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RESEARCH ARTICLE

Numerical semigroups of genus seven and double coverings of curves of genus three

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Abstract The authors determine all possible numerical semigroups at ramification points of double coverings of curves when the covered curve is of genus three and the covering curve is of genus seven, and prove that all of such numerical semigroups are actually of double covering type.

Keywords Weierstrass semigroups · Numerical semigroups · Double coverings · Plane quartics

1 Introduction

This paper is a sequel of our previous work [2] on numerical semigroups obtained by double coverings of curves of genus three.

A *numerical semigroup* is a submonoid of $\mathbb{N}_0$, the additive monoid of non-negative integers, such that its complement is a finite set. The *genus* $g(H)$ of a numerical semigroup H is defined as the cardinality of its complement. We denote by $a \to b$ a sequence $a,\ a+1,\ a+2, \ldots, b$ of non-negative integers and $\langle n_1, n_2, \ldots, n_r\rangle$ stands for the numerical semigroup generated by positive integers $n_1, n_2, \ldots, n_r$.

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In this paper a *curve* means a projective, smooth and irreducible curve over an algebraically closed field of characteristic zero unless otherwise mentioned. We denote by $X^{(r)}$ the variety of effective divisors of degree r on X. We simply write X instead of $X^{(1)}$.

For a point P on a curve X, the semigroup

$$H(P) = \{m \in \mathbb{N}_0 \mid \text{there exists a rational function } f \text{ on } X \text{ with } (f)_\infty = mP\}$$

is a numerical semigroup of genus $g(X)$. A numerical semigroup H is said to be *Weierstrass* if there exist a curve X and a point P on X such that $H(P) = H$.

For a numerical semigroup $\tilde{H}$, we define the set $d_2(\tilde{H})$ by

$$d_2(\tilde{H}) = \{h \in \mathbb{N}_0 \mid 2h \in \tilde{H}\}.$$

This is also a numerical semigroup.

Definition 1.1 A numerical semigroup $\tilde{H}$ is said to be *of double covering type*, or simply *DC*, if there exists a double covering $\pi : X \to Y$ of curves with a ramification point $\tilde{P} \in X$ such that $\tilde{H} = H(\tilde{P})$.

It is easy to verify that $d_2(\tilde{H}) = H(\pi(\tilde{P}))$ and $g(\tilde{H}) \geq 2g(d_2(\tilde{H}))$ for the semigroup of double covering type in the definition.

In this article we are interested in numerical semigroups of genus seven whose image by the d_2 map is of genus three. Let $\tilde{H}$ be such a numerical semigroup of genus seven, i.e., $H := d_2(\tilde{H})$ is of genus three. We shall show that $\tilde{H}$ is DC (see Theorem 1.3 for the precise statement).

Remark 1.2 (1) Every numerical semigroup $\tilde{H}$ such that $H = d_2(\tilde{H})$ is of genus at most two is known to be DC unless $\tilde{H} = \langle 3, 5, 7\rangle$ (see [4, 7–9] and [3] for details).
(2) Every numerical semigroup $\tilde{H}$ such that $g(\tilde{H}) \geq 8$ and $g(H) = 3$ is DC (see [5] and [2]).

Our main result is the following theorem:

Theorem 1.3 *Let H be a numerical semigroup of genus three, i.e., $H = \langle 2, 7\rangle$, $\langle 3, 4\rangle$, $\langle 3, 5, 7\rangle$ or $\langle 4, 5, 6, 7\rangle$. Let $\tilde{H}$ be a numerical semigroup of genus seven such that $d_2(\tilde{H}) = H$. Then $\tilde{H}$ is of double covering type.*

The assertion is known to hold if $H = \langle 2, 7\rangle$ (cf. [7, Main Theorem]). Thus we shall exclude the case from consideration.

2 Determination of numerical semigroups

In this section we determine the numerical semigroups under consideration. See Table 1 for the result. First we note the following:

Lemma 2.1 *Let $\tilde{H}$ be any numerical semigroup.*

Table 1 The classification of $\tilde{H}$

n	H: $\langle 3, 4\rangle$	$\langle 3, 5, 7\rangle$	$\langle 4, 5, 6, 7\rangle$
3	$\langle 3, 8\rangle$	$\langle 3, 10, 14\rangle$	
5		$\langle 5, 6, 9\rangle$	$\langle 5, 7, 8\rangle$
		$\langle 5, 6, 13, 14\rangle$	$\langle 5, 8, 9, 12\rangle$
			$\langle 5, 8, 11, 12, 14\rangle$
7	$\langle 6 \to 9\rangle$	$\langle 6, 7, 9, 10\rangle$	$\langle 7 \to 12\rangle$
	$\langle 6, 7, 8, 11\rangle$	$\langle 6, 7, 10, 11, 15\rangle$	$\langle 7 \to 10, 12, 13\rangle$
			$\langle 7, 8, 10 \to 13\rangle$
9	$\langle 6, 8, 9, 11, 13\rangle$	$\langle 6, 9, 10, 11, 13, 14\rangle$	$\langle 8 \to 15\rangle$

(i) $\tilde{H} \subseteq d_2(\tilde{H})$.
(ii) $2d_2(\tilde{H}) = \{2h \mid h \in d_2(\tilde{H})\} = \tilde{H} \cap 2\mathbb{N}_0$.

