Semigroup Forum (2014) 89:571–581
DOI 10.1007/s00233-014-9590-3

RESEARCH ARTICLE

Numerical semigroups of genus eight and double coverings of curves of genus three

Takeshi Harui · Jiryo Komeda

Received: 25 November 2013 / Accepted: 13 February 2014 / Published online: 20 March 2014

Abstract The authors determine all possible numerical semigroups at ramification points of double coverings of curves when the covered curve is of genus three and the covering curve is of genus eight. Moreover, it is shown that all of such numerical semigroups are actually of double covering type.

Keywords Weierstrass semigroups · Numerical semigroups · Double coverings · Plane quartics

1 Introduction

A *numerical semigroup* is a submonoid of $\mathbb{N}_0$, the additive monoid of non-negative integers, such that its complement is a finite set. The *genus* $g(H)$ of a numerical semigroup H is defined by $g(H) = |\mathbb{N}_0 \backslash H|$. We denote by $a \to b$ a sequence $a, a+1, a+2, \ldots, b$ of non-negative integers and $\langle n_1, n_2, \ldots, n_r \rangle$ stands for the numerical semigroup generated by the positive integers $n_1, n_2, \ldots, n_r$.

In this paper a *curve* means a projective, smooth and irreducible curve over an algebraically closed field of characteristic zero unless otherwise mentioned. We denote

Communicated by Fernando Torres.

T. Harui (✉)
Academic Support Center, Kogakuin University, 2665-1 Nakano,
Hachioji, Tokyo 192-0015, Japan
e-mail: takeshi@cwo.zaq.ne.jp; kt13459@ns.kogakuin.ac.jp

J. Komeda
Department of Mathematics, Center for Basic Education and Integrated Learning,
Kanagawa Institute of Technology, Atsugi, Kanagawa 243-0292, Japan
e-mail: komeda@gen.kanagawa-it.ac.jp

by $X^{(r)}$ the variety of effective divisors of degree r on X. We simply write X instead of $X^{(1)}$.

For a point P on a curve X, the semigroup

$$H(P) = \{m \in \mathbb{N}_0 \mid \text{there exists a rational function } f \text{ on } X \text{ with} (f)_\infty = mP\}$$

is a numerical semigroup of genus $g(X)$. A numerical semigroup H is said to be *Weierstrass* if there exist a curve X and a point P on X such that $H(P) = H$.

For a numerical semigroup $\tilde{H}$, we define the set $d_2(\tilde{H})$ by

$$d_2(\tilde{H}) = \{h \in \mathbb{N}_0 \mid 2h \in \tilde{H}\}.$$

This is also a numerical semigroup.

Definition 1.1 A numerical semigroup $\tilde{H}$ is said to be *of double covering type*, or simply *DC*, if there exists a double covering $\pi : X \to Y$ of curves with a ramification point $\tilde{P} \in X$ such that $\tilde{H} = H(\tilde{P})$.

It is easy to verify that $d_2(\tilde{H}) = H(\pi(\tilde{P}))$ and $g(\tilde{H}) \geq 2g(d_2(\tilde{H}))$ for the semigroup of double covering type in the definition. A numerical semigroup of double covering type is Weierstrass.

It is an interesting and important problem to determine whether a given numerical semigroup is Weierstrass, or moreover, of double covering type or not. Let $\tilde{H}$ be a numerical semigroup with $d_2(\tilde{H}) = H$. In this article we shall show that $\tilde{H}$ is DC if $g(H) = 3$ and $g(\tilde{H}) = 8$ (see Theorem 1.2 for the precise statement).

First we briefly collect the results when $g(H) \leq 3$.

If $g(H) = 0$, i.e., $H = \mathbb{N}_0$ then $2 \in \tilde{H}$. In this case $\tilde{H}$ appears as a Weierstrass semigroup of a ramification point of a hyperelliptic covering ([3, Proposition 2.2]).

If $g(H) = 1, 2$ then $\tilde{H}$ is DC unless $\tilde{H} = \langle 3, 5, 7\rangle$, which is clearly not DC. (see [5, Main Theorem], [3, Theorem 3.5, Theorem 4.11], [2, Theorem 1], [1, Lemma 7, Lemma 9] and also [6, Theorem 4.1, Theorem 4.3] when $g(\tilde{H})$ is large).

If $g(H) = 3$ then some partial results are already known (see [4, Main Theorem] and also [7, Example 3.5, Example 3.6, Example 3.9]) when $g(\tilde{H})$ is large.

In general our problem can be treated by a systematic method if the genus of $\tilde{H}$ is large with respect to that of $d_2(\tilde{H})$. Indeed, it is proved in [4, Main Theorem] that a semigroup $\tilde{H}$ with $g(d_2(\tilde{H})) = 3$ is DC if $g(\tilde{H}) \geq 9$. Unfortunately, the argument in [4] is not directly applicable to the cases of lower genera. Thus we treat such cases separately according to the genus of the semigroup $\tilde{H}$.

