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RESEARCH ARTICLE

Numerical semigroups of genus six and double coverings of curves of genus three

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Abstract This is a sequel of previous papers of the authors on Weierstrass semigroups at ramification points of double coverings of algebraic curves of genus three. In this paper they give a list of possible numerical semigroups when the covering curve is of genus six and show that all of such semigroups are actually of double covering type. This result completes a classification of numerical semigroups of double covering type obtained by ramified double coverings of curves of genus three.

Keywords Weierstrass semigroups · Numerical semigroups · Double coverings · Plane quartics

1 Introduction

Notation and Conventions

We denote by $\mathbb{N}_0$ the additive monoid of non-negative integers. A submonoid of $\mathbb{N}_0$ is called a *numerical semigroup* if its complement is a finite set. For a numerical semigroup H, a positive integer m is called a *gap* of H if $m \notin H$. The number of gaps of H is called its *genus* and is denoted by $g(H)$.

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In this paper $a \to b$ denotes a sequence $a, a+1, a+2, \dots, b$ of non-negative integers.

A *curve* always means a projective irreducible curve over an algebraically closed field of characteristic zero and it is assumed to be smooth unless otherwise mentioned. For a curve X, we denote by $X^{(r)}$ the variety of effective divisors of degree r on X.

For a point P on a curve X, the semigroup

$$H(P) = \{m \in \mathbb{N}_0 \mid \text{there exists a rational function } f \text{ on } X \text{ such that } (f)_\infty = mP\}$$

is a numerical semigroup of genus $g(X)$. This is called the *Weierstrass semigroup* at P. A numerical semigroup H is said to be *Weierstrass* if there exist a curve X and a point P on X such that $H(P) = H$.

When $\tilde{H}$ is a numerical semigroup, the set $d_2(\tilde{H}) := \{h \in \mathbb{N}_0 \mid 2h \in \tilde{H}\}$ is also a numerical semigroup such that

(i) $\tilde{H} \subseteq d_2(\tilde{H})$.
(ii) $2d_2(\tilde{H}) = \{2h \mid h \in d_2(\tilde{H})\} = \tilde{H} \cap 2\mathbb{N}_0$.

A numerical semigroup $\tilde{H}$ is said to be *of double covering type*, or simply *DC*, if there exist a ramified double covering $\pi : X \to Y$ of curves and a ramification point $\tilde{P} \in X$ such that $\tilde{H} = H(\tilde{P})$. Then it is easy to verify that $d_2(\tilde{H}) = H(\pi(\tilde{P}))$ and $g(\tilde{H}) \geq 2g(d_2(\tilde{H}))$ since π has a ramification point. By definition a numerical semigroup of double covering type is Weierstrass.

Is the converse true? That is to say, if $\tilde{H}$ is a numerical semigroup with $d_2(\tilde{H}) = H$ such that $g(\tilde{H}) \geq 2g(H)$, then is it of double covering type? This is an interesting and important problem. In this paper we give a complete answer for this problem when $g(\tilde{H}) = 6$ and $g(H) = 3$, which is the last piece for a classification of numerical semigroups of double covering type obtained by a ramified double covering of a curve of genus three.

For the results when $g(H) \leq 2$, see [1,5,7–9] and [4]. In short, we have the affirmative answer for the above problem. Precisely, every numerical semigroup $\tilde{H}$ such that $H = d_2(\tilde{H})$ is of genus at most two and $g(\tilde{H}) \geq 2g(H)$ is known to be DC.

Assume that $g(H) = 3$. Then $g(\tilde{H}) \geq 6$ and $H = \langle 2, 7\rangle$, $\langle 3, 4\rangle$, $\langle 3, 5, 7\rangle$ or $\langle 4, 5, 6, 7\rangle$, all of which are Weierstrass. It is known that $\tilde{H}$ is DC if one of the following holds:

(i) $g(\tilde{H}) \geq 7$ (see [6, Main Theorem] and [9, Examples 3.6 and 3.9] for $g(\tilde{H}) \geq 9$, [2, Theorem 1.2] and [3, Theorem 1.3] for $g(\tilde{H}) = 8$ and $g(\tilde{H}) = 7$, respectively).
(ii) $H = \langle 2, 7\rangle$ (cf. [7, Main Theorem]).