In what follows let $\tilde{H}$ be a numerical semigroup of genus seven such that $H = d_2(\tilde{H})$ is $\langle 3, 4\rangle$, $\langle 3, 5, 7\rangle$ or $\langle 4, 5, 6, 7\rangle$. Let $\tilde{G} = \mathbb{N}_0 \setminus \tilde{H}$ be the set of gaps of $\tilde{H}$ and n the minimum odd non-gap of $\tilde{H}$. Then it is clear that $\tilde{G} \supseteq \{1, 2, 4\}$. Furthermore, note that $n \leq 9$. Indeed, if $n \geq 11$ then $\tilde{G} \supseteq \{1 \to 5, 7, 9\}$. In fact equality holds, which implies that $\tilde{H} \supseteq \{6, 8, 10\}$. Then $H \supseteq \langle 3, 4, 5\rangle$, a contradiction.

We classify the numerical semigroups of our interest according to H and the value of n.

Case(a) $H = \langle 3, 4\rangle$. Note that $\tilde{G} \supseteq \{1, 2, 4, 5, 10\}$ in this case.
Subcase(a-i) If $n = 3$, then $\tilde{H} \supseteq \langle 3, 8\rangle$, which shows that $\tilde{H} = \langle 3, 8\rangle$.
Subcase(a-ii) If $n = 7$, then $\tilde{H} \supseteq \langle 6, 7, 8\rangle$, or equivalently, $\tilde{G} \subseteq \{1 \to 5, 9, 10, 11, 17\}$. Therefore $\tilde{H}$ contains exactly two elements in $\{9, 11, 17\}$. Note that $\tilde{H}$ must contain 17 since $6 \in \tilde{H}$. Hence $\tilde{H} = \langle 6 \to 9\rangle$ or $\langle 6, 7, 8, 11\rangle$ in this case.
Subcase(a-iii) If $n = 9$, then $\tilde{G} \supseteq \{1 \to 5, 7, 10\}$. Therefore $\tilde{H} = \langle 6, 8, 9, 11, 13\rangle$.
Case(b) $H = \langle 3, 5, 7\rangle$. Then $\tilde{G} \supseteq \{1, 2, 4, 8\}$.
Subcase(b-i) If $n = 3$, then $\tilde{H} \supseteq \langle 3, 10, 14\rangle$. In fact equality holds, since both semigroups have the same genus.
Subcase(b-ii) If $n = 5$ then $\tilde{H} \supseteq \langle 5, 6, 14\rangle$, or equivalently, $\tilde{G} \subseteq \{1 \to 4, 7, 8, 9, 13\}$. Thus $\tilde{H}$ contains 9 or 13, that is to say, $\tilde{H} = \langle 5, 6, 9\rangle$ or $\langle 5, 6, 13, 14\rangle$.
Subcase(b-iii) If $n = 7$ then $\tilde{H} \supseteq \langle 6, 7, 10\rangle$. In other words $\tilde{G} \subseteq \{1 \to 5, 8, 9, 11, 15\}$. Therefore $\tilde{H}$ contains exactly two elements in $\{9, 11, 15\}$. If $\tilde{H}$ contains 9 then $\tilde{H}$ also contains 15 since $6 \in \tilde{H}$. It follows that $\tilde{H} = \langle 6, 7, 9, 10\rangle$ or $\langle 6, 7, 10, 11, 15\rangle$.
Subcase(b-iv) If $n = 9$ then $\tilde{G} \supseteq \{1 \to 5, 7, 8\}$, which is in fact equality. It follows that $\tilde{H} = \langle 6, 9, 10, 11, 13, 14\rangle$.
Case(c) $H = \langle 4, 5, 6, 7\rangle$. Note that $\tilde{G} \supseteq \{1 \to 4, 6\}$.
Subcase(c-i) If $n = 5$ then $\tilde{H} \supseteq \langle 5, 8, 12, 14\rangle$, i.e., $\tilde{G} \subseteq \{1 \to 4, 6, 7, 9, 11\}$. Hence there exists another non-gap of $\tilde{H}$ in $\{7, 9, 11\}$. It follows that $\tilde{H} = \langle 5, 7, 8\rangle$, $\langle 5, 8, 9, 12\rangle$ or $\langle 5, 8, 11, 12, 14\rangle$.

Subcase(c-ii) If $n = 7$ then $\tilde{H} \supseteq \langle 7, 8, 10, 12\rangle$. Hence $\tilde{G} \subseteq \{1 \to 6, 9, 11, 13\}$, which implies that $\tilde{H}$ contains exactly two elements in $\{9, 11, 13\}$. Thus $\tilde{H} = \langle 7 \to 12\rangle$, $\langle 7 \to 10, 12, 13\rangle$ or $\langle 7, 8, 10 \to 13\rangle$.
Subcase(c-iii) If $n = 9$ then $\tilde{G} \supseteq \{1 \to 7\}$, which shows that $\tilde{H} = \langle 8 \to 15\rangle$.