Back to our situation, let H be a numerical semigroup of genus three. Then $H = \langle 2, 7\rangle$, $\langle 3, 4\rangle$, $\langle 3, 5, 7\rangle$ or $\langle 4, 5, 6, 7\rangle$. Each of them is Weierstrass. More precisely, for a curve Γ of genus three and a point P on Γ, $H(P)$ is determined as follows:

If Γ is hyperelliptic, then the Weierstrass semigroup $H(P)$ is $\langle 2, 7\rangle$ (resp. $\langle 4, 5, 6, 7\rangle$) when P is a ramification point (resp. a non-ramification point) of the hyperelliptic covering from Γ to $\mathbb{P}^1$.

If Γ is non-hyperelliptic, we identify Γ with a smooth plane quartic. Then $H(P)$ is $\langle 3, 4\rangle$, $\langle 3, 5, 7\rangle$ or $\langle 4, 5, 6, 7\rangle$ according as P is a hyperflex, an ordinary flex or a non-flex of Γ.

The following is the main result of this article:

Theorem 1.2 *Let H be a numerical semigroup of genus three, i.e., $H = \langle 2, 7\rangle$, $\langle 3, 4\rangle$, $\langle 3, 5, 7\rangle$ or $\langle 4, 5, 6, 7\rangle$. Let $\tilde{H}$ be a numerical semigroup of genus eight such that $d_2(\tilde{H}) = H$. Then $\tilde{H}$ is of double covering type.*

For $H = \langle 2, 7\rangle$ the assertion follows from [5, Main Theorem]. Thus we shall exclude it from consideration.

2 Determination of numerical semigroups

To begin with, we determine all numerical semigroups under consideration. See Table 1 for the result of the argument below.

First of all we recall the following basic facts on numerical semigroup:

Lemma 2.1 *Let $\tilde{H}$ be any numerical semigroup.*

(i) $\tilde{H} \subset d_2(\tilde{H})$.
(ii) $2d_2(\tilde{H}) = \{2h \mid h \in d_2(\tilde{H})\} = \tilde{H} \cap 2\mathbb{N}_0$.

In what follows let $\tilde{H}$ be a numerical semigroup of genus eight such that $H = d_2(\tilde{H})$ is $\langle 3, 4\rangle$, $\langle 3, 5, 7\rangle$ or $\langle 4, 5, 6, 7\rangle$. Let $\tilde{G} = \mathbb{N}_0\backslash\tilde{H}$ be the complement of $\tilde{H}$ and n the minimum odd element of $\tilde{H}$. Then we obtain two facts:

Claim. *The following hold.*

(i) $\tilde{G} \supset \{1 \to 4\}$.
(ii) $5 \le n \le 11$.

Proof of the claim (i) It is a direct consequence of Lemma 2.1 that $\tilde{G} \ni 1, 2$ and 4. Suppose that $3 \in \tilde{H}$. If $H = \langle 3, 4\rangle$ or $\langle 4, 5, 6, 7\rangle$ then $\tilde{H} \supset \langle 3, 8\rangle$, which implies that $g(\tilde{H}) \le g(\langle 3, 8\rangle) = 7$, a contradiction. In the same way, if $H = \langle 3, 5, 7\rangle$ then $\tilde{H} \supset \langle 3, 10, 14\rangle$, which implies that $g(\tilde{H}) \le g(\langle 3, 10, 14\rangle) = 7$, a contradiction again. It follows that $3 \in \tilde{G}$.

(ii) If $n \ge 13$, then $\tilde{G} \supset \{1 \to 5, 7, 9, 11\}$. Comparing the numbers of elements in these sets, we see that $\tilde{G} = \{1 \to 5, 7, 9, 11\}$. Then, however, $\tilde{H} \ni 6, 8$ and 10, which implies that $H \supset \langle 3, 4, 5\rangle$. In particular $g(H) \le g(\langle 3, 4, 5\rangle) = 2$, a contradiction. □

We divide our cases according to H and the value of n.

Case(a) $H = \langle 3, 4\rangle$. First observe that $\tilde{G} \supset \{1 \to 5, 10\}$ from Claim (i) and Lemma 2.1. In particular $n = 7, 9$ or 11.

Subcase(a–i) If $n = 7$, then $\tilde{H} \supset \langle 6, 7, 8\rangle$, namely, $\tilde{G} \subset \{1 \to 5, 9, 10, 11, 17\}$. Hence $\tilde{H}$ contains exactly one element in $\{9, 11, 17\}$. Then $\tilde{H}$ must contain 17 since $6, 8 \in \tilde{H}$. Thus $\tilde{H} = \langle 6, 7, 8, 17\rangle$ in this case.

Subcase(a–ii) If $n = 9$, then $\tilde{H} \supset \langle 6, 8, 9\rangle$ or equivalently, $\tilde{G} \subset \{1 \to 5, 7, 10, 11, 13, 19\}$. Therefore $\tilde{H}$ contains two elements in $\{11, 13, 19\}$. It follows that $\tilde{H} = \langle 6, 8, 9, 11\rangle$ or $\langle 6, 8, 9, 13\rangle$.