The main result of this article is the following theorem:

Theorem 1.1 *Let H be a numerical semigroup of genus three, i.e., $H = \langle 2, 7\rangle$, $\langle 3, 4\rangle$, $\langle 3, 5, 7\rangle$ or $\langle 4, 5, 6, 7\rangle$. Let $\tilde{H}$ be a numerical semigroup of genus at least six such that $d_2(\tilde{H}) = H$. Then $\tilde{H}$ is of double covering type.*

Noting the previous results mentioned above, we may assume that $g(\tilde{H}) = 6$ and $H \neq \langle 2, 7\rangle$.

Table 1 the classification of $\tilde{H}$

n	H		
	$\langle 3, 4\rangle$	$\langle 3, 5, 7\rangle$	$\langle 4, 5, 6, 7\rangle$
3	$\langle 3, 8, 13\rangle$	$\langle 3, 7\rangle$	
		$\langle 3, 10, 11\rangle$	
5		$\langle 5, 6, 7\rangle$	$\langle 5, 7, 8, 9\rangle$
		$\langle 5, 6, 9, 13\rangle$	$\langle 5, 7, 8, 11\rangle$
			$\langle 5, 8, 9, 11, 12\rangle$
7	$\langle 6 \to 9, 11\rangle$	$\langle 6, 7, 9, 10, 11\rangle$	$\langle 7 \to 13\rangle$

2 Numerical semigroups under consideration

Let $\tilde{H}$ be a numerical semigroup of genus six such that $H = d_2(\tilde{H})$ is $\langle 3, 4\rangle$, $\langle 3, 5, 7\rangle$ or $\langle 4, 5, 6, 7\rangle$. Let $\tilde{G} = \mathbb{N}_0 \setminus \tilde{H}$ be the set of gaps of $\tilde{H}$ and n the minimum odd non-gap of $\tilde{H}$. First note that $\tilde{G} \supseteq \{1, 2, 4\}$ and $n \leq 7$. Indeed, if $n \geq 9$ then $\tilde{G} \supseteq \{1 \to 5, 7\}$. In fact equality holds, which implies that $\tilde{H} \supseteq \{6, 8, 10\}$. Then $H \supseteq \langle 3, 4, 5\rangle$, a contradiction.

In this section we determine numerical semigroups of our interest according to H and the value of n. See Table 1 for the result.

Case(a) $H = \langle 3, 4\rangle$. Note that $\tilde{G} \supseteq \{1, 2, 4, 5, 10\}$ in this case. Hence the extra gap is 3 or 7.

Subcase(a-i) If $n = 3$, then $\tilde{G} = \{1, 2, 4, 5, 7, 10\}$, i.e., $\tilde{H} = \langle 3, 8, 13\rangle$.

Subcase(a-ii) If $n = 7$, then $\tilde{G} = \{1 \to 5, 10\}$, i.e., $\tilde{H} = \langle 6 \to 9, 11\rangle$.

Case(b) $H = \langle 3, 5, 7\rangle$. Then $\tilde{G} \supseteq \{1, 2, 4, 8\}$.

Subcase(b-i) If $n = 3$, then $5 \in \tilde{G}$. If furthermore $7 \in \tilde{H}$ then $\tilde{H} \supseteq \langle 3, 7\rangle$, which is in fact an equality since these semigroups have the same genus. Otherwise we see that $\tilde{G} = \{1, 2, 4, 5, 7, 8\}$, i.e., $\tilde{H} = \langle 3, 10, 11\rangle$.

Subcase(b-ii) If $n = 5$ then $\tilde{G} \supseteq \{1 \to 4, 8\}$. If furthermore $7 \in \tilde{H}$ then $\tilde{H} \supseteq \langle 5, 6, 7\rangle$, which is in fact an equality since both semigroups have the same genus. Otherwise $\tilde{G} = \{1 \to 4, 7, 8\}$, i.e., $\tilde{H} = \langle 5, 6, 9, 13\rangle$.