3 Preliminary results

In this section we show several results on numerical semigroups and curves needed later.

Let H be any numerical semigroup of genus q and $\tilde{H}$ a numerical semigroup of genus g with $d_2(\tilde{H}) = H$. We consider a subsemigroup $\tilde{H}_0 := 2H + n\mathbb{N}_0$ of $\tilde{H}$, where n is the minimum odd element of $\tilde{H}$. First we have a trivial inequality for the genus of $\tilde{H}$ (cf. [2, Lemma 3.1], [9, Lemma 2.4(1)]):

Lemma 3.1 *Keep the notation as above. Then* $g \leq g(\tilde{H}_0) = 2q + (n-1)/2$. *Equality holds if and only if* $\tilde{H} = \tilde{H}_0$.

In the rest of this article we fix the following notation for semigroups and curves. Let H always denote $\langle 3, 4\rangle$, $\langle 3, 5, 7\rangle$ or $\langle 4, 5, 6, 7\rangle$ and $\tilde{H}$ a numerical semigroup of genus 7 such that $d_2(\tilde{H}) = H$. Let n be the minimal odd element of $\tilde{H}$.

For a point P on a smooth plane quartic, the Weierstrass semigroup $H(P)$ is $\langle 3, 4\rangle$, $\langle 3, 5, 7\rangle$ or $\langle 4, 5, 6, 7\rangle$ according as P is a hyperflex, an ordinary flex or a non-flex of the quartic. Let Γ always denote a curve of genus three and P a point on Γ such that $H(P) = H$. Assume that Γ is non-hyperelliptic and is identified with a smooth plane quartic unless otherwise stated. Note that the canonical divisor K_Γ is cut out by a line in $\mathbb{P}^2$.

For an irreducible plane curve X we shall denote the tangent line of X at a point p by $T_p(X)$, or simply T_p.

First we show a lemma on the freeness of linear systems of degree five on curves of genus three. For this purpose, we recall three elementary facts on algebraic curves.

Proposition 3.2 *Let X be a smooth curve of genus g.*

(1) A pencil without base points has at most finitely many non-reduced members.
(2) For a positive integer m and a divisor A on X, there exist at most finitely many divisors B_i such that $mB_i \sim A$ for any i and $B_i \not\sim B_j$ for $i \neq j$.
(3) A complete linear system $|D|$ of degree $2g - 1$ is free from base points unless $|D| = |K_X| + Q$ for a point $Q \in X$. In particular, for any point P on X, the linear system $|D|$ has a reduced member not containing P unless $|D| = |K_X| + P$.

The following observation is very useful for our argument.

Lemma 3.3 *Let Γ be any curve of genus three, E a divisor on Γ and P a point on Γ. Let S_r $(r = 1, 2, 3)$ be an infinite subset of $\Gamma^{(r)}$. Assume that $h^0(\Gamma, \Delta_r) = 1$ for any $\Delta_r \in S_r$.*

(1) If $\deg E = 7$, then $|E - 2\Delta_1|$ is free from base points except for finitely many $\Delta_1 \in S_1$.

(2) If $r = 2$ or 3 and $\deg E = 2r + 5$, then $|E - 2\Delta_r|$ has a reduced member not containing P except for finitely many $\Delta_r \in S_r$.

Proof (1) We use reduction to absurdity. Suppose that $|E - 2\Delta_1|$ has a base point for infinitely many $\Delta_1 \in S_1$. Since $\deg(E - 2\Delta_1) = 5$, there exists a point P_{Δ_1} on Γ such that $E - 2\Delta_1 \sim K_\Gamma + P_{\Delta_1}$ by Proposition 3.2 (3). Then $2\Delta_1 + P_{\Delta_1}$ is linearly equivalent to a fixed divisor $E - K_\Gamma$ for infinitely many $\Delta_1 \in S_1$. In particular, $\dim|2\Delta_1 + P_{\Delta_1}| \geq 1$. Thus $|2\Delta_1 + P_{\Delta_1}| = g^1_3$, a pencil of degree three. The variable part of g^1_3 contains infinitely many non-reduced members of the form $2\Delta_1$, which contradicts Proposition 3.2 (1).

(2) Let r be two or three. Suppose that there exists an infinite subset S'_r of S_r such that $|E - 2\Delta_r|$ does not have a reduced member not containing P for any $\Delta_r \in S'_r$. Since $\deg(E - 2\Delta_r) = 5$, it follows from Proposition 3.2 (3) that $E - 2\Delta_r \sim K_\Gamma + P$ for any $\Delta_r \in S'_r$. In other words, $2\Delta_r$ is linearly equivalent to a fixed divisor $E - K_\Gamma - P$. Then $|\Delta_r|$ is the same linear system for infinitely many $\Delta_r \in S'_r$ by virtue of Proposition 3.2 (2). In particular $\dim|\Delta_r| \geq 1$, which conflicts with our assumption. Thus we finish the proof.

