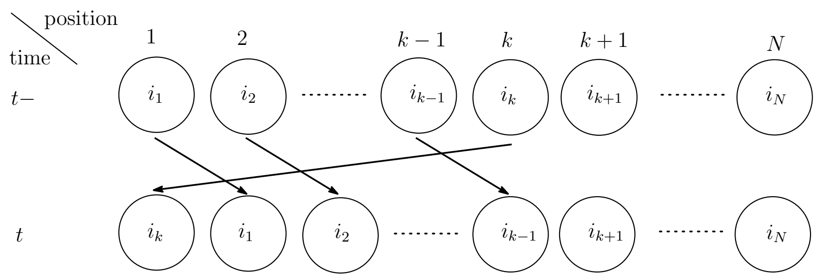
STOCHASTIC RANKING PROCESS WITH SPACE-TIME DEPENDENT INTENSITIES

TETSUYA HATTORI AND SEIICHIRO KUSUOKA

Abstract. We consider the stochastic ranking process with space-time dependent jump rates for the particles. The process is a simplified model of the time evolution of the rankings such as sales ranks at online bookstores. We prove that the joint empirical distribution of jump rate and scaled position converges almost surely to a deterministic distribution, and also the tagged particle processes converge almost surely, in the infinite particle limit. The limit distribution is characterized by a system of inviscid Burgers-like integral-partial differential equations with evaporation terms, and the limit process of a tagged particle is a motion along a characteristic curve of the differential equations except at its Poisson times of jumps to the origin.

1. Introduction.

A stochastic ranking process is a model of ranking system, such as the sales ranks found at online bookstores. Let us consider N particles, which we label by $1, 2, \dots, N$, and each of which are exclusively located at one of the positions $1, 2, \dots, N$. We denote the position of particle i at time t by $X_i^{(N)}(t)$, and its initial position by $x_i^{(N)} = X_i^{(N)}(0)$. Each particle jumps to position 1 according to its Poisson clock. When a jump of the particle at position k occurs, the particle moves to position 1 and the locations of the particles at $1, 2, \dots, k-1$ are shifted by $+1$. The figure below is the time evolution at time t when the particles $i_1, i_2, \dots, i_N$ are aligned from left-hand side to right-hand side at time $t-$ and a jump of the particle at position k occurs.



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For example, if $N=4$ and $(x_1^{(N)}, x_2^{(N)}, x_3^{(N)}, x_4^{(N)}) = (4,1,3,2)$, namely, the particles are aligned in the order of $2,4,3,1$ (corresponding to i_1, i_2, i_3, i_4 in the figure) at time 0, and if the clock of particle $i=3$ rings first, then the particles realign in the order of $3,2,4,1$, or equivalently, $X^{(N)}(t) = (X_1^{(N)}(t), X_2^{(N)}(t), X_3^{(N)}(t), X_4^{(N)}(t)) = (4,2,1,3)$. Particles whose Poisson clocks rang recently have small $X_i^{(N)}$'s, and the others have large $X_i^{(N)}$'s. We regard the number for each particle as the particle's rank. This system enables us to give ranks to N particles, and we call the time evolution of the particles given by this ranking system a *stochastic ranking process*.

In this paper we consider a hydrodynamic limit of the stochastic ranking processes whose jump rates depend not only on time but also on their positions. Here we used the term hydrodynamic limit in the sense that we scale the length so that the N particles aligned are contained in a (macroscopic) unit length, and take the limit as the number of the particles N to ∞.

A Poisson clock, or a Poisson random measure, is determined by its intensity measure, which represents how often the Poisson clock rings (i.e. how often the jumps occur in our model). Our main concern in this paper is to consider mathematically the case where the intensity measures have position dependence as well as time dependence. Since the position $X_i^{(N)}$ of the particle i is a random variable, the intensity measure is also random. To avoid mathematical complexity in applying a general theory of stochastic integration (see Ikeda and Watanabe (1989)), we introduce a Poisson random measure $\nu_i(d\xi\, ds)$ on $[0,\infty)\times[0,\infty)$ with the uniform intensity measure $d\xi\, ds$. We do not give Poisson clocks to each position. Instead, we incorporate the space-time dependence of the Poisson jumps of the particle i through a function $w_i^{(N)}(k,t)$ and denote the number of times the jumps to rank 1 occurred for the particle i in the time interval $a < s \leqq b$ by

$$\int_{s\in(a,b]} \int_{\xi\in[0,\infty)} \mathbf{1}_{\xi\in[0,w_i^{(N)}(X_i^{(N)}(s-),s))}\, \nu_i(d\xi ds) \tag{1.1}$$

where $\mathbf{1}_B$ is the indicator function of an event B. If the jump rate is a constant $w_i^{(N)}(k,t) = w$, then (1.1) is equal to $\nu_i([0,w]\times(a,b])$ which further is equal in distribution to the Poisson distribution with mean $(b-a)\,w$, hence we see that the function $w_i^{(N)}(k,t)$ works as a 'density function' for the intensity measure of the Poisson random measure representing the random jump to rank 1 of the particle i. A precise definition of the stochastic ranking processes introduced so far is summarized in the stochastic differential equation (2.2) of Section 2.

In taking an infinite particle limit, we scale the ranking (regarded as position). We therefore assume $w_i^{(N)}(k,t) = w_i(\frac{k-1}{N}, t)$ for $N=1,2,3,\ldots$ with some function w_i on $[0,1]\times[0,\infty)$, and introduce the scaled rank $Y_i^{(N)}(t) = \frac{1}{N}(X_i^{(N)}(t)-1)$. Consider the joint empirical distribution of w_i and the normalized position, given by

$$\mu_t^{(N)} = \frac{1}{N}\sum_{i=1}^{N} \delta_{(w_i, Y_i^{(N)}(t))}\,,$$

where δ_c is a unit measure concentrated at c. $\mu_\cdot^{(N)}$ is a stochastic process taking values in the set of Borel probability measures.

The main result of this paper, stated informally, is the following. Assume that the initial configuration $\mu_0^{(N)}$ converges as $N \to \infty$ to a probability measure μ_0 . Then $\mu_\cdot^{(N)}$ converges almost surely as $N \to \infty$ to a deterministic probability-measure-valued time-evolution $\mu_\cdot$. Furthermore, for each integer L, the tagged particle system $(Y_1^{(N)}(t), Y_2^{(N)}(t), \dots, Y_L^{(N)}(t))$ converges to a limit process uniformly in $t \in [0,T]$ as $N \to \infty$, if the system of the initial scaled positions converges. The components of the limit are independent of each other. This fact implies that the propagation of chaos occurs in our model. A precise statement is given in Theorem 2.2 of Section 2.

The limit of the tagged-particle system is characterized by a stochastic differential equation which contains a quantity determined by the distribution function $U(dw, y, t) = \mu_t(dw \times [y, 1))$. (See (2.28) and (2.18) of Section 2.) The distribution function U, and consequently the limit measure μ_t, is characterized by a global Lipschitz solution to a system of quasilinear integral partial differential equations. A precise form of the system of equations which characterizes the limit measure is given in Theorem 2.1 of Section 2.

It has been found that the ranking numbers such as those found in the webpages of online retails, e.g., the sales ranks of books at the Amazon online bookstore, follow the predictions of the stochastic ranking processes (see Hattori and Hattori (2009a, 2010); Hattori (2011b,a)). In the ranking of books, at each time when a book is sold, its ranking spontaneously jumps to small numbers (relatively close to 1), regardless of how bad its previous position was (large $X_i^{(N)}(t)$, in our notation), and regardless of how unpopular (small $w_i^{(N)}$, in our notation) the book is. The stochastic process which we consider here: at each time when a book is sold its ranking jumps to 1 instantaneously, is a mathematical simplification of this observation. In view that the process is a model of such online and in real time rankings of a large number of items according to their popularity, we call the model the stochastic ranking processes.

One might guess that such a naive ranking rules of spontaneous jump to 1 at each sale, as in the definition of the stochastic ranking processes, will not be a good index for the popularity of books. But with a closer look, one notices that the well-sold books are dominant near the top position, while books near the tail position are rarely sold. Though the rankings of each book are stochastic with sudden jumps, the spatial distribution of the jump rates is more stable. On the side of bookstores, what matters is not a specific book, but the total of sales. This observation motivates us to consider the evolution of the joint empirical distribution of position and jump rates similarly to the hydrodynamic limit of realistic fluid, and to prove mathematically the expectation that the spatial distribution of the jump rates is deterministic at the limit as $N \to \infty$.

If the model is independent of spatial position, i.e., if $w_i(x,t)$'s are independent of x, then the law of the process reduces to that of Hariya et al. (2011, eq. (2)) and Nagahata (2010, eq. (1)), and the stochastic ranking process with time-dependent (but position-independent) intensities. Thus our process is an extension of Hariya et al. (2011); Nagahata (2010) to the case where the dynamics is dependent on the value of $X_i^{(N)}(t)$, i.e., to the position-dependent case.

If, furthermore, w_i's are positive constants, our process further reduces to the homogeneous case considered in Hattori and Hattori (2009b,a). A discrete-time version of the homogeneous case has been known in Tsetlin (1963), and has been extensively studied and is called move-to-front (MTF) rules in McCabe (1965); Hendricks (1972); Burville and Kingman (1973); Letac (1974); Kingman (1975). The process and its generalization have, in particular, been extensively studied in the field of information theory as a model of least-recently-used (LRU) caching (see Rivest (1976); Fagin (1977); Bitner (1979); Chung et al. (1988); Blom and Holst (1991); Rodrigues (1976); Fill (1996a); Fill and Holst (1996); Fill (1996b); Jelenković (1999); Sugimoto and Miyoshi (2006); Jelenković and Radovanović (2008); Barrera and Fontbona (2010); Hirade and Osogami (2010)), and also is noted as a time-reversed process of top-to-random shuffling.

In Hattori and Hattori (2009b,a); Hariya et al. (2011); Nagahata (2010), the explicit formula of the limit distributions of the joint empirical distributions of scaled position and the jump rate are found in the cases of position-independent jump rates. The limit is characterized by a solution to a system of inviscid Burgers-like equations with a term representing evaporation, in the terminology of fluid dynamics. The limit formula is successfully applied to the time developments of ranking numbers such as those found in the webpages of online bookstores (see Hattori and Hattori (2009a, 2010); Hattori (2011b)). Furthermore, convergence of the joint empirical distributions as a process and convergence of tagged-particle processes are proved in Nagahata (2010).

In the present paper, we mathematically extend the previous results to the case where the jump rates are both position and time dependent. A motivation for an online web-retail store to provide the sales ranks, in their webpages for public access, would be to give information on the popularity of each products which the store provides, to attract consumers' attention on popular products. We extend the previous results to the case of position-dependent jump rates corresponds to providing a mathematical framework for considering a possibility of such expected effect of popular products receiving extra attention and effectively increase their jump rates according to their rankings. For the case of position-dependent jump rates which we consider in this paper, the non-locality of the equation characterizing the limit is inevitable, because of the position-dependence of the jump rate functions. Hence, we need to consider a harder problem of a system of integral–differential equations compared with the previous cases.

The plan of the present paper is as follows. In Section 2 we give a mathematical formulation of the stochastic ranking process with space-time dependent intensities and state the main results. In Section 3 we prove Theorem 2.1, and in Section 4 we prove Theorem 2.2.

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2. Formulation and main results.

The precise formulation of the stochastic ranking process which we consider in this paper is as follows.

Let $\{\nu_i(d\xi ds)\}_{i=1,2,3,\dots}$ be independent Poisson random measures on $[0,\infty)\times[0,\infty)$ with the intensity measure $d\xi ds$. Let W be a set of non-negative valued C^1 functions

$$w:\ [0,1]\times[0,\infty)\to[0,\infty),$$

such that, for each $T>0$,

$$R_w(T):=\sup_{w\in W}\sup_{(y,t)\in[0,1]\times[0,T]}\max\left\{w(y,t),\ \left|\frac{\partial w}{\partial y}(y,t)\right|\right\}<\infty. \tag{2.1}$$

Let w_i, $i=1,2,\dots$ be a sequence in W, and for a positive integer N, put

$$w_i^{(N)}(k,t):=w_i(\frac{k-1}{N},t),\quad k=1,2,\dots,N,\ t\in[0,\infty),\ i=1,2,\dots,N.$$

Also, let $x_1^{(N)}$, $x_2^{(N)}$, ..., $x_N^{(N)}$ be a permutation of $1,2,\dots,N$. Let $(\Omega,\mathcal{F},P)$ be a probability space, and define a process

$$X^{(N)}=(X_1^{(N)},\dots,X_N^{(N)})$$

by

$$\begin{aligned}
&X_i^{(N)}(t)=x_i^{(N)}\\
&+\sum_{j=1}^{N}\int_{s\in(0,t]}\int_{\xi\in[0,\infty)}\mathbf{1}_{X_i^{(N)}(s-)<X_j^{(N)}(s-)}\,\mathbf{1}_{\xi\in[0,w_j^{(N)}(X_j^{(N)}(s-),s))}\,\nu_j(d\xi ds)\\
&+\int_{s\in(0,t]}\int_{\xi\in[0,\infty)}(1-X_i^{(N)}(s-))\,\mathbf{1}_{\xi\in[0,w_i^{(N)}(X_i^{(N)}(s-),s))}\,\nu_i(d\xi ds),\\
&i=1,2,\dots,N,\ t\geqq 0,
\end{aligned} \tag{2.2}$$

where, $\mathbf{1}_B$ is the indicator function of event B. The integrands in the (2.2) are predictable, hence the right hand side of (2.2) is well-defined as the Ito–integrals (see Ikeda and Watanabe (1989, §IV.9)).

As mentioned in Section 1, the function w_i on $[0,1]\times[0,\infty)$ is introduced to control the jump rate of $X_i^{(N)}$. Indeed, the mass of $\nu_i(d\xi ds)$ on the complement of $[0,w_i)\times(0,t]$ is ignored on the right-hand side of (2.2).

$X^{(N)}(t)$ is a permutation of $1,2,\dots,N$ for all $t\geqq 0$, which we regard as ranks or positions of particles $1,2,\dots,N$ at time t. Moreover, for $i=1,2,\dots,N$, and $t>t_0\geqq 0$, let

$$J_i^{(N)}(t_0,t)=\left\{\int_{s\in(t_0,t]}\int_{\xi\in[0,\infty)}\mathbf{1}_{\xi\in[0,w_i^{(N)}(X_i^{(N)}(s-),s))}\,\nu_i(d\xi ds)>0\right\}. \tag{2.3}$$

Then, the last term on the right hand side of (2.2) implies that $J_i^{(N)}(t_0,t)$ denotes the event that the particle i jumps to the top position ($X_i^{(N)}(s)=1$) in the time interval $(t_0,t]$. In other words, the last term of (2.2) represents the definition of the stochastic ranking process that the particle i jumps to the top rank $X_i^{(N)}(t)=1$ with jump rate determined by $w_i^{(N)}$. Also, the second term on the right hand side of (2.2) implies that on the complement $J_i^{(N)}(t_0,t)^c$ of $J_i^{(N)}(t_0,t)$,

$$X_i^{(N)}(s)-X_i^{(N)}(s-)=0\ \text{ or }\ 1, \tag{2.4}$$

for $t_0<s\leqq t$, where the latter occurs if and only if a particle k at tail side ($X_k^{(N)}(s)>X_i^{(N)}(s)$) jumps to top at time s. In other words, the second term of

(2.2) represents the increase in the ranking number of particle i, when the other particles at the lower rank (larger ranking number) jumps to rank 1.