Subcase(b-iii) If $n = 7$ then $\tilde{G} = \{1 \to 5, 8\}$, i.e., $\tilde{H} = \langle 6, 7, 9, 10, 11\rangle$.

Case(c) $H = \langle 4, 5, 6, 7\rangle$. Note that $\tilde{G} \supseteq \{1 \to 4, 6\}$.

Subcase(c-i) If $n = 5$ then $\tilde{H} \supseteq \langle 5, 8, 12, 14\rangle$, i.e., $\tilde{G} \subseteq \{1 \to 4, 6, 7, 9, 11\}$. Hence the remaining gap is 7, 9 or 11. It follows that $\tilde{H} = \langle 5, 8, 9, 11, 12\rangle$, $\langle 5, 7, 8, 11\rangle$ or $\langle 5, 7, 8, 9\rangle$.

Subcase(c-ii) If $n = 7$ then $\tilde{G} = \{1 \to 6\}$, i.e., $\tilde{H} = \langle 7 \to 13\rangle$.

3 Preliminary results

In this section we recall our method used in [2] and [3], please see these papers for the details. Let H be any Weierstrass semigroup of genus q, $\tilde{H}$ a numerical semigroup of genus g such that $d_2(\tilde{H}) = H$. Define an integer r by the equation $g = 2q + \frac{n-1}{2} - r$, where n is the minimum odd element of $\tilde{H}$. Then $r \geq 0$ and we obtain a sequence of numerical semigroups $\tilde{H}_0 \subset \tilde{H}_1 \subset \cdots \subset \tilde{H}_s = \tilde{H}$ as follows:

(i) $\tilde{H}_0 := 2H + n\mathbb{N}_0$.
(ii) For $j = 1, 2, \ldots, s$, $\tilde{H}_j := \tilde{H}_{j-1} + (n + 2l_j)\mathbb{N}_0$, where $n + 2l_j$ is the minimum odd element of $\tilde{H}$ not belonging to $\tilde{H}_{j-1}$.

Theorem 3.1 ([5, Theorem 2.2]) *Let H and $\tilde{H}$ be as above. Take any curve Γ of genus q and a point on Γ such that $H(P) = H$. Assume that there exists an effective divisor $\Delta_r \in \Gamma^{(r)}$ satisfying the following conditions:*

(C1) *Δ_r does not contain P.*
(C2) $h^0(\Gamma, \Delta_r) = 1$.
(C3) $h^0(\Gamma, l_j P + \Delta_r) = h^0(\Gamma, (l_j - 1)P + \Delta_r) + 1$ *for* $j = 1, 2, \ldots, s$.
(C4) *$|nP - 2\Delta_r|$ has a reduced member not containing P.*

Then there exists a double covering $\pi : C \to \Gamma$ with a ramification point $\tilde{P}$ over P such that $H(\tilde{P}) = \tilde{H}$. In particular $\tilde{H}$ is of double covering type.

Remark 3.2 In our case $g(\tilde{H}) = 6$, $q = 3$ and $r = \frac{n-1}{2}$. Note that (C1) is satisfied if $r = 1$ and $\Delta_1 \neq P$ and (C2) is trivial if $r \leq 2$ and Γ is non-hyperelliptic.

In the rest of this paper $\tilde{H}$ is a numerical semigroup of genus six and $H = d_2(\tilde{H})$ is $\langle 3, 4\rangle$, $\langle 3, 5, 7\rangle$ or $\langle 4, 5, 6, 7\rangle$. Note that there exists a smooth plane quartic Γ with a point P such that $H(P) = H$. For a point Q on Γ, we denote by T_Q the tangent line of Γ at Q.

