On automorphism groups and plane models of Riemann surfaces *

Takeshi Harui

Academic Support Center, Kogakuin University

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1 Introduction

The group of automorphisms of a Riemann surface is an old subject of research in algebraic geometry. In this article we investigate automorphism groups of Riemann surfaces via its plane models, i.e., plane algebraic curves birationally equivalent to it.

Except for the last section we consider smooth plane curves, in other words, Riemann surfaces embedded into a projective plane. Automorphism groups of smooth plane curves of degree at most three are classically well understood. So we study the cases of higher degree and consider the following problems:

Problem. (1) Classify automorphism groups of smooth plane curves.
(2) Give a sharp upper bound for the order of automorphism groups of such curves.
(3) Determine smooth plane curves with the group of automorphisms of large order.

We shall give a complete answer for each problem in Main Theorem 1, Main Theorem 2 and Main Theorem 3, respectively. These are the main results of this article.

Roughly speaking, Main Theorem 1 gives a classification of automorphism groups of smooth plane curves as follows: They are divided into five kinds. Groups of the first kind are cyclic and have a fixed point on the curves. The second kind consists of the central extension of finite subgroups of Möbius group $\mathrm{PGL}(2,\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1)$ by cyclic groups. Groups of the third (resp. the fourth) kind are subgroups of the full automorphism group of Fermat (resp. Klein) curves. The fifth kind consists of primitive subgroups of $\mathrm{PGL}(3,\mathbb{C})$.

*Automorphism groups of smooth plane curves, arXiv:math.AG/1306.5842 [H1]

For the order of automorphism groups of smooth plane curves, it is natural to expect that there exists a stronger upper bound than Hurwitz's one. We obtain such a bound in Main Theorem 2, which is a by-product of Main Theorem 1. We show that the order of the full automorphism group of a smooth plane curve $d \neq 4, 6$ is at most $6d^2$, which is attained by Fermat curves only. Moreover, a smooth plane curve with the full automorphism group of maximum order is unique up to projective equivalence for each degree.

Main Theorem 3, which is another by-product of Main Theorem 1, is a classification of smooth plane curves whose full automorphism group has large order in terms of defining equations.

In the last section we investigate automorphisms of Riemann surfaces induced by projective transformations of their plane models and we propose a theorem on the order of linear automorphism groups of irreducible plane curves, which is a generalization of Main Theorem 2.

2 Main results

In this article C denotes a smooth plane curve of degree $d \geq 4$ and G is a subgroup of $\mathrm{Aut}(C)$ unless otherwise mentioned. First note that C has a unique embedding into $\mathbb{P}^2$ up to projective equivalence, which implies that G is naturally considered as a subgroup of $\mathrm{PGL}(3, \mathbb{C}) = \mathrm{Aut}(\mathbb{P}^2)$.

Let F_d be Fermat curve $X^d + Y^d + Z^d = 0$ of degree d. In this article we denote by K_d a smooth plane curve defined by the equation $XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$, which is called *Klein curve* of degree d.

For a non-zero monomial $cX^iY^jZ^k$ we define its *exponent* as $\max\{i, j, k\}$. For a homogeneous polynomial F, the *core* of F is defined as the sum of all terms of F with the greatest exponent. A term of F is said to be *low* if it does not belong to the core of F.

Let C_0 be a smooth plane curve of degree at least four with a defining polynomial F_0. Then the pair (C, G) is said to be a *descendant* of C_0 if C is defined by a homogeneous polynomial whose core coincides with F_0 and G acts on C_0 under a suitable coordinate system. We simply call C a descendant of C_0 if $(C, \mathrm{Aut}(C))$ is a descendant of C_0.