We introduce the normalized position for each particle i at time t

$$Y_i^{(N)}(t) = \frac{1}{N}(X_i^{(N)}(t) - 1), \tag{2.5}$$

and consider the joint empirical distribution of jump rate and normalized position, given by

$$\mu_t^{(N)} = \frac{1}{N}\sum_{i=1}^{N} \delta_{(w_i, Y_i^{(N)}(t))}\,. \tag{2.6}$$

(We will denote a unit measure on any space by δ_c.) For each $T > 0$, $\mu_t^{(N)}$, $t \in [0,T]$, is regarded as a stochastic process taking values in the set of probability measures on $C^{1,0}([0,1]\times[0,T])\times[0,1]$. Here $C^{1,0}([0,1]\times[0,T])$ is the total set of functions $f \in C([0,1]\times[0,T])$ such that $\dfrac{\partial f}{\partial y} \in C([0,1]\times[0,T])$. Since $C^{1,0}([0,1]\times[0,T])$ is a Polish space with norm

$$\sup_{(y,t)\in[0,1]\times[0,T]} \left\{ |w(y,t)|,\ \left|\frac{\partial w}{\partial y}(y,t)\right| \right\},$$

so is $C^1([0,1]\times[0,T])\times[0,1]$ (see Bauer (2001, Example 26.2)). We assume a standard topology of weak convergence of probability measures on $C^1([0,1]\times[0,T])\times[0,1]$.

To prove convergence of measures, we work with a distribution function. For each integer N define

$$U^{(N)}(dw,y,t) = \mu_t^{(N)}(dw \times [y,1)) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{X_i^{(N)}(t) \geqq Ny+1}\ \delta_{w_i}(dw)\,, \qquad 0 \leqq y \leqq 1,\ t \geqq 0. \tag{2.7}$$

For each (y,t), $U^{(N)}(\cdot,y,t)$ is a Borel measure on W. Note that $U^{(N)}(dw,y,t)$ is non-increasing in y and satisfies

$$\int_W U^{(N)}(dw,y,t) = \frac{[N\,(1-y)]}{N}\,, \quad 0 \leqq y \leqq 1,\ t \geqq 0, \tag{2.8}$$

where, for real z, $[z]$ is the largest integer not exceeding z.

As an analog of the corresponding results in Hattori and Hattori (2009a); Hariya et al. (2011), the infinite-particle scaling limit U of $U^{(N)}$ turns out to be characterized by a system of inviscid Burgers-like integral–partial differential equations with evaporation terms. Denote the set of 'boundary points' and of 'initial points' by

$$\Gamma_b = \{(0,t_0) \mid t_0 \geqq 0\}, \tag{2.9}$$

and

$$\Gamma_i = \{(y_0,0) \mid 0 \leqq y_0 \leqq 1\}, \tag{2.10}$$

respectively, and put

$$\Gamma = \Gamma_b \cup \Gamma_i\,. \tag{2.11}$$

Also, for $t \geqq 0$ put

$$\Gamma_t = \{(y_0,t_0) \in \Gamma \mid t_0 \leqq t\} = \Gamma_i \cup \{(0,t_0) \mid 0 \leqq t_0 \leqq t\}. \tag{2.12}$$

Theorem 2.1. *Let λ be a Borel probability measure on W, and $\rho:\ W\times[0,1]\to[0,1]$ be a non-negative Borel measurable function continuous in y, such that $\dfrac{\partial\rho}{\partial y}(w,y)$ exists for almost all y, is bounded, satisfying*

$$\frac{\partial\rho}{\partial y}(w,y)\leqq 0,\ \text{for almost all } (y,w)\in[0,1]\times W, \tag{2.13}$$

$\rho(w,0)=1$ and $\rho(w,1)=0$ for $w\in W$. Define a Borel measure on W with parameter $y\in[0,1]$ by

$$U_0(dw,y)=\rho(w,y)\,\lambda(dw),\quad y\in[0,1],\ w\in W. \tag{2.14}$$

In particular, $U_0(dw,0)=\lambda(dw)$. Assume also

$$U_0(W,y)=\int_W U_0(dw,y)=1-y,\ 0\leqq y\leqq 1. \tag{2.15}$$

Then there exists a unique pair of functions

$$y_C:\ \{(\gamma,t)\in\Gamma\times[0,\infty)\mid\gamma\in\Gamma_t\}\to[0,1],$$

and $U=U(dw,y,t)$ on $[0,1]\times[0,\infty)$ taking values in the non-negative Borel measures on W such that

(i) *$y_C(\gamma,t)$ and $\dfrac{\partial y_C}{\partial t}(\gamma,t)$ are continuous,*
(ii) *for each $t>0$, $y_C(\cdot,t):\ \Gamma_t\to[0,1]$ is surjective,*
(iii) *for all bounded continuous $h:\ W\to\mathbb{R}$, $U(h,y,t):=\int_W h(w)U(dw,y,t)$ is Lipschitz continuous in $(y,t)\in[0,1]\times[0,T]$ for any $T>0$, and non-increasing in y, and*
(iv) *the following (2.16), (2.17), (2.19), and (2.20) hold:*

$$y_C(\gamma,t_0)=y_0\quad\text{and}\quad U(dw,y_0,t_0)=U_0(dw,y_0),\quad\gamma=(y_0,t_0)\in\Gamma, \tag{2.16}$$

$$U(h,y_C(\gamma,t),t)=U_0(h,y_0)-\int_{t_0}^t V(h,y_C(\gamma,s),s)\,ds,\quad t\geqq t_0,\ \gamma=(y_0,t_0)\in\Gamma, \tag{2.17}$$

for all bounded continuous function $h:\ W\to\mathbb{R}$, where

$$U(h,y,t):=\int_W h(w)U(dw,y,t),$$

and

$$V(h,y,t)=\int_W h(w)\,w(y,t)\,U(dw,y,t)+\int_y^1\int_W h(w)\,\frac{\partial w}{\partial z}(z,t)\,U(dw,z,t)\,dz, \tag{2.18}$$

and

$$\frac{\partial y_C}{\partial t}(\gamma,t)=V(\mathbf{1}_W,y_C(\gamma,t),t),\quad t\geqq t_0,\ \gamma=(y_0,t_0)\in\Gamma, \tag{2.19}$$

where $\mathbf{1}_W(w)=1$ for all $w\in W$, and

$$U(\mathbf{1}_W,y,t)=1-y,\quad 0\leqq y\leqq 1,\ t\geqq 0. \tag{2.20}$$

◇

The claim (2.20), together with continuity and monotonicity of U, implies that U determines a Borel probability measure μ_t on the direct product $W \times [0,1]$ with parameter t by

$$U(dw,y,t) = \mu_t(dw \times [y,1)), \quad 0 \leqq y \leqq 1,\ t \geqq 0. \tag{2.21}$$

If $U(h,y,t)$ in Theorem 2.1 is C^1 in a neighborhood of $(y,t) \in (0,1) \times (0,\infty)$, then differentiating (2.17) by t, using (2.19), and noting that $y_C(\cdot,t) :\ \Gamma_t \to [0,1]$ is surjective, we have

$$\frac{\partial U}{\partial t}(h,y,t) + V(\mathbf{1}_W,y,t)\,\frac{\partial U}{\partial y}(h,y,t) = -V(h,y,t), \tag{2.22}$$

where V is as in (2.18). y_C in (2.19) determines the characteristic curves for (2.22). In terms of Bressan (2005, §3.4), we can therefore say that Theorem 2.1 claims global existence of the Lipschitz solution (broad solution which is Lipschitz continuous) to the system of quasilinear partial differential equations (2.22), with components parametrized by (possibly continuous) w. To be more precise, we have extended the definition in Bressan (2005, §3.4) of Lipschitz solution for (2.22) to the non-local case (see (2.18)), and for the case where $V(\mathbf{1}_W,y,t)$ in the left-hand side of Theorem 2.1 is common for all h. We have also generalized the notion of domain of determinancy defined in Bressan (2005, §3.4), which in the present case corresponds to

$$\{(y,t) \in [0,1] \times [0,\infty) \mid y \geqq y_C((0,0),t)\},$$

to the domain determined by boundary conditions

$$\{(y,t) \in [0,1] \times [0,\infty) \mid y < y_C((0,0),t)\},$$

with initial data $U(h,\cdot,0) = U_0(h,\cdot)$ and the boundary condition $U(h,0,t) = U_0(h,0)$, $t \geqq 0$, as obtained in (2.16).

As an example, where the jump rates are finitely many space-time constants, we can identify W (the space of jump rate functions) with the finite set of the constant jump rates $\tilde{W} = \{w^{(1)}, w^{(2)}, \ldots, w^{(A)}\}$ for some positive integer A, and the distribution of the jump rates $U_0(\cdot,0) = \lambda(\cdot)$ can be identified with

$$\lambda = \sum_{a=1}^{A} r^{(a)}\,\delta_{w^{(a)}}, \tag{2.23}$$

for some positive constants $r^{(a)}$, $a = 1,2,\ldots,A$, satisfying $\displaystyle\sum_{a=1}^{A} r^{(a)} = 1$. In this example, (2.22) reduces to

$$\frac{\partial U_a}{\partial t}(y,t) + \sum_{b=1}^{A} w^{(b)} U_b(y,t)\,\frac{\partial U_a}{\partial y}(y,t) = -w^{(a)} U_a(y,t), \tag{2.24}$$

where we wrote $U_a(y,t) = U(h_a,y,t)$ and $V_a(y,t) = V(h_a,y,t) = w^{(a)}U_a(y,t)$. Here, $h_a :\ W \to \mathbb{R}$ is defined by $h_a(w^{(a)}) = 1$ and $h_a(w^{(b)}) = 0$, if $b \neq a$, where we identified W with $\tilde{W}$. In particular, $\displaystyle\sum_{a=1}^{A} h_a = \mathbf{1}_W$, so that $V(\mathbf{1}_W,y,t) = \displaystyle\sum_{a=1}^{A} w^{(a)} U_a(y,t)$.

The example of the limit distribution determined by (2.23) can be realized from the stochastic ranking process as follows. To simplify the notation, we give the case $A = 2$ and $r^{(1)} = \dfrac{q}{p}$ for positive integers p and q satisfying $p > q$. Given $i \geqq 1$, consider the decomposition $i = m\,p + r$, where m and r are non-negative integers satisfying $0 \leqq r < p$. Then let $w_i = w_{m\,p+r} = w^{(1)}$ for $m = 0, 1, 2, \ldots$ and $r = 0, 1, \ldots, q-1$, and $w_i = w_{m\,p+r} = w^{(2)}$ for $m = 0, 1, 2, \ldots$ and $r = q, q+1, \ldots, p-1$. This determines the jump rates for all the particles $i = 1, 2, \ldots$, hence, with an initial configuration $(x_1, x_2, \ldots, x_N)$ the stochastic ranking process is defined. We see that the limit $U_0(\cdot, 0) = \lambda(\cdot)$ of the distribution of jump rates is given in this example by (2.23) for $A = 2$, $r^{(1)} = \dfrac{q}{p}$ and $r^{(2)} = 1 - \dfrac{q}{p}$. An extension to general A and $(r^{(1)}, r^{(2)}, \ldots, r^{(A)})$ should be clear.

If $A = 1$ and the right hand side of (2.24) is 0, the partial differential equation is known as the inviscid Burgers equation, in the terminology of fluid dynamics. In terms of fluid dynamics, the right hand side of (2.24) could be interpreted as the evaporation of the fluid. That this term is equal (with $h = \mathbf{1}_W$) to the negative velocity of the fluid, the coefficient to the y-derivative of U in the left hand side of (2.22), implies that the motion of fluid is fully driven by the evaporation. This intuitively implies that there are no 'shock waves' in our model (see Hattori and Hattori (2009a)). This is perhaps in contrast with a general interest where Burgers equations are noted for the existence of shock waves.

The case of constant jump rates (2.24) can be solved explicitly by using characteristic curves (see Hattori and Hattori (2009a)). The results of the case that the jump rates are time-dependent, i.e. the jump rates $w^{(b)}$ are changed to the functions $w^{(b)}(t)$ in (2.24), which also can be solved by using characteristic curves, and the solution is given in Hariya et al. (2011). For the case (2.22) which we consider in this paper, the non-locality of interaction is inevitable, because of the position dependence of the jump rate functions. Hence, we need to consider a harder problem of a system of differential–integral equations compared with the previous cases. In fact, the case corresponding to (2.24) with position dependence on $w^{(a)}$ leads to (assuming regularity on solution)

$$\frac{\partial U_a}{\partial t}(y,t) - \sum_{b=1}^{A} \int_y^1 w^{(b)}(z,t)\, \frac{\partial U_b}{\partial z}(z,t)\, dz\, \frac{\partial U_a}{\partial y}(y,t) = \int_y^1 w^{(a)}(z,t)\, \frac{\partial U_a}{\partial z}(z,t)\, dz. \tag{2.25}$$

Though we have no hope to obtain solutions explicitly, we will show in Section 3 that we can nevertheless prove uniqueness and existence of the global solution, using the characteristic curves y_C.

Now we give a norm of measures in order to state the limit theorem. Let $||\cdot||_{\mathrm{var}}$ be the total variation norm for Borel measures on W, i.e. for a signed measure μ on W define $||\mu||_{\mathrm{var}}$ by

$$||\mu||_{\mathrm{var}} = \mu^+(W) + \mu^-(W),$$

where μ^+ and μ^- are the positive part and the negative part obtained by Hahn-Jordan decomposition of μ respectively.

We consider a scaling limit of the stochastic ranking process as $N \to \infty$, the limit for the number of particles to infinity. The result obtained in this paper is a non-trivial limit theorem of the law of large numbers for dependent variables.

A non-trivial dependence is suggested by the fact, which we state in the following Theorem, that the limit distribution satisfies (2.17) or more intuitively, non-linear equations (2.22). The previous results in Hattori and Hattori (2009b); Hariya et al. (2011) are also the law of large numbers for dependent variables, but in the previous results, where the jump rate functions are independent of spatial positions, a special combination of quantities $(U^{(N)}(B, Y_C^{(N)}(y_0, t_0, t), t)$, in terms of notations in Section 4, turns out to be a sum of independent random variables. However, the position-dependence of jump rates, as considered in the present paper, implies that the dependence of random variables are built-in in the model, so that the proofs in Hattori and Hattori (2009b); Hariya et al. (2011) do not work in the present case. Inspired partly by Nagahata (2010), where the case of finite types of position independent particles are proved (see Nagahata (2010, Prop. 1.1 and Thm. 1.2)), we extend his result to our position-dependent case, and obtain a convergence of empirical distribution and also the limiting dynamics of fixed finite particles (tagged particles) for the case of jump rate functions with space-time dependence as follows.