Subcase(a–iii) If $n = 11$, then $\tilde{G} \supset \{1 \to 5, 7, 9, 10\}$. Comparing the numbers of elements in these sets, we see that $\tilde{G} = \{1 \to 5, 7, 9, 10\}$, i.e., $\tilde{H} = \langle 6, 8, 11, 13, 15\rangle$.

Table 1 The classification of $\tilde{H}$

	$\tilde{H}$		
n	$H = \langle 3, 4\rangle$	$H = \langle 3, 5, 7\rangle$	$H = \langle 4, 5, 6, 7\rangle$
5		$\langle 5, 6, 14\rangle$	$\langle 5, 8, 12, 14\rangle$
7	$\langle 6, 7, 8, 17\rangle$	$\langle 6, 7, 10, 11\rangle$	$\langle 7 \to 10, 12\rangle$
		$\langle 6, 7, 10, 15\rangle$	$\langle 7, 8, 10, 11, 12\rangle$
			$\langle 7, 8, 10, 12, 13\rangle$
9	$\langle 6, 8, 9, 11\rangle$	$\langle 6, 9, 10, 11, 14\rangle$	$\langle 8 \to 14\rangle$
	$\langle 6, 8, 9, 13\rangle$	$\langle 6, 9, 10, 13, 14, 17\rangle$	$\langle 8 \to 12, 14, 15\rangle$
			$\langle 8 \to 10, 12 \to 15\rangle$
11	$\langle 6, 8, 11, 13, 15\rangle$	$\langle 6, 10, 11, 13, 14, 15\rangle$	$\langle 8, 10 \to 15, 17\rangle$

Case(b) $H = \langle 3, 5, 7\rangle$. Note that $\tilde{G} \supset \{1 \to 4, 8\}$ in this case.
Subcase(b–i) If $n = 5$ then $\tilde{H} \supset \langle 5, 6, 14\rangle$. We see that $\tilde{H} = \langle 5, 6, 14\rangle$ by comparing the genera of these numerical semigroups.
Subcase(b–ii) If $n = 7$ then $\tilde{H} \supset \langle 6, 7, 10\rangle$. In other words $\tilde{G} \subset \{1 \to 5, 8, 9, 11, 15\}$. Hence $\tilde{H}$ contains exactly one element in $\{9, 11, 15\}$. If $\tilde{H}$ contains 9 then $\tilde{H}$ also contains 15 since $6 \in \tilde{H}$. It follows that $\tilde{H} = \langle 6, 7, 10, 11\rangle$ or $\langle 6, 7, 10, 15\rangle$.
Subcase(b–iii) If $n = 9$ then $\tilde{H} \supset \langle 6, 9, 10, 14\rangle$, namely, $\tilde{G} \subset \{1 \to 5, 7, 8, 11, 13, 17\}$. Therefore $\tilde{H}$ contains exactly two elements in $\{11, 13, 17\}$. Furthermore, if 11 is a non-gap then 17 is also a non-gap since $6 \in \tilde{H}$. Thus $\tilde{H} = \langle 6, 9, 10, 11, 14\rangle$ or $\tilde{H} = \langle 6, 9, 10, 13, 14, 17\rangle$.
Subcase(b–iv) If $n = 11$ then $\tilde{G} \supset \{1 \to 5, 7, 8, 9\}$, which implies that $\tilde{G} = \{1 \to 5, 7, 8, 9\}$ by comparing the numbers of elements in these sets. That is to say, $\tilde{H} = \langle 6, 10, 11, 13, 14, 15\rangle$.
Case(c) $H = \langle 4, 5, 6, 7\rangle$. Note that $\tilde{G} \supset \{1 \to 4, 6\}$ by virtue of Claim and an easy observation.
Subcase(c–i) If $n = 5$ then $\tilde{H} \supset \langle 5, 8, 12, 14\rangle$. In fact $\tilde{H} = \langle 5, 8, 12, 14\rangle$ since both of these numerical semigroups are of genus eight.
Subcase(c–ii) If $n = 7$ then $\tilde{H} \supset \langle 7, 8, 10, 12\rangle$. Hence $\tilde{G} \subset \{1 \to 6, 9, 11, 13\}$, which implies that there exists an extra non-gap belonging to $\{9, 11, 13\}$. Thus $\tilde{H} = \langle 7 \to 10, 12\rangle$, $\langle 7, 8, 10, 11, 12\rangle$ or $\langle 7, 8, 10, 12, 13\rangle$.
Subcase(c–iii) If $n = 9$ then $\tilde{H} \supset \langle 8, 9, 10, 12, 14\rangle$, namely, $\tilde{G} \subset \{1 \to 7, 11, 13, 15\}$. Hence there exists exactly one gap in $\{11, 13, 15\}$, which implies that $\tilde{H} = \langle 8 \to 14\rangle$, $\langle 8 \to 12, 14, 15\rangle$ or $\langle 8, 9, 10, 12 \to 15\rangle$.
Subcase(c–iv) If $n = 11$ then $\tilde{G} \subset \{1 \to 7, 9\}$. In fact $\tilde{G} = \{1 \to 7, 9\}$ since these two sets has the same cardinality. Hence $\tilde{H} = \langle 8, 10 \to 15, 17\rangle$.

