ON SMOOTH PLANE CURVES WHOSE AUTOMORPHISM GROUP HAS FIXED POINTS

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This article is based on my talk on December 16, 2017. The main result is a classification of automorphism groups of smooth plane curves with fixed points in the projective plane. This is a joint work with A. Ohbuchi and it is a generalization of my previous talk in 2016.

1. Introduction

Notation

- $\mathbb{Z}_m$: a cyclic group of order m;
- ζ_m: a primitive m-th root of unity.
- $D_{2m} = \langle a, b \mid a^m = b^2 = 1, bab^{-1} = a^{-1}\rangle$: the dihedral group of order $2m$;
- $\overline{D}_{2m} = Q_{4m} = \langle a, b \mid a^{2m} = 1, b^2 = a^m, bab^{-1} = a^{-1}\rangle$: the binary dihedral subgroup of $SL(2,\mathbb{C})$ (the dicyclic group of order $4m$);
- $Q_8 = \langle a, b \mid a^4 = 1, b^2 = a^2, bab^{-1} = a^{-1}\rangle = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1\rangle$: the quaternion group;
- $T \simeq A_4$, $O \simeq S_4$, $I \simeq A_5$: the tetrahedral, octahedral, icosahedral subgroups of $PGL(2,\mathbb{C})$;
- $\overline{T} \simeq SL(2,3)$, $\overline{O}$, $\overline{I} \simeq SL(2,5)$: the binary tetrahedral, binary octahedral, binary icosahedral subgroup of $SL(2,\mathbb{C})$.

In this article

$$\mathrm{PBD}(2,1) := \left\{ \begin{pmatrix} \begin{matrix} & A & \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \alpha \end{pmatrix} \;\middle|\; A \in GL(2,\mathbb{C}), \alpha \in \mathbb{C}^\times \right\} \Big/ \mathbb{C}^\times$$
$$= \left\{ \begin{pmatrix} \begin{matrix} & A & \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \alpha \end{pmatrix} \;\middle|\; A \in SL(2,\mathbb{C}), \alpha \in \mathbb{C}^\times \right\} \Big/ \{\pm E_3\}$$
$$\subset PGL(3,\mathbb{C})$$

and $\rho : \mathrm{PBD}(2,1) \to PGL(2,\mathbb{C})$ is the natural homomorphism.

First we recall a classification of automorphism groups of smooth plane curves:

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Theorem 1. *Let C be a smooth plane curve of degree $d \geq 4$, G a subgroup of $\mathrm{Aut}(C)$. Then one of the following holds:*

(a-i) *G fixes a point on C and G is a cyclic group whose order is at most $d(d-1)$. Furthermore, if $d \geq 5$ and $|G| = d(d-1)$, then C is projectively equivalent to the curve $YZ^{d-1} + X^d + Y^d = 0$.*

(a-ii) *G fixes a point not lying on C and there exists a commutative diagram*

$$\begin{array}{ccccccccc} 1 \to & \mathbb{C}^* & \to & \mathrm{PBD}(2,1) & \overset{\rho}{\to} & PGL(2,\mathbb{C}) & \to 1 & \text{(exact)} \\ & \uparrow & & \uparrow & & \uparrow & & \\ 1 \to & \mathbb{Z}_n & \longrightarrow & G & \longrightarrow & G' & \to 1 & \text{(exact)}, \end{array}$$

where n is a factor of d and G' is conjugate to $\mathbb{Z}_m$, D_{2m} $(m \leq d-1)$, T, O or I. Furthermore, if $n \neq 1$ and $G' \simeq D_{2m}$ then $m \mid d-2$. In particular $|G| \leq \max\{2d(d-2), 60d\}$.

