for the spacially dependent $\{w_\alpha\}$, and that it is determined as the fixed point to a map. The map is on the collection of the characteristic curves parametrized by its intersection point with the initial/boundary points, the totality of which we introduce as flow in § 4.

These notions were absent in the preceding work [11], and it is to clarify such mathematical structure of the solution that mainly motivated the present paper.

The plan of the paper is as follows. In § 2 we give the precise statement of our result, where we generalize (5) to allow also for uncountable number of fluid components, by generalizing the unknown functions to measure valued function. In § 3 we introduce the underlying stochastic process and its elementary properties, with which we give an expression of the solution in § 4 (see (88)), assuming existence of a fixed point to a certain map (Theorem 9). The existence of the fixed point is proved in § 5, which completes the existence proof of the solution. A uniqueness proof of the solution is given in § 6. As a remark concerning the condition in (9) on the spacial derivatives of w_α, we apply Schauder's fixed point theorem in § A to the map defined by (77) and (78), with the condition on derivative relaxed to a global bound on oscillation of w_α.

2 Main Result.

Throughout this paper we fix $T > 0$, $W \subset C^1([0,1]\times[0,T];[0,\infty))$ a set of non-negative valued C^1 functions on $[0,1]\times[0,T]$, and a Borel probability measure λ supported on the Borel measurable space $(W, \mathcal{B}(W))$. $\mathcal{B}(W)$ is the σ-algebra generated by open sets with the topology from the space of continuous functions $C^0([0,1]\times[0,T];[0,\infty)) \supset C^1([0,1]\times[0,T];[0,\infty))$ with the metric given by the supremum norm

$$\|w\|_{\mathrm{T}} = \sup_{(y,t)\in[0,1]\times[0,T]} |w(y,t)|. \tag{11}$$

We assume that

$$M_W := \int_W \|w\|_{\mathrm{T}}\, \lambda(dw) < \infty \tag{12}$$

and

$$C_W := \sup_{w\in W} \left\| \frac{\partial\, w}{\partial y} \right\|_{\mathrm{T}} < \infty \tag{13}$$

hold.

Denote the sets of 'initial ($t = 0$) points' in the space-time $[0,1]\times[0,T]$, the set of 'upper stream boundary ($y = 0$) points', and their union, the set of initial/boundary points, respectively by

$$\begin{aligned} &\Gamma_b = \{0\}\times[0,T] = \{(0,s) \mid 0 \leqq s \leqq T\}, \\ &\Gamma_i = [0,1]\times\{0\} = \{(z,0) \mid 0 \leqq z \leqq 1\}, \\ &\Gamma = \Gamma_b \cup \Gamma_i\,. \end{aligned} \tag{14}$$

For $t \in [0,T]$, denote the set of initial/boundary points up to time t by

$$\Gamma_t = \{(z,s)\in\Gamma \mid t_0 \leqq t\} = \Gamma_i \cup \{(0,t_0)\in\Gamma_b \mid 0 \leqq t_0 \leqq t\}, \tag{15}$$

and the set of admissible pairs of the initial/boundary point γ and time t by

$$\Delta_T := \{(\gamma,t)\in\Gamma_T\times[0,T] \mid \gamma\in\Gamma_t\}. \tag{16}$$

To state the initial condition, let $\mu_0 = \mu_0(dw\times dz)$ be a Borel probability measure on the measurable space $(W\times[0,1], \mathcal{B}(W\times[0,1]))$ of the product space of W and $[0,1]$. We assume that μ_0 is absolutely continuous with respect to the product measure $\lambda\times dz$, where dz denotes the standard Lebesgue measure on $\mathbb{R}$. Denote the density function by σ, namely,

$$\mu_0(dw\times dz) = \sigma(w,z)\,\lambda(dw)\,dz,\ \ (w,z)\in W\times[0,1]. \tag{17}$$

We assume $\mu_0(W\times dz) = dz$ and $\mu_0(dw\times[0,1)) = \lambda$, or equivalently, in terms of σ, we assume

$$\int_W \sigma(w,y)\,\lambda(dw) = 1,\ \ y\in[0,1], \tag{18}$$

and

$$\int_0^1 \sigma(w,z)\,dz = 1,\ \ w\in W. \tag{19}$$

We now state the main result we prove in this paper. For notational convenience, in the following, and throughout the paper, we use a notation such as $\mu(dw) = \nu(dw)$ to indicate the equality of measures, $\mu(B) = \nu(B)$, for all $B\in\mathcal{B}(W)$.

Theorem 1 *There exists a unique pair of functions y_C and $\mu_t(dw \times dz)$, where y_C is a function of $(\gamma, t) \in \Delta_T$ taking values in $[0,1]$, and $\mu_t(dw \times dz)$ is a function of $t \in [0,T]$ taking values in the probability measures on $W \times [0,1]$, such that the following hold.*

(i) $y_C((y_0, 0), t)$ *is non-decreasing in* y_0, $y_C((0, t_0), t)$ *is non-increasing in* t_0, *and* $y_C(\gamma, t)$ *is non-decreasing in* t.

(ii) $y_C(\gamma, t)$ *and* $\dfrac{\partial y_C}{\partial t}(\gamma, t)$ *are continuous, and for each* $t \in [0,T]$, $y_C(\cdot, t) : \ \Gamma_t \to [0,1]$ *is surjective.*

(iii) For all bounded measurable $h : \ W \to \mathbb{R}$, $\int_W h(w)\mu_t(dw \times [y,1))$ *is Lipschitz continuous in* $(y,t) \in [0,1] \times [0,T]$, *with Lipschitz constant uniform in* h *satisfying*

$$\sup_{w \in W} |h(w)| \leqq 1. \tag{20}$$

More precisely,

$$\begin{aligned} &\Big| \int_W h(w)\mu_{t'}(dw \times [y',1))) - \int_W h(w)\mu_t(dw \times [y,1)) \Big| \\ &\leqq |y' - y| + M_W e^{2C_W T} |t' - t|, \end{aligned} \tag{21}$$

for h *satisfying (20).*

(iv) The following equation of motion and initial and boundary conditions hold.

$$y_C((y_0, t_0), t_0) = y_0\,, \ (y_0, t_0) \in \Gamma, \qquad \text{and} \ \ \mu_0(dw \times dy) \ \ \text{as in (17)}, \tag{22}$$

$$\mu_t(dw \times [0,1)) = \lambda(dw), \quad t \in [0,T], \tag{23}$$

$$\mu_t(W \times [y,1)) = 1 - y, \quad (y,t) \in [0,1] \times [0,T], \tag{24}$$

$$\begin{aligned} &\mu_t(dw \times [y_C((y_0,t_0),t),1)) \\ &= \mu_{t_0}(dw \times [y_0,1)) - \int_{t_0}^t \int_{z \in [y_C((y_0,t_0),s),1)} w(z,s)\mu_s(dw \times dz)\, ds, \\ &\ ((y_0,t_0),t) \in \Delta_T\,. \end{aligned} \tag{25}$$

$\diamond$

Note that a substitution $y = y_C(\gamma, t)$ in (24) implies

$$y_C(\gamma, t) = 1 - \mu_t(W \times [y_C(\gamma,t),1)), \tag{26}$$

with which (25) and (24) imply

$$y_C(\gamma, t) = y_0 + \int_{t_0}^t \int_{W \times [y_C(\gamma,s),1)} w(z,s)\mu_s(dw \times dz)\, ds. \tag{27}$$

If W is a countable set $W = \{w_1, w_2, \ldots\}$, denote the distribution functions by

$$U_\alpha(y,t) = \mu_t(\{w_\alpha\} \times [y,1)). \tag{28}$$

Assume further that the functions $U_\alpha :\ [0,1]\times[0,T]\to[0,\infty)$ are in C^1. Differentiating (27) by t we reproduce (8) in § 1. Differentiating (25) by t, substituting (8), and then changing the notation from $y_C(\gamma,t)$ to y, we can eliminate the dependence on initial/boundary parameter γ, and we reproduce (5) in § 1. With $\lambda(\{w_\alpha\})=r_\alpha$, (23) and (24) respectively correspond to (2) and (3), and the conditions (12) and (13) imply (9). Thus Theorem 1 contains a solution to the problem introduced in § 1.

In Theorem 1 we claim differentiability for $y_C(\gamma,t)$ in t, while we formulated (25) so that differentiability assumptions on $U_\alpha(y,t)$ or $\mu_t(dw\times[y,1))$ are absent. In fact, at (y,t) with $y=y_C((0,0),t)$, where the characteristic curves starting at initial points $\gamma\in\Gamma_i$ and those starting at boundary points $\gamma\in\Gamma_b$ meet, the differentiability with respect to variables which cross the curve are lost in general. Loss of regularity across the characteristic curves is common for the quasilinear partial differential equations [2]. In terms of [2, §3.4], we may therefore say that Theorem 1 claims global existence of the Lipschitz solution (broad solution which is Lipschitz continuous) to the system of quasilinear partial differential equations (5), where we extended the definition of Lipschitz solution in [2, §3.4], to include the non-local (integration) terms, and also generalized the notion of domain of determinancy defined in [2, §3.4], which in the present case corresponds to $\{(y,t)\in[0,1]\times[0,\infty)\mid y\geqq y_C((0,0),t)\}$, to the boundary condition dependent domain $\{(y,t)\in[0,1]\times[0,\infty)\mid y<y_C((0,0),t)\}$. By formulating Theorem 1 in terms of probability measures on $W\times[0,1]$ we also included uncountably many components parametrized by the evaporation rates w, which are componentwise bounded but may be unbounded as a total fluid.

3 Point process with last-arrival-time dependent intensity.

Let $N=N(t)$, $t\geqq 0$, be a non-decreasing, right-continuous, non-negative integer valued stochastic process on a measurable space with $N(0)=0$ (point process, or counting process), and for each non-negative integer k define its k-th arrival time τ_k by

$$\tau_k=\inf\{t\geqq 0\mid N(t)\geqq k\},\quad k=1,2,\ldots,\quad \text{and}\quad \tau_0=0. \tag{29}$$

The arrival times τ_k are non-decreasing in k, because N is non-decreasing, and since N is also right-continuous, the arrival times are stopping times; if we denote the associated filtration by $\mathcal{F}_t=\sigma[N(s),\ s\leqq t]$, then $\{\tau_k\leqq t\}\in\mathcal{F}_t$, $t\geqq 0$.

Let ω be a non-negative valued bounded continuous function of (s,t) for $0\leqq s\leqq t$, and for $k=1,2,\ldots$ assume that

$$\mathrm{P}[\,t<\tau_k\mid\mathcal{F}_{\tau_{k-1}}\,]=\exp(-\int_{\tau_{k-1}}^t\omega(\tau_{k-1},u)\,du)\quad\text{on}\quad t\geqq\tau_{k-1}\,. \tag{30}$$

In particular, (30) with $k=1$ implies

$$\mathrm{P}[\,N(t)=0\,]=\mathrm{P}[\,\tau_1>t\,]=\exp(-\int_0^t\omega(0,u)\,du),\quad t\geqq 0. \tag{31}$$

Note that the function ω has different dependence on the variables from the evaporation rate function w in the other sections of this paper. (We will relate ω to w by (57) in § 4, namely, we will introduce an intensity function as a composite function of the evaporation rate function and a flow.) If ω is independent of the first variable, then (30) implies that N is the (inhomogeneous) Poisson process with intensity function ω. We are considering a generalization of the Poisson process such that the intensity function depends on the latest arrival time.

A process $N(t)$ satisfying (30) can be constructed in terms of a standard Poisson random measure on $[0,\infty)^2$. See [9, §1.2].

Let us turn to basic formulas to be used in this paper. For a continuously differentiable function f vanishing at ∞, integration by parts and the Fubini's theorem and (30) imply

$$\begin{aligned}
&\int_{\tau_{k-1}}^{\infty} f(t)\,\omega(\tau_{k-1},t)\,\exp(-\int_{\tau_{k-1}}^{t}\omega(\tau_{k-1},s)\,ds)\,dt \\
&= \int_{\tau_{k-1}}^{\infty} f'(t)\,\exp(-\int_{\tau_{k-1}}^{t}\omega(\tau_{k-1},s)\,ds)\,dt + f(\tau_{k-1}) \\
&= \int_{\tau_{k-1}}^{\infty} f'(t)\,\mathrm{P}[\,t<\tau_k \mid \mathcal{F}_{\tau_{k-1}}\,]\,dt + f(\tau_{k-1}) \\
&= \mathrm{E}[\int_{\tau_{k-1}}^{\infty} f'(t)\,\mathbf{1}_{t<\tau_k}\,dt \mid \mathcal{F}_{\tau_{k-1}}\,] + f(\tau_{k-1}) \\
&= \mathrm{E}[\int_{\tau_{k-1}}^{\tau_k} f'(t)\,dt \mid \mathcal{F}_{\tau_{k-1}}\,] + f(\tau_{k-1}) \\
&= \mathrm{E}[\,f(\tau_k) \mid \mathcal{F}_{\tau_{k-1}}\,],\quad k=1,2,\ldots.
\end{aligned} \tag{32}$$

Approximating by a series of smooth functions, (32) holds for any $f \in L_0([0,\infty))$, where $L_0([0,\infty))$ is the space of bounded measurable functions $f:\ [0,\infty)\to\mathbb{R}$ vanishing at infinity, equipped with the supremum norm.