3 Preliminary results

In the rest of this article we show that every numerical semigroup in Table 1 is of double covering type. In this section we show several results on numerical semigroups and curves needed later.

Let H be any Weierstrass semigroup of genus q, $\tilde{H}$ a numerical semigroup of genus g with $d_2(\tilde{H}) = H$. We consider a subsemigroup $\tilde{H}_0 := 2H + n\mathbb{N}_0$ of $\tilde{H}$, where n is the minimum odd element of $\tilde{H}$.

Lemma 3.1 *Keep the notation as above. Then* $g \leq g(\tilde{H}_0) = 2q + (n-1)/2$.

Proof We only have to verify that $g(\tilde{H}_0) = 2q + (n-1)/2$. Let G (resp. $\tilde{G}_0$) be the complement of H (resp. $\tilde{H}_0$). It is easy to show that

$$\tilde{G}_0 \cap 2\mathbb{N}_0 = 2G \quad \text{and} \quad \tilde{G}_0 \cap (n + 2\mathbb{N}_0) = n + 2G.$$

Thus the number of even elements of $\tilde{G}_0$ and that of odd elements of $\tilde{G}_0$ greater than n are equal to q respectively. Furthermore, the rest of elements of $\tilde{G}_0$ are the positive odd integer less than n. Hence $g(\tilde{H}_0) = 2q + (n-1)/2$. □

Remark 3.2 Claim (ii) and Lemma 3.1 are particular cases of a more general result (see, for example, [7, Lemma 2.4]).

Define a non-negative integer r by the equation $g = 2q + (n-1)/2 - r$. Then we obtain a sequence of numerical semigroups $\tilde{H}_0 \subset \tilde{H}_1 \subset \cdots \subset \tilde{H}_s = \tilde{H}$ inductively by setting $\tilde{H}_j := \tilde{H}_{j-1} + (n + 2l_j)\mathbb{N}_0$ for $j = 1, 2, \ldots, s$, where $n + 2l_j$ is the minimum odd element of $\tilde{H}$ not belonging to $\tilde{H}_{j-1}$. Then it is easy to verify that $s \leq r$ since $g(\tilde{H}_0) = 2q + (n-1)/2$.

To prove Theorem 1.2, it suffices to show the existence of certain effective divisor on a curve because of the following theorem.

Theorem 3.3 ([4, Theorem 2.2]) *Take any curve* Γ *of genus* q *and a point* P *on* Γ *with* $H(P) = H$. *Assume that there exists an effective divisor* $\Delta_r \in \Gamma^{(r)}$ *satisfying the following conditions:*

(C1) Δ_r *does not contain* P.
(C2) $h^0(\Gamma, \Delta_r) = 1$.
(C3) $h^0(\Gamma, l_j P + \Delta_r) = h^0(\Gamma, (l_j - 1)P + \Delta_r) + 1$ *for* $j = 1, 2, \ldots, s$.
(C4) *the linear system* $|nP - 2\Delta_r|$ *has a reduced member not containing* P.

Then there exists a double covering $\pi : C \to \Gamma$ *with a ramification point* $\tilde{P}$ *over* P *such that* $H(\tilde{P}) = \tilde{H}$, *in other words,* $\tilde{H}$ *is of double covering type.*

Though the fourth condition is replaced with the freeness of $|nP - 2\Delta_r|$ in [4, Theorem 2.2], the same proof applies under our assumption as well.

Remark 3.4 In our case $q = 3$ and $g = 8$, which implies that $r = (n-5)/2$. Note that (C1) is satisfied if $r = 1$ and $\Delta_1 \neq P$ and (C2) is trivial if $r = 1$, or $r = 2$ and Γ is non-hyperelliptic. If $|nP - 2\Delta_r|$ is free from base points, then (C4) is satisfied.

In the rest of this article we fix the following notation for semigroups and curves. Let H always denote $\langle 3, 4\rangle$, $\langle 3, 5, 7\rangle$ or $\langle 4, 5, 6, 7\rangle$, $\tilde{H}$ a numerical semigroup of genus eight such that $d_2(\tilde{H}) = H$ unless otherwise mentioned. Let n be the minimum odd element of $\tilde{H}$.

Let Γ be a curve of genus three, P a point on Γ such that $H(P) = H$. Note that we can take a hyperelliptic curve as Γ only when $H = \langle 4, 5, 6, 7\rangle$. Assume that Γ is non-hyperelliptic unless otherwise stated. Thus Γ is identified with a smooth plane quartic. Let T_P denote the tangent line of Γ at P.

First we show a lemma on the property of linear systems of degree five on Γ. For this purpose, we recall three elementary facts on algebraic curves.