(b-i) *(C, G) is a descendant of Fermat curve $F_d : X^d + Y^d + Z^d = 0$. In this case $G \subset \mathrm{Aut}(F_d)$. In particular, $|G| \leq 6d^2$.*

(b-ii) *(C, G) is a descendant of Klein curve $K_d : XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$. In this case $G \subset \mathrm{Aut}(K_d)$. In particular, $|G| \leq 3(d^2 - 3d + 3)$ if $d \geq 5$. On the other hand, $|G| \leq 168$ if $d = 4$.*

(c) *G is conjugate to one of the following subgroups of $PGL(3,\mathbb{C})$: the alternating group A_5 or A_6, the Klein group $PSL(2,7)$, the Hessian group H_{216} of order 216 or its subgroup of order 36 or 72. In particular $|G| \leq 360$.*

In this article we consider the case (a-ii). Let P is a fixed point of G. By the commutative diagram above, we may assume that $P = (0:0:1)$, G is a subgroup of $\mathrm{PBD}(2,1)$ and $\mathbb{Z}_n$ is generated by a projective reflection $[1, 1, \zeta_n]$. If $n = 1$ then $G = G'$, whose structure is well known. Thus we assume that $n \geq 2$. When $G = \mathrm{Aut}(C)$, such a point P is called a *quasi Galois point of order n* for C.

Remark 1. Put $G[Q] := \{\sigma \in \mathrm{Aut}(C) \mid \pi_Q \circ \sigma = \pi_Q\}$ for $Q \in \mathbb{P}^2$, where $\pi_Q \colon C \to \mathbb{P}^1$ is the projection with center Q. Then

$$Q \text{ is quasi Galois point for } C \quad \underset{\text{def}}{\Longleftrightarrow} \quad |G[Q]| \geq 2.$$

If $|G[Q]| = d$ then Q is called a *Galois point* for C (see [Y] for details).

Since $[1, 1, \zeta_n]$ acts on C, the curve C has a defining equation of the form

$$Z^d + \sum_{j=1}^{k} F_j(X,Y) Z^{d-jn} = 0,$$

where $k = d/n$ and $F_j(X,Y)$ is a homogeneous polynomial of degree jn. We formally set $F_0(X,Y) := 1$.

Remark 2. Since $G_P = \{\,\sigma \in G \mid \sigma(P) = P\,\} = \mathbb{Z}_n$, we see that $\gcd\{\, d - jn \mid F_j \neq 0, j = 0, 1, \ldots, k-1\,\} = n$. Hence

$$\gcd\{\, k - j \mid F_j \neq 0, j = 0, 1, \ldots, k-1\,\} = 1$$

holds.

Note that there exists a natural commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & SL(2,\mathbb{C}) \times \mathbb{C}^\times & \longrightarrow & SL(2,\mathbb{C}) & \longrightarrow & 1 \\
 & & \downarrow \wr & \circlearrowleft & \downarrow \varpi & \circlearrowleft & \downarrow \pi & & \\
1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathrm{PBD}(2,1) & \underset{\rho}{\longrightarrow} & PGL(2,\mathbb{C}) & \longrightarrow & 1,
\end{array}$$

where π and ϖ are natural projections. This diagram induces another commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}_n & \longrightarrow & \widetilde{G} & \overset{\tilde{\rho}}{\longrightarrow} & \widetilde{G'} & \longrightarrow & 1 \\
 & & \downarrow \wr & \circlearrowleft & \downarrow \varpi & \circlearrowleft & \downarrow \pi & & \\
1 & \longrightarrow & \mathbb{Z}_n & \longrightarrow & G & \underset{\rho}{\longrightarrow} & G' & \longrightarrow & 1,
\end{array}$$

where $\widetilde{G} := \varpi^{-1}(G)$ and $\widetilde{G'} := \pi^{-1}(G')$. Then $\operatorname{Ker}\varpi = \{\pm E_3\}$, $\operatorname{Ker}\pi = \{\pm E_2\}$ and

$$\widetilde{G'} = \begin{cases} \mathbb{Z}_{2m} & (\text{if } G' = \mathbb{Z}_m) \\ \overline{D}_{2m} & (\text{if } G' = D_{2m}) \\ \overline{T} & (\text{if } G' = T) \\ \overline{O} & (\text{if } G' = O) \\ \overline{I} & (\text{if } G' = I). \end{cases}$$

Furthermore, $\widetilde{G'}$ is presented as follows.