Theorem 2.2. *Assume that with probability* 1,

$$\lim_{N\to\infty} \sup_{y\in[0,1)} ||U^{(N)}(\cdot, y, 0) - U_0(\cdot, y)||_{\mathrm{var}} = 0, \tag{2.26}$$

where $U_0(dw, y)$ satisfies all the assumptions in Theorem 2.1. Then the following hold.

(i) *With probability* 1, *for all $T > 0$, $\lim_{N\to\infty} U^{(N)}(dw, y, t) = U(dw, y, t)$, uniformly in $y \in [0, 1)$ and $t \in [0, T]$, where U is the solution claimed in Theorem 2.1.*

(ii) *Assume in addition that,*

$$\lim_{N\to\infty} \frac{1}{N} x_i^{(N)} = y_i\,, \quad i = 1, 2, \ldots, L, \tag{2.27}$$

for a positive integer L and $y_i \in [0, 1)$, $i = 1, 2, \ldots, L$. Then, with probability 1, *for all $T > 0$, the tagged particle system*

$$(Y_1^{(N)}(t), Y_2^{(N)}(t), \ldots, Y_L^{(N)}(t))$$

converges as $N \to \infty$, uniformly in $t \in [0, T]$ to a limit $(Y_1(t), Y_2(t), \ldots, Y_L(t))$. Here, for each $i = 1, 2, \ldots, L$, Y_i is the unique solution to

$$\begin{aligned} Y_i(t) = y_i &+ \int_0^t V(\mathbf{1}_W, Y_i(s-), s)ds \\ &- \int_{s\in(0,t]} \int_{\xi\in[0,\infty)} Y_i(s-)\, \mathbf{1}_{\xi\in[0,w_i(Y_i(s-),s))}\, \nu_i(d\xi ds), \end{aligned} \tag{2.28}$$

where, V is as in (2.18).

◇

Theorem 2.2 implies propagation of chaos for the stochastic ranking processes. For each N all of $\{Y_i^{(N)}\}$ are random and interact with each other and $U^{(N)}(dw, y, t)$ is also random. However, the limit $U(dw, y, t)$ is deterministic. Furthermore, the randomness of the limit process Y_i of the tagged particles depends only on its own Poisson random measure ν_i, and is independent of Y_j or ν_j with $j \neq i$. Indeed, in the proof of Theorem 2.2 (Section 4) we focus at martingale terms $M_U^{(N)}$ and $M_i^{(N)}$ (definitions are in (4.8) and (4.29), respectively) and show that they converge to 0 in sense of square martingales. This implies that the fluctuation disappears as the number of particles goes to infinity. In this sense, Theorem 2.2 can be regarded as a law of large numbers.

We give one more remark on Theorem 2.2. By taking the role of V for y_C into account, we have that the tagged particle Y_i in the limit system behaves just as the characteristic curve $y_C((y_i,0),\cdot)$ before the particle's own jump occurs. So, we can approximately know y_C by checking the behavior of one particle $Y_i^{(N)}$ in the N-particle system. Note also that a discrete correspondence $Y_C^{(N)}$ of the characteristic curves y_C is defined in (4.1), which has been a key quantity of the limit theorems since Hattori and Hattori (2009b).

When $\{w_i; i=1,2,3,\ldots\}$ is a finite set of W, because of Proposition 5.1 in Appendix, we obtain the following corollary easily.

Corollary 2.3. *When $w_i \in \{\tilde{w}_\alpha \in W; \alpha = 1,2,\ldots,A\}$ for $i=1,2,3,\ldots$, the assumption (2.26) of Theorem 2.2 is relaxed as follows:*

$$\lim_{N\to\infty} U^{(N)}(\{\tilde{w}_\alpha\},y,0) = U(\{\tilde{w}_\alpha\},y,0), \quad \textit{for each} \ \ y\in[0,1)$$

with probability 1 *for* $\alpha = 1,2,\ldots,A$.

3. Proof of Theorem 2.1.

A basic idea of the proof of existence of the solution $U(dw,y,t)$ is, as in the standard quasilinear partial differential equations (see Bressan (2005)), to construct the solution along a characteristic curve $y = y_C(\gamma,t)$, which is a curve that a 'fluid particle' starting from an initial point γ moves along under the dynamics of the partial differential equation. For each starting point γ we put $f(y_0,t) = y_C(\gamma,t)$, if the starting point is in the initial line $t=0$; and put $\gamma = (y_0,0)$ or $g(t_0,t) = y_C(\gamma,t)$, if the starting point is in the boundary, i.e. $\gamma = (0,t_0)$ (see (3.21)). Since we deal with differential–integral equations containing non-local terms (for terms with integration, see (2.18)), the characteristic curves are not obtained explicitly. This makes the argument technically complicated, and much of the argument in this section deals with existence proof of the characteristic curves y_C by using iteration methods, and derivation of basic properties of y_C. In particular, $g(t_0,t)$, the characteristic curve starting from the boundary, is dependent on all the other characteristic curves. Hence we need to prepare some lemmas before proving existence of $g(t_0,t)$ in Lemma 3.4. Once y_C is proved to exist, we are more or less along a standard line to find $\varphi(dw,\gamma,t) = U(dw,y_C(\gamma,t),t)$ the solution observed along the characteristic curve (see (3.25)).

Consider first the case $(y_0,t_0)\in\Gamma_i$, namely, the case $t_0 = 0$.

Lemma 3.1. *There exists a unique C^1 function $f:\ [0,1]\times[0,\infty)\to[0,1]$ which satisfies*

$$f(y,t) = 1 + \int_W \left(\int_y^1 \frac{\partial\rho}{\partial z}(w,z)\,\exp(-\int_0^t w(f(z,s),s)\,ds)\,dz \right) U_0(dw,0), \qquad (3.1)$$
$$y\in[0,1],\ t \geqq 0,$$

where ρ and U_0 are as in the assumptions of Theorem 2.1. ◇

Proof. For $k\in\mathbb{Z}_+$, define $f_k:\ [0,1]\times[0,\infty)\to[0,1]$ inductively by

$$f_0(y,t) = 1 + \int_W \left(\int_y^1 \frac{\partial\rho}{\partial z}(w,z)\,\exp(-\int_0^t w(z,s)\,ds)\,dz \right) U_0(dw,0),$$

and

$$
\begin{aligned}
&f_{k+1}(y,t) = 1 \\
&+ \int_W \left(\int_y^1 \frac{\partial \rho}{\partial z}(w,z) \, \exp(- \int_0^t w(f_k(z,s),s)\, ds)\, dz \right) U_0(dw,0), \qquad (3.2)\\
&k \in \mathbb{Z}_+.
\end{aligned}
$$

Assume that f_k is continuous and takes values in $[0,1]$. Then (3.2) is well-defined. This with (2.13) implies $f_{k+1} \leqq 1$. Similarly, using also (2.14) and (2.15),

$$
\begin{aligned}
&f_{k+1}(y,t) \\
&\geqq 1 + \int_W \left(\int_y^1 \frac{\partial \rho}{\partial z}(w,z)\, dz \right) U_0(dw,0) = 1 + \int_W U_0(dw,1) - \int_W U_0(dw,y) = y \\
&\geqq 0.
\end{aligned}
$$

By assumption of Theorem 2.1, $\dfrac{\partial \rho}{\partial z}(w,z)$ is bounded almost surely, hence (3.2) implies that f_{k+1} is continuous. By induction, f_k is continuous and takes values in $[0,1]$, for all k.

For $k \in \mathbb{Z}_+$, put $F_k(y,t) = |f_{k+1}(y,t) - f_k(y,t)|$. Then, using (2.1) and the assumptions of Theorem 2.1 as above, we have

$$
F_{k+1}(y,t) \leqq R_w(T) \int_y^1 \int_0^t F_k(z,s)\, ds\, dz, \quad y \in [0,1],\ t \in [0,T],\ k \in \mathbb{Z}_+, \qquad (3.3)
$$

for any $T > 0$. Since all f_k's are continuous and take values in $[0,1]$, F_k, $k = 1,2,\ldots$, are also continuous and take values in $[0,1]$. Then it holds by the argument of Bressan (2005, §3.8, Lemma 3.4), that

$$
0 \leqq F_k(y,t) \leqq e^{2R_w(T)t} 2^{-k}, \quad y \in [0,1],\ t \in [0,T],\ k \in \mathbb{Z}_+. \qquad (3.4)
$$

In fact, since F_0 takes values in $[0,1]$, (3.4) holds for $k = 0$. Assume (3.4) holds for some k. Then (3.3) implies

$$
F_{k+1}(y,t) \leqq 2^{-k} R_w(T) \int_y^1 \int_0^t e^{2R_w(T)s}\, ds \leqq e^{2R_w(T)t} 2^{-k-1}, \quad 0 \leqq y < 1,\ 0 \in [0,T].
$$

By induction, (3.4) holds for all $k \in \mathbb{Z}_+$. In particular, $f_0(y,t) + \sum_{k=0}^{\infty} F_k(y,t)$ converges uniformly in (y,t) for any bounded range of t. Hence, $f_k(y,t) = f_0(y,t) + \sum_{j=0}^{k-1} (f_{j+1}(y,t) - f_j(y,t))$ converges as $k \to \infty$ to a function, continuous in y and t. Let

$$
f(y,t) = \lim_{k\to\infty} f_k(y,t), \quad y \in [0,1],\ t \geqq 0.
$$

Then (3.2) implies that f satisfies (3.1). Also, $0 \leqq f_k \leqq 1$ implies

$$
0 \leqq f(y,t) \leqq 1,\ 0 \leqq y \leqq 1,\ t \geqq 0. \qquad (3.5)
$$

The right hand side of (3.1), with the assumptions in Theorem 2.1 implies that $f(y,t)$ is C^1.

Next, we prove the uniqueness. Suppose for $i = 1, 2$, $f^{(i)} : [0,1] \times [0,\infty) \to [0,1]$ are continuous functions which satisfy (3.1). Then $|f^{(1)}(y,0) - f^{(2)}(y,0)| = 0$ and, as above, for each $T > 0$,

$$|f^{(1)}(y,t) - f^{(2)}(y,t)| \leqq R_w(T) \int_y^1 \int_0^t |f^{(1)}(z,s) - f^{(2)}(z,s)|\, ds\, dz,$$
$$y \in [0,1],\ t \in [0,T],$$

which implies $f^{(1)} = f^{(2)}$. □

Next, consider the case $(y_0, t_0) \in \Gamma_b$, namely, the case $y_0 = 0$.

Lemma 3.2. *For each continuous function* $\tilde{g} : \{(s,t) \in [0,\infty)^2 \mid 0 \leqq s \leqq t\} \to [0,1]$*, there exists a unique non-negative function* $\eta : W \times [0,\infty) \to [0,\infty)$*, integrable with respect to* $U_0(dw,0)$*, continuous in the second variable, which satisfy, for each* $w \in W$*,*

$$\begin{aligned}\eta(w,t) = &\int_0^t \eta(w,u)\, w(\tilde{g}(u,t),t)\, \exp(-\int_u^t w(\tilde{g}(u,v),v)\, dv)\, du \\ &- \int_0^1 \frac{\partial \rho}{\partial z}(w,z)\, w(f(z,t),t)\, \exp(-\int_0^t w(f(z,v),v)\, dv)\, dz, \\ t \geqq 0,&\end{aligned} \tag{3.6}$$

where ρ *is as in the assumption of Theorem 2.1, and* f *is the function given by Lemma 3.1.*

Moreover, it holds that

$$\begin{aligned}&\int_0^t \eta(w,u)\, \exp(-\int_u^t w(\tilde{g}(u,v),v)\, dv)\, du \\ &= 1 + \int_0^1 \frac{\partial \rho}{\partial z}(w,z) \exp(-\int_0^t w(f(z,s),s)\, ds)\, dz.\end{aligned} \tag{3.7}$$

In particular, for any $T > 0$*, there exists* $C(T) > 0$*, which is independent of* $\tilde{g}$*, such that*

$$0 \leqq \int_W \eta(w,t)\, U_0(dw,0) \leqq C(T),\ 0 \leqq t \leqq T. \tag{3.8}$$

◇

Proof. Define a sequence of continuous functions $\eta_k : W \times [0,\infty) \to [0,\infty)$, $k = 0, 1, 2, \ldots$, inductively, by

$$\eta_0(w,t) = 0, \quad w \in W,\ t \geqq 0,$$

and

$$\begin{aligned}\eta_{k+1}(w,t) = &\int_0^t \eta_k(w,u)\, w(\tilde{g}(u,t),t)\, \exp(-\int_u^t w(\tilde{g}(u,v),v)\, dv)\, du \\ &- \int_0^1 \frac{\partial \rho}{\partial z}(w,z)\, w(f(z,t),t)\, \exp(-\int_0^t w(f(z,v),v)\, dv)\, dz.\end{aligned} \tag{3.9}$$

For $k \in \mathbb{Z}_+$ put $H_k(t) = \displaystyle\int_W |\eta_{k+1}(w,t) - \eta_k(w,t)|\, U_0(dw,0)$. Non-negativity of $w \in W$ and (2.1) imply

$$H_{k+1}(t) \leqq R_w(T) \int_0^t H_k(u)\, du,\ 0 \leqq t \leqq T.$$

Bressan (2005, §3.8, Lemma 3.4) implies that there exists a positive constant $C(T)$ such that

$$H_k(t) \leqq C(T)\, 2^{-k},\ t \in [0,T],\ k \in \mathbb{Z}_+,$$

hence, as in the proof of Lemma 3.1, $\eta(w,t) = \lim_{k\to\infty} \eta_k(w,t)$ exists, is continuous, non-negative, and satisfies (3.6). Integrability inductively follows from (3.9) by

$$\begin{aligned}
&\sup_{t\in[0,T]} \int_W \eta_{k+1}(w,t)\, U_0(dw,0) \\
&\leqq R_w(T) \sup_{t\in[0,T]} \int_W \int_0^t \eta_k(w,u)\, du\, U_0(dw,0) \\
&+ R_w(T) \int_W \int_0^1 \left(-\frac{\partial \rho}{\partial z}(w,z) \right) dz\, U_0(dw,0) \\
&= R_w(T) \sup_{t\in[0,T]} \int_W \int_0^t \eta_k(w,u)\, du\, U_0(dw,0) + R_w(T),
\end{aligned}$$

where we also used (2.13), (2.14) and (2.15).

Next, we prove the uniqueness. Suppose for $i = 1,2$, $\eta^{(i)} : W \times [0,\infty) \to [0,\infty)$ are functions, continuous in the second variable and satisfy (3.6). Then $|\eta^{(1)}(w,0) - \eta^{(2)}(w,0)| = 0$ and, as above, for each $T > 0$,

$$|\eta^{(1)}(w,t) - \eta^{(2)}(w,t)| \leqq R_w(T) \int_0^t |\eta^{(1)}(w,s) - \eta^{(2)}(w,s)|\, ds \quad t \in [0,T],$$

which implies $\eta^{(1)} = \eta^{(2)}$.