□

Lemma 3.4 *Let Γ be a smooth plane quartic, E a divisor on Γ and P a point on Γ. Let S be an infinite subset of Γ. If $\deg E = 5$ and $|E|$ is free from base points, then $|E - 2Q|$ has a reduced member not containing P except for finitely many $Q \in S$.*

Proof By our assumption $|E| = g^2_5$, which induces a birational morphism $\varphi : \Gamma \to \Gamma_0 \subseteq \mathbb{P}^2$, where $\Gamma_0 = \varphi(\Gamma)$ is a plane quintic. For any point $Q \in \Gamma$ such that $q = \varphi(Q)$ is a smooth point of Γ_0, the pull-back of the tangent line of Γ_0 at q cuts out an effective divisor $D_Q \in |E|$ of degree five on Γ. Then $D_Q - 2Q$ is reduced and does not contain P except for finitely many $Q \in S$, since Γ_0 has at most finitely many bitangents and hyperflexes.

Let H be any Weierstrass semigroup of genus q and $\tilde{H}$ a numerical semigroup of genus g with $d_2(\tilde{H}) = H$. Define an integer r by the equation $g = 2q + (n-1)/2 - r$, where n is the minimal odd element of $\tilde{H}$. Then we obtain a sequence of numerical semigroups $\tilde{H}_0 \subseteq \tilde{H}_1 \subseteq \cdots \subseteq \tilde{H}_s = \tilde{H}$ as follows:

(i) $\tilde{H}_0 := 2H + n\mathbb{N}_0$.

(ii) For $j = 1, 2, \ldots, s$, $\tilde{H}_j := \tilde{H}_{j-1} + (n + 2l_j)\mathbb{N}_0$, where $n + 2l_j$ is the minimal odd element of $\tilde{H}$ not belonging to $\tilde{H}_{j-1}$.

Then it is easy to verify that $s \leq r$ since $g(\tilde{H}_0) = 2q + (n-1)/2$.

To prove Theorem 1.3, it suffices to show the existence of certain effective divisor on Γ from the following theorem.

Theorem 3.5 *[5, Theorem 2.2] Let H and $\tilde{H}$ be as above. Take any curve Γ of genus q and a point on Γ with $H(P) = H$. Assume that there exists an effective divisor $\Delta_r \in \Gamma^{(r)}$ satisfying the following conditions:*

(C1) Δ_r does not contain P.

(C2) $h^0(\Gamma, \Delta_r) = 1$.

(C3) $h^0(\Gamma, l_j P + \Delta_r) = h^0(\Gamma, (l_j - 1)P + \Delta_r) + 1$ *for* $j = 1, 2, \ldots, s$.
(C4) the linear system $|nP - 2\Delta_r|$ *has a reduced member not containing* P.

Then there exists a smooth curve C *admitting a double covering* $\pi : C \to \Gamma$ *whose branch locus is linearly equivalent to* $P + \Delta_r$. *In particular* π *has a ramification point* $\tilde{P}$ *over* P *such that* $H(\tilde{P}) = \tilde{H}$, *in other words,* $\tilde{H}$ *is of double covering type.*

Though the fourth condition is replaced with the freeness of $|nP - 2\Delta_r|$ in [5, Theorem 2.2], the same proof applies in our situation as well.

Remark 3.6 In our case $q = 3$ and $r = (n-3)/2$. Note that (C1) is satisfied if $r \leq 1$ and $\Delta_1 \neq P$ and (C2) is trivial if $r \leq 1$, or $r = 2$ and Γ is non-hyperelliptic. If $|nP - 2\Delta_r|$ is free from base points, then (C4) is satisfied.

We define a subset $S(\Gamma, P, \tilde{H})$ of $\Gamma^{(r)}$ as follows:

$$S(\Gamma, P, \tilde{H}) := \{\Delta_r \in \Gamma^{(r)} \mid \Delta_r \text{ satisfies the conditions (C1), (C2) and (C3)}\}.$$

In our case $r = 1, 2$ or 3. We shall use the following lemma for proving Theorem 1.3 in many cases.

Lemma 3.7 *Assume that* $g = g(\tilde{H}) = 7$. *Further assume that, for a general divisor* $D \in \Gamma^{(r-1)}$ *and a general point* $Q \in \Gamma$, $\Delta_r = D + Q$ *belongs to* $S(\Gamma, P, \tilde{H})$. *Then there exists an effective divisor in* $S(\Gamma, P, \tilde{H})$ *that satisfies the condition (C4). In particular* $\tilde{H}$ *is DC.*

Proof Consider the divisor $E := nP - 2D$ for a fixed general divisor $D \in \Gamma^{(r-1)}$. Note that $\deg E = n - 2(r-1) = n - 2r + 2 = 5$, since $g = 7$. From Proposition 3.2 (3) we may assume that $|E|$ is free from base points. Then it follows from our assumption and Lemma 3.4 that, for a general point $Q \in \Gamma$, $\Delta_r := D + Q$ belongs to $S(\Gamma, P, \tilde{H})$ and $|nP - 2\Delta_r| = |(E + 2D) - 2(D + Q)| = |E - 2Q|$ has a reduced member not containing P, which is nothing but the condition (C4).

