AUTOMORPHISM GROUPS OF SMOOTH PLANE CURVES WITH OUTER GALOIS POINTS

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1. Introduction

This article is based on my talk on December 18, 2016 and it concerns several works on automorphism groups of smooth plane curves with outer Galois points. These are joint works with A. Ohbuchi and partially with K. Miura.

Notation

- $\mathbb{Z}_m$: a cyclic group of order m;
- ζ_m: a primitive m-th root of unity.
- D_{2m}: the dihedral group of order $2m$;
- $\overline{D}_{2m} = Q_{4m} = \langle a, b \mid a^{2m} = 1, b^2 = a^m, b^{-1}ab = a^{-1}\rangle$: the binary dihedral subgroup of $SL(2,\mathbb{C})$ (the dicyclic group of order $4m$);
- $Q_8 = \langle a, b \mid a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1}\rangle = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1\rangle$: the quaternion group;
- $T \simeq A_4$, $O \simeq S_4$, $I \simeq A_5$: the tetrahedral, octahedral, icosahedral subgroups of $PGL(2,\mathbb{C})$;
- $\overline{T} \simeq SL(2,3)$, $\overline{O}$, $\overline{I} \simeq SL(2,5)$: the binary tetrahedral, binary octahedral, binary icosahedral subgroup of $SL(2,\mathbb{C})$;

In this paper

$$\begin{aligned}\mathrm{PBD}(2,1) &:= \left\{ \begin{pmatrix} A & \\ & \alpha \end{pmatrix} \;\middle|\; A \in GL(2,\mathbb{C}), \alpha \in \mathbb{C}^\times \right\} \Big/ \mathbb{C}^\times \\ &= \left\{ \begin{pmatrix} A & \\ & \alpha \end{pmatrix} \;\middle|\; A \in SL(2,\mathbb{C}), \alpha \in \mathbb{C}^\times \right\} \Big/ \{\pm E_3\} \\ &\subset PGL(3,\mathbb{C})\end{aligned}$$

and $\rho : \mathrm{PBD}(2,1) \to PGL(2,\mathbb{C})$ is the natural homomorphism.

Remark 1. The group ID of $\overline{O}$ is $[48, 28]$, which is not isomorphic to $GL(2,3)$ (its group ID is $[48, 29]$).

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First we state a classification of automorphism groups of smooth plane curves:

Theorem 2. [Ha, Theorem 2.1] *Let C be a smooth plane curve of degree $d \geq 4$, $G = \mathrm{Aut}(C)$. Then one of the following holds:*

(a-i) *G fixes a point on C and G is a cyclic group whose order is at most $d(d-1)$. Furthermore, if $d \geq 5$ and $|G| = d(d-1)$, then C is projectively equivalent to the curve $YZ^{d-1} + X^d + Y^d = 0$.*

(a-ii) *G fixes a point not lying on C and there exists a commutative diagram*

$$\begin{array}{ccccccccc} 1 \to & \mathbb{C}^* & \to & \mathrm{PBD}(2,1) & \xrightarrow{\rho} & PGL(2,\mathbb{C}) & \to 1 & \text{(exact)} \\ & \uparrow & & \uparrow & & \uparrow & & \\ 1 \to & N & \longrightarrow & G & \longrightarrow & G' & \to 1 & \text{(exact)}, \end{array}$$

where N is a cyclic group whose order is a factor of d and G' is conjugate to $\mathbb{Z}_m$, D_{2m}, T, O or I, where m is an integer at most $d-1$. Moreover, if $G' \simeq D_{2m}$, then $m|\, d-2$ or N is trivial. In particular $|G| \leq \max\{2d(d-2), 60d\}$.