Changing the variable t in (3.6) to s, and then integrating from 0 to t, and changing the order of integration in the first term on the right hand side, we have

$$\begin{aligned}
&\int_0^t \eta(w,s)\, ds \\
&\quad = -\int_0^t \eta(w,u) \left(\int_u^t \frac{\partial}{\partial s} \exp(-\int_u^s w(\tilde{g}(u,v),v)\, dv) ds \right) du \\
&\qquad + \int_0^1 \frac{\partial \rho}{\partial z}(w,z) \left(\int_0^t \frac{\partial}{\partial s} \exp(-\int_0^s w(f(z,v),v)\, dv) ds \right) dz \\
&\quad = \int_0^t \eta(w,u) \left(1 - \exp(-\int_u^t w(\tilde{g}(u,v),v)\, dv) \right) du \\
&\qquad - \int_0^1 \frac{\partial \rho}{\partial z}(w,z) \left(1 - \exp(-\int_0^t w(f(z,v),v)\, dv) \right) dz,
\end{aligned}$$

which, with $\rho(w,0) = 1$ and $\rho(w,1) = 0$, proves (3.7).

Combining (3.6) and (2.1), together with $\dfrac{\partial \rho}{\partial z}(w,z) \leqq 0$, $\rho(w,0) = 1$ and $\rho(w,1) = 0$, we see that

$$\int_W \eta(w,t)\, U_0(dw,0) \leqq R_w(T) \int_0^t \int_W \eta(w,u)\, U_0(dw,0)\, du + R_w(T).$$

Bressan (2005, §3.8, Lemma 3.4) again implies that there exists $C(T) > 0$, independent of $\tilde{g}$, such that $\displaystyle\int_W \eta(w,t)\, U_0(dw,0) \leqq C(T)$, $0 \leqq t \leqq T$. □

Corollary 3.3. *For $i = 1, 2$, let η_i be η in Lemma 3.2 with g_i in place of $\tilde{g}$, respectively. Then, for each $T > 0$ there exists a positive constant $C(T)$ such that*

$$\int_W |\eta_1(w,t) - \eta_2(w,t)|\, U_0(dw,0) \leqq C(T) \int_0^t \sup_{v\in[u,T]} |g_1(u,v) - g_2(u,v)|\, du. \quad (3.10)$$

◇

Proof. Put

$$\Delta\eta(t) = \int_W |\eta_1(w,t) - \eta_2(w,t)|\, U_0(dw,0)$$

and

$$\Delta g(u) = \sup_{v\in[u,T]} |g_1(u,v) - g_2(u,v)|.$$

Lemma 3.2, in particular, (3.6), (3.8), and (2.1), implies that

$$\Delta\eta(t) \leqq C_1(T) \int_0^t \Delta\eta(u)\, du + C_2(T) \int_0^t \Delta g(u)\, du, \;\; t \in [0,T],$$

for each T and for positive constants $C_i(T)$, $i = 1, 2$. Hence

$$\Delta\eta(t) \leqq C_2(T) \int_0^t e^{C_1(T)(t-s)} \Delta g(s)\, ds \leqq C_2(T) e^{TC_1(T)} \int_0^t \Delta g(s)\, ds,$$

which implies (3.10). □

Lemma 3.4. *There exists a unique C^1 function $g : \{(s,t) \in [0,\infty)^2 \mid 0 \leqq s \leqq t\} \to [0,1]$ such that*

$$\begin{aligned} g(s,t) = 1 &+ \int_W \int_0^1 \frac{\partial \rho}{\partial z}(w,z) \exp(-\int_0^t w(f(z,u),u)\, du)\, dz\, U_0(dw,0) \\ &- \int_W \int_0^s \eta(w,u) \exp(-\int_u^t w(g(u,v),v)\, dv)\, du\, U_0(dw,0), \end{aligned} \quad (3.11)$$

$0 \leqq s \leqq t.$

Here, $f(s,t)$ is defined in (3.1) and η is the function given by Lemma 3.2 with g in place of $\tilde{g}$. ◇

Proof. For $k \in \mathbb{Z}_+$, define a sequence of functions, g_k and η_k, inductively by $g_0(s,t) = 1$, $0 \leqq s \leqq t$, and, for $k \in \mathbb{Z}_+$, η_k the function η in Lemma 3.2 with g_k in place of $\tilde{g}$, and

$$\begin{aligned} g_{k+1}(s,t) = 1 &+ \int_W \int_0^1 \frac{\partial \rho}{\partial z}(w,z) \exp(-\int_0^t (f(z,u),u)\, du)\, dz\, U_0(dw,0) \\ &- \int_W \int_0^s \eta_k(u) \exp(-\int_u^t w(g_k(u,v),v)\, dv)\, du\, U_0(dw,0), \end{aligned} \quad (3.12)$$

$0 \leqq s \leqq t.$

Note that (2.13) and $\eta_k(w,z) \geqq 0$ implies $g_k(s,t) \leqq 1$, and that (3.7) and (2.15), with $\eta_k(w,z) \geqq 0$ imply

$$1 - g_k(s,t) \leqq \int_W \rho(w,0)\, U_0(dw,0) = 1.$$

Hence by $0 \leqq g_k(s,t) \leqq 1$, η_k is well-defined.

Put $\Delta g_k = |g_{k+1} - g_k|$ and $\Delta \eta_k = |\eta_{k+1} - \eta_k|$. Repeating the arguments in Lemma 3.1 or Lemma 3.2, we see that (3.12) implies, with (3.8),

$$\Delta g_{k+1}(s,t) \leqq \int_W \int_0^s \Delta\eta_k(w,u)\,du\,U_0(dw,0) + C_1(T)\int_0^s \left(\int_u^t \Delta g_k(u,v)\,dv\right)du,$$

for $0 \leqq s \leqq t \leqq T$, where $C_1(T)$ is a positive constant. Putting $G_k(s) = \sup_{t\in[s,T]} \Delta g_k(s,t)$, we have, with Corollary 3.3,

$$\begin{aligned} G_{k+1}(s) &\leqq C_2(T)\int_0^s \left(\int_0^u G_k(v)\,dv\right)du + T\,C_1(T)\int_0^s G_k(u)\,du \\ &\leqq (C_2(T) + C_1(T))\,T\int_0^s G_k(u)\,du, \end{aligned}$$

where $C_2(T)$ is a positive constant. As in the proof of Lemma 3.1 or Lemma 3.2, this implies that the limit $g = \lim_{k\to\infty} g_k$ exists and is continuous. Also, $0 \leqq g_k(s,t) \leqq 1$ implies

$$0 \leqq g(s,t) \leqq 1,\ t \geqq s \geqq 0. \tag{3.13}$$

Then $\eta = \lim_{k\to\infty} \eta_k$ also exist and are continuous, and these functions satisfy (3.6) with g in place of $\tilde{g}$, and (3.11). C^1 properties follow from the right hand side of (3.11), and uniqueness also follows as in the proof of Lemma 3.2. □

Corollary 3.5. *The following hold.*

$$f(y,0) = y, \quad y \in [0,1]. \tag{3.14}$$

$$\frac{\partial f}{\partial y}(y,t) > 0,\ \frac{\partial f}{\partial t}(y,t) \geqq 0, \quad (y,t) \in [0,1]\times[0,\infty). \tag{3.15}$$

$$\frac{\partial g}{\partial s}(s,t) \leqq 0,\ \frac{\partial g}{\partial t}(s,t) \geqq 0, \quad 0 \leqq s \leqq t. \tag{3.16}$$

$$g(t,t) = 0, \quad t \geqq 0. \tag{3.17}$$

$$g(0,t) = f(0,t),\ t \geqq 0. \tag{3.18}$$

◇

Proof. The claims on f, (3.14) and (3.15), are consequences of (3.1) and the assumptions in Theorem 2.1. The only perhaps less obvious claim is that the derivative of f in y cannot be 0 in (3.15), which follows from (2.13) and (2.1), with

$$\begin{aligned} &\frac{\partial f}{\partial y}(y,t) \\ &\geqq -e^{-T\,R_w(T)}\int_W \frac{\partial \rho}{\partial y}(w,y)\,U_0(dw,0) = -e^{-T\,R_w(T)}\frac{\partial}{\partial y}U_0(\mathbf{1}_W, y) = e^{-T\,R_w(T)} \\ &> 0. \end{aligned}$$

Differentiating (3.7) with $\tilde{g}$ replaced by g,

$$\begin{aligned} &\eta(w,t) - \int_0^t \eta(w,u)\,w(g(u,t),t)\,\exp\Big(-\int_u^t w(g(u,v),v)\,dv\Big)\,du \\ &= -\int_0^1 \frac{\partial \rho}{\partial z}(w,z)\,w(f(z,t),t)\,\exp\Big(-\int_0^t w(f(z,s),s)\,ds\Big)\,dz. \end{aligned} \tag{3.19}$$

Integrating (3.19) over W with measure $U_0(dw,0)$, and recalling that η and $w \in W$ are non-negative, and using (3.1) and (3.15),

$$\begin{aligned}&\int_W \eta(w,t)\, U_0(dw,0)\\ &\geqq -\int_W \int_0^1 \frac{\partial \rho}{\partial z}(w,z)\, w(f(z,t),t)\, \exp(-\int_0^t w(f(z,s),s)\, ds)\, dz\, U_0(dw,0)\\ &= \frac{\partial f}{\partial t}(0,t) \geqq 0.\end{aligned} \tag{3.20}$$

Differentiating (3.11) by s and using (3.20), we then have

$$\frac{\partial g}{\partial s}(s,t) = -\int_W \eta(w,s) \exp(-\int_s^t w(g(s,v),v)\, dv)\, U_0(dw,0) \leqq 0.$$

Similarly, differentiating $g(s,t)$ by t and using (3.11) and (3.15),

$$\frac{\partial g}{\partial t}(s,t) \geqq \frac{\partial f}{\partial t}(0,t) \geqq 0.$$

The rest of the claims are obtained easily. Indeed, (3.17) follows from (2.15), (3.11) and (3.7), and (3.18) from (3.11) and (3.1). □

We are ready to define the characteristic curves $y = y_C(\gamma,t)$ for (2.22). For $\gamma = (y_0,t_0) \in \Gamma$ and $t \geqq t_0$, put

$$y_C(\gamma,t) := \begin{cases} f(y_0,t) & \text{if } \gamma \in \Gamma_i\,, \ \text{ i.e., } t_0 = 0, \\ g(t_0,t) & \text{if } \gamma \in \Gamma_b\,, \ \text{ i.e., } y_0 = 0. \end{cases} \tag{3.21}$$

Note that (3.18) implies that (3.21) is well-defined on $(y_0,t_0) = (0,0) \in \Gamma_i \cap \Gamma_b$. Lemma 3.1 and Lemma 3.4 imply continuity of $y_C(\gamma,t)$, and C^1 property in t. (In fact, it is also C^1 in (γ,t) except on $y = y_C((0,0),t)$.) Also, (3.14) and (3.17) imply the first equality in (2.16).

Note also that, for each $t \geqq 0$, $y_C(\cdot,t) :\ \Gamma_t \to [0,1]$ is surjective. In fact, f and g are continuous, (3.14) and (3.15) imply $f(1,t) \geqq f(1,0) = 1$. These and (3.17) and (3.18) imply that y_C is surjective:

$$\{y_C(\gamma,t) \mid \gamma \in \Gamma_t\} = [0,1].$$

Note that (3.15) implies that there exists a unique C^1, increasing, one-to-one onto inverse function $\hat{f} :\ [f(0,t),1] \to [0,1]$ of $f(y,t)$ with respect to y. For $y < y_C(0,0,t) = f(0,t) = g(0,t)$ we define $\hat{g} :\ [0,g(0,t)] \to [0,t]$ by

$$\hat{g}(y,t) = \inf\{s \geqq 0 \mid g(s,t) = y\}. \tag{3.22}$$

Since, as noted above, $g(\cdot,t) :\ [0,t] \to [0,g(0,t)]$ is surjective, $\hat{g}$ is well-defined, and since g is continuous, $g(\hat{g}(y,t),t) = y$. Also (3.16) implies that $\hat{g}(y,t)$ is non-increasing with respect to y. Put

$$\hat{\gamma}(y,t) = \begin{cases} (\hat{f}(y,t),0) \in \Gamma_i & \text{if } f(0,t) \leqq y \leqq 1, \\ (0,\hat{g}(y,t)) \in \Gamma_b \cap \Gamma_t & \text{if } 0 \leqq y \leqq g(0,t). \end{cases} \tag{3.23}$$

The definition implies

$$y_C(\hat{\gamma}(y,t),t) = y, \quad y \in [0,1], \quad \text{and} \quad \hat{\gamma}(y_C(\gamma,t),t) = \gamma, \quad \gamma \in \Gamma_i. \tag{3.24}$$

Note that the second equality may fail on $\gamma \in \Gamma_b$.

For $t \geqq 0$, define a measure valued function

$$\varphi(dw,\cdot,t) :\ \Gamma_t \to [0,\infty)$$

as follows. If $\gamma=(y_0,0)\in\Gamma_i$,

$$\varphi(dw,\gamma,t):=-\int_{y_0}^{1}\frac{\partial\rho}{\partial z}(w,z)\exp(-\int_0^t w(f(z,s),s)\,ds)\,dz\,U_0(dw,0), \tag{3.25}$$

where f is as in Lemma 3.1; and if $\gamma=(0,t_0)\in\Gamma_b\cap\Gamma_t$,

$$\begin{aligned}&\varphi(dw,\gamma,t)\\ &:=-\int_0^1\frac{\partial\rho}{\partial z}(w,z)\exp(-\int_0^t w(f(z,s),s)\,ds)\,dz\,U_0(dw,0)\\ &\quad+\int_0^{t_0}\eta(w,u)\exp(-\int_u^t w(g(u,v),v)dv)\,du\,U_0(dw,0),\end{aligned} \tag{3.26}$$

where, f is as in Lemma 3.1, and η and g are as in Lemma 3.4. Let

$$\varphi(h,\gamma,t):=\int_W h(s)\,\varphi(dw,\gamma,t)$$

for a continuous bounded function h, $\gamma\in\Gamma$ and $t\in[0,\infty)$.