We define a subset $S(\Gamma, P, \tilde{H})$ of $\Gamma^{(r)}$ as follows:

$$S(\Gamma, P, \tilde{H}) := \{\Delta_r \in \Gamma^{(r)} \mid \Delta_r \text{ satisfies the conditions } (C1), (C2) \text{ and } (C3)\}.$$

In our case $r = 1, 2$ or 3. To prove Theorem 1.1, it suffices to find an effective divisor $\Delta_r \in S(\Gamma, P, \tilde{H})$ satisfying the condition $(C4)$, i.e., $nP - 2\Delta_r$ is linearly equivalent to a point on Γ different from P.

4 When $n = 3$ and $\tilde{H} = \langle 3, 8, 13\rangle$, $\langle 3, 7\rangle$ or $\langle 3, 10, 11\rangle$

We choose a smooth plane quartic Γ with three points Q_1, Q_2 and Q_3 satisfying the following conditions:

(i) Q_1 is a hyperflex.
(ii) Q_2 and Q_3 are ordinary flexes and both T_{Q_2} and T_{Q_3} pass through Q_1.

In other words

$$K_\Gamma \sim 4Q_1 \sim Q_1 + 3Q_2 \sim Q_1 + 3Q_3.$$

(1) For $\tilde{H} = \langle 3, 8, 13\rangle$ we set $P := Q_1$. Then $H(P) = \langle 3, 4\rangle$ and the condition (C3) is the equality

$$h^0(\Gamma, 5P + \Delta_1) = h^0(\Gamma, 4P + \Delta_1) + 1,$$

which holds for any $\Delta_1 \in \Gamma$ because both divisors are non-special. Then $\Delta_1 := Q_2$ belongs to $S(\Gamma, P, \tilde{H})$ and

$$3P - 2\Delta_1 \sim (K_\Gamma - Q_1) - 2Q_2 \sim Q_2 \neq Q_1 = P,$$

which implies that $\tilde{H}$ is DC.

(2) For $\tilde{H} = \langle 3, 7\rangle$ we set $P := Q_2$. Then $H(P) = \langle 3, 5, 7\rangle$ and the condition (C3) is the equality

$$h^0(\Gamma, 2P + \Delta_1) = h^0(\Gamma, P + \Delta_1) + 1 = 2,$$

which holds if and only if Δ_1 is contained in the divisor of Γ cut out by T_P. Hence $\Delta_1 := Q_1$ belongs to $S(\Gamma, P, \tilde{H})$ and

$$3P - 2\Delta_1 \sim (K_\Gamma - Q_1) - 2Q_1 \sim Q_1 \neq Q_2 = P,$$

which shows that $\tilde{H}$ is DC.

(3) For $\tilde{H} = \langle 3, 10, 11\rangle$ we set $P := Q_2$. Then $H(P) = \langle 3, 5, 7\rangle$ and the condition (C3) is the equality

$$h^0(\Gamma, 3P + \Delta_1) = h^0(\Gamma, 4P + \Delta_1) - 1 = 2,$$

which holds if and only if Δ_1 is not contained in the divisor of Γ cut out by T_P. Hence $\Delta_1 := Q_3$ belongs to $S(\Gamma, P, \tilde{H})$ and

$$3P - 2\Delta_1 \sim (K_\Gamma - Q_1) - 2Q_3 \sim Q_3 \neq Q_2 = P,$$

which implies that $\tilde{H}$ is DC.

5 When $n = 5$ and $\tilde{H} = \langle 5, 6, 7\rangle, \langle 5, 6, 9, 13\rangle, \langle 5, 7, 8, 9\rangle, \langle 5, 7, 8, 11\rangle$ or $\langle 5, 8, 9, 11, 12\rangle$

(1) For $\tilde{H} = \langle 5, 6, 7\rangle$, we choose a smooth plane quartic Γ that has four distinct points P, P', Q_1 and Q_2 satisfying the following:

(i) P is an ordinary flex and T_P passes through P', i.e., $K_\Gamma \sim 3P + P'$. In particular $H := H(P) = \langle 3, 5, 7\rangle$.
(ii) $T_{P'}$ passes through Q_1, i.e., $K_\Gamma \sim 2P' + Q_1 + R$ for some point R on Γ.
(iii) T_{Q_2} passes through P and Q_1, i.e., $K_\Gamma \sim P + Q_1 + 2Q_2$.