(b-i) *(C, G) is a descendant of Fermat curve $F_d : X^d + Y^d + Z^d = 0$. In this case $G \subset \mathrm{Aut}(F_d)$. In particular, $|G| \leq 6d^2$.*

(b-ii) *(C, G) is a descendant of Klein curve $K_d : XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$. In this case $G \subset \mathrm{Aut}(K_d)$. In particular, $|G| \leq 3(d^2 - 3d + 3)$ if $d \geq 5$. On the other hand, $|G| \leq 168$ if $d = 4$.*

(c) *G is conjugate to one of the following subgroups of $PGL(3,\mathbb{C})$: the alternating group A_5 or A_6, the Klein group $PSL(2, 7)$, the Hessian group H_{216} of order 216 or its subgroup of order 36 or 72. In particular $|G| \leq 360$.*

In this article we consider smooth curves with outer Galois points. Most of them correspond to the special case of (a-ii).

We recall basic facts on smooth plane curves with Galois points from [Y].

Let C be a smooth plane curve of degree $d \geq 4$ with an outer Galois point P. Then C has a defining equation of the form $Z^d = F(X, Y)$ and $P = (0 : 0 : 1)$ under a suitable coordinate system. The polynomial F has no multiple factors.

Throughout this article we assume that C is not isomorphic to Fermat curve. Then P is the unique outer Galois point (see [Y, Proposition 4']). In particular $G = \mathrm{Aut}(C)$ fixes P. Hence the case (a-ii) occurs, i.e., $G \subset \mathrm{PBD}(2,1)$ and $G' := \rho(G)$ is a finite subgroup of $PGL(2,\mathbb{C})$. Furthermore, G' is one of the following groups: a cyclic group $\mathbb{Z}_m$ ($m \leq d-1$), a dihedral group D_{2m} ($m \mid d-2$), the tetrahedral group T, the octahedral group O or the icosahedral group I.

In this article we assume that G' is not cyclic, i.e., $G' = D_{2m}, T, O$ or I. Note that there exists a natural commutative diagram

$$\begin{array}{ccccccccc} 1 \longrightarrow & \mathbb{C}^\times & \longrightarrow & SL(2,\mathbb{C}) \times \mathbb{C}^\times & \longrightarrow & SL(2,\mathbb{C}) & \longrightarrow 1 \\ & \downarrow \wr & \circlearrowleft & \downarrow \varpi & \circlearrowleft & \downarrow \pi & \\ 1 \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathrm{PBD}(2,1) & \xrightarrow[\rho]{} & PGL(2,\mathbb{C}) & \longrightarrow 1, \end{array}$$

where π and ϖ are natural projections. This diagram induces another commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}_d & \longrightarrow & \widetilde{G} & \xrightarrow{\tilde{\rho}} & \widetilde{G'} & \longrightarrow & 1 \\
 & & \downarrow \wr & \circlearrowleft & \downarrow \varpi & \circlearrowleft & \downarrow \pi & & \\
1 & \longrightarrow & \mathbb{Z}_d & \longrightarrow & G & \xrightarrow[\rho]{} & G' & \longrightarrow & 1,
\end{array}$$

where $\widetilde{G} := \varpi^{-1}(G)$ and $\widetilde{G'} := \pi^{-1}(G')$. Then $\operatorname{Ker}\varpi = \{\pm E_3\}$, $\operatorname{Ker}\pi = \{\pm E_2\}$ and

$$\widetilde{G'} = \begin{cases} \overline{D}_{2m} & (\text{if } G' = D_{2m}) \\ \overline{T} & (\text{if } G' = T) \\ \overline{O} & (\text{if } G' = O) \\ \overline{I} & (\text{if } G' = I). \end{cases}$$

Furthermore, these groups are presented as follows:

- $\overline{D}_{2m} = Q_{4m} = \langle a_{2m}, b\rangle$, where
$$a_{2m} := \begin{pmatrix} \zeta_{2m} & 0 \\ 0 & \zeta_{2m}^{-1} \end{pmatrix}, \quad b := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$
- $\overline{T} = \langle a_4, b, c\rangle$, where
$$a_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad c := \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon^7 & \epsilon^7 \\ \epsilon_8^5 & \epsilon \end{pmatrix}$$
(ϵ is a primitive 8th root of unity).
- $\overline{O} = \langle a_8, b, c\rangle$, where
$$a_8 = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}.$$
- $\overline{I} = \langle a', b', c'\rangle$, where
$$a' := -\begin{pmatrix} \eta^3 & 0 \\ 0 & \eta^2 \end{pmatrix}, \quad b' := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$c' := \frac{1}{\eta^2 - \eta^3} \begin{pmatrix} \eta + \eta^{-1} & 1 \\ 1 & -(\eta + \eta^{-1}) \end{pmatrix}$$
(η is a primitive 5th root of unity).