Proposition 3.6. *The following hold.*

$$y_C(\gamma,t)=1-\varphi(\mathbf{1}_W,\gamma,t):=1-\int_W\varphi(dw,\gamma,t),\quad \gamma\in\Gamma_t,\ t\geqq 0. \tag{3.27}$$

$$\varphi(dw,\gamma,t_0)=U_0(dw,y_0),\quad \gamma=(y_0,t_0)\in\Gamma. \tag{3.28}$$

For bounded continuous $h:\ W\to\mathbb{R}$ and $t>0$,

$$\begin{aligned}&\frac{\partial\varphi}{\partial t}(h,(y_0,0),t)=\int_W\int_{y_0}^1 w(y_C((z,0),t),t)\frac{\partial\varphi}{\partial z}(h,(z,0),t)\,dz\,U_0(dw,0),\\ &0\leqq y_0\leqq 1,\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}&\frac{\partial\varphi}{\partial t}(h,(0,t_0),t)=\frac{\partial\varphi}{\partial t}(h,(0,0),t)\\ &-\int_W\int_0^{t_0} w(y_C((0,u),t),t)\frac{\partial\varphi}{\partial u}(h,(0,u),t)\,du\,U_0(dw,0),\\ &0\leqq t_0\leqq t.\end{aligned} \tag{3.30}$$

◇

Proof. The definitions (3.21), (3.25) and (3.26), with Lemma 3.1 and Lemma 3.4 imply (3.27), and (3.28) follows from (3.7), (3.25) and (3.26). The definitions (3.21) and (3.25) imply that both hand sides of (3.29) are equal to

$$\int_W h(w)\int_{y_0}^1\frac{\partial\rho}{\partial z}(w,z)\,w(f(z,t),t)\ \exp(-\int_0^t w(f(z,s),s)\,ds)\,dz\,U_0(dw,0).$$

Similarly, (3.21) and (3.26) imply that both hand sides of (3.30) are equal to

$$\begin{aligned}&\frac{\partial\varphi}{\partial t}(h,(0,0),t)\\ &-\int_W h(w)\int_0^{t_0}\eta(w,u)\,w(g(u,t),t)\ \exp(-\int_u^t w(g(u,v),v)\,dv)\,du\,U_0(dw,0).\end{aligned}$$

□

For $(y,t)\in[0,1]\times[0,\infty)$ put

$$U(dw,y,t) := \varphi(dw,\hat\gamma(y,t),t) = \left\{ \begin{array}{ll} \varphi(dw,(\hat f(y,t),0),t) & f(0,t)\leqq y\leqq 1, \\ \varphi(dw,(0,\hat g(y,t)),t) & 0\leqq y\leqq g(0,t), \end{array}\right. \tag{3.31}$$

where $\hat\gamma$ is defined in (3.23).

Theorem 3.7. *It holds that*

$$\varphi(dw,\gamma,t) = U(dw,y_C(\gamma,t),t),\quad \gamma\in\Gamma_t,\ t\geqq 0. \tag{3.32}$$

Furthermore, for bounded continuous function $h:\ W\to\mathbb{R}$, $U(h,\cdot,\cdot):\ [0,1]\times[0,\infty)\to[0,\infty)$ is Lipschitz continuous in $(y,t)\in[0,1]\times[0,T]$ for any $T>0$, and satisfies the second equality in (2.16), (2.17), (2.19), and (2.20). $\diamond$

Proof. For $\gamma\in\Gamma_i$, (3.32) follows from (3.24). The point is the case $\gamma\in\Gamma_b$, where $y_C((0,s),t)=g(s,t)$, as a function of s, may fail to be one-to-one. Suppose $g(s,t)=g(s',t)$ for some s and s' satisfying $0\leqq s<s'\leqq t$. Then (3.11) and non-negativity of $\eta(w,u)$ implies

$$\int_s^{s'} \eta(w,u)\exp(-\int_u^t w(g(u,v),v)dv)\,du = 0,\ \ U_0(dw,0)\text{–almost surely.}$$

Hence (3.26) implies $\varphi(dw,(0,s'),t)=\varphi(dw,(0,s),t)$. On the other hand, the first equality of (3.24) implies

$$y_C(\hat\gamma(y_C(\gamma,t),t),t) = y_C(\gamma,t),\ \ \gamma\in\Gamma_b.$$

Therefore, $\varphi(dw,\hat\gamma(y_C(\gamma,t),t),t)=\varphi(dw,\gamma,t)$, with which (3.31) implies

$$U(dw,y_C(\gamma,t),t) = \varphi(dw,\hat\gamma(y_C(\gamma,t),t),t) = \varphi(dw,\gamma,t),$$

so that (3.32) holds.

The Lipschitz continuity of $U(h,y,t)$ for $f(0,t)\leqq y\leqq 1$, $0\leqq t\leqq T$ is obvious, since the definitions (3.31), (3.25), and the definition of $\hat f$ stated just before (3.22) imply that $U(h,y,t)$ is C^1. To prove the Lipschitz continuity of $U(h,y,t)$ for $0\leqq g(0,t)=f(0,t)\leqq y\leqq 1$, $0\leqq t\leqq T$, let (y,t) and (y',t') be 2 points in this domain. Use (3.31) to decompose

$$\begin{aligned} &|U(h,y',t')-U(h,y,t)| \\ &\leqq |\varphi(h,\hat\gamma(y',t'),t')-\varphi(h,\hat\gamma(y',t'),t)| + |\varphi(h,\hat\gamma(y',t'),t)-\varphi(h,\hat\gamma(y,t),t)|. \end{aligned}$$

Since by definition (3.26) $\varphi(h,\gamma,t)$ is C^1 in t, the first term on the right-hand side is bounded by a global constant times $|t'-t|$. To evaluate the second term, let M be a positive constant such that $|h(w)|\leqq M$, $w\in W$, and denote by h_+ and h_- the positive and negative part of h, respectively, so that $h=h_+-h_-$, $0\leqq h_\pm\leqq M$. Definitions (3.23) and (3.26), and the non-negativity of η imply

$$\begin{aligned} &|\varphi(h,\hat\gamma(y',t'),t)-\varphi(h,\hat\gamma(y,t),t)| \\ &= \left|\int_{\hat g(y',t')}^{\hat g(y,t)}\int_W h(w)\,\eta(w,u)\exp(-\int_u^t w(g(u,v),v)dv)\,U_0(dw,0)\,du\right| \\ &\leqq 2M\left|\int_{\hat g(y',t')}^{\hat g(y,t)}\int_W \mathbf{1}_W(w)\,\eta(w,u)\exp(-\int_u^t w(g(u,v),v)dv)\,U_0(dw,0)\,du\right|, \end{aligned}$$

which, by (3.23), (3.26), and (3.27), is equal to

$$2M\,|y_C(\hat\gamma(y',t'),t)-y_C(\hat\gamma(y,t),t)|.$$

This with (3.24) implies

$$
\begin{aligned}
&|\varphi(h,\hat\gamma(y',t'),t)-\varphi(h,\hat\gamma(y,t),t)|\\
&\leqq 2M\,|y_C(\hat\gamma(y',t'),t')-y_C(\hat\gamma(y',t'),t)|+2M\,|y_C(\hat\gamma(y',t'),t')-y_C(\hat\gamma(y,t),t)|\\
&\quad = 2M\,|y_C(\hat\gamma(y',t'),t')-y_C(\hat\gamma(y',t'),t)|+2M\,|y'-y|.
\end{aligned}
$$

Since $y_C(\gamma,t)$ is C^1 in t, we have the global Lipschitz continuity.

The property (2.20) follows from (3.27) and (3.24). The second equality in (2.16) then follows from (3.28), (3.32), (3.32), and the first equality in (2.16). (Note that the first equality in (2.16) and other claims in Theorem 2.1 for y_C is proved below (3.21).)

To prove (2.17) for $(y_0,t_0)\in\Gamma_i$, namely, for $t_0=0$, use (2.18), (3.31), (3.32), and (3.25), and change the order of integration, to find

$$
\begin{aligned}
&-V(h,y_C((y_0,0),t),t)\\
&=-\int_W h(w)\,w(y_C((y_0,0),t),t)\,\varphi(dw,(y_0,0),t)\\
&\quad+\int_W h(w)\int_{y_C((y_0,0),t)}^1 \frac{\partial\,w}{\partial z}(z,t)\left(\int_{\hat f(z,t)}^1 \frac{\partial\,\rho}{\partial z'}(w,z')\exp(-\int_0^t w(f(z',s),s)\,ds)\,dz'\right)dz\\
&\qquad\times U_0(dw,0)\\
&=-\int_W h(w)\,w(y_C((y_0,0),t),t)\,\varphi(dw,(y_0,0),t)\\
&\quad+\int_W h(w)\int_{y_0}^1\left(\int_{y_C((y_0,0),t)}^{y_C((z',0),t)} \frac{\partial\,w}{\partial z}(z,t)dz\right)\frac{\partial\,\rho}{\partial z'}(w,z')\exp(-\int_0^t w(f(z',s),s)\,ds)\,dz'\\
&\qquad\times U_0(dw,0),
\end{aligned}
$$

which, by the definition (3.25), is equal to $\dfrac{\partial\,\varphi}{\partial t}(h,\gamma,t)$. Integrating from t_0 to t and using (3.32) and (3.28), we have (2.17).

To prove (2.17) for $(y_0,t_0)\in\Gamma_b$, namely, for $y_0=0$, first decompose the integration range in (2.18) with $y=y_C((0,t_0),t)$ as

$$
[y_C((0,t_0),t),1]=[g(t_0,t),g(0,t)]\cup[f(0,t),1],
$$

then use the definitions (3.31) and (3.25) or (3.26), and change the order of integration, to find

$$\begin{aligned}
&-V(h, y_C((0,t_0),t),t)\\
&= -\int_W h(w)\, w(y_C((0,t_0),t),t)\varphi(dw,(0,t_0),t)\\
&\quad -\int_W h(w) \int_{g(t_0,t)}^{g(0,t)} \frac{\partial\, w}{\partial z}(z,t) \Big(-\int_0^1 \frac{\partial\, \rho}{\partial z'}(w,z') \exp(-\int_0^t w(f(z',s),s)\,ds)\,dz'\\
&\qquad\qquad + \int_0^{\hat g(z,t)} \eta(w,u) \exp(-\int_u^t w(g(u,v),v)dv)\,du \Big)\, dz\, U_0(dw,0)\\
&\quad + \int_W h(w) \int_{f(0,t)}^1 \frac{\partial\, w}{\partial z}(z,t) \Big(\int_{\hat f(z,t)}^1 \frac{\partial\, \rho}{\partial z'}(w,z') \exp(-\int_0^t w(f(z',s),s)\,ds)\,dz' \Big)\, dz\\
&\qquad \times U_0(dw,0),\\
&= -\int_W h(w)\, w(y_C((0,t_0),t),t)\varphi(dw,(0,t_0),t)\\
&\quad + \int_W h(w)\, (w(g(0,t),t) - w(g(t_0,t),t)) \int_0^1 \frac{\partial\, \rho}{\partial z'}(w,z') \exp(-\int_0^t w(f(z',s),s)\,ds)\,dz'\\
&\qquad \times U_0(dw,0)\\
&\quad - \int_W h(w) \int_0^{t_0} \eta(w,u) \exp(-\int_u^t w(g(u,v),v)dv) \Big(\int_{g(t_0,t)}^{g(u,t)} \frac{\partial\, w}{\partial z}(z,t)\,dz \Big)\, du\, U_0(dw,0)\\
&\quad + \int_W h(w) \int_0^1 \frac{\partial\, \rho}{\partial z'}(w,z') \exp(-\int_0^t w(f(z',s),s)\,ds) \Big(\int_{f(0,t)}^{f(z',t)} \frac{\partial\, w}{\partial z}(z,t)\,dz \Big)\, dz'\\
&\qquad \times U_0(dw,0)
\end{aligned}$$

By using (3.26), this is further simplified as

$$\begin{aligned}
&\int_W h(w) \Big(-\int_0^{t_0} \eta(u) w(g(u,t),t) \exp(-\int_u^t w(g(u,v),v)dv)\,du\\
&\qquad + \int_0^1 \frac{\partial\, \rho}{\partial z'}(w,z') w(f(z',t),t) \exp(-\int_0^t w(f(z',s),s)\,ds)\,dz' \Big)\, U_0(dw,0),
\end{aligned}$$

which, by using (3.26), is seen to be equal to $\dfrac{\partial\, \varphi}{\partial t}(h,\gamma,t)$. Integrating from t_0 to t and using (3.32) and (3.28), we have (2.17).

Substituting $h = \mathbf{1}_W$ in (2.17), and using (2.20), (3.27) and (3.32), we have (2.19). □

To complete a proof of Theorem 2.1, it only remains to prove uniqueness. Besides the pair y_C and U which we constructed and proved so far to satisfy the properties stated in Theorem 2.1, assume that there are another such pair $\tilde y_C$ and $\tilde U$. For $T > 0$, let $L(T) > 0$ be a positive constant such that

$$\begin{aligned}
&\max\{|U(h,y,t) - U(h,y',t')|,\ |\tilde U(h,y,t) - \tilde U(h,y',t')|\}\\
&\leqq L(T)\ \|(y,t) - (y',t')\|\,,\quad (y,t),(y',t') \in [0,1]\times[0,T],\\
&h:\ W \to [-1,1];\ \text{continuous}.
\end{aligned}$$

Put

$$I(t) = \sup_{h:\ W\to[-1,1];\ \text{conti.}}\ \sup_{y\in[0,1]} |U(h,y,t) - \tilde U(h,y,t)|$$

and

$$J(t)=\sup_{\gamma\in\Gamma_t}|\tilde{y}_C(\gamma,t)-y_C(\gamma,t)|.$$

Then (2.16) and its correspondence for $\tilde{U}$ imply $I(0)=0$. Since $y_C(\cdot,t):\ \Gamma_t\to[0,1]$ is onto,

$$I(t)=\sup_{h:\ W\to[-1,1];\ \text{conti.}}\ \sup_{\gamma\in\Gamma_t}|U(h,y_C(\gamma,t),t)-\tilde{U}(h,y_C(\gamma,t),t)|. \tag{3.33}$$

Note also that since $\tilde{U}(dw,y,t)$ is, by assumption, a non-negative measure, for h with $|h(w)|\leqq 1$, $w\in W$, we have

$$\tilde{U}(h,y,t)\leqq\tilde{U}(\mathbf{1}_W,y,t)=1-y\leqq 1,$$

where we also used (2.20).

It holds that

$$I(t)\leqq L(T)J(t)$$

Subtracting

$$\tilde{U}(h,\tilde{y}_C(\gamma,t),t)=U_0(h,y_0)-\int_{t_0}^t\tilde{V}(h,\tilde{y}_C(\gamma,s),s)\,ds, \tag{3.34}$$

from (2.17), and using (3.33), (2.18) and (2.1), we have

$$\begin{aligned}&|\tilde{y}_C(\gamma,t)-y_C(\gamma,t)|=|\tilde{U}(\mathbf{1}_W,\tilde{y}_C(\gamma,t),t)-U(\mathbf{1}_W,y(\gamma,t),t)|\\&\leqq 2R_w(T)\int_{t_0}^t J(s)\,ds+R_w(T)\int_{t_0}^t I(s)\,ds+R_w(T)L(T)\int_{t_0}^t J(s)\,ds.\end{aligned}$$

Therefore,

$$J(t)\leqq 2R_w(T)\int_{t_0}^t J(s)\,ds+R_w(T)\int_{t_0}^t I(s)ds+R_w(T)L(T)\int_{t_0}^t J(s)\,ds.$$

Then,

$$I(t)\leqq L(T)\left(2R_w(T)\int_{t_0}^t J(s)\,ds+R_w(T)\int_{t_0}^t I(s)ds+R_w(T)L(T)\int_{t_0}^t J(s)\,ds\right),$$

so that if we put $K(t)=\max\{I(t),J(t)\}$, then there exists $C(T)$ such that

$$K(t)\leqq C(T)\int_{t_0}^t K(s)\,ds,\ \ K(t_0)=0.$$

This implies $K(t)=0$. Hence $\tilde{U}=U$ and $\tilde{y}_C=y_C$.