We shall prove Theorem 1.3 in the subsequent sections according to the value of n, the minimal odd element of $\tilde{H}$.

4 When $n = 3$ and $\tilde{H} = \langle 3, 8\rangle$ or $\langle 3, 10, 14\rangle$

In the following sections we assume that $g = 7$. In the former (resp. the latter) case $K_\Gamma \sim 4P$ (resp. $3P + P'$ ($P' \neq P$)), which implies that $|3P| = |K_\Gamma - P|$ (resp. $|K_\Gamma - P'|$) is free from base points, since K_Γ is very ample. Thus there exists nothing to prove in these cases.

5 When $n = 5$ and $\tilde{H} = \langle 5, 6, 9\rangle$, $\langle 5, 6, 13, 14\rangle$, $\langle 5, 7, 8\rangle$, $\langle 5, 8, 9, 12\rangle$ or $\langle 5, 8, 11, 12, 14\rangle$

Note that $H = \langle 3, 5, 7\rangle$ or $\langle 4, 5, 6, 7\rangle$ by the assumption.

(1) First consider the case where $\tilde{H} = \langle 5, 6, 9 \rangle$. In this case $H = \langle 3, 5, 7 \rangle$, or equivalently, $K_\Gamma \sim 3P + P'\,(P' \neq P)$. Hence $|3P| = |K - P'|$ is free from base points. We choose a smooth plane quartic Γ with an ordinary flex P such that $K_\Gamma \sim 3P + P'$, where P' is a hyperflex, i.e., $K_\Gamma \sim 4P'$, which implies that $3P \sim 3P'$. We put $\Delta_1 := P'$. Then it satisfies (C1), (C2) and (C3) and

$$5P - 2\Delta_1 \sim 2P + 3P - 2P' \sim 2P + 3P' - 2P' \sim 2P + P' \sim K_\Gamma - P.$$

Hence $|5P - 2\Delta_1|$ has no base points. In particular Δ_1 satisfies (C4), which implies that $\tilde{H} = \langle 5, 6, 9 \rangle$ is DC.

(2) Secondly, consider the case where $\tilde{H} = \langle 5, 8, 9, 12 \rangle$. We take a nodal plane quintic Γ_0 with three nodes q_j $(j = 1, 2, 3)$ as its singularity satisfying the following conditions (see Fig. 1):
   (i) Γ_0 has a total inflection point p.
   (ii) Three points p, q_1 and q_2 are collinear.
   (iii) There exists a smooth non-flex q of Γ_0 on the line $\overline{pq_3}$ such that the tangent line T_q is not a bitangent and does not pass through p.

   Let $\varphi : S \to \mathbb{P}^2$ be the composite of the blow-ups at q_j $(j = 1, 2, 3)$. Then Γ, the strict transform of Γ_0, is a smooth curve of genus three. This is non-hyperelliptic, since it has a g^1_3 without base points corresponding to a projection from Γ_0 to $\mathbb{P}^1$ whose center is a node.

   Note that $K_S \sim -3l + \sum_{j=1}^3 e_j$ and $\Gamma \sim 5l - 2\sum_{j=1}^3 e_j$, where l is the pull-back of a line and e_j is the pull-back of the exceptional curve corresponding to q_j.

   We set $\Delta_1 := Q = \varphi^{-1}(q)$, which satisfies the conditions (C1) and (C2). By using the adjunction formula and the above conditions (ii) and (iii) we see that

$$\begin{aligned} K_\Gamma \sim (\Gamma + K_S)|_\Gamma &\sim \left(2l - \sum_{j=1}^{3} e_j\right)\Big|_\Gamma \\ &\sim (l - e_1 - e_2)|_\Gamma + (l - e_3)|_\Gamma \\ &\sim P + (P + Q + R), \end{aligned}$$

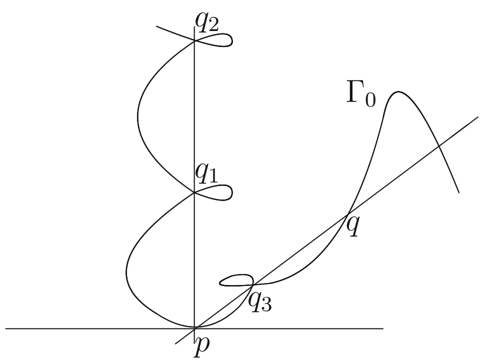

Fig. 1 A nodal plane quintic Γ_0 in (2)

where R is a point on Γ. Therefore $h^0(\Gamma, 2P+Q) = h^0(\Gamma, K_\Gamma - R) = 2$, which implies the condition (C3).
Furthermore, it follows from the above conditions (i) and (iii) that $5P \sim 2Q + D$ on Γ, where D is an effective reduced divisor of Γ of degree three not containing P. Thus $\Delta_1 = Q$ also satisfies (C4). Hence $\tilde{H}$ is DC by virtue of Theorem 3.5.