2. Defining equations of curves

Our first main result is the following:

Theorem 3. *The polynomial $F(X,Y)$ is invariant under a subgroup $\widetilde{H} \subset \widetilde{G'}$ and invariant under $\widetilde{G'}$ up to multiplication by k-th roots of unity, where*

$$\widetilde{H} = \begin{cases} \langle a^2 \rangle & (\text{if } G' = D_{2m}) \\ \overline{V} = Q_8 & (\text{if } G' = T) \\ \overline{T} & (\text{if } G' = O) \\ \overline{I} & (\text{if } G' = I) \end{cases}$$

(a is one of generators of $\overline{D}_{2m} = Q_{4m}$ and $\overline{V}$ is the only normal subgroup of index three in $\overline{T}$) and

$$k = \begin{cases} 2 & (\textit{if } G' = D_{2m} \textit{ and } m \textit{ is even}) \\ 4 & (\textit{if } G' = D_{2m} \textit{ and } m \textit{ is odd}) \\ 3 & (\textit{if } G' = T) \\ 2 & (\textit{if } G' = O) \\ 1 & (\textit{if } G' = I). \end{cases}$$

Let R_0 be the invariant ring of $\widetilde{G'}$, i.e., $R_0 = \mathbb{C}[X,Y]^{\widetilde{G'}}$ and R the invariant ring of $\widetilde{H}$, i.e., $R = \mathbb{C}[X,Y]^{\widetilde{H}}$. Then $F(X,Y) \in R$. Note that $R_0 \subset R$ and R is an R_0-module. It is well known that

$$R_0 = \mathbb{C}[X,Y]^{\widetilde{G'}} = \begin{cases} \mathbb{C}[X^{2m}+Y^{2m}, (XY)^2, XY(X^{2m}-Y^{2m})] \\ \qquad (\text{if } G' = D_{2m},\ m \text{ is even}) \\ \mathbb{C}[X^{2m}-Y^{2m}, (XY)^2, XY(X^{2m}+Y^{2m})] \\ \qquad (\text{if } G' = D_{2m},\ m \text{ is odd}) \\ \mathbb{C}[f,g,h] \quad (\text{if } G' = T) \\ \mathbb{C}[g,f^2,fh] \quad (\text{if } G' = O) \\ \mathbb{C}[u,v,w] \quad (\text{if } G' = I), \end{cases}$$

where

$$\begin{aligned} & f = XY(X^4 - Y^4), \quad g = X^8 + 14X^4Y^4 + Y^8, \\ & h = X^{12} - 33(X^8Y^4 + X^4Y^8) + Y^{12}, \\ & \deg f = 6, \quad \deg g = 8, \quad \deg h = 12 \quad \text{and} \quad h^2 = g^3 - 108f^4, \\ & u = XY(X^{10} + 11X^5Y^5 - Y^{10}), \\ & v = X^{20} - 228(X^{15}Y^5 - X^5Y^{15}) + 494X^{10}Y^{10} + Y^{20}, \\ & w = X^{30} + 522(X^{25}Y^5 - X^5Y^{25}) - 10005(X^{20}Y^{10} + X^{10}Y^{20}) + Y^{30}, \\ & \deg u = 12, \quad \deg v = 20, \quad \deg w = 30 \quad \text{and} \quad w^2 = 1728u^5 - v^3. \end{aligned}$$

Furthermore, there exist two homogeneous polynomials g_1 and g_2 of degree 4 such that $g = g_1g_2$ and $h = g_1^3$. Concretely

$$g_1 = X^4 + 2\sqrt{3}iX^2Y^2 + Y^4, \quad g_2 = X^4 - 2\sqrt{3}iX^2Y^2 + Y^4$$

and $12\sqrt{3}if^2 - g_1^3 + g_2^3 = 0$ holds.