This completes a proof of Theorem 2.1.

4. Proof of Theorem 2.2.

Let Γ be as in (2.11). To simplify the notation, for $\gamma=(y_0,t_0)\in\Gamma$, we will write $y_C((y_0,t_0),t)$ defined in (3.21) as $y_C(y_0,t_0,t)$.

We define a stochastic process $(Y_C^{(N)}(y_0,t_0,t);t\geqq t_0)$ by

$$\begin{aligned}&Y_C^{(N)}(y_0,t_0,t)=y_0+\frac{1}{N}\sum_{i;\ X_i^{(N)}(t_0)\geqq Ny_0+1}\mathbf{1}_{J_i^{(N)}(t_0,t)}\,,\\&(y_0,t_0)\in[0,1)\times[0,\infty),\ \ t>t_0\,,\end{aligned} \tag{4.1}$$

where $J_i^{(N)}$ is defined in (2.3). The process $(Y_C^{(N)}(y_0,t_0,t); t \geqq t_0)$ is an analogue to $y_C(t) = y_C(y_0,t_0,t)$. Indeed, $Y_C^{(N)}(y_0,t_0,t)$ is non-decreasing in t and increases when the jumps occur for the particles whose numbers of their position are larger than $Y_C^{(N)}(y_0,t_0,t-)$. One can find similarity for $Y_C^{(N)}(y_0,t_0,t)$ to $y_C(t) = y_C(y_0,t_0,t)$ by regarding the jumps of particles as evaporation. Later it will be shown that $Y_C^{(N)}(y_0,t_0,t)$ converges as $N \to \infty$ to $y_C(t) = y_C(y_0,t_0,t)$, for $(y_0,t_0) \in \Gamma$, $t \geqq t_0$.

We put, as an analogue to (2.5),

$$y_i^{(N)} = \frac{1}{N}(x_i^{(N)} - 1), \quad i = 1,2,\dots,N. \tag{4.2}$$

Then, (2.3) and (2.4) imply

$$Y_i^{(N)}(t) \geqq Y_C^{(N)}(y_0,t_0,t) \quad \Leftrightarrow \quad Y_i^{(N)}(t_0) \geqq y_0 \ \text{ and } \ J_i^{(N)}(t_0,t) \ \text{ does not hold.} \tag{4.3}$$

Hence, we have

$$\begin{aligned} &Y_C^{(N)}(y_0,t_0,t) = y_0 + \frac{1}{N} \\ &\sum_i \int_{s\in(t_0,t]} \int_{\xi\in[0,\infty)} \mathbf{1}_{Y_i^{(N)}(s-)\geqq Y_C^{(N)}(y_0,t_0,s-)}\, \mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}\, \nu_i(d\xi ds). \end{aligned} \tag{4.4}$$

For the spatially homogeneous case, $Y_A^{(N)}(t_0,t)$ in Hariya et al. (2011) is equal to $Y_C^{(N)}(0,t-t_0,t)$, $Y_B^{(N)}(y_0,t)$ to $Y_C^{(N)}(y_0,0,t)$, and $Y_C^{(N)}(t)$ in Hariya et al. (2011) is equal to $Y_C^{(N)}(0,0,t)$ of (4.1).

Let Γ be as in (2.11). Let $(y_0,t_0) \in \Gamma$, $t \geqq t_0$. The definition (2.7) and the properties (2.3), (2.4), and (4.3) imply that for $B \in \mathrm{B}(W)$, $U^{(N)}(B, Y_C^{(N)}(y_0,t_0,t),t)$ as a function of t changes its value if and only if $J_i^{(N)}(t_0,t)$ occurs for some i satisfying $y_i^{(N)} \geqq y_0$ and $w_i \in B$. Therefore, for $B \in \mathrm{B}(W)$

$$\begin{aligned} &U^{(N)}(B, Y_C^{(N)}(y_0,t_0,t),t) - U^{(N)}(B,y_0,t_0) \\ &= -\frac{1}{N} \sum_{i;\ w_i\in B} \int_{s\in(t_0,t]} \int_{\xi\in[0,\infty)} \mathbf{1}_{Y_i^{(N)}(s-)\geqq Y_C^{(N)}(y_0,t_0,s-)}\, \mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}\, \nu_i(d\xi ds). \end{aligned}$$

In analogy to (2.18) define for $B \in \mathrm{B}(W)$

$$V^{(N)}(B,y,t) = \int_B w(y,t)\, U^{(N)}(dw,y,t) + \int_y^1 \int_B \frac{\partial w}{\partial z}(z,t)\, U^{(N)}(dw,z,t)\, dz. \tag{4.5}$$

By definition (2.7), for $B \in \mathrm{B}(W)$

$$\begin{aligned} V^{(N)}(B,y,t) &= \frac{1}{N} \sum_{m\geqq Ny+1} \sum_{i;\ w_i\in B} w_i(\frac{m-1}{N},t)\, \mathbf{1}_{X_i^{(N)}(t-)=m} \\ &= \frac{1}{N} \sum_{i;\ w_i\in B,\, Y_i^{(N)}(t-)\geqq y} w_i(Y_i^{(N)}(t-),t). \end{aligned} \tag{4.6}$$

Denote the compensated Poisson process by

$$\tilde\nu_i(d\xi ds) = \nu_i(d\xi ds) - d\xi ds, \tag{4.7}$$

and put for $B \in \mathrm{B}(W)$

$$\begin{aligned} &M_U^{(N)}(B, y_0, t_0, t) = -\frac{1}{N} \sum_{i;\ w_i \in B} \\ &\int_{s \in (t_0, t]} \int_{\xi \in [0,\infty)} \mathbf{1}_{Y_i^{(N)}(s-) \geqq Y_C^{(N)}(y_0,t_0,s-)} \mathbf{1}_{\xi \in [0, w_i(Y_i^{(N)}(s-), s))} \tilde{\nu}_i(d\xi ds). \end{aligned} \tag{4.8}$$

This martingale $M_U^{(N)}(B, y_0, t_0, t)$ means the oscillation term for $U^{(N)}(B, Y_C^{(N)}(y_0, t_0, t), t)$ generated by the random jumps of other particles. Indeed, we have for $B \in \mathrm{B}(W)$

$$\begin{aligned} &U^{(N)}(B, Y_C^{(N)}(y_0, t_0, t), t) \\ &= U^{(N)}(B, y_0, t_0) + M_U^{(N)}(B, y_0, t_0, t) \\ &\quad -\frac{1}{N} \sum_{i;\ w_i \in B} \int_{s \in (t_0, t]} \int_{\xi \in [0,\infty)} \mathbf{1}_{Y_i^{(N)}(s-) \geqq Y_C^{(N)}(y_0,t_0,s-)} \mathbf{1}_{\xi \in [0, w_i(Y_i^{(N)}(s-), s))} d\xi ds \\ &= U^{(N)}(B, y_0, t_0) + M_U^{(N)}(B, y_0, t_0, t) \\ &\quad -\frac{1}{N} \sum_{i;\ w_i \in B} \int_{t_0}^{t} w_i \left(Y_i^{(N)}(s-), s \right) \mathbf{1}_{Y_i^{(N)}(s-) \geqq Y_C^{(N)}(y_0,t_0,s-)} ds \\ &= U^{(N)}(B, y_0, t_0) + M_U^{(N)}(B, y_0, t_0, t) - \int_0^t V^{(N)}(B, Y_C^{(N)}(y_0, t_0, s), s) ds. \end{aligned}$$

Later we will show that $M_U^{(N)}(B, y_0, t_0, t)$ vanishes as N goes to infinity. The vanishment implies that the limit processes become almost deterministic. Combining this equality with (2.17), we have for $B \in \mathrm{B}(W)$

$$\begin{aligned} &U^{(N)}(B, Y_C^{(N)}(y_0, t_0, t), t) - U(B, y_C(y_0, t_0, t), t) \\ &= U^{(N)}(B, y_0, t_0) - U(B, y_0, t_0) + M_U^{(N)}(B, y_0, t_0, t) \\ &\quad - \int_{t_0}^{t} \Big(V^{(N)}(B, Y_C^{(N)}(y_0, t_0, s), s) - V(B, y_C(y_0, t_0, s), s) \Big) ds. \end{aligned} \tag{4.9}$$

Put

$$\begin{aligned} W^{(N)}(t) = &\sup_{(y_0,t_0) \in \Gamma; t_0 \leqq t} \ \sup_{s \in [t_0, t]} |Y_C^{(N)}(y_0, t_0, s) - y_C(y_0, t_0, s)| \\ \vee &\sup_{(y_0,t_0) \in \Gamma; t_0 \leqq t} \ \sup_{s \in [t_0, t]} ||U^{(N)}(\cdot, Y_C^{(N)}(y_0, t_0, s), s) - U(\cdot, y_C(y_0, t_0, s), s)||_{\mathrm{var}} \\ \vee &\sup_{B \in \mathrm{B}(W)} \ \sup_{(y_0,t_0) \in \Gamma; t_0 \leqq t} \ \sup_{s \in [t_0, t]} |V^{(N)}(B, Y_C^{(N)}(y_0, t_0, s), s) - V(B, y_C(y_0, t_0, s), s)|. \end{aligned} \tag{4.10}$$

Now we prepare some estimates in order to apply Gronwall's inequality to $W^{(N)}(t)$. First we consider an estimate for

$$\sup_{(y_0,t_0) \in \Gamma; t_0 \leqq t} \ \sup_{s \in [t_0, t]} ||U^{(N)}(\cdot, Y_C^{(N)}(y_0, t_0, s), s) - U(\cdot, y_C(y_0, t_0, s), s)||_{\mathrm{var}}.$$

Since for all $z \in [0,1)$ there exist $(y_0,t_0) \in \Gamma$ such that $y_C(y_0,t_0,t) = z$, for $B \in \mathrm{B}(W)$

$$\begin{aligned}
&\sup_{z\in[0,1]} |U^{(N)}(B,z,t) - U(B,z,t)| \\
&\leqq \sup_{(y_0,t_0)\in\Gamma;t_0\leqq t} \sup_{s\in[t_0,t]} |U^{(N)}(B,y_C(y_0,t_0,s),s) - U(B,y_C(y_0,t_0,s),s)| \\
&\leqq \sup_{(y_0,t_0)\in\Gamma;t_0\leqq t} \sup_{s\in[t_0,t]} |U^{(N)}(B,Y_C^{(N)}(y_0,t_0,s),s) - U(B,y_C(y_0,t_0,s),s)| \\
&\quad + \sup_{(y_0,t_0)\in\Gamma;t_0\leqq t} \sup_{s\in[t_0,t]} |U^{(N)}(B,Y_C^{(N)}(y_0,t_0,s),s) - U^{(N)}(B,y_C(y_0,t_0,s),s)|.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sup_{z\in[0,1]} |U^{(N)}(B,z,t) - U(B,z,t)| \\
&\leqq W^{(N)}(t) + \sup_{(y_0,t_0)\in\Gamma;t_0\leqq t} \sup_{s\in[t_0,t]} |U^{(N)}(B,Y_C^{(N)}(y_0,t_0,s),s) \\
&\qquad\qquad - U^{(N)}(B,y_C(y_0,t_0,s),s)|.
\end{aligned} \tag{4.11}$$

By (2.7) it holds that

$$\begin{aligned}
&\sup_{(y_0,t_0)\in\Gamma;t_0\leqq t} \sup_{s\in[t_0,t]} |U^{(N)}(B,Y_C^{(N)}(y_0,t_0,s),s) - U^{(N)}(B,y_C(y_0,t_0,s),s)| \\
&\qquad \leqq \sup_{(y_0,t_0)\in\Gamma;t_0\leqq t} \sup_{s\in[t_0,t]} |Y_C^{(N)}(y_0,t_0,s) - y_C(y_0,t_0,s)| + \frac{1}{N}.
\end{aligned} \tag{4.12}$$

By (4.11) and (4.12) we obtain

$$\sup_{z\in[0,1]} |U^{(N)}(B,z,t) - U(B,z,t)| \leqq 2W^{(N)}(t) + \frac{1}{N}. \tag{4.13}$$

This implies

$$\sup_{z\in[0,1]} ||U^{(N)}(\cdot,z,t) - U(\cdot,z,t)||_{\mathrm{var}} \leqq 4W^{(N)}(t) + \frac{2}{N}. \tag{4.14}$$

By (4.9) and (4.10), we have for $B \in \mathrm{B}(W)$

$$\begin{aligned}
&\left|U^{(N)}(B,Y_C^{(N)}(y_0,t_0,t),t) - U(B,y_C(y_0,t_0,t),t)\right| \\
&\leqq \left|U^{(N)}(B,y_0,t_0) - U(B,y_0,t_0)\right| + \left|M_U^{(N)}(B,y_0,t_0,t)\right| + \int_{t_0}^t W^{(N)}(s)ds.
\end{aligned} \tag{4.15}$$

Next we consider an estimate for

$$\sup_{(y_0,t_0)\in\Gamma;t_0\leqq t} \sup_{s\in[t_0,t]} |Y_C^{(N)}(y_0,t_0,s) - y_C(y_0,t_0,s)|.$$

By using (4.4), (4.6) and (4.8), we have

$$Y_C^{(N)}(y_0,t_0,t) = y_0 + M_U^{(N)}(W,y_0,t_0,t) + \int_{t_0}^t V^{(N)}(W,Y_C^{(N)}(y_0,t_0,s),s)\,ds. \tag{4.16}$$

Combining with (2.19),

$$\begin{aligned} &Y_C^{(N)}(y_0,t_0,t) - y_C(y_0,t_0,t) \\ &= M_U^{(N)}(W,y_0,t_0,t) + \int_{t_0}^t \left[V^{(N)}(W, Y_C^{(N)}(y_0,t_0,s),s) - V(W, y_C(y_0,t_0,s),s) \right] ds. \end{aligned} \tag{4.17}$$

Hence, by (4.10) we have

$$\left| Y_C^{(N)}(y_0,t_0,t) - y_C(y_0,t_0,t) \right| \leqq |M_U^{(N)}(W,y_0,t_0,t)| + \int_{t_0}^t W^{(N)}(s) ds. \tag{4.18}$$

Finally we consider an estimate for

$$\sup_{B \in \mathrm{B}(W)} \sup_{(y_0,t_0)\in\Gamma; t_0 \leqq t} \sup_{s\in[t_0,t]} |V^{(N)}(B, Y_C^{(N)}(y_0,t_0,s),s) - V(B, y_C(y_0,t_0,s),s)|.$$