For $t \geqq t_0$ put

$$\Omega(t_0,t) = \int_{t_0}^{t}\omega(t_0,u)\,du, \tag{33}$$

and define a linear map $A_\omega :\ L_0([0,\infty)) \to L_0([0,\infty))$ by

$$(A_\omega f)(t) = \int_t^{\infty} f(u)\,\omega(t,u)\,e^{-\Omega(t,u)}\,du. \tag{34}$$

Then (32) implies

$$\mathrm{E}[\,f(\tau_k)\mid\mathcal{F}_{\tau_{k-1}}\,] = (A_\omega f)(\tau_{k-1}),\quad f\in L_0([0,\infty)). \tag{35}$$

By induction and $\tau_0 = 0$ we have

$$\begin{aligned}
\mathrm{E}[\,f(\tau_k)\,] &= \mathrm{E}[\,\mathrm{E}[\,\cdots\mathrm{E}[\,\mathrm{E}[\,f(\tau_k)\mid\mathcal{F}_{\tau_{k-1}}\,]\mid\mathcal{F}_{\tau_{k-2}}\,]\cdots\mid\mathcal{F}_{\tau_1}\,]\,] \\
&= (A_\omega^k f)(0) \\
&= \int_{0\leqq u_1\leqq u_2\leqq\cdots\leqq u_k<\infty} f(u_k)\prod_{i=1}^{k}\omega(u_{i-1},u_i)\,e^{-\Omega(u_{i-1},u_i)}\,du_i\,,
\end{aligned} \tag{36}$$

where we put $u_0 = 0$ to simplify notations.

For example, by choosing $f(u) = \mathbf{1}_{u \leqq t}$, (35) implies

$$\text{(37)} \qquad \mathrm{P}[\ \tau_k \leqq t \mid \mathcal{F}_{\tau_{k-1}}\ ] = (1 - e^{-\Omega(\tau_{k-1},t)})\ \mathbf{1}_{\tau_{k-1} \leqq t}\,,$$

and (36) implies

$$\text{(38)} \qquad \mathrm{P}[\ N(t) \geqq k\ ] = \mathrm{E}[\ \tau_k \leqq t\ ] = (A_\omega^k f)(0),\ \ k = 1, 2, \ldots.$$

Then (37) and (36) with $f(u) = e^{-\Omega(u,t)}\ \mathbf{1}_{u \leqq t}$ imply

$$\text{(39)} \qquad \begin{aligned} \mathrm{P}[\ N(t) = k\ ] &= \mathrm{P}[\ \tau_k \leqq t < \tau_{k+1}\ ] \\ &= \mathrm{E}[\ \mathbf{1}_{\tau_k \leqq t}\,(1 - \mathrm{P}[\ \tau_{k+1} \leqq t \mid \mathcal{F}_{\tau_k}\ ])\ ] \\ &= \mathrm{E}[\ f(\tau_k)\ ] = (A_\omega^k\, f)(0), \quad k \in \mathbb{Z}_+,\ \ t > 0. \end{aligned}$$

Hence, as in the last line of (36),

$$\text{(40)} \qquad \begin{aligned} &\mathrm{P}[\ N(t) = k\ ] = (A_\omega^k f)(0) \\ &= \begin{cases} \displaystyle\int_{\substack{0 \leqq u_1 \leqq \cdots \\ \leqq u_k \leqq t}} e^{-\Omega(u_k,t)} \prod_{i=1}^{k} \omega(u_{i-1}, u_i) e^{-\Omega(u_{i-1},u_i)} du_i, & k \in \mathbb{N}, \\ e^{-\Omega(0,t)}, & k = 0. \end{cases} \end{aligned}$$

In particular, $\mathrm{P}[\ N(t) \geqq 0\ ] = 1$ implies a sum rule

$$\text{(41)} \qquad \begin{aligned} &e^{-\Omega(0,t)} \\ &+ \sum_{k=1}^{\infty} \int_{0 \leqq u_1 \leqq u_2 \leqq \cdots \leqq u_k \leqq t} e^{-\Omega(u_k,t)}\ \prod_{i=1}^{k} \omega(u_{i-1}, u_i)\, e^{-\Omega(u_{i-1},u_i)}\, du_i \\ &= 1, \quad t > 0, \end{aligned}$$

where $u_0 = 0$, as in (36).

Similarly, given s and t satisfying $0 \leqq s < t$, the probability that there is no arrival in the interval $(s, t]$ is

$$\text{(42)} \qquad \begin{aligned} \mathrm{P}[\ N(t) = N(s)\ ] &= \sum_{k=0}^{\infty} \mathrm{P}[\ N(t) = N(s) = k\ ] \\ &= \sum_{k=0}^{\infty} \mathrm{P}[\ \tau_k \leqq s,\ \ t < \tau_{k+1}\ ] \\ &= \sum_{k=0}^{\infty} \mathrm{E}[\ \mathbf{1}_{\tau_k \leqq s}\,(1 - \mathrm{P}[\ \tau_{k+1} \leqq t \mid \mathcal{F}_{\tau_k}\ ])\ ] \\ &= \sum_{k=0}^{\infty} \mathrm{E}[\ \mathbf{1}_{\tau_k \leqq s}\ e^{-\Omega(\tau_k,t)}\ ]. \end{aligned}$$

With $f(u) = \mathbf{1}_{u \leqq s}\ e^{-\Omega(u,t)}$ in (36), we also have an explicit formula

$$\text{(43)} \qquad \begin{aligned} &\mathrm{P}[\ N(t) = N(s) = k\ ] = \mathrm{E}[\ \mathbf{1}_{\tau_k \leqq s}\ e^{-\Omega(\tau_k,t)}\ ] = (A_\omega^k\, f)(0) \\ &= \begin{cases} e^{-\Omega(0,t)}, & k = 0, \\ \displaystyle\int_{\substack{0 \leqq u_1 \leqq u_2 \leqq \cdots \\ \leqq u_k \leqq s}} e^{-\Omega(u_k,t)} \prod_{i=1}^{k} \omega(u_{i-1}, u_i) e^{-\Omega(u_{i-1},u_i)} du_i, & k \in \mathbb{N}, \end{cases} \end{aligned}$$

for $t \geqq s > 0$, where $u_0 = 0$, as in (36). Note that the explicit formula implies that the quantity $\mathrm{P}[\ N(t) = N(s)\ ]$ is C^1 in s and t. The following property relates the s and t dependences of this quantity.

Proposition 2 *For $k = 1, 2, \ldots$,*

$$\text{(44)} \qquad \begin{aligned} &\frac{\partial}{\partial t} \mathrm{P}[\ N(t) = N(s) = k\ ] \\ &= -\int_0^s \omega(u,t)\, \frac{\partial}{\partial u} \mathrm{P}[\ N(t) = N(u) = k\ ]\, du, \\ &0 \leqq s < t. \end{aligned}$$

◇

Proof. First we prove

$$\text{(45)} \qquad \mathrm{E}[\ f(\tau_k)\, g(\tau_k)\ \mathbf{1}_{\tau_k \leqq s}\ ] = \int_0^s f(u)\, Q'(u)\, du$$

for locally bounded and measurable f and g such that

$$\text{(46)} \qquad Q(s) := \mathrm{E}[\ g(\tau_k)\ \mathbf{1}_{\tau_k \leqq s}\ ]$$

is absolutely continuous with respect to the Lebesgue measure (so that the derivative Q' almost surely exists). Approximating by a series of smooth functions, it suffices to prove (45) for $f \in C^1$. By Fubini's theorem and partial integration, and noting that $\tau_k > 0$ for $k > 0$ implies $Q(0) = 0$,

$$\begin{aligned} \mathrm{E}[\ f(\tau_k)\, g(\tau_k)\ \mathbf{1}_{\tau_k \leqq s}\ ] &= \mathrm{E}[\ \Big(-\int_{\tau_k}^s f'(u)\, du + f(s) \Big)\, g(\tau_k)\ \mathbf{1}_{\tau_k \leqq s}\ ] \\ &= f(s)\, Q(s) - \int_0^s f'(u)\, \mathrm{E}[\ \mathbf{1}_{\tau_k \leqq u}\ g(\tau_k)\ ]\, du \\ &= f(s)\, Q(s) - \int_0^s f'(u)\, Q(u)\, du \\ &= \int_0^s f(u)\, Q'(u)\, du. \end{aligned}$$

Thus (45) is proved.

Now for a positive integer k, let $f(u) = \omega(u,t)$ and $g(u) = e^{-\Omega(u,t)}$ in (45). Note that for this choice (43) implies

$$Q(s) = \mathrm{E}[\ e^{-\Omega(\tau_k,t)}\ \mathbf{1}_{\tau_k \leqq s}\ ] = \mathrm{P}[\ N(t) = N(s) = k\ ].$$

Then (45) implies

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{P}[\,N(t)=N(s)=k\,] &= -\mathrm{E}[\,\omega(\tau_k,t)\,e^{-\Omega(\tau_k,t)}\ \mathbf{1}_{\tau_k\leqq s}\ ] \\ &= -\mathrm{E}[\,f(\tau_k)\,g(\tau_k)\ \mathbf{1}_{\tau_k\leqq s}\ ] = -\int_0^s f(u)\,Q'(u)\,du \\ &= -\int_0^s \omega(u,t)\,\frac{\partial}{\partial u}\mathrm{P}[\,N(t)=N(u)=k\,]\,du,\end{aligned}$$

which proves (44). □

Using (41), (43), and Proposition 2, it is also easy to deduce bounds on the derivatives.

Corollary 3 *Put* $\|\omega\| = \sup_{0\leqq s\leqq t\leqq T}|\omega(s,t)|$. *Then*

$$\begin{aligned}0 &\leqq -\frac{\partial}{\partial t}\mathrm{P}[\,N(t)=N(s)\,] \\ &\leqq \|\omega\|\,(\mathrm{P}[\,N(t)=N(s)\,]-\mathrm{P}[\,N(t)=N(0)\,]) \leqq \|\omega\|\,, \\ 0 &\leqq \frac{\partial}{\partial s}\mathrm{P}[\,N(t)=N(s)\,] \leqq \|\omega\|\,\mathrm{P}[\,N(t)=N(s)\,] \leqq \|\omega\|\,,\end{aligned} \tag{47}$$

for $0\leqq s<t\leqq T$. ◇

So far we prepared basic properties for the probabilities of $N(t)$. We can also find explicit formula for expectations. As an example, note that for $t>s$,

$$\begin{aligned}\mathrm{E}[\,N(t)-N(s)\,] &= \sum_{k=1}^{\infty}\mathrm{P}[\,N(t)\geqq N(s)+k\,] \\ &= \sum_{k=1}^{\infty}\sum_{\ell=0}^{\infty}\mathrm{P}[\,\tau_{k+\ell}\leqq t,\ \tau_\ell\leqq s<\tau_{\ell+1}\,] \\ &= \sum_{k=1}^{\infty}\sum_{\ell=0}^{\infty}\mathrm{P}[\,\tau_{k+\ell}\leqq t,\ \tau_\ell\leqq s\,] - \sum_{k=1}^{\infty}\sum_{\ell'=1}^{\infty}\mathrm{P}[\,\tau_{k+\ell'-1}\leqq t,\ \tau_{\ell'}\leqq s\,] \\ &= \sum_{k=1}^{\infty}\sum_{\ell=0}^{\infty}\mathrm{P}[\,\tau_{k+\ell}\leqq t,\ \tau_\ell\leqq s\,] - \sum_{k'=0}^{\infty}\sum_{\ell'=1}^{\infty}\mathrm{P}[\,\tau_{k'+\ell'}\leqq t,\ \tau_{\ell'}\leqq s\,] \\ &= \sum_{k\geqq 1}\mathrm{P}[\,s<\tau_k\leqq t\,].\end{aligned}$$

Applying (36) with $f(u)=\mathbf{1}_{s<u\leqq t}$, we arrive at

$$\begin{aligned}\mathrm{E}[\,N(t)-N(s)\,] &= \sum_{k=1}^{\infty}(A_\omega^k\,f)(0) \\ &= \sum_{k=1}^{\infty}\int_{\substack{0\leqq u_1\leqq\cdots\leqq u_k\leqq t \\ s<u_k}}\prod_{i=1}^{k}\omega(u_{i-1},u_i)e^{-\Omega(u_{i-1},u_i)}du_i\,.\end{aligned} \tag{48}$$

Before closing this section, we note relations of the distribution of the process N with that of the Poisson process. In contrast to the Poisson process, the process N is in general not of independent increment, i.e., $N(t) - N(s)$ depends in general on $\{N(u),\ u \leqq s\}$. In fact, (43) depends on the intensity function ω at times before s. We however have a following Poisson bound for the expectations. In analogy to (33), put

$$\bar{\omega}(t) = \max_{s\in[0,t]} \omega(s,t), \quad \bar{\Omega}(t_0,t) = \int_{t_0}^{t} \bar{\omega}(u)du,\ t \geqq t_0 \geqq 0, \tag{49}$$

and in analogy to (34), define a linear map $\bar{A}_\omega :\ L_0([0,\infty)) \to L_0([0,\infty))$ by

$$(\bar{A}_\omega f)(t) := \int_t^\infty f(u)\,\bar{\omega}(u)\,e^{-\bar{\Omega}(t,u)}\,du. \tag{50}$$

Lemma 4 *If $g \in L_0([0,\infty))$ is non-increasing and non-negative valued,*

$$(A_\omega^k\, g)(t) \leqq (\bar{A}_\omega^k\, g)(t) = \int_t^\infty g(s)\bar{\omega}(s)\frac{\bar{\Omega}(t,s)^{k-1}}{(k-1)!}\,e^{-\bar{\Omega}(t,s)}\,ds,$$

holds for all $t \geqq 0$ and $k \in \mathbb{N}$. ◇

Proof. Note that the definition (49) implies

$$\bar{\Omega}(u,s) \geqq \Omega(u,s), \quad s \geqq u \geqq 0. \tag{51}$$

The definition (34), integration by parts, (51), the assumptions on g, and (50) imply

$$(A_\omega\, g)(t) = g(t) - \int_{s\in[t,\infty)} e^{-\Omega(t,s)}\,dg(s) \leqq g(t) - \int_{s\in[t,\infty)} e^{-\bar{\Omega}(t,s)}\,dg(s) = (\bar{A}_\omega\, g)(t),$$

which proves the claim for $k = 1$.