The invariant ring R of $\widetilde{H}$ is as follows:

$$R = \mathbb{C}[X,Y]^{\widetilde{H}} = \begin{cases} \mathbb{C}[X^m, XY, Y^m] = \mathbb{C}[p,q,r] & (\text{if } G' = D_{2m}) \\ \mathbb{C}[f, g_1, g_2] & (\text{if } G' = T) \\ \mathbb{C}[f,g,h] & (\text{if } G' = O) \\ \mathbb{C}[u,v,w] \ (= R_0) & (\text{if } G' = I), \end{cases}$$

where $p = X^m + Y^m$, $q = X^m - Y^m$, $r = XY$.

Our second main result is the following decomposition of R as an R_0-module.

Theorem 4. (1) *If $G' = D_{2m}$ (m is odd), T or O then R has a direct sum decomposition*

$$R = R_0 \oplus M_1 \oplus M_2 \oplus \cdots \oplus M_{k-1},$$

where

- $M_1 = R_0 p + R_0 qr$, $M_2 = R_0 q + R_0 pr + R_0 p^3$, $M_3 = R_0 r + R_o p^2$ (*if $G' = D_{2m}$ and m is odd*),
- $M_1 = R_0 g_1 + R_0 g_2^2$, $M_2 = R_0 g_2 + R_0 g_1^2$ (*if $G' = T$*),
- $M_1 = R_0 f + R_0 h$ (*if $G' = O$*).

(2) *If $G' = D_{2m}$ (m is even) then R has the direct sum decomposition*

$$R = R_0 \oplus (R_0 p + R_0 qr) \oplus (R_0 q + R_0 pr) \oplus (R_0 r + R_0 pq).$$

In each case, the polynomial $F(X,Y)$ is a homogeneous element of some direct summand.

Remark 5. In Theorem 4, all polynomials in a direct summand are multiplied by the same root of unity under the action of each element of $\widetilde{G'}$.

Corollary 6. *If d is odd, then G' is dihedral.*

For each direct summand in the above theorem, we can choose a homogeneous polynomial F without multiple factors such that the smooth plane curve C defined by the equation $Z^d = F(X,Y)$ has an outer Galois point and there exists an exact sequence

$$1 \to \mathbb{Z}_d \to \mathrm{Aut}(C) \to G' \to 1.$$

In general this exact sequence is not split.

3. The structure of automorphism groups: when $G' = I$

In the rest of this article we determine the structure of automorphism groups when G' is a polyhedral group, i.e., $G' = T, O$ or I.

In this section we assume that $G' = I$. Recall the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}_d & \longrightarrow & \widetilde{G} & \xrightarrow{\tilde{\rho}} & \overline{I} & \longrightarrow & 1 \\
 & & \downarrow \wr & \circlearrowleft & \downarrow \varpi & \circlearrowleft & \downarrow \pi & & \\
1 & \longrightarrow & \mathbb{Z}_d & \longrightarrow & G & \xrightarrow[\rho]{} & I & \longrightarrow & 1,
\end{array}$$

where $\operatorname{Ker} \varpi = \{\pm E_3\}$ and $\operatorname{Ker} \pi = \{\pm E_2\}$.

Let d_0 be the odd part of d, i.e., $d = 2^e d_0$ $(e \geqq 1)$ and $2 \nmid d_0$.

Let $\widetilde{G}_0$ be the subgroup of $\widetilde{G}$ generated by

$$\begin{pmatrix} a' & \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} b' & \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} c' & \\ & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & & \\ & 1 & \\ & & \zeta_{2^e} \end{pmatrix}.$$

Then it is clear that $-E_3 \in \widetilde{G}_0$ and $\widetilde{G} = \mathbb{Z}_{d_0} \times \widetilde{G}_0$. Hence $G = \widetilde{G}/\{\pm E_3\} \simeq \mathbb{Z}_{d_0} \times G_0$, where $G_0 := \widetilde{G}_0/\{\pm E_3\}$.