(3) Thirdly, consider the case where $\tilde{H} = \langle 5, 7, 8\rangle$. We choose a plane quintic Γ_0 with a unique triple point q as its only singularity satisfying the following conditions:
 (i) Γ_0 has a total inflection point p.
 (ii) The tangent line $T_{p'}$ of Γ_0 is not a bitangent, where p' is the remaining intersection point of Γ_0 and the line $\overline{pq}$.

Note that $T_{p'} \neq \overline{pq}$, which implies that $T_{p'}$ does not pass through p. Let $\varphi : \Gamma \to \Gamma_0$ be the desingularization of Γ_0. Then Γ is hyperelliptic, since it has a g_2^1 corresponding to the projection from Γ_0 with the center q. Set $P := \varphi^{-1}(p)$ and $P' := \varphi^{-1}(p')$. Then $|P + P'| = g_2^1$, in other words, $h^0(\Gamma, P+P') = 2$ and P is not a ramification point of the hyperelliptic covering of Γ, which implies that $H(P) = \langle 4, 5, 6, 7\rangle$. Thus $\Delta_1 := P'$ satisfies the conditions (C1), (C2) and (C3). Furthermore, it follows from (ii) that $5P \sim 2P' + D$ for a reduced divisor D of Γ of degree three. Note that D does not contain P, since $T_{p'}$ does not pass through p. Hence $\Delta_1 = P'$ also satisfies the condition (C4), which implies that $\tilde{H}$ is DC.

(4) For the remaining cases where $\tilde{H} = \langle 5, 6, 13, 14\rangle$ and $\tilde{H} = \langle 5, 8, 11, 12, 14\rangle$, the condition (C3) is as follows:

$$h^0(\Gamma, 4P+\Delta_1) = h^0(\Gamma, 3P+\Delta_1) + 1 \quad \text{if } \tilde{H} = \langle 5, 6, 13, 14\rangle \text{ and}$$
$$h^0(\Gamma, 3P+\Delta_1) = h^0(\Gamma, 2P+\Delta_1) + 1 \quad \text{if } \tilde{H} = \langle 5, 8, 11, 12, 14\rangle.$$

Hence a general $\Delta_1 \in \Gamma$ satisfies the conditions (C1), (C2) and (C3) in each case. Thus we obtain the assertion that $\tilde{H}$ is DC by virtue of Lemma 3.7.

6 When $n = 7$ and $\tilde{H} = \langle 6 \to 9\rangle, \langle 6, 7, 8, 11\rangle, \langle 6, 7, 9, 10\rangle, \langle 6, 7, 10, 11, 15\rangle, \langle 7 \to 12\rangle, \langle 7 \to 10, 12, 13\rangle$ or $\langle 7, 8, 10 \to 13\rangle$

(1) First consider the case where $\tilde{H} = \langle 6 \to 9\rangle$. Then $H = \langle 3, 4\rangle$, i.e., $K_\Gamma \sim 4P$. We choose two distinct points Q_1 and Q_2 on Γ different from P such that T_{Q_1} passes through P and Q_2, in other words, $K_\Gamma \sim P + 2Q_1 + Q_2$. Then $3P \sim 2Q_1 + Q_2$ and the divisor $\Delta_2 := Q_1 + Q_2$ satisfies (C1) and (C2). Furthermore

$$h^0(\Gamma, P+\Delta_2) = h^0(\Gamma, P+Q_1+Q_2) = h^0(\Gamma, K_\Gamma - Q_1) = 2.$$

Thus Δ_2 satisfies (C3) as well, since $h^0(\Gamma, \Delta_2) = 1$. In addition, we have the relation that

$$7P - 2\Delta_2 \sim 4P + (2Q_1 + Q_2) - 2(Q_1 + Q_2) \sim K - Q_2,$$

which implies that $|7P - 2\Delta_2|$ is free from base points. In particular Δ_2 satisfies (C4). Hence $\tilde{H}$ is DC.

(2) Secondly, consider the case where $H = \langle 3, 5, 7\rangle$ and $\tilde{H} = \langle 6, 7, 9, 10\rangle$. We can take a smooth plane quartic Γ with three different ordinary flexes P, Q_1 and Q_2 satisfying the following conditions (see Example 6.1):
  (i) Three points $P,\ Q_1$ and Q_2 are collinear.
  (ii) Three tangent lines $T_P,\ T_{Q_1}$ and T_{Q_2} meet at a point P' on Γ.

  Let R be the remaining intersection point of Γ and the line passing through P, Q_1 and Q_2. Then we have the following relations:

$$K_\Gamma \sim 3P + P' \sim 3Q_1 + P' \sim 3Q_2 + P' \sim P + Q_1 + Q_2 + R.$$

  In particular $H := H(P) = \langle 3, 5, 7\rangle$ and it is clear that $\Delta_2 := Q_1 + Q_2$ satisfies (C1) and (C2). Furthermore

$$h^0(\Gamma, P + \Delta_2) = h^0(\Gamma, P + Q_1 + Q_2) = h^0(\Gamma, K_\Gamma - R) = 2.$$

  Thus Δ_2 satisfies (C3), since $h^0(\Gamma, \Delta_2) = 1$. In addition, we have the relation that

$$\begin{aligned} 7P - 2\Delta_2 &\sim P + 3P + 3P - 2(Q_1 + Q_2) \sim P + 3Q_1 + 3Q_2 - 2(Q_1 + Q_2) \\ &\sim P + Q_1 + Q_2 \sim K_\Gamma - R, \end{aligned}$$

  which implies that $|7P - 2\Delta_2|$ is free from base points. In particular Δ_2 satisfies (C4). Hence $\tilde{H}$ is DC.