- $\mathbb{Z}_{2m}$ generated by
$$a_{2m} := \begin{pmatrix} \zeta_{2m} & 0 \\ 0 & \zeta_{2m}^{-1} \end{pmatrix}.$$
- $\overline{D}_{2m} = Q_{4m} = \langle a, b \mid a^{2m} = b^4 = 1, bab^{-1} = a^{-1} \rangle$, where
$$a := a_{2m}, \quad b := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

For $\overline{T}, \overline{O}$ we put

$$s := a_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad t := \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon & -\epsilon \\ \epsilon^{-1} & \epsilon^{-1} \end{pmatrix} \quad \text{and} \quad u := a_8 = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}$$

(ϵ is a primitive 8th root of unity).

Then

$$\overline{T} = \langle s, t \mid s^2 = t^3 = (st^{-1})^3 \rangle, \quad \overline{O} = \langle t, u \mid (tu)^2 = t^3 = u^4 \rangle.$$

Furthermore, put $u' = tut^{-1}$. Then $\overline{V} = \langle u, u' \rangle \simeq Q_8$ is the only normal subgroup of index three in $\overline{T}$.

Finally

$$\overline{I} = \langle v, w \mid v^2 = (w^{-1}v)^3 = w^5 \rangle,$$

where

$$v := \frac{1}{\eta^2 - \eta^3} \begin{pmatrix} \eta + \eta^{-1} & 1 \\ 1 & -(\eta + \eta^{-1}) \end{pmatrix}, \quad w := \begin{pmatrix} -\eta & 0 \\ 0 & -\eta^{-1} \end{pmatrix}$$

(η is a primitive 5th root of unity).

2. Defining equations and automorphism groups of curves

In what follows, we assume that G' is not cyclic.

The special linear group $SL(2, \mathbb{C})$ naturally acts on $\mathbb{C}[X, Y]$, i.e., $A \in SL(2, \mathbb{C})$ induces a homomorphism $F(X, Y) \mapsto F((X, Y)A)$.

Theorem 2. *Every $F_j(X, Y)$ $(j = 1, 2, \dots, k)$ is invariant under a subgroup $\widetilde{H} \subset \widetilde{G'}$ and invariant under $\widetilde{G'}$ up to multiplication by l-th roots of unity, where*

$$\widetilde{H} = \begin{cases} \langle a^2 \rangle \quad (\text{if } G' = D_{2m}) \\ \langle u, u' \rangle = \overline{V} \simeq Q_8 \quad (\text{if } G' = T) \\ \langle s, t \rangle = \overline{T} \quad (\text{if } G' = O) \\ \overline{I} \quad (\text{if } G' = I), \end{cases}$$

and

$$l = \begin{cases} 2 & (\text{if } G' = D_{2m} \text{ and } m \text{ is even}) \\ 4 & (\text{if } G' = D_{2m} \text{ and } m \text{ is odd}) \\ 3 & (\text{if } G' = T) \\ 2 & (\text{if } G' = O) \\ 1 & (\text{if } G' = I). \end{cases}$$

Let R_0 be the invariant ring of $\widetilde{G'}$, i.e., $R_0 = \mathbb{C}[X, Y]^{\widetilde{G'}}$ and R the invariant ring of $\widetilde{H}$, i.e., $R = \mathbb{C}[X, Y]^{\widetilde{H}}$. Then $F_j(X, Y) \in R$ $(j = 1, 2, \dots, k)$. Note that $R_0 \subset R$ and R is R_0-module. It is well known that