The structure of G_0 is as follows:

Theorem 7. *The group $G_0 = \mathbb{Z}_{2^e}{}^{\bullet} I$ is a non-split extension of I by $\mathbb{Z}_{2^e}$ and it is an extension of a cyclic group $\mathbb{Z}_{2^{e-1}}$ by $\overline{I}$. Precisely*

$$G_0 = \begin{cases} \overline{I} \quad (\text{if } e = 1) \\ \overline{I} \rtimes \mathbb{Z}_2 \quad (\text{if } e = 2) \\ \overline{I}{}^{\bullet} \mathbb{Z}_{2^{e-1}} \quad (\text{if } e \geq 3). \end{cases}$$

4. The structure of automorphism groups: when $G' = O$

In this section we assume that $G' = O$. Recall the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}_d & \longrightarrow & \widetilde{G} & \xrightarrow{\tilde{\rho}} & \overline{O} & \longrightarrow & 1 \\
 & & \downarrow \wr & \circlearrowleft & \downarrow \varpi & \circlearrowleft & \downarrow \pi & & \\
1 & \longrightarrow & \mathbb{Z}_d & \longrightarrow & G & \xrightarrow[\rho]{} & O & \longrightarrow & 1,
\end{array}$$

where $\operatorname{Ker} \varpi = \{\pm E_3\}$ and $\operatorname{Ker} \pi = \{\pm E_2\}$.

Let d_0 be the odd part of d, i.e., $d = 2^e d_0$ $(e \geqq 1)$ and $2 \nmid d_0$.

Note that $F((X,Y)a_8) = \mu F(X,Y)$ $(\mu = \pm 1)$. First we verify that $e = 1$ or 2 (i.e., $8 \nmid d$) if $\mu = -1$.

Suppose that $8 \mid d$. It follows from $F \in R = \mathbb{C}[f,g,h]$ and F is smooth that $d \equiv 6l + 8m + 12n \pmod{24}$ for some $l, m, n \in \mathbb{N}$ with $n \leq 1$. Since $8 \mid d$, we see that $8 \mid 6(l + 2n)$, which implies that $4 \mid l + 2n$. Thus $n = 0$ and $4 \mid l$ or $n = 1$ and $l \geq 2$. If the latter is the case, then F is singular. Hence $n = 0$ and $4 \mid l$, which shows that F is invariant under a_8, i.e., $\mu = 1$. We put

$$\nu := \begin{cases} 1 & (\text{if } \mu = 1) \\ i & (\text{if } \mu = -1 \text{ and } e = 1) \\ \epsilon & (\text{if } \mu = -1 \text{ and } e = 2). \end{cases}$$

Furthermore, if $\mu = 1$ and $e = 2$, then each term of F divided by f^2 or $(fh)^2$. Thus F is singular. Hence we can exclude the case.

Put

$$\widetilde{a} := \begin{pmatrix} a_8 & \\ & \nu \end{pmatrix}, \quad \widetilde{b} := \begin{pmatrix} b & \\ & 1 \end{pmatrix}, \quad \widetilde{c} := \begin{pmatrix} c & \\ & 1 \end{pmatrix} \quad \text{and} \quad \widetilde{h} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \zeta_{2^e} \end{pmatrix}.$$

Let

$$\widetilde{G}_0 := \begin{cases} \langle \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{h} \rangle & (\text{if } \nu = 1) \\ \langle \widetilde{a}, \widetilde{b}, \widetilde{c} \rangle & (\text{if } \nu = i \text{ or } \epsilon). \end{cases}$$

Note that $\widetilde{h} = (\widetilde{a}^2 \widetilde{c}^{-1})^9 \in \widetilde{G}_0$ if $\nu = i$ or ϵ. Then it is clear that $-E_3 \in \widetilde{G}_0$ and $\widetilde{G} = \mathbb{Z}_{d_0} \widetilde{G}_0$. Furthermore, $\mathbb{Z}_{d_0} \cap \widetilde{G}_0 = \{E_3\}$ since the order of $(3,3)$ elements of $\mathbb{Z}_{d_0}$ is odd and that of $\widetilde{G}_0$ is a power of two. Hence $\widetilde{G} = \mathbb{Z}_d \times \widetilde{G}_0$ and $G = \widetilde{G}/\{\pm E_3\} \simeq \mathbb{Z}_{d_0} \times G_0$, where $G_0 := \widetilde{G}_0/\{\pm E_3\}$.