Note that R is different from P. The condition (C3) is the equality

$$h^0(\Gamma, P + \Delta_2) = h^0(\Gamma, \Delta_2) + 1 = 2.$$

We set $\Delta_2 := Q_1 + Q_2 \sim K_\Gamma - P - Q_2$. Then it belongs to $S(\Gamma, P, \tilde{H})$ and

$$\begin{aligned} 5P - 2\Delta_2 &\sim 2P + (K_\Gamma - P') - 2(Q_1 + Q_2) \\ &\sim 2P + (P + Q_1 + 2Q_2 - P') - 2(Q_1 + Q_2) \\ &\sim 3P - P' - Q_1 \\ &\sim (K_\Gamma - P') - P' - Q_1 \\ &\sim R\ (\neq P), \end{aligned}$$

which implies that Δ_2 satisfies (C4). Hence $\tilde{H} = \langle 5, 6, 7\rangle$ is DC.

Example 5.1 Let Γ be the plane quartic defined by the polynomial

$$f(x, y) = x^4 + 2x^3y + y^4 - x^3 + 4x^2y - y^3 - 8xy - y^2 + y.$$

Then it is smooth and

$$\begin{aligned} f(x, 0) &= x^3(x - 1), \\ f(0, y) &= y(y - 1)^2(y + 1) \quad \text{and} \\ f(x, x - 1) &= 4x(x - 1)^2(x + 1). \end{aligned}$$

Hence Γ satisfies the above conditions (i)–(iii) for $P = (0, 0)$, $Q_1 = (0, -1)$, $Q_2 = (0, 1)$ and $P' = (1, 0)$. In this case $T_{P'}$ and T_{Q_2} are defined by $y = x - 1$ and $x = 0$, respectively, and $R = (-1, -2)$.

(2) For $\tilde{H} = \langle 5, 6, 9, 13\rangle$, we choose a smooth plane quartic Γ that has an ordinary flex P and a hyperflex P' such that T_P passes through P'. Then $H := H(P) = \langle 3, 5, 7\rangle$ and $K_\Gamma \sim 3P + P' \sim 4P'$. The condition (C3) is the equalities

$$\begin{aligned} h^0(\Gamma, 2P + \Delta_2) &= h^0(\Gamma, P + \Delta_2) + 1 \quad \text{and} \\ h^0(\Gamma, 4P + \Delta_2) &= h^0(\Gamma, 3P + \Delta_2) + 1. \end{aligned}$$

Note that the latter is trivial. Consider the pencil $|K_\Gamma - P|$. It induces a covering from $\Gamma \to \mathbb{P}^1$ branched at four points. Let R be its ramification point different from P and P'. Then there exists a point $R' \in \Gamma$ such that $K_\Gamma - P \sim 2R + R'$. Note that $R' \neq P$, or else $3P + P' \sim K_\Gamma \sim 2P + 2R$, which implies that $R = P' = P$, a contradiction. Then $\Delta_2 := P' + R$ belongs to $S(\Gamma, P, \tilde{H})$ and

$$\begin{aligned} 5P - 2\Delta_2 &\sim 2P + (K_\Gamma - P') - 2(P' + R) \\ &\sim 2P + 3P' - 2(P' + R) \\ &\sim 2P + P' - 2R \\ &\sim (K_\Gamma - P) - 2R \\ &\sim R'\ (\neq P), \end{aligned}$$

which implies that Δ_2 satisfies (C4). Hence $\tilde{H} = \langle 5, 6, 9, 13\rangle$ is DC.

(3) Next we consider the remaining cases: $\tilde{H} = \langle 5, 7, 8, 9 \rangle$, $\tilde{H} = \langle 5, 7, 8, 11 \rangle$ and $\tilde{H} = \langle 5, 8, 9, 11, 12 \rangle$. For these cases, we take a nodal plane quintic Γ_0 with three nodes p_i ($i = 1, 2, 3$) as its singularities satisfying the following conditions:

(a) Γ_0 has a total inflection point p.
(b) There exists a line l that is tangent to Γ_0 at two smooth points, say q_1 and q_2, and does not pass through p.