Similarly, combining (4.5) with (2.18), we have

$$\begin{aligned} &V^{(N)}(B, Y_C^{(N)}(y_0,t_0,t),t) - V(B, y_C(y_0,t_0,t),t) \\ &= \int_B w(Y_C^{(N)}(y_0,t_0,t),t) \left(U^{(N)}(dw, Y_C^{(N)}(y_0,t_0,t),t) - U(dw, y_C(y_0,t_0,t),t) \right) \\ &\quad + \int_B \left(w(Y_C^{(N)}(y_0,t_0,t),t) - w(y_C(y_0,t_0,t),t) \right) U(dw, y_C(y_0,t_0,t),t) \\ &\quad + \int_{Y_C^{(N)}(y_0,t_0,t)}^1 \int_B \frac{\partial w}{\partial z}(z,t) \left(U^{(N)}(dw,z,t) - U(dw,z,t) \right) dz \\ &\quad - \int_{y_C(y_0,t_0,t)}^{Y_C^{(N)}(y_0,t_0,t)} \int_B \frac{\partial w}{\partial z}(z,t)\, U(dw,z,t)\, dz. \end{aligned}$$

Hence, using this estimate, (2.1), (4.14) and the fact that $0 \leqq U^{(N)} \leqq 1$, we have for $B \in \mathrm{B}(W)$

$$\begin{aligned} &\left| V^{(N)}(B, Y_C^{(N)}(y_0,t_0,t),t) - V(B, y_C(y_0,t_0,t),t) \right| \\ &\leqq R_w(T) ||U^{(N)}(\cdot, Y_C^{(N)}(y_0,t_0,t),t) - U(\cdot, y_C(y_0,t_0,t),t)||_{\mathrm{var}} \\ &\quad + 2R_w(T) \left| Y_C^{(N)}(y_0,t_0,t) - y_C(y_0,t_0,t) \right| + R_w(T) \left(4 \int_{t_0+}^t W^{(N)}(s) ds + \frac{2}{N} \right). \end{aligned} \tag{4.19}$$

By (2.1), (4.15), (4.19), and (4.18), we obtain for $B \in \mathrm{B}(W)$

$$\begin{aligned} &\left| V^{(N)}(B, Y_C^{(N)}(y_0,t_0,t),t) - V(B, y_C(y_0,t_0,t),t) \right| \\ &\leqq R_w(T) ||U^{(N)}(\cdot, y_0,t_0) - U(\cdot, y_0,t_0)||_{\mathrm{var}} + 7R_w(T) \int_{t_0}^t W^{(N)}(s) ds \\ &\quad + 3R_w(T) \left| M_U^{(N)}(B,y_0,t_0,t) \right| + \frac{2R_w(T)}{N}. \end{aligned} \tag{4.20}$$

We are now ready to apply Gronwall's inequality to $W^{(N)}(t)$. By (4.10), (4.15), (4.18), and (4.20) we have

$$\begin{aligned} W^{(N)}(t) \leqq C_1 & \sup_{(y_0,t_0)\in\Gamma} ||U^{(N)}(\cdot,y_0,t_0) - U(\cdot,y_0,t_0)||_{\mathrm{var}} \\ & + (1+R_w(T)) \sup_{B\in \mathrm{B}(W)} \sup_{(y_0,t_0)\in\Gamma; t_0\leqq t} \left| M_U^{(N)}(B,y_0,t_0,t)\right| \\ & + C_2 \int_0^t W^{(N)}(s)ds + \frac{2R_w(T)}{N} \end{aligned}$$

where C_1 and C_2 are constants depending on $A, T, R_w(T)$. Hence, Gronwall's inequality implies

$$\begin{aligned} & \sup_{t\in[0,T]} W^{(N)}(t) \\ & \leqq e^{C_2 T} \left[C_1 \sup_{(y_0,t_0)\in\Gamma} ||U^{(N)}(\cdot,y_0,t_0) - U(\cdot,y_0,t_0)||_{\mathrm{var}} \right. \\ & \left. + (1+R_w(T)) \sup_{B\in \mathrm{B}(W)} \sup_{(y_0,t_0)\in\Gamma; t_0\leqq t} \left| M_U^{(N)}(B,y_0,t_0,t)\right| + \frac{2R_w(T)}{N} \right]. \end{aligned} \tag{4.21}$$

Next we show that

$$\lim_{N\to\infty} E\left[\sup_{B\in \mathrm{B}(W)} \sup_{(y_0,t_0)\in\Gamma;\, t_0\leqq T} \sup_{t\in[t_0,T]} \left| M_U^{(N)}(B,y_0,t_0,t)\right|^2 \right] = 0.$$

By the definition of Γ, we have

$$\begin{aligned} & E\left[\sup_{B\in \mathrm{B}(W)} \sup_{(y_0,t_0)\in\Gamma, t_0\leqq T} \sup_{t\in[t_0,T]} \left| M_U^{(N)}(B,y_0,t_0,t)\right|^2 \right] \\ & \leqq E\left[\sup_{B\in \mathrm{B}(W)} \sup_{y_0\in[0,1)} \sup_{t\in[0,T]} \left| M_U^{(N)}(B,y_0,0,t)\right|^2 \right] \\ & \quad + E\left[\sup_{B\in \mathrm{B}(W)} \sup_{0\leqq t_0\leqq t\leqq T} \left| M_U^{(N)}(B,0,t_0,t)\right|^2 \right]. \end{aligned} \tag{4.22}$$

First we show the first term of the right-hand side of (4.22) vanishes as N goes to infinity. Note that $Y_i^{(N)}(t) \in \{0, 1/N, \dots (N-1)/N\}$ for $t \in [0,T]$ and $i = 1, 2, \dots, N$, $Y_C^{(N)}(y_0, t_0, \cdot)$ is a process of pure jumps by $1/N$ and for each $k = 1, 2, \dots, N$, $Y_C^{(N)}(y_0, t_0, \cdot) - y_0$ is independent of y_0 as long as $y_0 \in (k-1/N, k/N]$. Also, note that $\{M_U^{(N)}(B, \cdot, 0, \cdot); B \in \mathrm{B}(W)\} = \{M_U^{(N)}(B, \cdot, 0, \cdot); B \in 2^{\{w_i; i=1,2,\dots,N\}}\}$. By

(4.8)

$$
\begin{aligned}
&E\left[\sup_{B\in \mathrm{B}(W)}\sup_{y_0\in[0,1)}\sup_{t\in[0,T]}\left|M_U^{(N)}(B,y_0,0,t)\right|^2\right]\\
&=\frac{1}{N^2}E\left[\sup_{B\in \mathrm{B}(W)}\sup_{y_0\in[0,1)}\sup_{t\in[0,T]}\left|\sum_{i;\,w_i\in B}\int_{s\in(0,t]}\int_{\xi\in[0,\infty)}\right.\right.\\
&\qquad\left.\left.\mathbf{1}_{Y_i^{(N)}(s-)\geqq Y_C^{(N)}(y_0,0,s-)}\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}\,\tilde{\nu}_i(d\xi ds)\right|^2\right]\\
&=\frac{1}{N^2}\sup_{B\in 2^{\{w_i;i=1,2,\dots,N\}}}\max_{k=0,1,\dots,N-1}E\left[\sup_{t\in[0,T]}\left|\sum_{i;\,w_i\in B}\int_{s\in(0,t]}\int_{\xi\in[0,\infty)}\right.\right.\\
&\qquad\left.\left.\mathbf{1}_{Y_i^{(N)}(s-)\geqq Y_C^{(N)}(k/N,0,s-)}\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}\,\tilde{\nu}_i(d\xi ds)\right|^2\right].
\end{aligned}
$$

Hence, by Doob's martingale inequality (see (6.16) of Chapter I in Ikeda and Watanabe (1989)) and (3.9) of Chapter II in Ikeda and Watanabe (1989) we have

$$
\begin{aligned}
&E\left[\sup_{B\in \mathrm{B}(W)}\sup_{y_0\in[0,1)}\sup_{t\in[0,T]}\left|M_U^{(N)}(B,y_0,0,t)\right|^2\right]\\
&\leqq\frac{C_3}{N^2}\max_{k=0,1,\dots,N-1}E\left[\sum_i\int_{s\in(0,T]}\int_{\xi\in[0,\infty)}\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}\,d\xi ds\right]\\
&\leqq\frac{C_3R_w(T)T}{N}
\end{aligned}
$$

where C_3 is a positive constant. Thus, it holds that

$$
\lim_{N\to\infty}E\left[\sup_{B\in \mathrm{B}(W)}\sup_{y_0\in[0,1)}\sup_{t\in[0,T]}\left|M_U^{(N)}(B,y_0,0,t)\right|^2\right]=0. \tag{4.23}
$$

Next we show the second term of the right-hand side of (4.22) vanishes as N goes to infinity. By (4.8) again, similarly to the case of $t_0 = 0$

$$
\begin{aligned}
&E\left[\sup_{B\in \mathrm{B}(W)}\ \sup_{0\leqq t_0\leqq t\leqq T}\left|M_U^{(N)}(B,0,t_0,t)\right|^2\right] \\
&= \frac{1}{N^2}\sup_{B\in 2^{\{w_i; i=1,2,\dots,N\}}} E\left[\sup_{0\leqq t_0\leqq t\leqq T}\left|\sum_{i;\, w_i\in B}\int_{s\in(t_0,t]}\int_{\xi\in[0,\infty)}\right.\right. \\
&\qquad\qquad \left.\left. \mathbf{1}_{Y_i^{(N)}(s-)\geqq Y_C^{(N)}(0,t_0,s-)}\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}\,\tilde{\nu}_i(d\xi ds)\right|^2\right] \\
&= \frac{1}{N^2}\sup_{B\in 2^{\{w_i; i=1,2,\dots,N\}}} E\left[\sup_{0\leqq t_0\leqq t\leqq T}\left|\sum_{i;\, w_i\in B}\int_{s\in(0,t]}\int_{\xi\in[0,\infty)}\right.\right. \\
&\qquad\qquad \mathbf{1}_{Y_i^{(N)}(s-)\geqq Y_C^{(N)}(0,t_0,s-)}\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}\,\tilde{\nu}_i(d\xi ds) \\
&\qquad \left.\left. -\sum_{i;\, w_i\in B}\int_{s\in(0,t_0]}\int_{\xi\in[0,\infty)}\mathbf{1}_{Y_i^{(N)}(s-)\geqq Y_C^{(N)}(0,t_0,s-)}\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}\,\tilde{\nu}_i(d\xi ds)\right|^2\right] \\
&\leqq \frac{1}{N^2}\sup_{B\in 2^{\{w_i; i=1,2,\dots,N\}}} E\left[\left(2\sup_{0\leqq t\leqq T}\left|\sum_{i;\, w_i\in B}\int_{s\in(0,t]}\int_{\xi\in[0,\infty)}\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. \mathbf{1}_{Y_i^{(N)}(s-)\geqq Y_C^{(N)}(0,t_0,s-)}\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}\,\tilde{\nu}_i(d\xi ds)\right|\right)^2\right].
\end{aligned}
$$

Hence, by Doob's martingale inequality and (3.9) of Chapter II in Ikeda and Watanabe (1989) imply

$$
\begin{aligned}
&E\left[\sup_{B\in \mathrm{B}(W)}\ \sup_{0\leqq t_0\leqq t\leqq T}\left|M_U^{(N)}(B,0,t_0,t)\right|^2\right] \\
&\leqq \frac{4C_3}{N^2}E\left[\sum_i\int_{s\in(0,T]}\int_{\xi\in[0,\infty)}\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}\,d\xi ds\right] \\
&\leqq \frac{4C_3R_w(T)T}{N}.
\end{aligned}
$$

Thus, we obtain

$$
\lim_{N\to\infty} E\left[\sup_{B\in \mathrm{B}(W)}\ \sup_{0\leqq t_0\leqq t\leqq T}\left|M_U^{(N)}(B,0,t_0,t)\right|^2\right] = 0. \tag{4.24}
$$

Combining (4.22), (4.23) and (4.24), we have

$$
\lim_{N\to\infty} E\left[\sup_{B\in \mathrm{B}(W)}\ \sup_{(y_0,t_0)\in\Gamma;\, t_0\leqq T}\ \sup_{t\in[t_0,T]}\left|M_U^{(N)}(B,y_0,t_0,t)\right|^2\right] = 0. \tag{4.25}
$$

On the other hand, by (2.26) we have

$$\lim_{N\to\infty} ||U^{(N)}(\cdot,0,t_0) - U(\cdot,0,t_0)||_{\mathrm{var}} = \lim_{N\to\infty} ||U^{(N)}(\cdot,0,0) - U_0(\cdot,0)||_{\mathrm{var}} = 0. \tag{4.26}$$

Thus, (4.26) and (2.26) implies

$$\lim_{N\to\infty} \sup_{(y_0,t_0)\in\Gamma} ||U^{(N)}(\cdot,y_0,t_0) - U(\cdot,y_0,t_0)||_{\mathrm{var}} = 0. \tag{4.27}$$

(4.21), (4.25) and (4.27) yields

$$\lim_{N\to\infty} E\left[\sup_{t\in[0,T]} W^{(N)}(t)^2\right] = 0.$$

Hence, there exists a subsequence $\{N(k)\}$ such that

$$\lim_{k\to\infty} \sup_{t\in[0,T]} W^{(N(k))}(t) = 0$$

almost surely. However, the argument above is also available even if we replace N by any subsequence $N(k)$. Therefore, we have

$$\lim_{N\to\infty} \sup_{t\in[0,T]} W^{(N)}(t) = 0$$

almost surely. This proves the first assertion of Theorem 2.2.

We turn to a proof of the second assertion of Theorem 2.2. First, we show the uniqueness of the stochastic differential equation (2.28). Note that

$$E\left[\int_{s\in(0,t]}\int_{\xi\in[0,R_w(T)]} \nu_i(d\xi ds)\right] = tR_w(T)$$

and that for all i

$$\begin{aligned}&\int_{s\in(0,t]}\int_{\xi\in[0,\infty)} Y_i(s-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i(s-),s))}\,\nu_i(d\xi ds)\\ &= \int_{s\in(0,t]}\int_{\xi\in[0,R_w(T)]} Y_i(s-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i(s-),s))}\,\nu_i(d\xi ds).\end{aligned}$$

Moreover, there exists a constant C_T such that

$$\sup_{s\in[0,T]} |V(W,x,s) - V(W,y,s)| \leqq C_T|x-y|,\ \ x,y\in[0,1).$$

The proof of Theorem 9.1 in Chapter IV of Ikeda and Watanabe (1989) is available by taking $U := [0,R_w(T)]$ and $U_0 := \emptyset$. Thus, we obtain the uniqueness.