The structure of G_0 is as follows:

Theorem 8. *The group $G_0 = \mathbb{Z}_{2^e}{}^\bullet O$ is a non-split extension of O by $\mathbb{Z}_{2^e}$ and*

$$G_0 = \begin{cases} \overline{O} & (\textit{if } \nu = 1 \textit{ and } e = 1) \\ (((\mathbb{Z}_{2^e} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2 & (\textit{if } \nu = 1 \textit{ and } e \geq 3) \\ GL(2,3) = \overline{T} \rtimes \mathbb{Z}_2 & (\textit{if } \nu = i) \\ \overline{T} \rtimes \mathbb{Z}_4 & (\textit{if } \nu = \epsilon). \end{cases}$$

Remark 9. If $\nu = 1$ and $e = 3$, then G_0 is also isomorphic to $(\overline{T} \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_2$.

5. The structure of automorphism groups: when $G' = T$

In this section we assume that $G' = T$. Recall the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}_d & \longrightarrow & \widetilde{G} & \xrightarrow{\tilde{\rho}} & \overline{T} & \longrightarrow & 1 \\ & & \downarrow \wr & \circlearrowleft & \downarrow \varpi & \circlearrowleft & \downarrow \pi & & \\ 1 & \longrightarrow & \mathbb{Z}_d & \longrightarrow & G & \xrightarrow[\rho]{} & T & \longrightarrow & 1, \end{array}$$

where $\operatorname{Ker} \varpi = \{\pm E_3\}$ and $\operatorname{Ker} \pi = \{\pm E_2\}$.

Let d_0 be the odd part of d, i.e., $d = 2^e d_0$ ($e \geqq 1$) and $2 \nmid d_0$. First note that $F((X,Y)c) = \omega F(X,Y)$ ($\omega^3 = 1$) and $\omega \neq 1$ since $F \in \mathbb{C}[f,g,h]$ and $G' = O$ if $\omega = 1$. Furthermore, F has a term in $\mathbb{C}[f,h]g_1$ or $\mathbb{C}[f,h]g_2$, which implies that $d \equiv 1 \pmod 3$.

Let $\widetilde{G}_0$ be the subgroup of $\widetilde{G}$ generated by

$$\begin{pmatrix} a_4 & \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} b & \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} c & \\ & \omega \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & & \\ & 1 & \\ & & \zeta_{2^{e_2}} \end{pmatrix}.$$

Then it is clear that $-E_3 \in \widetilde{G}_0$ and $\widetilde{G} = \mathbb{Z}_{d_0}\widetilde{G}_0$. Furthermore, $\mathbb{Z}_d \cap \widetilde{G}_0 = \{E_3\}$ since the order of $(3,3)$ elements of $\mathbb{Z}_{d_0}$ is 1 or 3 and that of $\widetilde{G}_0$ is a power of two. Hence $\widetilde{G} = \mathbb{Z}_d \times \widetilde{G}_0$ and $G = \widetilde{G}/\{\pm E_3\} \simeq \mathbb{Z}_{d_0} \times G_0$, where $G_0 := \widetilde{G}_0/\{\pm E_3\}$.