The proof for general k follows by induction in k. By the definition (50), $\bar{A}_\omega\, g$ is non-increasing and non-negative valued, and satisfies $\lim_{u\to\infty} \bar{A}_\omega\, g(u) = 0$, if g does. Hence by induction and the claim for $k = 1$, we have $(A_\omega^k\, g)(u) \leqq (\bar{A}_\omega^k\, g)(u)$ for all k and u.

The claimed explicit formula for $(\bar{A}_\omega^k\, g)(u)$ follows also by induction in k. In fact, noting $\bar{\Omega}(u,s) + \bar{\Omega}(s,v) = \bar{\Omega}(u,v)$, it follows that

$$\begin{aligned}
(\bar{A}_\omega^{k+1}\, g)(u) &= \int_u^\infty (\bar{A}_\omega^k\, g)(s)\,\bar{\omega}(s)\,e^{-\bar{\Omega}(u,s)}ds \\
&= \int_{u\leqq s\leqq v} g(v)\bar{\omega}(v)\frac{\bar{\Omega}(s,v)^{k-1}}{(k-1)!}\,e^{-\bar{\Omega}(s,v)}\,\bar{\omega}(s)\,e^{-\bar{\Omega}(u,s)}\,dv\,ds \\
&= \int_u^\infty g(v)\bar{\omega}(v)\frac{\bar{\Omega}(u,v)^k}{k!}\,e^{-\bar{\Omega}(u,v)}\,dv,
\end{aligned}$$

which proves the formula. □

Theorem 5 *It holds that*

$$\mathrm{E}[\,N(t)\,(N(t)-1)\,(N(t)-2)\cdots(N(t)-p+1)\,] \leqq (\bar{\Omega}(0,t))^p,$$

for all positive integer p and $t \geqq 0$. ◇

Proof. Using an elementary formula

$$\mathrm{E}[\, h(N(t))\,] = h(0) + \sum_{k\geqq 1} (h(k) - h(k-1))\,\mathrm{P}[\, N(t) \geqq k\,]$$

with $h(k) = k(k-1)\cdots(k-p+1)$, we have, from (38) with $f(u) = \mathbf{1}_{u\leqq t}$,

$$\mathrm{E}[\, N(t)\,(N(t)-1)\,(N(t)-2)\cdots(N(t)-p+1)\,] = p\sum_{k\geqq p} \frac{(k-1)!}{(k-p)!}(A_\omega^k\, f)(0). \tag{52}$$

Note that $f(u) = \mathbf{1}_{u\leqq t}$ is non-increasing and non-negative valued, and satisfies $\lim_{u\to\infty} f(u) = 0$. Lemma 4 then implies

$$\begin{aligned}&\mathrm{E}[\, N(t)\,(N(t)-1)\,(N(t)-2)\cdots(N(t)-p+1)\,]\\ &\leqq p\sum_{k\geqq p} \frac{(k-1)!}{(k-p)!}(\bar{B}^k\, f)(0) = p\int_0^t \bar{\omega}(s)\bar{\Omega}(0,s)^{p-1}\, ds = (\bar{\Omega}(0,t))^p,\end{aligned}$$

which proves the claim. □

We also remark that various explicit formulas using multiple integeration in this section reduces to a simple formula when the process N is the Poisson process (with intensity function depending only on 1 variable as for $\bar{\omega}(t)$) using an elementary formula

$$\int_{0\leqq u_1\leqq u_2\leqq\cdots\leqq u_k\leqq s} \prod_{i=1}^k f(u_i) du_1\, du_2 \ldots du_k = \frac{1}{k!}\Big(\int_0^s f(v)dv\Big)^k, \tag{53}$$

valid for any integrable function $f:\ \mathbb{R}\to\mathbb{R}$, $s\geqq 0$, and $k = 1, 2, \ldots$, which can be proved by induction in k in a similar way as in the proof of Lemma 4. For example, if the intensity function of $N(t) = \bar{N}(t)$ is $\omega(s,t) = \bar{\omega}(t)$, then the right hand side of (43) is simplified, using (53), as

$$\mathrm{P}[\, \bar{N}(t) = \bar{N}(s)\,] = \Big(1 + \sum_{k\geqq 1} \frac{1}{k!}\,\Omega(0,s)^k\Big)\, e^{-\Omega(0,t)} = e^{-\Omega(s,t)},$$

which reproduces a standard result for the Poisson distribution. In the general case of processes we consider in this paper, such simple relations to Poisson distributions are absent.

4 Flows and construction of solution.

The key quantities for the solution to the functional equations in Theorem 1 are the characteristic curves y_C and the associated measure $\varphi(dw,\gamma,t) = \mu_t(dw\times[y_C(\gamma,t),1))$. We will find y_C as a unique solution to a non-linear map on a space Θ_T of flows, a non-decreasing function in time t and in initial/boundary points $\gamma\in\Gamma$. To simplify the definition of Θ_T we first define a total order $\succeq$ on Γ by

$$\begin{aligned}&s\leqq t,\ z\leqq y\\ &\Leftrightarrow\ (0,T)\succeq(0,t)\succeq(0,s)\succeq(0,0)\succeq(z,0)\succeq(y,0)\succeq(1,0).\end{aligned} \tag{54}$$

We now define the set of flows Θ_T on $[0,1]\times[0,T]$ by

$$\begin{aligned}\Theta_T := \{\theta :\ \Delta_T \to [0,1] \mid \theta((y_0,t_0),t_0) = y_0,\ (y_0,t_0)\in\Gamma_T,\\ \text{continuous, surjective and non-increasing in } \gamma \text{ for each } t,\\ \text{non-decreasing in } t \text{ for each } \gamma\ \}.\end{aligned} \tag{55}$$

For example,

$$\theta((1,0),t) = 1, \quad t\in[0,T],\ \theta\in\Theta_T. \tag{56}$$

Let W, the set of evaporation rates, be as in Theorem 1, and let $\theta\in\Theta_T$. For each $w\in W$ and $z\in[0,1)$ define $\omega=\omega_{\theta,w,z}$, a non-negative valued continuous function of (s,t) satisfying $0\leqq s\leqq t\leqq T$, by

$$\omega_{\theta,w,z}(s,t) = \begin{cases} w(\theta((z,0),t),t), & \text{if } s=0,\\ w(\theta((0,s),t),t), & \text{if } s>0.\end{cases} \tag{57}$$

Note that $\omega_{\theta,w,z}$ is independent of z if $s>0$. Let $\{N_{\theta,w,z} \mid z\in[0,1),\ w\in W\}$ be a set of processes, with each $N_{\theta,w,z}$ being a point process N introduced in § 3 with the intensity function in (30) determined by $\omega=\omega_{\theta,w,z}$. The quantity in (33) for the choice (57) is

$$\begin{aligned}\Omega_{\theta,w,z}(0,t) &= \int_0^t w(\theta((z,0),u),u)\,du,\\ \Omega_{\theta,w}(s,t) &= \int_s^t w(\theta((0,s),u),u)\,du, \quad 0<s\leqq t.\end{aligned} \tag{58}$$

Let μ_0 be as in Theorem 1, and define a function $\varphi_\theta(dw,\gamma,t)$ on $(\gamma,t)\in\Delta_T$ taking values in the measures on W, by

$$\begin{aligned}\varphi_\theta(dw,\gamma,t) &= \int_{z\in[y_0,1)} \mathrm{P}[\ N_{\theta,w,z}(t) = N_{\theta,w,z}(t_0)\ ]\,\mu_0(dw\times dz),\\ \gamma &= (y_0,t_0)\in\Gamma,\ (\gamma,t)\in\Delta_T\,.\end{aligned} \tag{59}$$

The explicit form for (59) is simple for $\gamma=(y_0,0)\in\Gamma_i$, because $N_{\theta,w,z}(0)=0$, and (43) with $k=0$ imply

$$\varphi_\theta(dw,(y_0,0),t) = \int_{z\in[y_0,1)} e^{-\Omega_{\theta,w,z}(0,t)}\,\mu_0(dw\times dz). \tag{60}$$

For φ_θ in (59) define $\dfrac{\partial\varphi_\theta}{\partial\gamma}$, a measure valued function on Δ_T, by

$$\frac{\partial\varphi_\theta}{\partial\gamma}(dw,\gamma,t) = \begin{cases} -\dfrac{\partial\varphi_\theta}{\partial z}(dw,(z,0),t), & \text{if } \gamma=(z,0)\in\Gamma_i\,,\\ \dfrac{\partial\varphi_\theta}{\partial u}(dw,(0,u),t), & \text{if } \gamma=(0,u)\in\Gamma_b\,.\end{cases} \tag{61}$$

We keep non-negativity of the defined measure in determining the sign. Explicit calculation of the derivative at $\gamma\in\Gamma_i$ is straightforward from (60) and (17). The derivative at $\gamma=$

$(0,u) \in \Gamma_b$ is also calculated explicitly using (43) and (59), which is

$$
\begin{aligned}
&\frac{\partial \varphi_\theta}{\partial u}(dw,(0,u),t)\\
&= \int_{z\in[0,1)} \Big(w(\theta((z,0),u)\, e^{-\Omega_{\theta,w,z}(0,u)} \\
&\qquad + \sum_{k\geqq 2} \int_{0<u_1\leqq u_2\leqq\cdots\leqq u_{k-1}\leqq u} w(\theta((z,0),u_1)\, e^{-\Omega_{\theta,w,z}(0,u_1)}\, du_1 \\
&\qquad\quad \prod_{i=2}^{k-1}\Big(w(\theta((0,u_{i-1}),u_i)\, e^{-\Omega_{\theta,w}(u_{i-1},u_i)}\, du_i \Big) \\
&\qquad\quad w(\theta((0,u_{k-1}),u)\, e^{-\Omega_{\theta,w}(u_{k-1},u)} \Big)\, e^{-\Omega_{\theta,w}(u,t)}\, \mu_0(dw\times dz),
\end{aligned}
\tag{62}
$$

where we also used the notations (57) and (58) to make the z and u dependence explicit. Note that $w \in W$ are non-negative, hence $e^{-\Omega_{\theta,w}(u,t)} \leqq 1$. This and the sum rule (41) (with the replacements $t = u$ and $k = k' - 1$), with (12), (17), and (19) imply

$$
\begin{aligned}
&\frac{\partial \varphi_\theta}{\partial u}(W,(0,u),t)\\
&\leqq \int_{B\times[0,1)} \|w\|_{\mathrm{T}}\ \mu_0(dw\times dz) = \int_B \|w\|_{\mathrm{T}}\, \lambda(dw) = M_W < \infty,
\end{aligned}
$$

hence, $\dfrac{\partial \varphi_\theta}{\partial u}$ is well-defined.

For $f:\ \Gamma_T \to \mathbb{R}$ and a Borel subset $A \subset \Gamma_T$, define $\int_A f(\gamma)\, d\gamma$, a line integral on Γ_T, by

$$
\int_A f(\gamma)\, d\gamma = \int_{A_i} f(z,0)\, dz + \int_{A_b} f(0,u)\, du,
\tag{63}
$$

where, $A_i = \{z \in [0,1) \mid (z,0) \in A\}$ and $A_b = \{u \in [0,T] \mid (0,u) \in A\}$, and the integration in the right hand side of (63) are the standard one dimensional integrations.

Proposition 6 *It holds that*

$$
0 \leqq \int_{\gamma'\in\Gamma_t} \int_W \|w\|_{\mathrm{T}}\ \frac{\partial \varphi_\theta}{\partial \gamma}(dw,\gamma',t)\, d\gamma' \leqq M_W\, e^{2C_W t},
\tag{64}
$$

for all $t \in [0,T]$. ◇

Remark. Note the extra $\|w\|_{\mathrm{T}}$ in the integrand of (64). We are allowing W to contain unbounded functions w, so the finiteness of (64) is harder than that of (62). We use the condition (13) as well as (12) to prove (64).