$$R_0 = \mathbb{C}[X, Y]^{\widetilde{G'}} = \begin{cases} \mathbb{C}[X^{2m} + Y^{2m}, (XY)^2, XY(X^{2m} - Y^{2m})] \\ \qquad (\text{if } G' = D_{2m},\ m \text{ is even}) \\ \mathbb{C}[X^{2m} - Y^{2m}, (XY)^2, XY(X^{2m} + Y^{2m})] \\ \qquad (\text{if } G' = D_{2m},\ m \text{ is odd}) \\ \mathbb{C}[f, g, h] \quad (\text{if } G' = T) \\ \mathbb{C}[g, f^2, fh] \quad (\text{if } G' = O) \\ \mathbb{C}[\theta_1, \theta_2, \theta_3] \quad (\text{if } G' = I), \end{cases}$$

where

$$\begin{aligned} &f = XY(X^4 - Y^4), \quad g = X^8 + 14X^4Y^4 + Y^8, \\ &h = X^{12} - 33(X^8Y^4 + X^4Y^8) + Y^{12}, \\ &\deg f = 6, \quad \deg g = 8, \quad \deg h = 12 \quad \text{and} \quad h^2 = g^3 - 108f^4, \end{aligned}$$

$$\begin{aligned} &\theta_1 = XY(X^{10} + 11X^5Y^5 - Y^{10}), \\ &\theta_2 = X^{20} - 228(X^{15}Y^5 - X^5Y^{15}) + 494X^{10}Y^{10} + Y^{20}, \\ &\theta_3 = X^{30} + 522(X^{25}Y^5 - X^5Y^{25}) - 10005(X^{20}Y^{10} + X^{10}Y^{20}) + Y^{30}, \\ &\deg \theta_1 = 12, \quad \deg \theta_2 = 20, \quad \deg \theta_3 = 30 \quad \text{and} \quad \theta_3^2 = 1728\theta_1^5 - \theta_2^3. \end{aligned}$$

Furthermore $g = g_1g_2$, where

$$g_1 = X^4 + 2\sqrt{3}iX^2Y^2 + Y^4, \quad g_2 = X^4 - 2\sqrt{3}iX^2Y^2 + Y^4.$$

Then $h = g_1^3 + 6\sqrt{3}if^2 \in \mathbb{C}[f, g_1, g_2]$ and $12\sqrt{3}if^2 - g_1^3 + g_2^3 = 0$ holds. The structure of $R = \mathbb{C}[X, Y]^{\widetilde{H}}$ is also well-known:

$$R = \mathbb{C}[X, Y]^{\widetilde{H}} = \begin{cases} \mathbb{C}[X^m, XY, Y^m] = \mathbb{C}[p, q, r] \quad (\text{if } G' = D_{2m}) \\ \mathbb{C}[f, g_1, g_2] \quad (\text{if } G' = T) \\ \mathbb{C}[f, g, h] \quad (\text{if } G' = O) \\ \mathbb{C}[\theta_1, \theta_2, \theta_3] \ (= R_0) \quad (\text{if } G' = I), \end{cases}$$

where $p = X^m + Y^m$, $q = X^m - Y^m$ and $r = XY$. This implies the following fact.

Lemma 3. *The order n is even unless $G' = D_{2m}$ and m is odd.*

Proof. If $F_j \neq 0$ then $\deg F_j = jn$ is even unless $G' = D_{2m}$ and m is odd, since every nonzero polynomial in R is of even degree. In particular $d = \deg F_k$ is even. Hence $\gcd\{\, d - jn \mid F_j \neq 0, j = 0, 1, \ldots, k \,\} = n$ is also even. □

In the end of this section, we state a classification of G. First we consider a special case.

Proposition 4. *If n is odd, then $G' = D_{2m}$ and m is odd. In this case $G = \mathbb{Z}_n \times D_{2m}$.*

Proof. First it follows from Lemma 3 that $G' = D_{2m}$ and m is odd. Furthermore m and n are coprime, since m and n are both odd, $m \mid d - 2$ and $n \mid d$. Then $\mathbb{Z}_n$ and $G/\mathbb{Z}_n = D_{2m}$ have coprime orders, which implies that $G = \mathbb{Z}_n \times D_{2m}$. □

In what follows we assume that n is even.