The structure of G_0 is as follows:

Theorem 10. *The group $G_0 = \mathbb{Z}_{2^e}{}^{\bullet}T$ is a non-split extension of T by $\mathbb{Z}_{2^e}$ and*

$$G_0 = \begin{cases} \overline{T} \ (\simeq Q_8 \rtimes \mathbb{Z}_3) & (\text{if } e = 1) \\ ((\mathbb{Z}_{2^e} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3 & (\text{if } e \geq 2). \end{cases}$$

Remark 11. When $e = 1$, the group $G_0 = \overline{T}$ is not isomorphic to $((\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3$ since $\overline{V} = Q_8$ is the unique normal subgroup of index three in $\overline{T}$ and it does not contain $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a normal subgroup.

When $e = 2$, the group G_0 is also isomorphic to $\overline{T} \rtimes \mathbb{Z}_2$.

6. Examples

Example 12. Let C be the plane curve of degree 12 defined by

$$Z^{12} = XY(X^{10} + 11X^5Y^5 - Y^{10}) \ (= u).$$

Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbb{Z}_{12} \to G \to I \to 1$$

and $G = \mathbb{Z}_3 \times (\overline{I} \rtimes \mathbb{Z}_2)$.

Example 13. Let C be the plane curve of degree 20 defined by

$$Z^{20} = X^{20} - 228(X^{15}Y^5 - X^5Y^{15}) + 494X^{10}Y^{10} + Y^{20} \ (= v).$$

Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbb{Z}_{20} \to G \to I \to 1$$

and $G = \mathbb{Z}_5 \times (\overline{I} \rtimes \mathbb{Z}_2)$.

Example 14. Let C be the plane curve of degree 30 defined by

$$Z^{30} = X^{30} + 522(X^{25}Y^5 - X^5Y^{25}) - 10005(X^{20}Y^{10} + X^{10}Y^{20}) + Y^{30} \ (= w).$$

Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbb{Z}_{30} \to G \to I \to 1$$

and $G = \mathbb{Z}_{15} \times \overline{I}$.

Example 15. Let C be the plane curve of degree 32 defined by $Z^{32} = uv$. Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbb{Z}_{32} \to G \to I \to 1$$

and $G = \overline{I}^{\bullet}\mathbb{Z}_{16}$.

Example 16. Let C be the sextic curve defined by $Z^6 = XY(X^4 - Y^4)$. Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbb{Z}_6 \to G \to O \to 1$$

and $G = \mathbb{Z}_3 \times GL(2,3)$ $(GL(2,3) \not\simeq \overline{O})$.

Example 17. Let C be the octic curve defined by $Z^8 = X^8 + 14X^4Y^4 + Y^8$ $(= g)$. Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbb{Z}_8 \to G \to O \to 1$$

and $G = (((\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \simeq (\overline{T} \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_2$.

Example 18. Let C be the plane curve of degree 12 defined by $Z^{12} = X^{12} - 33(X^8Y^4 + X^4Y^8) + Y^{12}$ $(= h)$. Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbb{Z}_{12} \to G \to O \to 1$$

and $G = \mathbb{Z}_3 \times (\overline{T} \rtimes \mathbb{Z}_4)$.

Example 19. Let C be the plane curve of degree 18 defined by $Z^{18} = fh$. Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbb{Z}_{18} \to G \to O \to 1$$

and $G = \mathbb{Z}_9 \times \overline{O}$.

Example 20. Let C be the plane curve of degree 32 defined by $Z^{32} = gF$, where F is a polynomial of degree 24 in $\mathbb{C}[f, g, h]$ such that gF has no multiple factor. Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbb{Z}_{32} \to G \to O \to 1$$

and $G = (((\mathbb{Z}_{32} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$.

Example 21. Let C be the quartic curve defined by $Z^4 = X^4 + 2\sqrt{3}iX^2Y^2 + Y^4$ $(= g_1)$. Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbb{Z}_4 \to G \to T \to 1$$

and $G = \overline{T} \rtimes \mathbb{Z}_2$.

Example 22. Let C be the the plane curve of degree 16 defined by $Z^{16} = g_1(g_1^3 + g_2^3)$. Then $G = \mathrm{Aut}(C)$ satisfies the (non-split) exact sequence

$$1 \to \mathbb{Z}_{16} \to G \to T \to 1$$

and $G = ((\mathbb{Z}_{16} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3$.

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