Proposition 6 implies, in particular, that the integration (69) introduced later, is uniformly bounded in $(\gamma,t) \in \Delta_T$ and $B \in \mathcal{B}(W)$. ◇

Proof of Proposition 6. By the definitions (61) and (63),

$$\begin{aligned}&\int_{\gamma'\in\Gamma_t}\int_W \|w\|_{\mathrm{T}}\,\frac{\partial\,\varphi_\theta}{\partial\gamma}(dw,\gamma',t)\,d\gamma' \\ &= -\int_0^1\int_W \|w\|_{\mathrm{T}}\,\frac{\partial\,\varphi_\theta}{\partial z}(dw,(z,0),t)\,dz \\ &\quad+\int_0^t\int_W \|w\|_{\mathrm{T}}\,\frac{\partial\,\varphi_\theta}{\partial u}(dw,(0,u),t)\,du.\end{aligned}\tag{65}$$

The first term on the right hand side is explicitly calculated using (60) and (17). Using also $e^{-\Omega_{\theta,w,z}(0,t)} \leqq 1$, (19), and (12), we have an estimate

$$\begin{aligned}0 &\leqq -\int_0^1\int_W \|w\|_{\mathrm{T}}\,\frac{\partial\,\varphi_\theta}{\partial z}(dw,(z,0),t)\,dz \\ &= \int_{W\times[0,1)} \|w\|_{\mathrm{T}}\,e^{-\Omega_{\theta,w,z}(0,t)}\,\mu_0(dw\times dz) \\ &\leqq M_W < \infty.\end{aligned}\tag{66}$$

Next, for $t \geqq s \geqq 0$ and $w \in W$, put

$$\tilde\Omega_w(s,t) = \int_s^t w(1,u)\,du.\tag{67}$$

Then the condition (13) and the fact that θ takes values in $[0,1]$ imply

$$\begin{aligned}&|w(\theta(\gamma,t),t) - w(1,t)| \leqq C_W\,, \\ &|\Omega_{\theta,w}(s,t) - \tilde\Omega_w(s,t)| \leqq C_W\,(t-s), \quad\text{and}\quad |\Omega_{\theta,w,z}(0,t) - \tilde\Omega_w(0,t)| \leqq C_W\,t, \\ &(\gamma,t)\in\Delta_T\,,\ 0<s\leqq t,\ w\in W.\end{aligned}\tag{68}$$

Using (68) and (67) in (62), and then using (17) and (19), we have an estimate

$$\begin{aligned}&\int_0^t\int_W \|w\|_{\mathrm{T}}\,\frac{\partial\,\varphi_\theta}{\partial u}(dw,(0,u),t)\,du \\ &\leqq \int_W \|w\|_{\mathrm{T}}\,e^{-\tilde\Omega_w(0,t)+C_W\,t}\int_0^t (w(1,u)+C_W)\Big(1 \\ &\qquad + \sum_{k\geqq 1}\int_{0<u_1\leqq\cdots\leqq u_k\leqq u}\prod_{i=1}^k((w(1,u_i)+C_W)\,du_i)\Big)\,du\,\lambda(dw).\end{aligned}$$

The estimate is now reduced to that for the Poisson processes. and (53) implies

$$\begin{aligned}&\int_0^t\int_W \|w\|_{\mathrm{T}}\,\frac{\partial\,\varphi_\theta}{\partial u}(dw,(0,u),t)\,du \\ &\leqq \int_W \|w\|_{\mathrm{T}}\,e^{-\tilde\Omega_w(0,t)+C_W t}\int_0^t e^{\tilde\Omega_w(0,u)+C_W u}(w(1,u)+C_W u)\,du\,\lambda(dw) \\ &= \int_W \|w\|_{\mathrm{T}}\,e^{-\tilde\Omega_w(0,t)+C_W t}\Big[e^{\tilde\Omega_w(0,u)+C_W u}\Big]_{u=0}^{u=t}\lambda(dw) \\ &\leqq e^{2C_W t}\int_W \|w\|_{\mathrm{T}}\,\lambda(dw) \leqq M_W\,e^{2C_W t}.\end{aligned}$$

This proves (64). □

Proposition 7 *It holds that*

$$\frac{\partial\,\varphi_\theta}{\partial t}(B,\gamma,t) = -\int_{\gamma\succeq\gamma'}\int_B w(\theta(\gamma',t),t)\,\frac{\partial\,\varphi_\theta}{\partial\gamma}(dw,\gamma',t)\,d\gamma', \qquad (69)$$
$$(\gamma,t)\in\Delta_T,\;\; B\in\mathcal{B}(W),$$

where $\gamma\succeq\gamma'$ is defined in (54). ◇

Remark. If W consists of functions with no spatial dependence, namely, if $w(y,t)=w(1,t)$, then the factor w in the integrand of the right hand side of (69) is constant for γ' integration, and we have integration after differentiation, so that the right hand side is simplified as $-w(1,t)\varphi_\theta(B,\gamma,t)$, and the equation is solved easily, as remarked below (6) in § 1. ◇

Proof of Proposition 7. Consider first the case $\gamma=(y_0,0)\in\Gamma_i$. The explicit form (60), together with the definitions (61), (63), and (17), implies

$$\begin{aligned}
&-\int_{\gamma\succeq\gamma'}\int_B w(\theta(\gamma',t),t)\,\frac{\partial\,\varphi_\theta}{\partial\gamma}(dw,\gamma',t)\,d\gamma' \\
&= -\int_{z\in[y_0,1)}\int_B w(\theta((z,0),t),t)\,e^{-\Omega_{\theta,w,z}(0,t)}\,\sigma(w,z)\lambda(dw)\,dz \\
&= -\int_{z\in[y_0,1)}\int_B w(\theta((z,0),t),t)\,e^{-\Omega_{\theta,w,z}(0,t)}\,\mu_0(dw\times dz) \\
&= \frac{\partial\,\varphi_\theta}{\partial t}(B,(y_0,0),t),
\end{aligned}$$

which proves (69) for $\gamma\in\Gamma_i$.

To prove (69) for $\gamma\in\Gamma_b$, put, for $k\in\mathbb{Z}_+$,

$$\begin{aligned}
&\varphi_\theta^{(k)}(dw,(y_0,t_0),t) \\
&= \int_{z\in[y_0,1)} \mathrm{P}[\;N_{\theta,w,z}(t)=N_{\theta,w,z}(t_0)=k\;]\,\mu_0(dw\times dz).
\end{aligned} \qquad (70)$$

Then (59) implies

$$\varphi_\theta = \sum_{k=0}^{\infty}\varphi_\theta^{(k)}. \qquad (71)$$

We will prove, for $\gamma=(0,t_0)\in\Gamma_b$

$$\frac{\partial\,\varphi_\theta^{(k)}}{\partial t}(B,\gamma,t) = -\int_{\gamma\succeq\gamma'}\int_B w(\theta(\gamma',t),t)\,\frac{\partial\,\varphi_\theta^{(k)}}{\partial\gamma}(dw,\gamma',t)\,d\gamma', \qquad (72)$$
$$(\gamma,t)\in\Delta_T,\;\; B\in\mathcal{B}(W),$$

for all $k\in\mathbb{Z}_+$, where differentiation and integration with respect to γ are defined in accordance with (61) and (63). Then (71) and (72) prove (69). The changes in the order of series and integration and differentiation causes no problem, because all the terms and integrands are non-negative and the results of summation and integration are bounded by Proposition 6.

Consider first the case $k = 0$. Then (43) implies

$$\varphi_\theta^{(0)}(dw, (y_0, t_0), t) = \int_{z \in [y_0, 1)} e^{-\Omega_{\theta,w,z}(0,t)}\, \mu_0(dw \times dz). \tag{73}$$

Note that this is independent of t_0. Hence (61) and (17) imply

$$\frac{\partial \varphi_\theta^{(0)}}{\partial \gamma}(dw, \gamma', t) = \begin{cases} e^{-\Omega_{\theta,w,z}(0,t)} \sigma(w, z)\lambda(dw), & \text{if } \gamma' = (z, 0) \in \Gamma_i, \\ 0, & \text{if } \gamma' = (0, u) \in \Gamma_b. \end{cases} \tag{74}$$

For the case $\gamma = (0, t_0) \in \Gamma_b$,

$$\gamma \succeq \gamma' \;\Leftrightarrow\; \gamma' = (0, u),\; 0 \leqq u \leqq t_0, \quad \text{or} \quad \gamma' = (z, 0),\; 0 \leqq z \leqq 1.$$

The contribution, however, to (63) from the integration along Γ_b vanishes because the integrand (74) is 0 on Γ_b. Hence, (74), (63), (17), and (73) imply

$$\begin{aligned}
&\int_{\gamma \succeq \gamma'} \int_B w(\theta(\gamma', t), t)\, \frac{\partial \varphi_\theta^{(0)}}{\partial \gamma}(dw, \gamma', t)\, d\gamma' \\
&= \int_{z \in [0,1)} \int_B w(\theta((z, 0), t), t)\, e^{-\Omega_{\theta,w,z}(0,t)}\, \sigma(w, z)\lambda(dw)\, dz \\
&= \int_{B \times [0,1)} w(\theta((z, 0), t), t)\, e^{-\Omega_{\theta,w,z}(0,t)}\, \mu_0(dw \times dz) \\
&= -\frac{\partial \varphi_\theta^{(0)}}{\partial t}(B, \gamma, t),
\end{aligned}$$

which proves (72) for $k = 0$ and $\gamma \in \Gamma_b$.

To consider the case $k > 0$ and $\gamma = (0, t_0) \in \Gamma_b$, (70), Proposition 2, and (57) imply

$$\begin{aligned}
&\frac{\partial \varphi_\theta^{(k)}}{\partial t}(B, (0, t_0), t) \\
&= -\int_{B \times [0,1)} \int_0^{t_0} w(\theta(0, u), t), t) \\
&\qquad \times \frac{\partial}{\partial u} \mathrm{P}[\, N_{\theta,w,z}(t) = N_{\theta,w,z}(u) = k \,]\, du\, \mu_0(dw \times dz) \\
&= -\int_0^{t_0} \int_{w \in B} w(\theta(0, u), t), t)\, \frac{\partial \varphi_\theta^{(k)}}{\partial u}(dw, (0, u), t)\, du, \quad 0 < t_0 < t.
\end{aligned}$$

The contribution to (63) from $\gamma' \in \Gamma_i$ vanishes for the case $k > 0$ because $N_{\theta,w,z}(0) = 0$ and (70) then imply

$$\varphi_\theta^{(k)}(dw, \gamma, t) = 0, \quad \gamma \in \Gamma_i\,,\; t \geqq 0,\; k = 1, 2, \ldots, \tag{75}$$

hence (72) is proved for this case. This completes a proof of (72), and (69) follows. □

Next put

(76)
$$\begin{aligned}
G(\theta)(\gamma,t) &= 1-\varphi_\theta(W,\gamma,t)\\
&= 1-\int_{W\times[y_0,1)} \mathrm{P}[\, N_{\theta,w,z}(t)=N_{\theta,w,z}(t_0)\,]\,\mu_0(dw\times dz)\\
&= y_0+\int_{W\times[0,1)} \mathrm{P}[\, N_{\theta,w,z}(t)>N_{\theta,w,z}(t_0)\,]\,\mu_0(dw\times dz)\\
\gamma &= (y_0,t_0)\in\Gamma,\ \ (\gamma,t)\in\Delta_T\,.
\end{aligned}$$

With (17) and (43), we have an explicit formula

(77)
$$\begin{aligned}
&G(\theta)((y_0,0),t) = 1-\int_{W\times[y_0,1)} e^{-\Omega_{\theta,w,z}(0,t)}\,\sigma(w,z)\,\lambda(dw)\,dz,\\
&\gamma=(y_0,0)\in\Gamma_i\,,
\end{aligned}$$

and

(78)
$$\begin{aligned}
&G(\theta)((0,t_0),t)\\
&\quad = 1-\int_{W\times[0,1)} e^{-\Omega_{\theta,w,z}(0,t)}\,\sigma(w,z)\,\lambda(dw)\,dz\\
&\qquad -\int_{W\times[0,1)}\sum_{k\geqq 1}\int_{0\leqq u_1\leqq\cdots\leqq u_k\leqq t_0} w(\theta((z,0),u_1),u_1)\,e^{-\Omega_{\theta,w,z}(0,u_1)}\\
&\qquad\qquad \times\prod_{i=2}^{k}\Big(w(\theta((0,u_{i-1}),u_i),u_i)\,e^{-\Omega_{\theta,w}(u_{i-1},u_i)}\Big)\\
&\qquad\qquad \times e^{-\Omega_{\theta,w}(u_k,t)}\prod_{i=1}^{k}du_i\,\sigma(w,z)\,\lambda(dw)\,dz,\\
&\gamma=(0,t_0)\in\Gamma_b\cap\Gamma_t\,.
\end{aligned}$$

Using these explicit formula with (18) and (41), we see from (55) that $G(\theta)\in\Theta_T$. Namely, (76) defines a map

(79)
$$G:\ \Theta_T\to\Theta_T$$

on the set of flows Θ_T.

Proposition 8 *If $\theta\in\Theta_T$ is a fixed point of G in (79), namely, if $G(\theta)=\theta$, then φ_θ defined by (59) uniquely determines for each $t\in[0,T]$ a probability measure $\mu_{\theta,t}$ on $W\times[0,1]$ by the equation*

(80)
$$\varphi_\theta(dw,\gamma,t)=\mu_{\theta,t}(dw\times[\theta(\gamma,t),1)),\quad (\gamma,t)\in\Delta_T\,.$$

Furthermore, we have a following formula for a change of integration variables:

(81)
$$\begin{aligned}
&\int_{W\times\{\gamma'\in\Gamma_t\mid\gamma\succeq\gamma'\}} f(w,\theta(\gamma',t),t)\,\frac{\partial\varphi_\theta}{\partial\gamma}(dw,\gamma',t)\,d\gamma'\\
&=\int_{W\times[\theta(\gamma,t),1]} f(w,z,t)\,\mu_{\theta,t}(dw\times dz),
\end{aligned}$$

for integrable function $f:\ W\times[0,1]\times[0,T]\to\mathbb{R}$. $\diamond$

Remark. We are working with a generalization of characteristic curves, for which (γ, t) is a good coordinate. On the other hand, equation of motions are usually stated in the space time coordinates (y, t). This proposition relates the representations in these distinct coordinate systems. $\diamond$

Proof of Proposition 8. The definition (55) of Θ_T implies that $\theta(\cdot, t) : \Gamma_t \to [0,1]$ is continuous, surjective and non-decreasing. Hence to prove (80) we only need to prove consistency, namely,

$$\theta(\gamma, t) = \theta(\gamma', t) \;\Rightarrow\; \varphi_\theta(B, \gamma, t) = \varphi_\theta(B, \gamma', t),\; B \in \mathcal{B}(W). \tag{82}$$

If (82) holds, then (80) determines the distribution function $\mu_{\theta,t}(B \times [y,1))$, $y \in [0,1]$, on $[0,1]$, so that $\mu_{\theta,t}(B \times [a,b])$ is determined, and eventually $\mu_{\theta,t}$ is determined as a probability measure on the product space $W \times [0,1]$.