To determine the structure of the group G, we consider a group character on G. Let A be a matrix in $\widetilde{G'}$. Then there exists $\alpha \in \mathbb{C}^\times$ such that $\varpi((A, \alpha)) \in G$ and

α is unique up to multiplication by n-th roots of unity. Thus we obtain a group homomorphism

$$\widetilde{\chi} : \widetilde{G'} \to \mathbb{C}^\times \ (A \mapsto \alpha^n).$$

Lemma 5. *The image of the generators of $\widetilde{G'}$ under $\widetilde{\chi}$ is as follows.*

1. *If $G' = D_{2m}$ and m is even, then $\widetilde{\chi}(a) = \pm 1$ and $\widetilde{\chi}(b) = \pm 1$.*
2. *If $G' = D_{2m}$ and m is odd, then $\widetilde{\chi}(a) = 1$ and $\widetilde{\chi}(b) = \pm 1$.*
3. *If $G' = T$, then $\widetilde{\chi}(s) = 1$ and $\widetilde{\chi}(t) = \omega$ is a primitive third root of unity.*
4. *If $G' = O$, then $\widetilde{\chi}(t) = 1$ and $\widetilde{\chi}(u) = \pm 1$.*
5. *If $G' = I$, then $\widetilde{\chi} = 1$.*

When $G' \neq T$, let n_0 be the odd part of n, i.e., $n = 2^e n_0$ ($e \geq 1, 2 \nmid n_0$). When $G' = T$, put $n = 2^e 3^{e'} n_0$, where $e \geq 1, e' \geq 0$ and $2, 3 \nmid n_0$. Put the following numbers:

$$\widetilde{\kappa} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \zeta_{2^e} \end{pmatrix} \in \mathbb{Z}_n$$

and

(1) $\mu_1 := \begin{cases} 1 & (\text{if } \widetilde{\chi}(a) = 1) \\ \zeta_{2^{e+1}} & (\text{if } \widetilde{\chi}(a) = -1) \end{cases}$, $\mu_2 := \begin{cases} 1 & (\text{if } \widetilde{\chi}(b) = 1) \\ \zeta_{2^{e+1}} & (\text{if } \widetilde{\chi}(b) = -1) \end{cases}$
($G' = D_{2m}$ and m is even)

(2) $\mu := \begin{cases} 1 & (\text{if } \widetilde{\chi}(b) = 1) \\ \zeta_{2^{e+1}} & (\text{if } \widetilde{\chi}(b) = -1) \end{cases}$ ($G' = D_{2m}$ and m is odd)

(3) $\nu := \zeta_{3^{e'+1}}$ ($G' = T$)

(4) $\lambda := \begin{cases} 1 & (\text{if } \widetilde{\chi}(u) = 1) \\ \zeta_{2^{e+1}} & (\text{if } \widetilde{\chi}(u) = -1) \end{cases}$ ($G' = O$)

Then $\widetilde{G}$ contains a subgroup $\widetilde{G}_0$ generated by the following elements and $\widetilde{\kappa}$.

(1) $\begin{pmatrix} a & \\ & \mu_1 \end{pmatrix}$ and $\begin{pmatrix} b & \\ & \mu_2 \end{pmatrix}$ ($G' = D_{2m}$ and m is even)

(2) $\begin{pmatrix} a & \\ & 1 \end{pmatrix}$ and $\begin{pmatrix} b & \\ & \mu \end{pmatrix}$ ($G' = D_{2m}$ and m is odd)

(3) $\begin{pmatrix} s & \\ & 1 \end{pmatrix}$ and $\begin{pmatrix} t & \\ & \nu \end{pmatrix}$ ($G' = T$)

(4) $\begin{pmatrix} t & \\ & 1 \end{pmatrix}$ and $\begin{pmatrix} u & \\ & \lambda \end{pmatrix}$ ($G' = O$)

(5) $\begin{pmatrix} v & \\ & 1 \end{pmatrix}$ and $\begin{pmatrix} w & \\ & 1 \end{pmatrix}$ ($G' = I$)

Set $G_0 = \widetilde{G}_0/\{\pm E_3\}$. Then $G = \mathbb{Z}_{n_0} \times G_0$. Thus it suffices to classify the structure of G_0, which is given by the following theorems.