Let $\varphi : S \to \mathbb{P}^2$ be the composite of the blow-ups at p_i ($i = 1, 2, 3$), Γ the strict transform of Γ_0 and l (resp. e_i) the pull-back to S of a line (resp. the exceptional curve corresponding to p_i). Then Γ is a non-hyperelliptic curve of genus three.

Set $P := \varphi^{-1}(p)$ and $Q_i := \varphi^{-1}(q_i)$ ($i = 1, 2$). Then $|5P|$ is a simple net on Γ without base points and $\dim|3P| = 0$ since p is a smooth point of Γ, which implies that $H(P) = \langle 4, 5, 6, 7 \rangle$.

It is clear that $\Delta_2 := Q_1 + Q_2$ satisfies the conditions (C1) and (C2). Furthermore, from the above conditions (a) and (b) there exists a point $R_1 \neq P$ on Γ such that $5P \sim 2Q_1 + 2Q_2 + R_1$, or equivalently, $5P - 2\Delta_2 \sim R_1$. Hence Δ_2 satisfies (C4) as well.

We can impose any one of the following additional conditions on Γ_0:

(c-i) There exists a conic B_0 passing through all of the six points p, p_i ($i = 1, 2, 3$), q_1 and q_2 such that $i(B_0, \Gamma_0; p) = 2$.
(c-ii) There exists a conic B_0 passing through all of the six points p, p_i ($i = 1, 2, 3$), q_1 and q_2 such that $i(B_0, \Gamma_0; p) = 1$.
(c-iii) There exists no conic passing through all of the six points p, p_i ($i = 1, 2, 3$), q_1 and q_2.

Let B be the strict transform of B_0 by $\varphi : S \to \mathbb{P}^2$ under the condition (c-i) or (c-ii).

First, for $\tilde{H} = \langle 5, 7, 8, 9 \rangle$, we choose Γ_0 admitting (c-i). By using the adjunction formula we obtain the following:

$$K_\Gamma \sim \left(2l - \sum_{i=1}^{3} e_i \right)\Big|_\Gamma \sim B|_\Gamma \sim 2P + Q_1 + Q_2.$$

Hence $h^0(\Gamma, P + \Delta_2) = 2 = h^0(\Gamma, \Delta_2) + 1$, which is the condition (C3). Thus $\tilde{H} = \langle 5, 7, 8, 9 \rangle$ is DC.

Secondly, for $\tilde{H} = \langle 5, 7, 8, 11 \rangle$, we choose Γ_0 admitting (c-ii). By using the adjunction formula again we obtain the following:

$$K_\Gamma \sim \left(2l - \sum_{i=1}^{3} e_i \right)\Big|_\Gamma \sim B|_\Gamma \sim P + Q_1 + Q_2 + R_2,$$

where R_2 is a point of Γ different from P. Then

$$h^0(\Gamma, P + \Delta_2) = h^0(\Gamma, K_\Gamma - R_2) = 2 = h^0(\Gamma, \Delta_2) + 1 \quad \text{and}$$
$$h^0(\Gamma, 2P + \Delta_2) = h^0(\Gamma, K_\Gamma - R_2 + P) = 2 = h^0(\Gamma, 3P + \Delta_2) - 1.$$

Hence Δ_2 satisfies (C3). It follows that $\tilde{H} = \langle 5, 7, 8, 11 \rangle$ is DC.

Thirdly, for $\tilde{H} = \langle 5, 8, 9, 11, 12 \rangle$, we choose Γ_0 admitting (c-iii). Then there exists no canonical divisor of Γ containing $P + Q_1 + Q_2$, which implies that

$$h^0(\Gamma, 2P + \Delta_2) = 2 = h^0(\Gamma, P + \Delta_2) + 1 \quad \text{and}$$
$$h^0(\Gamma, 3P + \Delta_2) = 3 = h^0(\Gamma, 2P + \Delta_2) + 1.$$

Hence Δ_2 satisfies (C3). Thus $\tilde{H} = \langle 5, 8, 9, 11, 12 \rangle$ is DC.