Assume that $\theta(\gamma, t) = \theta(\gamma', t)$ for $\gamma = (y_0, t_0) \in \Gamma_t$ and $\gamma' = (y_0', t_0') \in \Gamma_t$. Since $(\Gamma, \succeq)$ is a totally ordered set, we may assume without loss of generality that $\gamma \succeq \gamma'$. Then (54) implies $t_0 \leqq t_0' \leqq t$ and $y_0 \geqq y_0'$. Non-decreasing property of the point process $N_{\theta,w,z}(t)$ in t and monotonicity of measures imply with the definition (59),

$$\varphi_\theta(B, \gamma', t) \geqq \varphi_\theta(B, \gamma, t), \quad \text{and} \quad \varphi_\theta(B^c, \gamma', t) \geqq \varphi_\theta(B^c, \gamma, t), \tag{83}$$

for any $B \in \mathcal{B}(W)$,

On the other hand, the assumption $G(\theta) = \theta$ and (76) and $\theta(\gamma, t) = \theta(\gamma', t)$ imply

$$\begin{aligned}&\varphi_\theta(B, \gamma, t) + \varphi_\theta(B^c, \gamma, t) = \varphi_\theta(W, \gamma, t) = 1 - G(\theta)(\gamma, t) = 1 - \theta(\gamma, t)\\ &= 1 - \theta(\gamma', t) = \varphi_\theta(B, \gamma', t) + \varphi_\theta(B^c, \gamma', t).\end{aligned}$$

Hence

$$\varphi_\theta(B, \gamma', t) - \varphi_\theta(B, \gamma, t) = -(\varphi_\theta(B^c, \gamma', t) - \varphi_\theta(B^c, \gamma, t)). \tag{84}$$

Combining (83) and (84), we see that (82) holds, which implies (80).

Next to prove (81), we first prove

$$\begin{aligned}&\int_{\gamma \succeq \gamma'} \frac{\partial \varphi_\theta}{\partial \gamma}(B, \gamma', t)\, d\gamma' = \mu_{\theta,t}(B \times [\theta(\gamma, t), 1)),\\ &(\gamma, t) \in \Delta_T\,,\; B \in \mathcal{B}(W).\end{aligned} \tag{85}$$

In fact, note that (59) implies

$$\varphi_\theta(B, (1,0), t) = 0, \quad B \in \mathcal{B}(W),\; t \geqq 0. \tag{86}$$

Then, if $\gamma = (y_0, 0) \in \Gamma_i$, The explicit formula (43), with the definitions (59) (61) (63), implies

$$\begin{aligned}\int_{\gamma \succeq \gamma'} \frac{\partial \varphi_\theta}{\partial \gamma}(B, \gamma', t)\, d\gamma' &= -\int_{y_0}^{1} \frac{\partial \varphi_\theta}{\partial z}(B, (z,0), t)\, dz\\ &= \varphi(B, (y_0, 0), t),\end{aligned}$$

where we used (86) in the last line. If $\gamma = (0, t_0) \in \Gamma_b$, the explicit formula (43) similarly implies

$$\begin{aligned}
&\int_{\gamma \succeq \gamma'} \frac{\partial \varphi_\theta}{\partial \gamma}(B, \gamma', t)\, d\gamma' \\
&= -\int_0^1 \frac{\partial \varphi_\theta}{\partial z}(B, (z, 0), t)\, dz + \int_0^{t_0} \frac{\partial \varphi_\theta}{\partial u}(B, (0, u), t)\, du \\
&= -(\varphi_\theta(B, (1, 0), t) - \varphi_\theta(B, (0, 0), t)) + (\varphi_\theta(B, (0, t_0), t) - \varphi_\theta(B, (0, 0), t)) \\
&= \varphi(B, (0, t_0), t),
\end{aligned}$$

hence (85) follows from (80).

Define a measure $\nu_{\theta,t}$ on $W \times \Gamma_t$ by

$$\nu_{\theta,t}(dw \times d\gamma) = \frac{\partial \varphi_\theta}{\partial \gamma}(dw, \gamma, t)\, d\gamma.$$

Then (85) implies

$$\nu_{\theta,t}(B \times \{\gamma' \mid \gamma \succeq \gamma'\}) = \mu_{\theta,t}(B \times [\theta(\gamma, t), 1)), \ \ \gamma \in \Gamma_t\,.$$

This implies that, if we put $X_{\theta,t} = (\mathrm{id}_W, \theta)$, where id_W is the identity map on W, then $\mu_{\theta,t}$ is the image measure of $\nu_{\theta,t}$ with respect to the map $X_{\theta,t} :\ W \times \Gamma_t \to W \times [0, 1]$:

$$\mu_{\theta,t} = \nu_{\theta,t} \circ X_{\theta,t}^{-1}\,.$$

Therefore (81) follows. □

Theorem 9 *Assume that G in (79) has a fixed point, and denote the fixed point by $y_C \in \Theta_T$;*

$$y_C = G(y_C) \tag{87}$$

Put $\varphi = \varphi_{y_C}$ and $\mu_t = \mu_{y_C,t}$, where φ_θ and $\mu_{\theta,t}$ are defined by (59) and (80) with $\theta = y_C$. Then the so defined y_C and μ_t satisfy all the properties stated in Theorem 1. ◇

Remark. With (59) and (80), the theorem implies an expression

$$\begin{aligned}
&\mu_{y_C,t}(dw \times [y_C(\gamma, t), 1)) \\
&\quad = \int_{z \in [y_0, 1)} \mathrm{P}[\ N_{y_C,w,z}(t) = N_{y_C,w,z}(t_0)\ ]\, \mu_0(dw \times dz), \\
&\gamma = (y_0, t_0) \in \Gamma_t\,,
\end{aligned} \tag{88}$$

for the solution to Theorem 1.

The properties of the solution claimed in Theorem 1 are mostly contained in the previous propositions and explicit formulas. The remaining point is the Lipschitz continuity of μ_t: If $w(y, t) = 0$ for certain time interval for all $w \in W$, then the characteristic curve y_C will remain constant for the interval, and a small change in $y = y_C(\gamma, t)$ may result in a large change in γ and hence in μ_t. It turns out that the situation causes no problem, because then the change in the quantity φ and eventually μ_t are small in the time interval, hence continuity follows. ◇

Proof of Theorem 9. By definition (76), we have

$$y_C(\gamma,t) = G(y_C)(\gamma,t) = 1 - \varphi_{y_C}(W,\gamma,t), \tag{89}$$

for which monotonicity properties stated in Theorem 1 are direct consequences of (59). Proposition 6 and Proposition 7 with explicit formula (77) and (78), or (59) and (62), imply that y_C and $\dfrac{\partial y_C}{\partial t}$ is continuous in (γ,t).

Let $h : W \to \mathbb{R}$ be a measurable function satisfying (20). For (y,t) and (y',t) in $[0,1]\times[0,T]$, choose $\gamma \in \Gamma_t$ and $\gamma' \in \Gamma_t$ such that $y = y_C(\gamma,t)$ and $y' = y_C(\gamma',t)$. We may assume $\gamma \succeq \gamma'$. Then monotonicity of y_C implies $y \leqq y'$, hence $h(w) \leqq 1$ and (24) imply

$$\begin{aligned}
&\int_W h(w)\mu_t(dw\times[y,1)) - \int_W h(w)\mu_t(dw\times[y',1))\\
&= \int_W h(w)\mu_t(dw\times[y,y'))\\
&\leqq \mu_t(W\times[y,y')) = y'-y.
\end{aligned}$$

Next, for $(y,t),\ (y,t') \in [0,1]\times[0,T]$, choose $\gamma\in\Gamma_t$ and $\gamma'\in\Gamma_{t'}$ such that $y = y_C(\gamma,t)$ and $y = y_C(\gamma',t')$. Then, since, by definition (59), $\varphi(B,\gamma,t)$ is monotone also in t,

$$\begin{aligned}
&\left|\int_W h(w)\mu_t(dw\times[y,1)) - \int_W h(w)\mu_{t'}(dw\times[y,1))\right|\\
&= \left|\int_W h(w)\left(\varphi(dw,\gamma,t) - \varphi(dw,\gamma',t')\right)\right|\\
&\leqq |\varphi(W,\gamma,t)-\varphi(W,\gamma',t)| + |\varphi(W,\gamma',t)-\varphi(W,\gamma',t')|\\
&= |y_C(\gamma,t)-y_C(\gamma',t)| + |y_C(\gamma',t)-y_C(\gamma',t')|\\
&= 2|y - y_C(\gamma',t)|\\
&= 2|y_C(\gamma',t') - y_C(\gamma',t)|\\
&\leqq 2\sup_{(\gamma'',t'')\in\Delta_T}\left|\frac{\partial y_C}{\partial t}(\gamma'',t'')\right| |t'-t|.
\end{aligned}$$

Using Proposition 7, monotonicity, and Proposition 6, with (89), we have

$$\begin{aligned}
&\frac{\partial y_C}{\partial t}(\gamma,t) = \int_{\gamma\succeq\gamma'}\int_W w(y_C(\gamma',t),t)\,\frac{\partial\varphi}{\partial\gamma}(dw,\gamma',t')\,d\gamma'\\
&\leqq \int_{\gamma'\in\Gamma_t}\int_W \|w\|_{\mathrm{T}}\,\frac{\partial\varphi}{\partial\gamma}(dw,\gamma',t)\,d\gamma'\\
&\leqq M_W e^{2C_W t},
\end{aligned}$$

which proves (21).

The initial conditions (22) and surjectivity for $y_C\in\Theta_T$ is contained in the definition (55) of Θ_T, and the initial condition (17), for for μ_t is in the definition (59) with Proposition 8.

For $t>0$ and $B\in\mathcal{B}(W)$, (80), (59), (17), and (19) imply

$$\mu_t(dw\times[0,1)) = \varphi(dw,(0,t),t) = \mu_0(dw\times[0,1)) = \lambda(dw),$$

which proves (23).

For $(y,t) \in [0,1] \times [0,T]$ choose $\gamma \in \Gamma_t$ such that $y = y_C(\gamma,t)$. Then (80) and (76) imply

$$\mu_t(W \times [y,1)) = \varphi(W,\gamma,t) = 1 - G(y_C)(\gamma,t) = 1 - y_C(\gamma,t) = 1 - y$$

which proves (24).

To prove (25), Proposition 8 implies, for $\gamma = (y_0, t_0) \in \Gamma_t$,

$$\mu_t(dw \times [y_C(\gamma,t),1)) - \mu_{t_0}(dw \times [y_0,1)) = \varphi(dw,\gamma,t) - \varphi(dw,\gamma,t_0),$$

for which Proposition 7 and (81) further imply

$$\begin{aligned}
&\mu_t(dw \times [y_C(\gamma,t),1)) - \mu_{t_0}(dw \times [y_0,1)) \\
&= -\int_{t_0}^{t} \Big(\int_{\gamma \succeq \gamma'} w(y_C(\gamma',s),s) \, \frac{\partial \varphi}{\partial \gamma}(dw,\gamma',s) \, d\gamma' \Big) \, ds \\
&= -\int_{t_0}^{t} \Big(\int_{y_C(\gamma,s)}^{1} w(z,s) \, \mu_s(dw \times dz) \Big) \, ds,
\end{aligned}$$

which proves (25). □

5 Fixed point and existence of solution.

In Theorem 9 we assumed existence of a fixed point $\theta = y_C$ of a map G defined in (76). To complete a proof of existence of a solution for Theorem 1, we prove that (76) has a fixed point. In fact, the assumptions (12) and (13) on W imply that that the fixed point is unique. This is the core of the existence proof for Theorem 1, and we heavily rely on the explicit formulas (77) and (78).

Theorem 10 *The map $G: \ \Theta \to \Theta$ in (79) has a unique fixed point $y_C \in \Theta_T$, namely, there is a unique y_C which satisfies (87).* ◇

Proof. For $t \geqq 0$ and θ and θ' in Θ_T define

(90) $$d(\theta',\theta,t) = \sup_{\gamma \in \Gamma_t} |\theta'(\gamma,t) - \theta(\gamma,t)|.$$

We first accumulate basic formulas for evaluating $\omega_{\theta,w,z}$ in (57) and $\Omega_{\theta,w,z}$ in (58). In the following lemma, we write $\Omega_{\theta,w,z}(s,t)$ also for $s > 0$ in (58) whenever it becomes notationally simpler, though the quantity is actually independent of z for $s > 0$. Recall the notation $\tilde{\Omega}_w(s,t)$ in (67).