Theorem 6. *When $G' = D_{2m}$ and m is even, the group $G_0 = \mathbb{Z}_{2^e}{}^\bullet D_{2m}$ is a non-split extension of D_{2m} by $\mathbb{Z}_{2^e}$.*

(i) *If $4 \mid m$ then $e = 1$ and*

$$G_0 \simeq \begin{cases} \overline{D}_{2m} = Q_{4m} & (\text{if } \mu_1 = \mu_2 = 1) \\ D_{4m} & (\text{if } \mu_1 = 1 \text{ and } \mu_2 = i) \\ \mathbb{Z}_{2m} \rtimes \mathbb{Z}_2 & (\text{if } \mu_1 = i). \end{cases}$$

(ii) *If $4 \nmid m$ and $e = 1$ then*

$$G_0 \simeq \begin{cases} \mathbb{Z}_{m/2} \rtimes Q_8 & (\text{if } \mu_1 = \mu_2 = 1) \\ D_{4m} & (\text{if } \mu_1 = 1 \text{ and } \mu_2 = i) \\ (\mathbb{Z}_m \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 & (\text{if } \mu_1 = i). \end{cases}$$

(iii) *If $4 \nmid m$ and $e \geq 2$ then*

$$G_0 \simeq \begin{cases} (\mathbb{Z}_{2^{e-1}m} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 & (\text{if } \mu_1 = \mu_2 = 1) \\ (\mathbb{Z}_{m/2} \rtimes \mathbb{Z}_{2^{e+1}}) \rtimes \mathbb{Z}_2 & (\text{if } \mu_1 = 1 \text{ and } \mu_2 = \zeta_{2^{e+1}}) \\ \mathbb{Z}_{2^e m} \rtimes \mathbb{Z}_2 & (\text{if } \mu_1 = \zeta_{2^{e+1}}). \end{cases}$$

Theorem 7. *When $G' = D_{2m}$ and m is odd, the group $G_0 = \mathbb{Z}_{2^e}{}_\bullet D_{2m}$ is an extension of D_{2m} by $\mathbb{Z}_{2^e}$. Furthermore,*

$$G_0 = \begin{cases} \mathbb{Z}_m \rtimes \mathbb{Z}_4 & (\text{if } e = 1 \text{ and } \mu = 1) \\ D_{4m} & (\text{if } e = 1 \text{ and } \mu = \zeta_{2^{e+1}}) \\ \mathbb{Z}_{2^e} \times D_{2m} & (\text{if } e \geq 2 \text{ and } \mu = 1) \\ \mathbb{Z}_m \rtimes \mathbb{Z}_{2^{e+1}} & (\text{if } e \geq 2 \text{ and } \mu = \zeta_{2^{e+1}}). \end{cases}$$

Theorem 8. *When $G' = T$, the group $G_0 = \mathbb{Z}_{2^e}{}^\bullet T$ is a non-split extension of T by $\mathbb{Z}_{2^e}$ and*

$$G_0 = \begin{cases} Q_8 \rtimes \mathbb{Z}_{3^{e'+1}} & (\text{if } e = 1) \\ ((\mathbb{Z}_{2^e} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_{3^{e'+1}} & (\text{if } e \geq 2). \end{cases}$$

Remark 3. If $e = 1$ and $e' = 0$ then $G_0 \simeq \overline{T}$.