Proposition 3.5 *Let X be a curve of genus $g \geq 2$.*

(1) *A pencil on X without base points has at most finitely many non-reduced members.*
(2) *For a positive integer m and a divisor A on X, there exist at most finitely many divisors B_i such that $mB_i \sim A$ for any i and $B_i \not\sim B_j$ for $i \neq j$.*
(3) *A complete linear system $|D|$ of degree $2g-1$ on X is free from base points unless $|D| = |K_X| + Q$ for a point $Q \in X$. In particular, for any point P on X, the linear system $|D|$ has a reduced member not containing P unless $|D| = |K_X| + P$.*

The following observation is very useful for our argument.

Lemma 3.6 *Let Γ be any curve of genus three, E a divisor on Γ and P a point on Γ. Let S_r $(r = 1, 2, 3)$ be an infinite subset of $\Gamma^{(r)}$. Assume that $h^0(\Gamma, \Delta_r) = 1$ for any $\Delta_r \in S_r$ if $r = 2$ or 3.*

(1) *If $\deg E = 7$, then $|E - 2\Delta_1|$ is free from base points except for finitely many $\Delta_1 \in S_1$.*
(2) *If $r = 2$ or 3 and $\deg E = 2r + 5$, then $|E - 2\Delta_r|$ has a reduced member not containing P except for finitely many $\Delta_r \in S_r$.*

Proof We use reduction to absurdity.

(1) Suppose that $|E - 2\Delta_1|$ has a base point for infinitely many $\Delta_1 \in S_1$. Since $\deg(E - 2\Delta_1) = 5$, there exists a point P_{Δ_1} on Γ such that $E - 2\Delta_1 \sim K_\Gamma + P_{\Delta_1}$ by Proposition 3.5 (3). Then $2\Delta_1 + P_{\Delta_1}$ is linearly equivalent to a fixed divisor $E - K_\Gamma$ for infinitely many $\Delta_1 \in S_1$. In particular, $\dim|2\Delta_1 + P_{\Delta_1}| \geq 1$. Thus $|2\Delta_1 + P_{\Delta_1}| = g^1_3$, a pencil of degree three. The variable part of the g^1_3 has infinitely many members containing non-reduced divisors of the form $2\Delta_1$, which contradicts Proposition 3.5 (1).
(2) Let r be two or three. Suppose that there exists an infinite subset T_r of S_r such that $|E - 2\Delta_r|$ does not have a reduced member not containing P for any $\Delta_r \in T_r$. Since $\deg(E - 2\Delta_r) = 5$, it follows from Proposition 3.5 (3) that $E - 2\Delta_r \sim K_\Gamma + P$ for any $\Delta_r \in T_r$. In other words, $2\Delta_r$ is linearly equivalent to a fixed divisor $E - K_\Gamma - P$. Then $|\Delta_r|$ is the same linear system for infinitely many $\Delta_r \in T_r$ by virtue of Proposition 3.5 (2). In particular $\dim|\Delta_r| \geq 1$, which conflicts with our assumption. Thus we finish the proof. □

We define a subset $S(\Gamma, P, \tilde{H})$ of $\Gamma^{(r)}$ as follows:

$$S(\Gamma, P, \tilde{H}) := \{\Delta_r \in \Gamma^{(r)} \mid \Delta_r \text{ satisfies the conditions } \text{(C1), (C2) and (C3) in Theorem 3.3}\}.$$

In our case $0 \leq r \leq 3$. We shall use the following lemma for proving Theorem 1.2 in many cases.

Lemma 3.7 *If $S(\Gamma, P, \tilde{H})$ is an infinite set, then there exists an effective divisor $\Delta_r \in S(\Gamma, P, \tilde{H})$ that satisfies the condition* (C4). *In particular $\tilde{H}$ is of double covering type.*

Proof Note that $\deg(nP - 2\Delta_r) = n - 2r = 5$ since $g = 8$. By virtue of Lemma 3.6 there exists an effective divisor $\Delta_r \in S(\Gamma, P, \tilde{H})$ such that $|nP - 2\Delta_r|$ has a reduced member not containing P, which is nothing but the condition (C4). □

We shall prove Theorem 1.2 in the subsequent sections according to the value of n, the minimum odd element of $\tilde{H}$. By virtue of the above lemma, we only have to show that $S(\Gamma, P, \tilde{H})$ is infinite. When the set is finite, we find a certain effective divisor on Γ satisfying the four conditions in Theorem 3.3 by ad hoc arguments.

4 The cases where $n = 5$

Note that $r = 0$ and $(H, \tilde{H}) = (\langle 3, 5, 7\rangle, \langle 5, 6, 14\rangle)$ or $(\langle 4, 5, 6, 7\rangle, \langle 5, 8, 12, 14\rangle)$. Then the conditions (C1), (C2) and (C3) are trivial. Thus it is enough to show that $|5P|$ is free from base points.