Lemma 11 *Let $\theta \in \Theta_T$ and $\theta' \in \Theta_T$. Then for $(\gamma,t) \in \Delta_T$ with $\gamma = (z,s)$, we have*

(i) $w(1,t) - C_W \leqq w(\theta(\gamma,t),t) \leqq w(1,t) + C_W$,

(ii) $0 < e^{-\Omega_{\theta,w,z}(s,t)} \leqq e^{-\tilde{\Omega}_w(s,t) + C_W\,(t-s)}$,

(iii) $|w(\theta'(\gamma,t),t) - w(\theta(\gamma,t),t)| \leqq C_W \, d(\theta',\theta,t)$,

(iv) $|e^{-\Omega_{\theta',w,z}(s,t)} - e^{-\Omega_{\theta,w,z}(s,t)}| \leqq C_W\, e^{-\tilde\Omega_w(s,t)+C_W\,(t-s)} \int_s^t d(\theta',\theta,v)\,dv.$

◇

Proof. The first estimate is an elementary consequence of (13) and the mean value theorem, if one notes that $\theta(\gamma,t)\in[0,1]$. This and the definitions (58) and (67), and non-negativity of $w\in W$ leads to the second estimate. With the definition (90), the third estimate is similarly proved as the first one. The last estimate follows from these estimates and

$$\begin{aligned}&|e^{-x'} - e^{-x}| = |e^{-(x'\vee x)} - e^{-(x'\wedge x)}| = e^{-(x'\wedge x)}\,(1-e^{-|x'-x|})\\ &\leqq e^{-(x'\wedge x)}|x'-x|.\end{aligned}$$

□

Lemma 12 *It holds that*

$$\begin{aligned}&d(G(\theta'),G(\theta),t) \leqq 2C_W\, e^{2C_W\,T}\textstyle\int_0^t d(\theta',\theta,v)\,dv,\\ &\theta,\,\theta\in\Theta_T\,,\ \ t\in[0,T].\end{aligned} \tag{91}$$

◇

Proof. If $\gamma=(y_0,0)\in\Gamma_i$, then applying Lemma 11 to (77), we have

$$\begin{aligned}&|G(\theta')(\gamma,t) - G(\theta)(\gamma,t)|\\ &\leqq C_W\,e^{C_W\,t}\int_{W\times[y_0,1]} e^{-\tilde\Omega_w(0,t)}\sigma(w,z)\lambda(dw)\,dz\int_0^t d(\theta',\theta,v)\,dv,\\ &\gamma\in\Gamma_i\ \ t\in[0,T].\end{aligned}$$

Non-negativity of $\tilde\Omega_w$ and (18) and the fact that λ is a probability measure further leads to

$$\sup_{\gamma\in\Gamma_i}|G(\theta')(\gamma,t) - G(\theta)(\gamma,t)| \leqq C_W\,e^{C_W\,T}\int_0^t d(\theta',\theta,v)\,dv,\quad t\in[0,T]. \tag{92}$$

The rest of the proof is for the case $\gamma=(0,t_0)\in\Gamma_t\cap\Gamma_b$. On applying Lemma 11 to each term of (78), we use an elementary equality

$$\prod_{i=1}^n b_i - \prod_{i=1}^n a_i = \sum_{j=1}^n\Big(\prod_{i=1}^{j-1} b_i\Big)\,(b_j-a_j)\,\Big(\prod_{i=j+1}^n a_i\Big), \tag{93}$$

where, and in the following, we adopt a notation

$$\prod_{i=1}^0 b_i = \prod_{i=n+1}^n a_i = 1$$

to simplify the formulas. We apply (93) to the difference of (78) and its analog, with θ replaced by θ', where b_i's are the factors depending on θ', and a_i's the factors depending on

θ. We then apply the last 2 estimates in Lemma 11 to the factor of the form $b_i - a_i$, and apply the first 2 estimates to other factors. We have

$$|G(\theta')((0,t_0),t) - G(\theta)((0,t_0),t)| \leqq I_1(t_0,t) + I_2(t_0,t), \tag{94}$$

where

$$\begin{aligned} &I_1(t_0,t) \\ &= C_W e^{C_W t} \int_W e^{-\tilde{\Omega}_w(0,t)} \Big(1 \\ &\qquad + \sum_{k\geqq 1} \int_{0\leqq u_1 \leqq \cdots \leqq u_k \leqq t_0} \prod_{i=1}^{k} (w(1,u_i) + C_W) \prod_{i=1}^{k} du_i \Big) \lambda(dw) \\ &\quad \times \int_0^t d(\theta',\theta,v)\, dv, \end{aligned} \tag{95}$$

and

$$\begin{aligned} &I_2(t_0,t) \\ &= C_W e^{C_W t} \int_W e^{-\tilde{\Omega}_w(0,t)} \times \sum_{k\geqq 1} \int_{0\leqq u_1 \leqq \cdots \leqq u_k \leqq t_0} \sum_{j=1}^{k} \Big(d(\theta',\theta,u_j) \\ &\qquad \times \prod_{i;\, 1\leqq i \leqq k,\, i\neq j} (w(1,u_i) + C_W) \Big) \prod_{i=1}^{k} du_i \lambda(dw). \end{aligned} \tag{96}$$

We apply (53) to (95), to find

$$\begin{aligned} &I_1(t_0,t) \\ &= C_W e^{C_W t} \int_W e^{-\tilde{\Omega}_w(0,t)} e^{\tilde{\Omega}_w(0,t_0) + C_W t_0} \lambda(dw) \times \int_0^t d(\theta',\theta,v)\, dv \\ &= C_W e^{C_W (t+t_0)} \int_W e^{-\tilde{\Omega}_w(t_0,t)} \lambda(dw) \int_0^t d(\theta',\theta,v)\, dv, \\ &\leqq C_W e^{2C_W T} \int_0^t d(\theta',\theta,v)\, dv. \end{aligned} \tag{97}$$

To evaluate (96), we first change an integration variable u_j to v and change the order of summation for j and k, to find

$$\begin{aligned} &I_2(t_0,t) \\ &= C_W e^{C_W t} \int_W e^{-\tilde{\Omega}_w(0,t)} \int_0^{t_0} d(\theta',\theta,v) \\ &\quad \times \Big(\sum_{j\geqq 1} \int_{0\leqq u_1 \leqq \cdots \leqq u_{j-1} \leqq v} \prod_{i=1}^{j-1} (w(1,u_i) + C_W) \prod_{i=1}^{j-1} du_i \Big) \\ &\quad \times \Big(\sum_{k\geqq j} \int_{v\leqq u_{j+1} \leqq \cdots \leqq u_k \leqq t_0} \prod_{i=j+1}^{k} (w(1,u_i) + C_W) \prod_{i=j+1}^{k} du_i \Big) \lambda(dw)\, dv. \end{aligned}$$

We apply (53) to the summation in j and to the summation in k, to find

$$
\begin{aligned}
&I_2(t_0,t)\\
&= C_W e^{C_W\,t}\int_W e^{-\tilde\Omega_w(0,t)}\int_0^{t_0} d(\theta',\theta,v)e^{\tilde\Omega_w(0,v)+C_W\,v}\\
&\qquad\qquad e^{\tilde\Omega_w(v,t_0)+C_W\,(t_0-v)}\,\lambda(dw)\,dv\\
&= C_W e^{C_W\,(t+t_0)}\int_W e^{-\tilde\Omega_w(t_0,t)}\,\lambda(dw)\times\int_0^{t_0} d(\theta',\theta,v)dv\\
&\leqq C_W e^{2C_W\,T}\int_0^t d(\theta',\theta,v)dv.
\end{aligned}
\tag{98}
$$

The equations (94), (97), and (98) imply

$$
\begin{aligned}
&\sup_{\gamma\in\Gamma_b\cap\Gamma_t}|G(\theta')((0,t_0),t)-G(\theta)((0,t_0),t)|\\
&\leqq 2C_W e^{2C_W\,T}\int_0^t d(\theta',\theta,v)dv.
\end{aligned}
\tag{99}
$$

The equations (90), (92), and (99) finally imply (91). □

Let us continue the proof of Theorem 10.

Define $\theta_0\in\Theta_T$ by

$$
\theta_0((y_0,t_0),t)=y_0\,,\quad ((y_0,t_0),t)\in\Delta_T\,,
\tag{100}
$$

and define a sequence of flows $\theta_k\in\Theta_T$, $k\in\mathbb{Z}_+$, inductively by (100) and

$$
\theta_{k+1}=G(\theta_k),\quad k\in\mathbb{Z}_+.
\tag{101}
$$

Lemma 12 implies

$$
\begin{aligned}
&d(\theta_{k+1},\theta_k,t)=d(G(\theta_k),G(\theta_{k-1}),t)\leqq C\int_0^t d(\theta_k,\theta_{k-1},v)dv,\\
&t\in[0,T],\ k=1,2,\ldots.
\end{aligned}
\tag{102}
$$

where $C=2C_W\,e^{2C_W\,T}$. Iterating, we obtain estimates which, by induction, is seen to have an expression

$$
\begin{aligned}
&d(\theta_{k+1},\theta_k,t)\\
&\leqq C^k\int_{0\leqq u_1\leqq\cdots\leqq u_k\leqq t} d(\theta_1,\theta_0,u_1)\prod_{i=1}^k du_i\\
&\quad = C^k\int_0^t\frac{(t-u)^{k-1}}{(k-1)!}\,d(\theta_1,\theta_0,u)\,du,\\
&t\in[0,T],\ k\in\mathbb{Z}_+.
\end{aligned}
\tag{103}
$$

Since $\theta\in\Theta_T$ takes values in $[0,1]$, the definition (90) implies

$$
d(\theta',\theta,t)\leqq 1,\quad \theta,\theta'\in\Theta_T\,,\ t\in[0,T].
\tag{104}
$$

Substituting (104) in (103),

$$d(\theta_{k+1},\theta_k,t) \leqq \frac{(C\,t)^k}{k!}\,,\ \ k\in\mathbb{Z}_+. \tag{105}$$

Since the summation in k of the right hand side of (105) converges to $e^{C\,t}$,

$$\theta_k(\gamma,t) = \theta_0(\gamma,t) + \sum_{i=0}^{k-1}(\theta_{i+1}(\gamma,t)-\theta_i(\gamma,t))$$

converges uniformly in $(\gamma,t)\in\Delta_T$. Denote the limit as

$$y_C(\gamma,t) = \lim_{k\to\infty}\theta_k(\gamma,t),\quad (\gamma,t)\in\Delta_T\,. \tag{106}$$

The equations (106), (101), and (105) imply (87). Since $\theta_k\in\Theta_T$ for all k, y_C also takes values in $[0,1]$, non-decreasing in γ for each t, and non-decreasing in t for each γ. Since the convergence (106) is uniform in (γ,t), and $\theta_k\in\Theta_T$ are continuous, y_C is also continuous. Also

$$y_C((y_0,t_0),t_0) = y_0\,,\quad (y_0,t_0)\in\Gamma_T\,,$$

holds.

In particular, $y_C((0,t),t)=0$ holds, and also (56) implies $y_C((1,0),t)=1$, hence with continuity, y_C is surjective in γ for each t. This proves $y_C\in\Theta_T$, namely, existence of a fixed point of G in Θ_T .

Suppose there is another fixed point $\tilde{y}_C\in\Theta_T$ of G. Then (87) and Lemma 12 imply

$$d(\theta',\theta,t) = d(G(\theta'),G(\theta),t) \leqq C\int_0^t d(\theta',\theta,v)\,dv,\quad t\in[0,T].$$

where $C=2C_W\,e^{2C_W\,T}$. Gronwall's inequality implies $d(\theta',\theta,t)=0$, $t\in[0,T]$. Namely, $\theta'=\theta$. This proves uniqueness of the fixed point of G. □

6 Uniqueness of the solution.

In previous sections we proved existence of a solution (y_C,μ_t) in Theorem 1. In this section we complete the proof of Theorem 1 by proving that the solution is unique. Assume that (y_C,μ_t) and $(\tilde{y}_C,\tilde{\mu}_t)$ are the pairs which satisfy all the properties stated in Theorem 1.

Fix $\gamma=(y_0,t_0)\in\Gamma$, and let $t\geqq t_0$. Since by assumption (μ_t,y_C) satisfy equation of motion (25) with initial and boundary conditions (22) and (23),

$$\begin{aligned}
&\mu_t(dw\times[y_C(\gamma,t),1))\\
&=\mu_{t_0}(dw\times[y_0,1))\\
&\quad+\int_{t_0}^t\int_{z\in[y_C(\gamma,s),1)}(w(1,s)-w(z,s))\mu_s(dw\times dz)\,ds\\
&\quad-\int_{t_0}^t w(1,s)\mu_s(dw\times[y_C(\gamma,s),1))\,ds,\quad t\geqq t_0\,.
\end{aligned} \tag{107}$$

Note that (23) and (12) do not rule out a possibility that μ_t has an unbounded support concerning $\|w\|_{\mathrm{T}}$. Therefore, a direct application of Gronwall type inequalities to the last term in the right hand side of (107) may lead to divergent expression upon integration with respect to w. We work around this problem by the following.