Theorem 9. *When $G' = O$, the group $G_0 = \mathbb{Z}_{2^e}{}^\bullet O$ is a non-split extension of O by $\mathbb{Z}_{2^e}$ and*

$$G_0 = \begin{cases} \overline{O} \quad (\text{if } e = 1 \text{ and } \lambda = 1) \\ GL(2,3) = \overline{T} \rtimes \mathbb{Z}_2 \quad (\text{if } \lambda = i) \\ (((\mathbb{Z}_{2^e} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \quad (\text{if } e \geq 2 \text{ and } \lambda = 1) \\ \overline{T} \rtimes \mathbb{Z}_4 \quad (\text{if } \lambda = \epsilon) \\ \overline{T}{}^\bullet \mathbb{Z}_{2^e} = \mathbb{Z}_{2^e}{}^\bullet S_4 \quad (\text{if } \lambda = \zeta_{2^{e+1}}\ (e \geq 3)). \end{cases}$$

Theorem 10. *When $G' = I$, the group $G_0 = \mathbb{Z}_{2^e}{}^\bullet I$ is a non-split extension of I by $\mathbb{Z}^{2^e}$ and it is an extension of a cyclic group $\mathbb{Z}_{2^{e-1}}$ by $\overline{I}$. Precisely*

$$G_0 = \begin{cases} \overline{I} \quad (\text{if } e = 1) \\ \overline{I} \rtimes \mathbb{Z}_2 \quad (\text{if } e = 2) \\ \overline{I}{}^\bullet \mathbb{Z}_{2^{e-1}} \quad (\text{if } e \geq 3). \end{cases}$$

3. Examples

Example 1. Let C be the plane curve of degree 10 defined by

$$Z^{10} = XY(X^8 + Y^8).$$

Then $G = \mathrm{Aut}(C)$ satisfies the non-split exact sequence

$$1 \to \mathbb{Z}_{10} \to G \to D_{16} \to 1$$

and $G = \mathbb{Z}_{80} \rtimes \mathbb{Z}_2$.

Example 2. Let C be the plane curve of degree 15 defined by

$$Z^{15} = XY(X^{13} + Y^{13}).$$

Then $G = \mathrm{Aut}(C)$ satisfies the split exact sequence

$$1 \to \mathbb{Z}_{15} \to G \to D_{26} \to 1,$$

in other words, $G = \mathbb{Z}_{15} \times D_{26}$.

Example 3. Let C be the plane curve of degree 16 defined by
$$Z^{16} = XY(X^{14} + Y^{14}).$$
Then $G = \mathrm{Aut}(C)$ satisfies the non-split exact sequence
$$1 \to \mathbb{Z}_{16} \to G \to D_{28} \to 1$$
and $G = \mathbb{Z}_{224} \rtimes \mathbb{Z}_2$.

Example 4. Let C be the plane curve of degree 16 defined by
$$Z^{16} = XY(X^{14} + X^7Y^7 + Y^{14}).$$
Then $G = \mathrm{Aut}(C)$ satisfies the split exact sequence
$$1 \to \mathbb{Z}_{16} \to G \to D_{14} \to 1,$$
in other words, $G = \mathbb{Z}_{16} \times D_{14}$.

Example 5. Let C be the plane curve of degree 16 defined by
$$Z^{16} + X^4Y^4Z^8 + XY(X^{14} + X^7Y^7 + Y^{14}) = 0.$$
Then $G = \mathrm{Aut}(C)$ satisfies the split exact sequence
$$1 \to \mathbb{Z}_8 \to G \to D_{14} \to 1,$$
in other words, $G = \mathbb{Z}_8 \times D_{14}$.

Example 6. Let C be the plane curve of degree 32 defined by
$$Z^{32} + X^3Y^3(X^{10} + Y^{10})Z^{16} + XY(X^{30} + Y^{30}) = 0.$$
Then $G = \mathrm{Aut}(C)$ satisfies the split exact sequence
$$1 \to \mathbb{Z}_{16} \to G \to D_{10} \to 1,$$
in other words, $G = \mathbb{Z}_{16} \times D_{10}$.