Remark 5.2 In the above argument we construct desired curves of genus three via nodal plane quintics. However, we can give a concrete pair of a smooth plane quartic and a point on the curve corresponding the numerical semigroup under consideration. For example, consider the smooth plane quartic

$$\Gamma : X^3Y + X^2YZ - XY^2Z + Y^3Z - X^2Z^2 - 2Y^2Z^2 + YZ^3 = 0$$

and set $P = (0 : 1 : 1)$. We verify that there exists a double covering of Γ with a ramification point $\tilde{P}$ over P such that $H(\tilde{P}) = \langle 5, 7, 8, 9 \rangle$ by using Theorem 3.1.

Set $P_1 = (0 : 0 : 1)$, $P_2 = (0 : 1 : 0)$, $S_1 = (-1 : 1 : 1)$, $S_2 = (1 : 1 : 1)$, $S_3 = (1 : 0 : 0)$ and $\Delta_2 := P_1 + P_2$. Note that T_P and T_{P_2} are defined by $X = 0$ and $Z = 0$, respectively. Therefore

$$\begin{aligned} K_\Gamma &\sim \Gamma.T_P = 2P + P_1 + P_2 \\ &\sim \Gamma.T_{P_2} = 3P_2 + S_3 \\ &\sim \Gamma.L = P + S_1 + S_2 + S_3, \end{aligned}$$

where L is the line defined by $Y - Z = 0$. Furthermore, $2K_\Gamma \sim \Gamma.C_2 = 4P_1 + 2P_2 + S_1 + S_2$, where C_2 is the conic $X^2 - YZ = 0$. Then

$$\begin{aligned} 5P - 2\Delta_2 &\sim 2(K_\Gamma - P_1 - P_2) + (K_\Gamma - S_1 - S_2 - S_3) - 2P_1 - 2P_2 \\ &\sim 3K_\Gamma - 4P_1 - 4P_2 - S_1 - S_2 - S_3 \\ &\sim (2K_\Gamma - 4P_1 - 2P_2 - S_1 - S_2) + (K_\Gamma - 3P_2 - S_3) + P_2 \\ &\sim P_2. \end{aligned}$$

Thus Theorem 3.1 shows that there exists a desired double covering.

6 When $n = 7$ and $\tilde{H} = \langle 6 \to 9, 11 \rangle, \langle 6, 7, 9, 10, 11 \rangle$ or $\langle 7 \to 13 \rangle$

(1) For $\tilde{H} = \langle 6 \to 9, 11 \rangle$, we choose a smooth plane quartic Γ with five points P, Q_1, Q_2, Q_3 and R satisfying the following conditions:

(i) P is a hyperflex, i.e., $H(P) = \langle 3, 4 \rangle$.
(ii) The line $\overline{Q_1Q_2}$ is tangent to Γ at both Q_1 and Q_2.
(iii) T_{Q_3} passes through P and R.

In other words, the relations

$$K_\Gamma \sim 4P \sim 2Q_1 + 2Q_2 \sim P + 2Q_3 + R$$

hold. We set $\Delta_3 := Q_1 + Q_2 + Q_3$. It satisfies the conditions (C1) and (C2), since Q_1, Q_2 and Q_3 are not collinear by (ii). Furthermore it is clear that

$$\begin{aligned} h^0(\Gamma, P + \Delta_3) &= 2 = h^0(\Gamma, \Delta_3) + 1 \quad \text{and} \\ h^0(\Gamma, 2P + \Delta_3) &= 3 = h^0(\Gamma, P + \Delta_3) + 1. \end{aligned}$$