Example 6.1 The smooth plane quartic $\Gamma : x^4 - x^3 - x^2y + y^3 + 2xy - y = 0$ satisfies the above conditions for $P = (0, 0)$, $Q_1 = (0, 1)$, $Q_2 = (0, -1)$ and $P' = (1, 0)$ (see Fig. 2).

(3) Thirdly, we consider the case where $\tilde{H} = \langle 7 \rightarrow 12\rangle$. We can choose a smooth plane quartic Γ with five points P, P_1, P_2, Q_1 and Q_2 satisfying the following conditions (see Fig. 3):

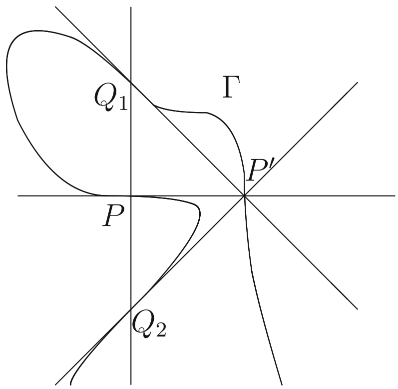

Fig. 2 $x^4 - x^3 - x^2y + y^3 + 2xy - y = 0$

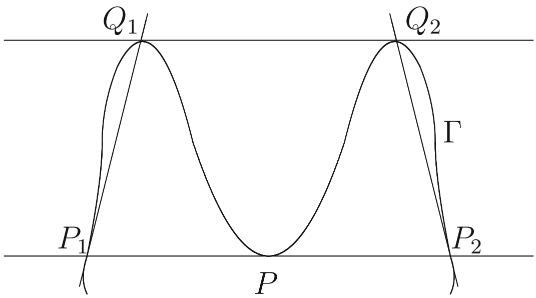

Fig. 3 A smooth plane quartic Γ in (3)

(i) T_P passes through P_1 and P_2.
(ii) P_i $(i = 1, 2)$ is an ordinary flex and T_{P_i} passes through Q_i.
(iii) The line $\overline{Q_1 Q_2}$ is tangent to Γ at both Q_1 and Q_2.

Then we obtain the relations that

$$K_\Gamma \sim 2P + P_1 + P_2 \sim 3P_1 + Q_1 \sim 3P_2 + Q_2 \sim 2Q_1 + 2Q_2.$$

Obviously $\Delta_2 := P_1 + P_2$ satisfies the conditions (C1) and (C2). Furthermore, $h^0(\Gamma, P + \Delta_2) = h^0(\Gamma, K_\Gamma - P) = 2$, which shows that Δ_2 satisfies (C3). Finally, we verify that $|7P - 2\Delta_2|$ is free from base points. Note that

$$4P + 2\Delta_2 = 2(2P + P_1 + P_2) \sim (3P_1 + Q_1) + (3P_2 + Q_2) = 3\Delta_2 + Q_1 + Q_2,$$

which implies that $\Delta_2 \sim 4P - Q_1 - Q_2$. Therefore

$$7P - 2\Delta_2 \sim 7P - 2(4P - Q_1 - Q_2) \sim 2Q_1 + 2Q_2 - P \sim K_\Gamma - P,$$

which shows that $|7P - 2\Delta_2|$ has no base points. Hence $\tilde{H}$ is DC.

(4) Next we consider the case where $\tilde{H} = \langle 7 \to 10, 12, 13 \rangle$. We can choose a smooth plane quartic Γ with six points P, P_1, P_2, Q_1, Q_2 and R satisfying the following conditions (see Example 6.2):
(i) T_P passes through P_1 and P_2.
(ii) Four points P, Q_1, Q_2 and R are collinear.
(iii) P_1 (resp. Q_1) is an ordinary flex and T_{P_1} (resp. T_{Q_1}) passes through Q_2 (resp. P_2).
(iv) The line $\overline{P_2 Q_2}$ is a bitangent of Γ.