Suppose that $|5P|$ has a base point. Then it is P, which implies that $\dim|4P| = \dim|5P| \geq 5 - 3 = 2$. Hence $|4P|$ is the canonical linear system of Γ. Then $H = H(P) = \langle 3, 4\rangle$, which contradicts our assumption. It follows that $|5P|$ has no base points.

5 The cases where $n = 7$

In this case $r = 1$.

(1) Assume that $H = \langle 3, 4\rangle$. Then $\tilde{H} = \langle 6, 7, 8, 17\rangle$. The condition (C3) is nothing but the equality $h^0(\Gamma, 5P + \Delta_1) = h^0(\Gamma, 4P + \Delta_1) + 1$, which holds for any $\Delta_1 \in \Gamma$. With Remark 3.4 we see that $S(\Gamma, P, \tilde{H}) = \Gamma \backslash \{P\}$. This is infinite, hence it follows from Lemma 3.7 that $\tilde{H}$ is DC.

(2) Assume that $H = \langle 3, 5, 7\rangle$. Then $K_\Gamma \sim 3P + P'$ $(P' \neq P)$ and $\tilde{H} = \langle 6, 7, 10, 11\rangle$ or $\langle 6, 7, 10, 15\rangle$.

First consider the case where $\tilde{H} = \langle 6, 7, 10, 11\rangle$. In this case we choose as Γ a general plane quartic. Then it is smooth and has no hyperflexes.

The condition (C3) is the equality $h^0(\Gamma, 2P + \Delta_1) = h^0(\Gamma, P + \Delta_1) + 1 = 2$, which is satisfied if and only if $\Delta_1 = P$ or P'. Thus $S(\Gamma, P, \tilde{H}) = \{P'\}$ holds. We only have to show that $\Delta_1 = P'$ satisfies (C4). Suppose, on the contrary, that the linear system $|7P - 2P'|$ does not have a reduced member not containing P. Then, using Proposition 3.5 (3), we obtain the relation that

$$7P - 2P' \sim K_\Gamma + P \sim 4P + P',$$

which implies that $3P \sim 3P'$. Then $4P' \sim 3P + P' \sim K_\Gamma$, that is to say, P' is a hyperflex of Γ. This contradicts our assumption. Thus we complete the proof in this case.

Secondly consider the case where $\tilde{H} = \langle 6, 7, 10, 15\rangle$. Then the condition (C3) is the equality $h^0(\Gamma, 4P + \Delta_1) = h^0(\Gamma, 3P + \Delta_1) + 1$, which is true if and only if $h^0(\Gamma, 3P + \Delta_1) = 2$, or equivalently, $\Delta_1 \neq P'$. Hence $S(\Gamma, P, \tilde{H}) = \Gamma\backslash\{P, P'\}$ by Remark 3.4. This is infinite, hence it follows from Lemma 3.7 that $\tilde{H}$ is DC.

(3) Finally assume that $H = \langle 4, 5, 6, 7\rangle$. Then $\tilde{H} = \langle 7 \rightarrow 10, 12\rangle$, $\langle 7, 8, 10, 11, 12\rangle$ or $\langle 7, 8, 10, 12, 13\rangle$.

First consider the case where $\tilde{H} = \langle 7 \rightarrow 10, 12\rangle$. In this case we choose as Γ a hyperelliptic curve of genus three. Take a general point $P \in \Gamma$. Then there exists a unique point $P'(\neq P)$ on Γ such that $P + P' \in g_2^1$. Then $H(P) = \langle 4, 5, 6, 7\rangle$ holds and $K_\Gamma \sim 2(P + P')$.

It is clear that $\Delta_1 := P'$ satisfies the conditions (C1) and (C2). It also satisfies the condition (C3), which is nothing but the trivial equality that $h^0(\Gamma, P + P') = h^0(\Gamma, P') + 1 = 2$.

Finally we verify the condition (C4). Suppose that Δ_1 fails to satisfy (C4). It follows from Proposition 3.5 (3) that $7P - 2P' \sim K_\Gamma + P \sim 2(P + P') + P$, which implies that $4(P - P') \sim 0$. By virtue of Proposition 3.5 (2), there exist infinitely many P_i's such that $P_i - P_i'$ is linearly equivalent one another, where $P_i + P_i' \in g_2^1$. Then $P_i - P_i' \sim P_j - P_j'$, which implies that $|P_i + P_j'| = |P_j + P_i'| = g_2^1$. Hence $P_i = P_j$ holds, a contradiction. Thus we conclude that Δ_1 satisfies (C4), which implies that $\tilde{H}$ is DC by Theorem 3.3.

For the remaining cases let Γ be a smooth plane quartic and we choose a non-flex $P \in \Gamma$ such that T_P is not a bitangent. Then $K_\Gamma \sim 2P + P' + P''$, where P' and P'' are distinct points different from P.