Lemma 13

$$\begin{aligned} &\mu_t(dw \times [y_C(\gamma,t),1)) = e^{-\tilde{\Omega}_w(t_0,t)} \mu_{t_0}(dw \times [y_0,1)) \\ &\quad + \int_{t_0}^t e^{-\tilde{\Omega}_w(s,t)} \int_{x\in[y_C(\gamma,s),1)} \frac{\partial w}{\partial z}(x,s) \mu_s(dw \times [y_C(\gamma,s),x))\, dx\, ds, \end{aligned} \tag{108}$$

where $\tilde{\Omega}_w(s,t)$ is as in (67). ◇

Proof. Iterating (107) and using Fubini's Theorem, we have

$$\begin{aligned} &\mu_t(dw \times [y_C(\gamma,t),1)) \\ &= \mu_{t_0}(dw \times [y_0,1)) \sum_{\ell=0}^{k} \frac{1}{\ell!} (-\tilde{\Omega}_w(t_0,t))^\ell \\ &\quad + \int_{t_0}^t \sum_{\ell=0}^{k} \frac{1}{\ell!} (-\tilde{\Omega}_w(s,t))^\ell \\ &\qquad \times \int_{z\in[y_C(\gamma,s),1)} (w(1,s) - w(z,s)) \mu_s(dw \times dz)\, ds \\ &\quad - \int_{t_0}^t w(1,s) \frac{1}{k!} (-\tilde{\Omega}_w(s,t))^k \mu_s(dw \times [y_C(\gamma,s),1))\, ds, \\ &t \geqq t_0\,,\ k = 0,1,2,\ldots. \end{aligned} \tag{109}$$

Since $w \in W$ are non-negative valued, so are $\tilde{\Omega}_w(s,t)$ for $s \leqq t$ and

$$0 \leqq \tilde{\Omega}_w(s,t) \leqq \|w\|_{\mathrm{T}}\, T, \quad 0 \leqq s \leqq t \leqq T. \tag{110}$$

Taylor's Theorem and (110) imply

$$\begin{aligned} &\left| e^{-\tilde{\Omega}_w(s,t)} - \sum_{\ell=0}^{k} \frac{1}{\ell!} (\tilde{\Omega}_w(s,t))^\ell \right| \leqq \frac{1}{(k+1)!} (\|w\|_{\mathrm{T}}\, T)^{k+1}, \\ &0 \leqq s \leqq t \leqq T, \quad k = 0,1,2,\ldots. \end{aligned} \tag{111}$$

Note also that (13) and the mean value theorem imply

$$|w(1,s) - w(z,s)| \leqq C_W\,, \quad z \in [0,1],\ s \in [0,T]. \tag{112}$$

Fix a constant $M > T$ (e.g., $M = 2T$), and let $B \in \mathcal{B}(W)$. Then, monotonicity of measures, (109), (110), (111), (112), and (23) imply

$$
\begin{aligned}
&\Big|\int_B e^{-M\,\|w\|_{\mathrm{T}}}\mu_t(dw\times[y_C(\gamma,t),1))\\
&\quad-\int_B e^{-M\,\|w\|_{\mathrm{T}}}\,e^{-\tilde\Omega_w(t_0,t)}\mu_{t_0}(dw\times[y_0,1))\\
&\quad-\int_{t_0}^t\int_B e^{-M\,\|w\|_{\mathrm{T}}}\,e^{-\tilde\Omega_w(s,t)}\\
&\qquad\times\int_{z\in[y_C(\gamma,s),1)}(w(1,s)-w(z,s))\mu_s(dw\times dz)\,ds\Big|\\
&\leqq\frac{T^{k+1}}{(k+1)!}\,(1+C_W\,T+k+1)\int_B e^{-M\,\|w\|_{\mathrm{T}}}\,\|w\|_{\mathrm{T}}^{k+1}\;\lambda(dw)\\
&\leqq\frac{T^{k+1}}{(k+1)!}\,(C_W\,T+k+2)\,\lambda(B)\,\sup_{x\geqq0}x^{k+1}\,e^{-M\,x}\\
&\leqq\frac{T^{k+1}}{(k+1)!}\,(C_W\,T+k+2)\,\sup_{x\geqq0}x^{k+1}\,e^{-M\,x},\\
&t\geqq t_0\,,\;\;k=0,1,2,\dots.
\end{aligned}
\tag{113}
$$

By elementary calculus,

$$
\begin{aligned}
\log\sup_{x\geqq0}x^{k+1}\,e^{-M\,x}&=(k+1)\,(\log\frac{k+1}{M}-1)\\
&=\int_0^{k+1}\log y\,dy-(k+1)\log M\\
&\leqq\sum_{\ell=1}^{k+1}\log\ell-(k+1)\log M=\log\frac{(k+1)!}{M^{k+1}}\,.
\end{aligned}
\tag{114}
$$

Combining (113) and (114), we have

$$
\begin{aligned}
&\Big|\int_B e^{-M\,\|w\|_{\mathrm{T}}}\mu_t(dw\times[y_C(\gamma,t),1))\\
&\quad-\int_B e^{-M\,\|w\|_{\mathrm{T}}}\,e^{-\tilde\Omega_w(t_0,t)}\mu_{t_0}(dw\times[y_0,1))\\
&\quad-\int_{t_0}^t\int_B e^{-M\,\|w\|_{\mathrm{T}}}\,e^{-\tilde\Omega_w(s,t)}\\
&\qquad\times\int_{z\in[y_C(\gamma,s),1)}(w(1,s)-w(z,s))\mu_s(dw\times dz)\,ds\Big|\\
&\leqq\Big(\frac{T}{M}\Big)^{k+1}(C_WT+k+2),\quad t\geqq t_0\,,\;B\in\mathcal{B}(W),\;k=0,1,2,\dots,
\end{aligned}
$$

which implies, by fixing $M > T$ and considering $k \to \infty$,

$$\begin{aligned}
&\int_B e^{-M\,\|w\|_{\mathrm{T}}}\mu_t(dw \times [y_C(\gamma,t),1)) \\
&\quad = \int_B e^{-M\,\|w\|_{\mathrm{T}}}\, e^{-\tilde{\Omega}_w(t_0,t)}\mu_{t_0}(dw \times [y_0,1)) \\
&\qquad + \int_{t_0}^t \int_B e^{-M\,\|w\|_{\mathrm{T}}}\, e^{-\tilde{\Omega}_w(s,t)} \\
&\qquad\qquad \times \int_{z\in[y_C(\gamma,s),1)} (w(1,s) - w(z,s))\mu_s(dw \times dz)\, ds, \\
&\quad t \geqq t_0\,,\ B \in \mathcal{B}(W).
\end{aligned}$$

This implies equality as a measure:

$$\begin{aligned}
\mu_t(dw \times [y_C(\gamma,t),1)) &= e^{-\tilde{\Omega}_w(t_0,t)}\mu_{t_0}(dw \times [y_0,1)) \\
&+ \int_{t_0}^t e^{-\tilde{\Omega}_w(s,t)} \int_{z\in[y_C(\gamma,s),1)} (w(1,s) - w(z,s))\mu_s(dw \times dz)\, ds.
\end{aligned} \tag{115}$$

Using

$$w(1,s) - w(z,s) = \int_z^1 \frac{\partial\, w}{\partial z}(x,s)\, dx$$

with Fubini's Theorem, we arrive at (108). □

Let us return to the proof of uniqueness in Theorem 1, and suppose there is another pair $(\tilde{\mu}, \tilde{y}_C)$ which satisfies all the properties stated in Theorem 1. Lemma 13 implies that $(\tilde{\mu}_t, \tilde{y}_C)$ satisfies an integral equation similar to (108),

$$\begin{aligned}
\tilde{\mu}_t(dw \times [\tilde{y}_C(\gamma,t),1)) &= e^{-\tilde{\Omega}_w(t_0,t)}\mu_{t_0}(dw \times [y_0,1)) \\
&+ \int_{t_0}^t e^{-\tilde{\Omega}_w(s,t)} \int_{x\in[\tilde{y}_C(\gamma,s),1)} \frac{\partial\, w}{\partial z}(x,s)\tilde{\mu}_s(dw \times [\tilde{y}_C(\gamma,s),x))\, dx\, ds.
\end{aligned} \tag{116}$$

The first term in the right hand side is equal to that of (108), because of the initial and boundary conditions (23) and (24).

Put

$$I(t) = \sup_h \sup_{y\in[0,1)} \Big| \int_W h(w)\tilde{\mu}_t(dw \times [y,1)) - \int_W h(w)\mu_t(dw \times [y,1)) \Big|, \tag{117}$$

and

$$\begin{aligned}
J(t) = \sup_h \sup_{\gamma\in\Gamma_t} \Big| &\int_W h(w)\tilde{\mu}_t(dw \times [\tilde{y}_C(\gamma,t),1)) \\
&- \int_W h(w)\mu_t(dw \times [y_C(\gamma,t),1)) \Big|,
\end{aligned} \tag{118}$$

where the supremum for h in the right hand sides of (117) and (118) are taken over measurable functions $h:\ W \to \mathbb{R}$ satisfying (20). In particular, (26) implies

$$\sup_{\gamma\in\Gamma_t} |\tilde{y}_C(\gamma,t) - y_C(\gamma,t)| \leqq J(t). \tag{119}$$

Fix s and x, and put

$$h_1(w) = \frac{1}{C_W}\, h(w)\, e^{-\tilde{\Omega}_w(s,t)}\, \frac{\partial\, w}{\partial z}(x,s).$$

Then $\tilde{\Omega}_w(s,t) \geqq 0$ and (13) imply that $h_1 : W \to \mathbb{R}$ satisfies (20) with $h = h_1$. The definitions (117) and (118) then imply

$$\begin{aligned}
&\Big|\int_W h(w)\, e^{-\tilde{\Omega}_w(s,t)}\, \frac{\partial\, w}{\partial z}(x,s)\, \tilde{\mu}_s(dw \times [\tilde{y}_C(\gamma,s),x)) \\
&\quad - \int_W h(w)\, e^{-\tilde{\Omega}_w(s,t)}\, \frac{\partial\, w}{\partial z}(x,s)\, \mu_s(dw \times [y_C(\gamma,s),x))\Big| \\
&\leqq C_W \Big|\int_W h_1(w)\tilde{\mu}_s(dw \times [\tilde{y}_C(\gamma,s),1)) \\
&\qquad - \int_W h_1(w)\mu_s(dw \times [y_C(\gamma,s),1))\Big| \\
&\quad + C_W \Big|\int_W h_1(w)\tilde{\mu}_s(dw \times [x,1)) - \int_W h_1(w)\mu_s(dw \times [x,1))\Big| \\
&\leqq C_W\, (I(s) + J(s)).
\end{aligned} \tag{120}$$

Also, monotonicity of measure implies

$$\begin{aligned}
&\Big|\int_W h(w)\, e^{-\tilde{\Omega}_w(s,t)} \frac{\partial\, w}{\partial z}(x,s)\tilde{\mu}_s(dw \times [\tilde{y}_C(\gamma,s),x))\Big| \\
&\leqq C_W \tilde{\mu}_s(W \times [0,1]) = C_W\,.
\end{aligned} \tag{121}$$

Substituting (108) and (116) in (118), using (120) and (121), and also (119), we have

$$\begin{aligned}
J(t) &\leqq \sup_h \sup_{\gamma\in\Gamma_t} \int_{t_0}^{t} \Big(\Big|\int_{\tilde{y}_C(\gamma,s)}^{y_C(\gamma,s)} C_W\, dx\Big| \\
&\qquad\qquad + \int_{x\in[y_C(\gamma,s),1)} \Big|C_W\,(I(s)+J(s))\Big|\, dx\Big)\, ds \\
&\leqq C_W \sup_h \sup_{\gamma\in\Gamma_t} \int_{t_0}^{t} (|\tilde{y}_C(\gamma,s) - y_C(\gamma,s)| + I(s) + J(s))\, ds \\
&\leqq \int_0^t (I(s) + 2J(s))\, ds.
\end{aligned} \tag{122}$$

Next, since by assumption, for each t, the map $\gamma \mapsto y_C(\gamma,t)$ is surjective, we have, from (117),

$$\begin{aligned}
I(t) = \sup_h \sup_{\gamma\in\Gamma_t} &\Big|\int_W h(w)\tilde{\mu}_t(dw \times [y_C(\gamma,t),1)) \\
&- \int_W h(w)\mu_t(dw \times [y_C(\gamma,t),1))\Big|.
\end{aligned} \tag{123}$$

Using Lipschitz continuity (21), the definition (118), and (119), we have

$$I(t) \leqq \sup_h \sup_{\gamma\in\Gamma_t} (|\tilde{y}_C(\gamma,t) - y_C(\gamma,t)| + J(t)) \leqq 2J(t). \tag{124}$$

The inequalities (123) and (124) imply

$$J(t) \leqq 4 \int_0^t J(s)\, ds, \;\; t \in [0,T],$$

which, by Gronwall's inequality, further implies $J(t) = 0$, $t \in [0,T]$. This proves $\tilde{y}_C = y_C$, and also (124) now implies $I(t) = 0$, $t \in [0,T]$, which proves $\tilde{\mu}_t = \mu_t$. This completes a proof of the uniqueness claim in Theorem 1.

A Application of Schauder's fixed point theorem.