Then we have the following relations:

$$K_\Gamma \sim 2P + P_1 + P_2 \sim P + Q_1 + Q_2 + R \sim 3P_1 + Q_2$$
$$\sim P_2 + 3Q_1 \sim 2P_2 + 2Q_2.$$

We set $\Delta_2 := Q_1 + Q_2$. It clearly satisfies the conditions (C1) and (C2). Moreover

$$h^0(\Gamma, P + \Delta_2) = h^0(\Gamma, K_\Gamma - R) = 2 \quad \text{and}$$
$$h^0(\Gamma, 2P + \Delta_2) = h^0(\Gamma, K_\Gamma - R + P) = 2 = h^0(\Gamma, 3P + \Delta_2) - 1.$$

Thus Δ_2 also satisfies (C3). Finally

$$\begin{aligned} 7P - 2\Delta_2 &\sim 3 \cdot 2P + P - 2(Q_1 + Q_2) \\ &\sim 3(K_\Gamma - P_1 - P_2) + (K_\Gamma - Q_1 - Q_2 - R) - 2(Q_1 + Q_2) \\ &= 4K_\Gamma - 3P_1 - 3P_2 - 3Q_1 - 3Q_2 - R \\ &= (K_\Gamma - 3P_1 - Q_2) + (K_\Gamma - P_2 - 3Q_1) + (K_\Gamma - 2P_2 - 2Q_2) \\ &\quad + (K_\Gamma - R) \\ &\sim K_\Gamma - R, \end{aligned}$$

which implies that $|7P - 2\Delta_2|$ is free from base points. Therefore $\tilde{H}$ is DC.

Example 6.2 The smooth plane quartic $\Gamma : x^4 + 3xy^3 + 3x^2y - xy^2 + y^3 - x^2 - y = 0$ satisfies the above conditions for $P = (0, 0)$, $P_1 = (-1, 0)$, $P_2 = (1, 0)$, $Q_1 = (0, -1)$, $Q_2 = (0, 1)$ and $R = (0 : 1 : 0)$ (see Fig. 4).

(5) For the remaining cases where $\tilde{H} = \langle 6, 7, 8, 11\rangle$, $\langle 6, 7, 10, 11, 15\rangle$ and $\langle 7, 8, 10 \rightarrow 13\rangle$, the condition (C3) is as follows:

$$h^0(\Gamma, 2P + \Delta_2) = h^0(\Gamma, P + \Delta_2) + 1,$$
$$h^0(\Gamma, 4P + \Delta_2) = h^0(\Gamma, 3P + \Delta_2) + 1 \quad \text{(only when } \tilde{H} = \langle 6, 7, 10, 11, 15\rangle\text{) and}$$
$$h^0(\Gamma, 3P + \Delta_2) = h^0(\Gamma, 2P + \Delta_2) + 1 \quad \text{(only when } \tilde{H} = \langle 7, 8, 10 \rightarrow 13\rangle\text{)}$$

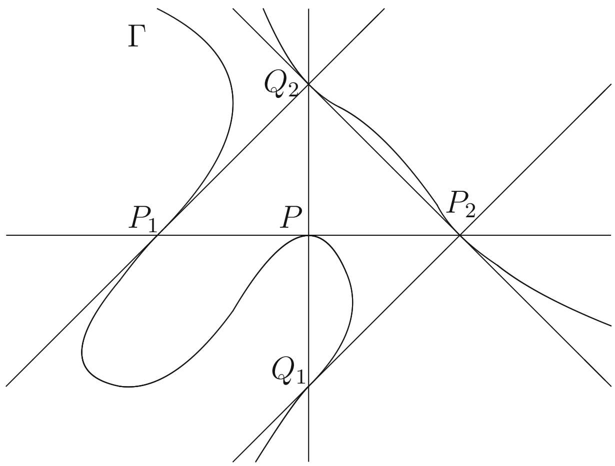

Fig. 4 $x^4 + 3xy^3 + 3x^2y - xy^2 + y^3 - x^2 - y = 0$

The second equality always holds. On the other hand, an effective divisor $\Delta_2 \in \Gamma^{(2)}$ satisfies the first one (and the third one if $\tilde{H} = \langle 7, 8, 10 \rightarrow 13\rangle$) unless Δ_2 is contained in a divisor of Γ cut out by a line passing through P. Hence, for two general points Q_1 and Q_2 on Γ, the divisor $\Delta_2 = Q_1 + Q_2 \in \Gamma^{(2)}$ satisfies (C1), (C2) and (C3). Thus we conclude from Lemma 3.7 that $\tilde{H}$ is DC in every case.

7 When $n = 9$ and $\tilde{H} = \langle 6, 8, 9, 11, 13\rangle$, $\langle 6, 9, 10, 11, 13, 14\rangle$ or $\langle 8 \rightarrow 15\rangle$

In these cases the condition (C3) is as follows:

$$\begin{aligned} h^0(\Gamma, P + \Delta_3) &= h^0(\Gamma, \Delta_3) + 1 = 2, \\ h^0(\Gamma, 2P + \Delta_3) &= h^0(\Gamma, P + \Delta_3) + 1 = 3 \quad \text{and also} \\ h^0(\Gamma, 3P + \Delta_3) &= h^0(\Gamma, 2P + \Delta_3) + 1 = 4 \quad (\text{only when } \tilde{H} = \langle 8 \rightarrow 15\rangle) \end{aligned}$$

Note that the first equality implies the others. Hence, for a general divisor $D \in \Gamma^{(2)}$ and a general point Q on Γ, the divisor $\Delta_3 = D + Q \in \Gamma^{(3)}$ belongs to $S(\Gamma, P, \tilde{H})$. Then it follows from Lemma 3.7 that $\tilde{H}$ is DC in every case.

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