Consider the case where $\tilde{H} = \langle 7, 8, 10, 11, 12\rangle$. The condition (C3) is the equality $h^0(\Gamma, 2P+\Delta_1) = h^0(\Gamma, P+\Delta_1)+1 = 2$, which is satisfied if and only if $2P+\Delta_1 \leq K_\Gamma$, in other words, $\Delta_1 = P'$ or P''. Then we only have to prove that either $\Delta_1 = P'$ or $\Delta_1 = P''$ satisfies the condition (C4). Suppose, on the contrary, that neither of linear systems $|7P - 2P'|$ and $|7P - 2P''|$ has a reduced member not containing P. Then, by virtue of Proposition 3.5 (3), we obtain the relation that

$$7P - 2P' \sim 7P - 2P'' \sim K_\Gamma + P,$$

which implies that $2P' \sim 2P''$ holds. Since $P' \neq P''$ by our assumption, $|2P'|$ is a pencil, a contradiction. Thus we finish the proof.

Lastly, consider the case where $\tilde{H} = \langle 7, 8, 10, 12, 13\rangle$. The condition (C3) is the equality $h^0(\Gamma, 3P + \Delta_1) = h^0(\Gamma, 2P + \Delta_1) + 1$, which is satisfied if $\Delta_1 \neq P', P''$ since $h^0(\Gamma, 3P + \Delta_1) = 2$ always holds. With Remark 3.4 we see that $S(\Gamma, P, \tilde{H}) = \Gamma\backslash\{P', P''\}$. This is infinite, hence it follows from Lemma 3.7 that $\tilde{H}$ is DC.

6 The cases where $n = 9$

In this case $r = 2$.

(1) Assume that $H = \langle 3, 4\rangle$. Note that $K_\Gamma \sim 4P$. By Table 1, $\tilde{H} = \langle 6, 8, 9, 11\rangle$ or $\langle 6, 8, 9, 13\rangle$.

In the former case the condition (C3) is the equality $h^0(\Gamma, P + \Delta_2) = h^0(\Gamma, \Delta_2) + 1 = 2$, or equivalently, Δ_2 is contained in a divisor of Γ cut out by a line passing through P. A general line passing through P meets Γ at other three distinct points, any two of which constitute an effective divisor of Γ of degree two satisfying (C1), (C2) and (C3). Thus the set $S(\Gamma, P, \tilde{H})$ is infinite. Hence $\tilde{H}$ is DC by virtue of Lemma 3.7.

In the latter case the condition (C3) is the equality $h^0(\Gamma, 2P + \Delta_2) = h^0(\Gamma, P + \Delta_2) + 1$. Hence a general member $\Delta_2 \in \Gamma^{(2)}$ satisfies the conditions (C1), (C2) and (C3). Thus $S(\Gamma, P, \tilde{H})$ is infinite, which implies that $\tilde{H}$ is DC from Lemma 3.7 again.

(2) Assume that $H = \langle 3, 5, 7\rangle$. Then $K_\Gamma \sim 3P + P'$ ($P' \neq P$). By Table 1, $\tilde{H} = \langle 6, 9, 10, 11, 14\rangle$ or $\langle 6, 9, 10, 13, 14, 17\rangle$.

In the former case, the condition (C3) is the equality $h^0(\Gamma, P + \Delta_2) = h^0(\Gamma, \Delta_2) + 1 = 2$, or equivalently, Δ_2 is contained in a divisor of Γ cut out by a line passing through P. Then we can show that the set $S(\Gamma, P, \tilde{H})$ is infinite and conclude that $\tilde{H}$ is DC by the same argument as in (1).

In the latter case, the condition (C3) is as follows:

$$h^0(\Gamma, 2P + \Delta_2) = h^0(\Gamma, P + \Delta_2) + 1, \quad h^0(\Gamma, 4P + \Delta_2) = h^0(\Gamma, 3P + \Delta_2) + 1.$$

The first equality is satisfied if $h^0(\Gamma, P + \Delta_2) = 1$. The second one always holds. It follows that a general divisor $\Delta_2 \in \Gamma^{(2)}$ satisfies conditions (C1), (C2) and (C3). Therefore $S(\Gamma, P, \tilde{H})$ is infinite again. Thus we conclude that $\tilde{H}$ is DC from Lemma 3.7.

(3) Lastly assume that $H = \langle 4, 5, 6, 7\rangle$. Then $K_\Gamma \sim 2P + P' + P''$ ($P', P'' \neq P$) for some $P', P'' \in \Gamma$. By Table 1, $\tilde{H} = \langle 8 \to 14\rangle$, $\langle 8 \to 12, 14, 15\rangle$ or $\langle 8 \to 10, 12 \to 15\rangle$.

In the first case, we choose as P a general point on Γ. Then T_P is not a bitangent, i.e., $P' \neq P''$. The condition (C3) is as follows:

$$h^0(\Gamma, P + \Delta_2) = h^0(\Gamma, \Delta_2) + 1 = 2, \quad h^0(\Gamma, 2P + \Delta_2) = h^0(\Gamma, P + \Delta_2) + 1 = 3,$$

which implies that $\Delta_2 = P' + P''$. It satisfies three conditions (C1), (C2) and (C3). We verify that it also satisfies the condition (C4). Suppose that $|9P - 2(P' + P'')|$ does not have a reduced member not containing P. Using Proposition 3.5 (3) we obtain the relation

$$9P - 2(P' + P'') \sim K_\Gamma + P \sim 3P + P' + P'',$$

which implies that

$$6P \sim 3(P' + P'') \sim 3(K_\Gamma - 2P) \sim 3K_\Gamma - 6P.$$

Hence $3K_\Gamma \sim 12P$ for a general point P. This is impossible by Proposition 3.5 (2). Thus we complete the proof.