In this section we consider the case where we keep the fundamental condition (12), but replace the global Lipschitz type condition (13) by a global bound condition on oscillation:

$$C'_W := \sup_{w\in W} \sup_{(y,t),\,(y',t')\in[0,1]\times[0,T]} |w(y,t) - w(y',t')| < \infty, \tag{125}$$

and consider the existence of fixed points to the map $G:\ \Theta_T \to \Theta_T$ defined by the explicit formula (77) and (78), where the notations are introduced in (18), (19), (14), (15), (16), (55), and (58). Note that $G(\theta) \in \Theta_T$ holds with the weaker assumption (125). This can be shown directly from the explicit expression (78). The only condition for Θ_T perhaps not obvious from the expression is the range condition $G(\theta)(\gamma,t) \in [0,1]$, which can be shown as follows. For $k = 1,2,\ldots$ put

$$\begin{aligned} I_k = \int_{0\leqq u_1\leqq\cdots\leqq u_k\leqq t_0} & w(\theta((z,0),u_1),u_1)\, e^{-\Omega_{\theta,w,z}(0,u_1)} \\ & \times \prod_{i=2}^{k} \Big(w(\theta((0,u_{i-1}),u_i),u_i)\, e^{-\Omega_{\theta,w}(u_{i-1},u_i)} \Big) \\ & \times e^{-\Omega_{\theta,w}(u_k,t)} \prod_{i=1}^{k} du_i \end{aligned}$$

and

$$\begin{aligned} J_k = \int_{0\leqq u_1\leqq\cdots\leqq u_k\leqq t_0} & w(\theta((z,0),u_1),u_1)\, e^{-\Omega_{\theta,w,z}(0,u_1)} \\ & \times \prod_{i=2}^{k} \Big(w(\theta((0,u_{i-1}),u_i),u_i)\, e^{-\Omega_{\theta,w}(u_{i-1},u_i)} \Big) \prod_{i=1}^{k} du_i\,. \end{aligned}$$

Note that non-negativity of $\Omega_{\theta,w}$ implies $I_k \leqq J_k$. We can perform the u_k integration in J_k to find $J_k = J_{k-1} - I_{k-1}$, which we can iterate to find

$$\sum_{i=1}^{k} I_i = J_1 - J_k + I_k \leqq J_1 = 1 - e^{-\Omega_{\theta,w,z}(0,t_0)}.$$

Substituting this in (78), and using monotonicity of $\Omega_{\theta,w,z}$, (19), and $\lambda(W) = 1$, we have $1 \geqq G(\theta)((0,t_0),t) \geqq 0$. A similar estimate for (77) is straightforward, hence we conclude $G(\theta) \in \Theta_T$.

Theorem 14 . *Under the conditions (12) and (125), the map* $G: \ \Theta_T \to \Theta_T$ *has a fixed point.* ◇

Remark. Since the proof relies on Schauder's fixed point theorem, our proof has no control of uniqueness of fixed points. ◇

The map G maps Θ_T into itself, and Θ_T is a subset of a Banach space (with the supremum norm) of continuous functions $C^0(\Delta_T; [0,1])$ taking values in a finite interval $[0,1] \subset \mathbb{R}$. The domain Δ_T is homeomorphic to a rectangle, since its parameterization in the definition (16) is homeomorphic to a trapezoid.

The Schauder fixed point theorem states [1, (2.4.3)] that a compact map of a closed bounded convex set in a Banach space into itself has a fixed point. (The notational correspondence between here and [1, §2.4] is given by $X = C^0(\Delta_T;\ [0,1])$, $K = U = \Theta_T$, and $f = G$.) We have shown $G(\Theta_T) \subset \Theta_T$ at the beginning of this section.

Concerning the required properties for the domain Θ_T of the map G, we have noted that $C^0(\Delta_T; [0,1])$ is a bounded set. For a sequence of continuous and monotone functions, the limit function with respect to the supremum norm also is continuous and monotone, and since for $\theta \in \Theta_T$, $\theta((0,t),t) = 0$ and $\theta((1,0),t) = 1$ holds, these properties are also preserved in the limit. This and continuity imply surjectivity of the limit function. Therefore Θ_T is a closed set. The continuity, monotonicity, the properties $\theta((0,t),t) = 0$ and $\theta((1,0),t) = 1$ are also preserved by convex linear combination, hence Θ_T is also convex. Thus Θ_T is a closed, bounded, convex set.

It remains to prove compactness of G. Since $C^0(\Delta_T;\ [0,1])$ is a bounded set with respect to the supremum norm, the Arzela-Ascoli theorem implies that it is sufficient to prove (i) that the map $G: \ \Theta_T \to \Theta_T$ is continuous, and (ii) that the functions in the image set $G(\Theta_T)$ are equicontinuous, which we prove in Lemma 16 and Lemma 17, respectively.

Note first that non-negativity of $w \in W$ obviously implies

$$0 < e^{-\tilde{\Omega}_w(s,t)} \leqq 1, \ \ s \leqq t, \tag{126}$$

where $\tilde{\Omega}_w(s,t)$ is as in (67).

Proposition 15 *For* θ *and* θ' *in* Θ_T *, and* $(\gamma, t) \in \Delta_T$ *with* $\gamma = (z,s)$*, we have*

(i) $w(1,t) - C'_W \leqq w(\theta(\gamma,t),t) \leqq w(1,t) + C'_W \leqq \|w\|_{\mathrm{T}} + C'_W$,

(ii) $0 < e^{-\Omega_{\theta,w,z}(s,t)} \leqq e^{-\tilde{\Omega}_w(s,t) + C'_W\,(t-s)}$,

(iii) $|e^{-\Omega_{\theta',w,z}(s,t)} - e^{-\Omega_{\theta,w,z}(s,t)}|$

$$\leqq e^{-\tilde{\Omega}_w(s,t) + C'_W\,(t-s)} \int_s^t |w(\theta'(\gamma,u),u) - w(\theta(\gamma,u),u)|\, du,$$

where C'_W *is as in (125), and* $\|w\|_{\mathrm{T}}$ *is defined by (11).* ◇

Proof. (125) implies

$$|w(y,t)-w(1,t)| \leqq C'_W,\ w\in W,\ (y,t)\in[0,1]\times[0,T],$$

which further implies

$$0< e^{-\Omega_{\theta,w,z}(s,t)} \leqq e^{-\int_s^t w(1,u)\,du + C'_W\,(t-s)},$$

These estimates imply the first 2 estimates. The last estimate follows from these estimates and

$$\begin{aligned}&|e^{-x'}-e^{-x}| = |e^{-(x'\vee x)}-e^{-(x'\wedge x)}| = e^{-(x'\wedge x)}|\,(1-e^{-|x'-x|})\\ &\leqq e^{-(x'\wedge x)}|x'-x|.\end{aligned}$$

□

Lemma 16 *$G:\ \Theta_T\to\Theta_T$ is a continuous map.* ◇

Proof. Let $\theta,\theta'\in\Theta_T$, and put $(\gamma,t)\in\Delta_T$ and $\gamma=(y_0,t_0)$.

If $\gamma=(y_0,0)\in\Gamma_i$ ($t_0=0$), (77), Proposition 15, (126), and (19) imply

$$\begin{aligned}&\sup_{y_0\in[0,1]}\sup_{t\in[0,T]}|G(\theta')((y_0,0),t)-G(\theta)((y_0,0),t)|\\ &\leqq \sup_{y_0\in[0,1]}\sup_{t\in[0,T]}\int_{W\times[y_0,1)} e^{-\tilde\Omega_w(0,t)+C'_W\,t}\\ &\qquad\int_0^t |w(\theta'((z,0),u),u)-w(\theta((z,0),u),u)|\,du\\ &\qquad\times\sigma(w,z)\,\lambda(dw)\,dz\\ &\leqq T\,e^{C'_W\,T}\int_{W\times[0,1)}\sup_{u\in[0,T]}\sup_{z\in[0,1]}|w(\theta'((z,0),u),u)\\ &\qquad - w(\theta((z,0),u),u)|\,\lambda(dw).\end{aligned} \tag{127}$$

Concerning the rightmost hand side, we have

$$\sup_{u\in[0,T]}\sup_{z\in[0,1]}|w(\theta'((z,0),u),u)-w(\theta((z,0),u),u)| \leqq 2\,\|w\|_{\mathrm{T}},$$

while (12) implies $\displaystyle\int_{W\times[0,1)}\|w\|_{\mathrm{T}}\,\lambda(dw)=M_W<\infty$. Hence the integrand in the right hand side of (127) is bounded, pointwise in $w\in W$, uniformly in θ' by an integrable function. Therefore, thanks to dominated convergence theorem we may interchange the order of integration and the limit $\theta'\to\theta$ in the right hand side of (127).

$W\subset C^1([0,1]\times[0,T];[0,\infty))$ and $[0,1]\times[0,T]$ is compact, hence each $w\in W$ is uniformly continuous. Hence for any $\epsilon>0$, there exists $\delta>0$ such that

$$(\forall y,y'\in[0,1];\ |y'-y|<\delta)(\forall u\in[0,T])\ |w(y,u)-w(y',u)|<\epsilon.$$

If the supremum norm of $\theta' - \theta$ is less than δ we have

$$\sup_{u\in[0,T]} \sup_{z\in[0,1]} |\theta'((z,0),u) - \theta((z,0),u)| < \delta,$$

which further implies

$$\sup_{u\in[0,T]} \sup_{z\in[0,1]} |w(\theta'((z,0),u),u) - w(\theta((z,0),u),u)| \leqq \epsilon,$$

hence (127) implies

$$\overline{\lim_{\theta'\to\theta}} \sup_{y_0\in[0,1]} \sup_{t\in[0,T]} |G(\theta')((y_0,0),t) - G(\theta)((y_0,0),t)| \leqq T\, e^{C'_W T} \epsilon.$$

Since $\epsilon > 0$ is arbitrary,

$$\lim_{\theta'\to\theta} \sup_{y_0\in[0,1]} \sup_{t\in[0,T]} |G(\theta')((y_0,0),t) - G(\theta)((y_0,0),t)| = 0. \tag{128}$$

Next if $\gamma = (0,t_0) \in \Gamma_t \cap \Gamma_b$ ($y_0 = 0$), we proceed as in exact analogy to the proof of Lemma 12, to find

$$\begin{aligned}&\sup_{t\in[0,T]} \sup_{t_0\in[0,t]} |G(\theta')((0,t_0),t) - G(\theta)((0,t_0),t)| \\ &\leqq T\, e^{2C'_W T} \int_W \sup_{(\gamma,u)\in\Delta_T} |w(\theta'(\gamma,u),u) - w(\theta(\gamma,u),u)|\, \lambda(dw).\end{aligned}$$

Therefore, as in the same reasoning as we derive (128) from (127),

$$\lim_{\theta'\to\theta} \sup_{t\in[0,T]} \sup_{t_0\in[0,t]} |G(\theta')((0,t_0),t) - G(\theta)((0,t_0),t)| = 0. \tag{129}$$

Finally, (128) and (129) imply

$$\lim_{\theta'\to\theta} \sup_{(\gamma,t)\in\Delta_T} |G(\theta')(\gamma,t) - G(\theta)(\gamma,t)| = 0,$$

which proves the continuity of $G: \ \Theta \to \Theta$. □

Lemma 17 *The functions in the set $G(\Theta_T)$ are equicontinuous.* ◇

Proof. We see from elementary calculus using the mean value theorem and triangular inequality that the following uniform estimates on the derivatives of $G(\theta)$ imply equicontinuity: for $t \in [0,T]$,

$$0 \leqq \frac{\partial}{\partial y_0} G(\theta)((y_0,0),t) \leqq 1, \quad y_0 \in [0,1], \tag{130}$$

$$0 \leqq \frac{\partial}{\partial t} G(\theta)((y_0,0),t) \leqq M_W + C'_W\,, \quad y_0 \in [0,1], \tag{131}$$

$$0 \leqq -\frac{\partial}{\partial t_0} G(\theta)((0,t_0),t) \leqq (M_W + C'_W)\, e^{2C'_W T}, \quad t_0 \in [0,t], \tag{132}$$

$$0 \leqq \frac{\partial}{\partial t} G(\theta)((0,t_0),t) \leqq (M_W + C'_W)\, e^{2C'_W T}, \quad t_0 \in [0,t]. \tag{133}$$

The remainder of the proof is devoted to proving these estimates.

To prove (130), differentiate the explicit formula (77) by t_0 and use (126) and (18). We have

$$\begin{aligned} 0 \leqq \frac{\partial}{\partial y_0} G(\theta)((y_0,0),t) &= \int_W e^{-\Omega_{\theta,w,y_0}(0,t)}\, \sigma(w,y_0)\, \lambda(dw) \\ &\leqq \int_W \sigma(w,y_0)\lambda(dw) = 1, \end{aligned}$$

which proves (130).

To prove (131), differentiate (77) by t, and use (58), Proposition 15, (126), (19), and (12), to find

$$\begin{aligned} 0 &\leqq \frac{\partial}{\partial t} G(\theta)((y_0,0),t) \\ &= \int_{W\times[y_0,1)} w(\theta((z,0),t),t)\, e^{-\Omega_{\theta,w,z}(0,t)}\, \sigma(w,z)\, \lambda(dw)\, dz \\ &\leqq \int_{W\times[0,1)} (\|w\|_{\mathrm{T}} + C'_W)\, \sigma(w,z)\, \lambda(dw)\, dz = M_W + C'_W\,, \end{aligned}$$

which proves (131).

Proofs of (132) and (133) are similar. We differentiate (78) by t_0 and t, respectively, and follow a similar line. The only new point is that we apply (53) in a similar way as in the proof of Lemma 12. This completes a proof of Lemma 17. □

As discussed in the beginning of this appendix, Lemma 16 and Lemma 17 prove Theorem 14.

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