Example 7. Let C be the quartic curve defined by $Z^4 = X^4 + 2\sqrt{3}iX^2Y^2 + Y^4$ $(= g_1)$. Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence
$$1 \to \mathbb{Z}_4 \to G \to T \to 1$$

and $G = \overline{T} \rtimes \mathbb{Z}_2$.

Example 8. Let C be the the plane curve of degree 16 defined by $Z^{16} = g_1(g_1^3 + g_2^3)$. Then $G = \text{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbb{Z}_{16} \to G \to T \to 1$$

and $G = ((\mathbb{Z}_{16} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3$.

Example 9. Let C be the sextic curve defined by $Z^6 = XY(X^4 - Y^4)$. Then $G = \text{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbb{Z}_6 \to G \to O \to 1$$

and $G = \mathbb{Z}_3 \times GL(2,3)$.

Example 10. Let C be the octic curve defined by $Z^8 = X^8 + 14X^4Y^4 + Y^8$ $(= g)$. Then $G = \text{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbb{Z}_8 \to G \to O \to 1$$

and $G = (((\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \simeq (\overline{T} \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_2$.

Example 11. Let C be the plane curve of degree 12 defined by $Z^{12} = X^{12} - 33(X^8Y^4 + X^4Y^8) + Y^{12}$ $(= h)$. Then $G = \text{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbb{Z}_{12} \to G \to O \to 1$$

and $G = \mathbb{Z}_3 \times (\overline{T} \rtimes \mathbb{Z}_4)$.

Example 12. Let C be the plane curve of degree 18 defined by $Z^{18} = fh$. Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbb{Z}_{18} \to G \to O \to 1$$

and $G = \mathbb{Z}_9 \times \overline{O}$.

Example 13. Let C be the plane curve of degree 32 defined by $Z^{32} = gF$, where F is a polynomial of degree 24 in $\mathbb{C}[f, g, h]$ such that gF has no multiple factor. Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbb{Z}_{32} \to G \to O \to 1$$

and $G = (((\mathbb{Z}_{32} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$.

Example 14. Let C be the plane curve of degree 24 defined by

$$Z^{24} + f^2 Z^{12} + F_{24} = 0,$$

where F_{24} is a general polynomial of degree 24 in $\mathbb{C}[f, g, h]$. Then $G = \mathrm{Aut}(C)$ satisfies the non-split exact sequence

$$1 \to \mathbb{Z}_{12} \to G \to O \to 1$$

and $G = \mathbb{Z}_3 \times ((((\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2)$.

Example 15. Let C be the plane curve of degree 12 defined by

$$Z^{12} = XY(X^{10} + 11X^5Y^5 - Y^{10}) \; (= \theta_1).$$

Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbb{Z}_{12} \to G \to I \to 1$$

and $G = \mathbb{Z}_3 \times (\overline{I} \rtimes \mathbb{Z}_2)$.

Example 16. Let C be the plane curve of degree 20 defined by

$$Z^{20} = X^{20} - 228(X^{15}Y^5 - X^5Y^{15}) + 494X^{10}Y^{10} + Y^{20} \; (= \theta_2).$$

Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbb{Z}_{20} \to G \to I \to 1$$

and $G = \mathbb{Z}_5 \times (\overline{I} \rtimes \mathbb{Z}_2)$.

Example 17. Let C be the plane curve of degree 30 defined by

$$Z^{30} = X^{30} + 522(X^{25}Y^5 - X^5Y^{25}) - 10005(X^{20}Y^{10} + X^{10}Y^{20}) + Y^{30} \ (= \theta_3).$$

Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbb{Z}_{30} \to G \to I \to 1$$

and $G = \mathbb{Z}_{15} \times \overline{I}$.

Example 18. Let C be the plane curve of degree 32 defined by $Z^{32} = \theta_1\theta_2$. Then $G = \mathrm{Aut}(C)$ satisfies the non-split exact sequence

$$1 \to \mathbb{Z}_{32} \to G \to I \to 1$$

and $G = \overline{I}^{\bullet}\mathbb{Z}_{16}$.

References

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