In the second case, where $\tilde{H} = \langle 8 \to 12, 14, 15\rangle$, the condition (C3) is as follows:

$$h^0(\Gamma, P + \Delta_2) = h^0(\Gamma, \Delta_2) + 1 = 2, \quad h^0(\Gamma, 3P + \Delta_2) = h^0(\Gamma, 2P + \Delta_2) + 1.$$

The former equality implies that Δ_2 is contained in a divisor of Γ cut out by a line passing through P. The latter one implies that $h^0(\Gamma, 2P + \Delta_2) = 2$, or equivalently, Δ_2 is not contained in the divisor of Γ cut out by T_P. Note that a general line passing through P cuts out a reduced divisor of degree four, from which we can take a reduced divisor $\Delta_2 \in S(\Gamma, P, \tilde{H})$. Thus the set $S(\Gamma, P, \tilde{H})$ is infinite. Hence we conclude that $\tilde{H}$ is DC using Lemma 3.7.

In the end, consider the case where $\tilde{H} = \langle 8 \to 10, 12 \to 15\rangle$. The condition (C3) is as follows:

$$h^0(\Gamma, 2P + \Delta_2) = h^0(\Gamma, P + \Delta_2) + 1, \quad h^0(\Gamma, 3P + \Delta_2) = h^0(\Gamma, 2P + \Delta_2) + 1.$$

Both of them are satisfied if Δ_2 is not contained in a divisor of Γ cut out by a line passing through P, since then $h^0(\Gamma, P+\Delta_2) = 1$, $h^0(\Gamma, 2P+\Delta_2) = 2$ and $h^0(\Gamma, 3P+\Delta_2) = 3$. Thus a general divisor Δ_2 of $\Gamma^{(2)}$ satisfies (C1), (C2) and (C3). Hence the set $S(\Gamma, P, \tilde{H})$ is infinite again, which implies that $\tilde{H}$ is DC from Lemma 3.7.

7 The cases where $n = 11$

In this case $r = 3$. By Table 1 $\tilde{H} = \langle 6, 8, 11, 13, 15\rangle$, $\langle 6, 10, 11, 13, 14, 15\rangle$ or $\tilde{H} = \langle 8, 10 \to 15, 17\rangle$. Under the condition (C2), the condition (C3) is as follows:

$$\begin{aligned} h^0(\Gamma, P + \Delta_3) &= h^0(\Gamma, \Delta_3) + 1 = 2, \\ h^0(\Gamma, 2P + \Delta_3) &= h^0(\Gamma, P + \Delta_3) + 1 = 3 \quad \text{and also} \\ h^0(\Gamma, 3P + \Delta_3) &= h^0(\Gamma, 2P + \Delta_3) + 1 = 4 \quad \text{when } \tilde{H} = \langle 8, 10 \to 15, 17\rangle. \end{aligned}$$

All of them hold for any $\Delta_3 \in \Gamma^{(3)}$ if $h^0(\Gamma, \Delta_3) = 1$. Hence a general divisor $\Delta_3 \in \Gamma^{(3)}$ satisfies (C1), (C2) and (C3). Thus the set $S(\Gamma, P, \tilde{H})$ is infinite. Hence $\tilde{H}$ is DC in these cases by virtue of Lemma 3.7.

References

1. Garcia, A.: Weights of Weierstrass points in double coverings of curves of genus one or two. Manuscripta Math. **55**, 419–432 (1986)
2. Harui, T., Komeda, J., Ohbuchi, A.: The Weierstrass semigroups on double covers of genus two curves (submitted)
3. Komeda, J.: A numerical semigroup from which the semigroup gained by dividing by two is either $\mathbb{N}_0$ or a 2-semigroup or $\langle 3, 4, 5\rangle$. Res. Rep. Kanagawa Inst. Technol. **B–33**, 37–42 (2009)
4. Komeda, J.: On Weierstrass semigroups of double coverings of genus three curves. Semigroup Forum **83**, 479–488 (2011)
5. Komeda, J., Ohbuchi, A.: Weierstrass points with first non-gap four on a double covering of a hyperelliptic curve II. Serdica Math. J. **34**, 771–782 (2008)

6. Oliveira, G., Pimentel, F.L.R.: On Weierstrass semigroups of double covering of genus two curves. Semigroup Forum **77**, 152–162 (2008)
7. Oliveira, G., Torres, F., Villanueva, J.: On the weight of numerical semigroups. J. Pure Appl. Algebra **214**, 1955–1961 (2010)