Thus Δ_3 satisfies (C3) as well. Finally

$$\begin{aligned} 7P - 2\Delta_3 &\sim (2K_\Gamma - P) - 2(Q_1 + Q_2 + Q_3) \\ &\sim (K_\Gamma - 2Q_1 - 2Q_2) + (K_\Gamma - P - 2Q_3) \\ &\sim R \ (\neq P). \end{aligned}$$

Hence Δ_3 satisfies (C4), which shows that $\tilde{H}$ is DC.
(2) Secondly, for $\tilde{H} = \langle 6, 7, 9, 10, 11 \rangle$, we choose a smooth plane quartic Γ with seven points P, P', Q_i $(i = 1, 2, 3)$ and R_j $(j = 1, 2)$ satisfying the following conditions:

- (i) P, Q_1 and Q_2 are ordinary flexes and three tangent lines T_P, T_{Q_1} and T_{Q_2} meets P'. In particular $H(P) = \langle 3, 5, 7 \rangle$.
- (ii) Four points P, Q_1, Q_2 and R_1 are collinear.
- (iii) T_{Q_3} passes through R_1 and R_2.

Then we obtain the relations that

$$K_\Gamma \sim 3P + P' \sim 3Q_1 + P' \sim 3Q_2 + P' \sim P + Q_1 + Q_2 + R_1 \sim 2Q_3 + R_1 + R_2.$$

Then $\Delta_3 := Q_1 + Q_2 + Q_3$ satisfies the conditions (C1) and (C2), since Q_1, Q_2 and Q_3 are not collinear by (ii). Furthermore

$$\begin{aligned} h^0(\Gamma, P + \Delta_3) &= 2 = h^0(\Gamma, \Delta_3) + 1 \quad \text{and} \\ h^0(\Gamma, 2P + \Delta_3) &= 3 = h^0(\Gamma, P + \Delta_3) + 1 \end{aligned}$$

hold, which shows that Δ_3 also satisfies (C3). Finally

$$\begin{aligned} 7P - 2\Delta_3 &\sim 2(K_\Gamma - P') + (K_\Gamma - Q_1 - Q_2 - R_1) - 2(Q_1 + Q_2 + Q_3) \\ &\sim 3K_\Gamma - 2P' - 3Q_1 - 3Q_2 - 2Q_3 - R_1 \\ &\sim (K_\Gamma - 3Q_1 - P') + (K_\Gamma - 3Q_2 - P') + (K_\Gamma - 2Q_3 - R_1) \\ &\sim R_2 \ (\neq P). \end{aligned}$$

Thus Δ_3 satisfies (C4). Hence $\tilde{H} = \langle 6, 7, 9, 10, 11 \rangle$ is DC.
(3) In the end, for $\tilde{H} = \langle 7 \to 13 \rangle$, let Γ be a smooth plane quartic without Galois points, P a non-flex of Γ. Then $H(P) = \langle 4, 5, 6, 7 \rangle$. It follows from [2, Lemma 3.6 (1)]

that $|7P - 2Q_1| = g^2_5$ is free from base points for a general point $Q_1 \in \Gamma$. Thus it gives a nodal plane quintic model Γ_0 of Γ. Let Q_2 be a point on Γ different from Q_1 corresponding to a node of Γ_0. Then $|7P - 2Q_1 - 2Q_2| = g^1_3$ is a pencil without base points. It induces a triple covering $\psi : \Gamma \to \mathbb{P}^1$ with at least six ramification points, since Γ has no Galois points. Hence we can choose a ramification point Q_3 of ψ satisfying the following:

(a) $\psi(Q_3) \neq \psi(P)$.
(b) Q_3 is distinct from Q_1 and Q_2.
(c) Q_1, Q_2 and Q_3 are not collinear.

Then $|7P - 2Q_1 - 2Q_2 - 2Q_3|$ consists of a point on Γ different from P by (a). Hence $\Delta_3 := Q_1 + Q_2 + Q_3$ satisfies (C1), (C2) and (C4). It is clear that Δ_3 also satisfies (C3). Thus $\tilde{H} = \langle 7 \to 13 \rangle$ is DC.

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