Next, we show that $(Y_1^{(N)}(t), Y_2^{(N)}(t), \dots, Y_L^{(N)}(t))$ converges to $(Y_1(t), Y_2(t), \dots, Y_L(t))$ uniformly in $t\in[0,T]$ almost surely and also converges in the sense of L^2. Let $i\in\{1,2,\dots,L\}$ be fixed. By (4.6) it is easy to see

$$\begin{aligned}Y_i^{(N)}(t) &= y_i^{(N)} + M_i^{(N)}(t) + \int_0^t V^{(N)}(W,Y_i^{(N)}(s-),s)ds\\ &\quad - \int_{s\in(0,t]}\int_{\xi\in[0,\infty)} Y_i^{(N)}(s-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(s-),s))}\,\nu_i(d\xi ds)\end{aligned} \tag{4.28}$$

where

$$M_i^{(N)}(t) := \frac{1}{N}\sum_{j=1}^{N}\int_{s\in(0,t]}\int_{\xi\in[0,\infty)} \mathbf{1}_{Y_i^{(N)}(s-)<Y_j^{(N)}(s-)}\,\mathbf{1}_{\xi\in[0,w_j(Y_j^{(N)}(s-),s))}\,\tilde{\nu}_j(d\xi ds). \tag{4.29}$$

Hence, (2.28) and (4.28) imply

$$\begin{aligned}
&E\left[\sup_{s\in[0,t]}|Y_i^{(N)}(s)-Y_i(s)|^2\right]\\
&\leqq 4|y_i^{(N)}-y_i|^2+4E\left[\sup_{s\in[0,t]}|M_i^{(N)}(s)|^2\right]\\
&\quad+4\int_0^t E\left[|V^{(N)}(W,Y_i^{(N)}(s-),s)-V(W,Y_i(s-),s)|^2\right]ds\\
&\quad+4E\left[\sup_{s\in[0,t]}\left|\int_{u\in(0,s]}\int_{\xi\in[0,\infty)}[Y_i^{(N)}(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(u-),u))}\right.\right.\\
&\qquad\qquad\left.\left.-Y_i(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i(u-),u))}]\nu_i(d\xi du)\right|^2\right].
\end{aligned}\tag{4.30}$$

Now we prepare some estimates in order to apply Gronwall's inequality to (4.30). First we estimate the third term of the right-hand side of (4.30). By (2.18) and (4.5), we have

$$\begin{aligned}
&V^{(N)}(W,Y_i^{(N)}(t-),t)-V(W,Y_i(t-),t)\\
&=\int_W w(Y_i^{(N)}(t-),t)\left(U^{(N)}(dw,Y_i^{(N)}(t-),t)-U(dw,Y_i(t-),t)\right)\\
&\quad+\int_W\left(w(Y_i^{(N)}(t-),t)-w_i(Y_i(t-),t)\right)U(dw,Y_i(t-),t)\\
&\quad+\int_{Y_i^{(N)}(t-)}^1\int_W\frac{\partial w}{\partial z}(z,t)\left(U^{(N)}(dw,z,t)-U(dw,z,t)\right)dz\\
&\quad-\int_{Y_i(t-)}^{Y_i^{(N)}(t-)}\int_W\frac{\partial w}{\partial z}(z,t)\,U(dw,z,t)\,dz.
\end{aligned}$$

Hence, noting that $0\leqq U^{(N)}\leqq 1$ and $0\leqq U\leqq 1$, we have positive constants C_4 and C_5 such that

$$\begin{aligned}
&|V^{(N)}(W,Y_i^{(N)}(t-),t)-V(W,Y_i(t-),t)|\\
&\leqq C_4|Y_i^{(N)}(t-)-Y_i(t-)|+C_5\sup_{z\in[0,1)}||U^{(N)}(\cdot,z,t)-U(\cdot,z,t)||_{\mathrm{var}}.
\end{aligned}\tag{4.31}$$

Next we estimate the fourth term of the right-hand side of (4.30). First we show that

$$\int_{\xi\in[0,\infty)}|x\,\mathbf{1}_{\xi\in[0,w_i(x,t))}-y\,\mathbf{1}_{\xi\in[0,w_i(y,t))}|^2d\xi\leqq C_4|x-y|,\quad x,y\in[0,1) \tag{4.32}$$

where C_4 is a positive constant. Let $x, y \in [0,1)$ and consider the case that $w_i(y,t) \leqq w_i(x,t)$. Then,

$$\begin{aligned}
&\int_{\xi\in[0,\infty)} |x\,\mathbf{1}_{\xi\in[0,w_i(x,t))} - y\,\mathbf{1}_{\xi\in[0,w_i(y,t))}|^2 d\xi \\
&= \int_{\xi\in[0,\infty)} |(x-y)\,\mathbf{1}_{\xi\in[0,w_i(x,t))} + y\,\mathbf{1}_{\xi\in[w_i(y,t),w_i(x,t))}|^2 d\xi \\
&\leqq \int_{\xi\in[0,\infty)} \left(2(x-y)^2\,\mathbf{1}_{\xi\in[0,w_i(x,t))} + 2y^2\,\mathbf{1}_{\xi\in[w_i(y,t),w_i(x,t))}\right) d\xi \\
&= 2(x-y)^2 w_i(x,t) + y^2(w_i(x,t) - w_i(y,t)).
\end{aligned}$$

Since w_i and the spatial derivative of w_i are bounded, we have

$$\int_{\xi\in[0,\infty)} |x\,\mathbf{1}_{\xi\in[0,w_i(x,t))} - y\,\mathbf{1}_{\xi\in[0,w_i(y,t))}|^2 d\xi \leqq C_4|x-y|,\ \ x, y \in [0,1),$$

where C_4 is a positive constant. Therefore, (4.32) holds for the case that $w_i(y,t) \leqq w_i(x,t)$. The case that $w_i(y,t) \geqq w_i(x,t)$ is shown similarly.

By (4.7), Doob's martingale inequality and (3.9) of Chapter II in Ikeda and Watanabe (1989), there exists a positive constant C_6 such that

$$\begin{aligned}
&E\left[\sup_{s\in[0,t]}\left|\int_{u\in(0,s]}\int_{\xi\in[0,\infty)}[Y_i^{(N)}(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(u-),u))} - Y_i(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i(u-),u))}]\nu_i(d\xi du)\right|^2\right] \\
&\leqq 2E\left[\sup_{s\in[0,t]}\left|\int_{u\in(0,s]}\int_{\xi\in[0,\infty)}[Y_i^{(N)}(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(u-),u))} - Y_i(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i(u-),u))}]\tilde{\nu}_i(d\xi du)\right|^2\right] \\
&\quad +2E\left[\sup_{s\in[0,t]}\left|\int_{u\in(0,s]}\int_{\xi\in[0,\infty)}[Y_i^{(N)}(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(u-),u))} - Y_i(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i(u-),u))}]d\xi du\right|^2\right] \\
&\leqq 2C_6E\left[\sup_{s\in[0,t]}\int_{u\in(0,s]}\int_{\xi\in[0,\infty)}\left|Y_i^{(N)}(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(u-),u))} - Y_i(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i(u-),u))}\right|^2 d\xi du\right] \\
&\quad +2E\left[\sup_{s\in[0,t]}\left|\int_0^s [Y_i^{(N)}(u-)w_i(Y_i^{(N)}(u-),u) - Y_i(u-)w_i(Y_i(u-),u)]du\right|^2\right].
\end{aligned}$$

By (4.32) and boundedness of the spatial derivative of w_i, there exists a positive constant C_7 such that

$$\begin{aligned}
&E\left[\sup_{s\in[0,t]}\left|\int_{u\in(0,s]}\int_{\xi\in[0,\infty)}[Y_i^{(N)}(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(u-),u))} - Y_i(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i(u-),u))}]\nu_i(d\xi du)\right|^2\right] \\
&\leqq 2C_4C_6E\left[\int_0^t \left|Y_i^{(N)}(u-) - Y_i(u-)\right| du\right] + 2C_7E\left[\sup_{s\in[0,t]}\left(\int_0^s \left|Y_i^{(N)}(u-) - Y_i(u-)\right| du\right)^2\right] \\
&\leqq 2C_4C_6\int_0^t E\left[\left|Y_i^{(N)}(u-) - Y_i(u-)\right|^2\right] du + 2C_7tE\left[\int_0^t \left|Y_i^{(N)}(u-) - Y_i(u-)\right|^2 du\right].
\end{aligned}$$

Thus, we obtain for $t \in [0,T]$

$$E\left[\sup_{s\in[0,t]}\left|\int_{u\in(0,s]}\int_{\xi\in[0,\infty)}[Y_i^{(N)}(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i^{(N)}(u-),u))} - Y_i(u-)\,\mathbf{1}_{\xi\in[0,w_i(Y_i(u-),u))}]\nu_i(d\xi du)\right|^2\right] \leqq (2C_4C_6+4TC_7)\int_0^t E\left[\left|Y_i^{(N)}(u-)-Y_i(u-)\right|^2\right]du. \tag{4.33}$$

We are now ready to apply Gronwall's inequality to (4.30). By (4.30), (4.31) and (4.33) we have, for $t \in [0,T]$,

$$\begin{aligned}
&E\left[\sup_{s\in[0,t]}|Y_i^{(N)}(s)-Y_i(s)|^2\right]\\
&\leqq 4|y_i^{(N)}-y_i|^2+4E\left[\sup_{s\in[0,t]}|M_i^{(N)}(s)|^2\right]\\
&\quad+4\int_0^t E\left[(C_4|Y_i^{(N)}(s-)-Y_i(s-)|+C_5\sup_{z\in[0,1),s\in[0,T]}||U^{(N)}(\cdot,z,s)-U(\cdot,z,s)||_{\mathrm{var}})^2\right]ds\\
&\quad+4(2C_4C_6+4TC_7)\int_0^t E\left[\left|Y_i^{(N)}(u-)-Y_i(u-)\right|^2\right]du\\
&\leqq 4|y_i^{(N)}-y_i|^2+4E\left[\sup_{s\in[0,t]}|M_i^{(N)}(s)|^2\right]+8C_5^2\sup_{z\in[0,1),s\in[0,T]}||U^{(N)}(\cdot,z,s)-U(\cdot,z,s)||_{\mathrm{var}}^2\\
&\quad+[8C_4^2+4(2C_4C_6+4TC_7)]\int_0^t E\left[\sup_{u\in[0,s]}|Y_i^{(N)}(u)-Y_i(u)|^2\right]ds.
\end{aligned}$$

By Gronwall's inequality, we obtain

$$\begin{aligned}
&E\left[\sup_{t\in[0,T]}|Y_i^{(N)}(t)-Y_i(t)|^2\right]\\
&\leqq 4e^{C_8T}\left(|y_i^{(N)}-y_i|^2+E\left[\sup_{t\in[0,T]}|M_i^{(N)}(t)|^2\right]\right.\\
&\qquad\left.+2C_5^2\sup_{z\in[0,1),s\in[0,T]}||U^{(N)}(\cdot,z,s)-U(\cdot,z,s)||_{\mathrm{var}})^2\right),
\end{aligned}\tag{4.34}$$

where C_8 is a positive constant.

To show that the right-hand side of (4.34) vanishes as N goes to infinity, we prove that $E\left[\sup_{t\in[0,T]}|M_i^{(N)}(t)|^2\right]$ converges to 0 as N goes to infinity. Doob's martingale inequality and (3.9) of Chapter II in Ikeda and Watanabe (1989) and (4.29) imply there exists a positive constant C_9 such that

$$\begin{aligned}
&E\left[\sup_{s\in[0,t]}|M_i^{(N)}(s)|^2\right]\\
&\leqq \frac{C_9}{N^2}E\left[\sum_{j=1}^N\int_{s\in(0,t]}\int_{\xi\in[0,\infty)}\mathbf{1}_{Y_i^{(N)}(s-)<Y_j^{(N)}(s-)}\,\mathbf{1}_{\xi\in[0,w_j(Y_j^{(N)}(s-),s))}\,d\xi ds\right]\\
&\leqq \frac{C_9R_w(T)t}{N}.
\end{aligned}$$

Hence,

$$\lim_{N\to\infty} E\left[\sup_{s\in[0,t]} |M_i^{(N)}(s)|^2\right] = 0. \tag{4.35}$$

Therefore, by the first assertion of Theorem 2.2, (2.27), (4.34) and (4.35) we obtain

$$\lim_{N\to\infty} E\left[\sup_{t\in[0,T]} |Y_i^{(N)}(t) - Y_i(t)|^2\right] = 0$$

for $i = 1, 2, \ldots, L$. This implies that $(Y_1^{(N)}(t), Y_2^{(N)}(t), \ldots, Y_L^{(N)}(t))$ converges to $(Y_1(t), Y_2(t), \ldots, Y_L(t))$ uniformly in $t \in [0, T]$ in the sense of L^2.

To show the almost sure convergence, see that there exists a subsequence $\{N(k)\}$ such that $(Y_1^{(N(k))}(t), Y_2^{(N(k))}(t), \ldots, Y_L^{(N(k))}(t))$ converges to $(Y_1(t), Y_2(t), \ldots, Y_L(t))$ uniformly in $t \in [0, T]$ almost surely. However, the argument above is also available even if we replace N by any subsequence $N(k)$. Therefore, we have $(Y_1^{(N)}(t), Y_2^{(N)}(t), \ldots, Y_L^{(N)}(t))$ converges to $(Y_1(t), Y_2(t), \ldots, Y_L(t))$ uniformly in $t \in [0, T]$ almost surely.

5. Appendix

Proposition 5.1. *Let $\{\phi_n\}$ be nondecreasing functions on $[0, 1]$ and ϕ be a continuous function on $[0, 1]$. Assume that $\phi_n(x)$ converges to $\phi(x)$ for all $x \in [0, 1]$. Then, $\phi_n(x)$ converges to $\phi(x)$ uniformly in $x \in [0, 1]$.* ◇

Proof. Let $\varepsilon > 0$. Since ϕ is uniformly continuous on $[0, 1]$, we can choose a positive integer N such that

$$|\phi(x) - \phi(y)| < \varepsilon, \quad |x - y| \leqq \frac{1}{N}.$$

By the assumption, there exists a integer n_0 such that

$$\left|\phi_n\left(\frac{k}{N}\right) - \phi\left(\frac{k}{N}\right)\right| < \varepsilon, \quad n \geqq n_0 \text{ and } k = 1, 2, \ldots, N.$$

For all $x \in [0, 1]$ we can choose $k_x \in \{1, 2, \ldots, N\}$ such that $0 \leqq x - k_x/N \leqq 1/N$. Hence, we have for all $x \in [0, 1]$ and $n \geqq n_0$

$$\begin{aligned}
&|\phi_n(x) - \phi(x)| \\
&\leqq \left|\phi_n(x) - \phi_n\left(\frac{k_x}{N}\right)\right| + \left|\phi_n\left(\frac{k_x}{N}\right) - \phi\left(\frac{k_x}{N}\right)\right| + \left|\phi\left(\frac{k_x}{N}\right) - \phi(x)\right| \\
&< \phi_n\left(\frac{k_x+1}{N}\right) - \phi_n\left(\frac{k_x}{N}\right) + 2\varepsilon \\
&\leqq \left|\phi_n\left(\frac{k_x+1}{N}\right) - \phi\left(\frac{k_x+1}{N}\right)\right| + \left|\phi\left(\frac{k_x+1}{N}\right) - \phi\left(\frac{k_x}{N}\right)\right| \\
&\qquad + \left|\phi\left(\frac{k_x}{N}\right) - \phi_n\left(\frac{k_x}{N}\right)\right| + 2\varepsilon \\
&\leqq 5\varepsilon.
\end{aligned}$$

This completes the proof. □

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