We denote by $\mathrm{PBD}(2, 1)$ the subgroup of $\mathrm{PGL}(3, \mathbb{C})$ that consists of all elements representable by a 3×3 complex matrix A of the form

$$\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{A'}} & 0 \\ & & 0 \\ 0 & 0 & \alpha \end{pmatrix} \; (A' \text{ is a regular } 2 \times 2 \text{ matrix, } \alpha \in \mathbb{C}^*) \,.$$

There exists a natural group homomorphism $\rho : \mathrm{PBD}(2, 1) \to \mathrm{PGL}(2, \mathbb{C})$ $([A] \mapsto [A'])$, where $[M]$ denotes the equivalence class of a regular matrix M. Using these concepts we state our first main result as follows:

Main Theorem 1. *Let C be a smooth plane curve of degree $d \geq 4$, G a subgroup of $\mathrm{Aut}(C)$. Then one of the following holds:*

(a-i) *G fixes a point on C and G is a cyclic group whose order is at most $d(d-1)$. Furthermore, if $d \geq 5$ and $|G| = d(d-1)$, then C is projectively equivalent to the curve $YZ^{d-1} + X^d + Y^d = 0$ and $G = \mathrm{Aut}(C)$.*

(a-ii) *G fixes a point not lying on C and there exists a commutative diagram*

$$\begin{array}{ccccccccc} 1 \to & \mathbb{C}^* & \to & \mathrm{PBD}(2,1) & \overset{\rho}{\to} & \mathrm{PGL}(2,\mathbb{C}) & \to 1 \\ & \uparrow & & \uparrow & & \uparrow & \\ 1 \to & N & \longrightarrow & G & \longrightarrow & G' & \to 1 \end{array}$$

whose rows are exact sequences, where N is a cyclic group whose order is a factor of d and G' is conjugate to a cyclic group $\mathbb{Z}_m$, a dihedral group D_{2m}, the tetrahedral group A_4, the octahedral group S_4 or the icosahedral group A_5. Furthermore, $m \leq d-1$ and if $G' \simeq D_{2m}$ then $m | \, d-2$ or N is trivial. In particular $|G| \leq \max\{2d(d-2), 60d\}$.

(b-i) *(C, G) is a descendant of Fermat curve $F_d : X^d + Y^d + Z^d = 0$. In this case $|G| \leq 6d^2$.*

(b-ii) *(C, G) is a descendant of Klein curve $K_d : XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$. In this case $|G| \leq 3(d^2 - 3d + 3)$ if $d \geq 5$. On the other hand, $|G| \leq 168$ if $d = 4$.*

(c) *G is conjugate to a finite primitive subgroup of $\mathrm{PGL}(3,\mathbb{C})$, namely, the icosahedral group A_5, the Klein group $\mathrm{PSL}(2,\mathbb{F}_7)$, the alternating group A_6, the Hessian group H_{216} of order 216 or its subgroup of order 36 or 72. In particular $|G| \leq 360$.*

We make some remarks on this theorem.

Remark 2.1. (1) In the cases (a-i) and (a-ii), G fixes a point, say P. In fact, $G \subset \mathrm{PGL}(3,\mathbb{C})$ also fixes a line not passing through P (cf. Theorem 3.5).
(2) A point P in $\mathbb{P}^2$ is called a *Galois point* for C if the projection π_P from C to a line with center P is a Galois covering. A Galois point P for C is said to be *inner* (resp. *outer*) if $P \in C$ (resp. $P \notin C$). In the case (a-ii), if $|N| = d$ then the fixed point of G is an outer Galois point for C.
(3) The Klein group in the case (c) is the full automorphism group of Klein quartic and the alternating group A_6 is that of Wiman sextic (see Main Theorem 2). The Hessian group of order 216 is generated by the four elements h_i ($i = 1, 2, 3, 4$) represented by the following matrices (cf. [Bl]):

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix},$$

where ω is a primitive third root of unity. This group is the full automorphism group of a smooth plane sextic (see Remark 2.2 (2)). Its primitive subgroups of order 36 and 72 are respectively equal to $\langle h_1, h_2, h_3 \rangle$ and $\langle h_1, h_2, h_3, u \rangle$, where $u = h_1^{-1} h_4^2 h_1$.

As a corollary of Main Theorem 1, we obtain a sharp upper bound for the order of automorphism groups of smooth plane curves and classify the extremal cases.

Main Theorem 2. *Let C be a smooth plane curve of degree $d \geq 4$. Then $|\mathrm{Aut}(C)| \leq 6d^2$ except the following cases:*

(i) *$d = 4$ and C is projectively equivalent to Klein quartic $XY^3+YZ^3+ZX^3 = 0$. In this case $\mathrm{Aut}(C)$ is the Klein group $\mathrm{PSL}(2, \mathbb{F}_7)$, which is of order 168.*

(ii) *$d = 6$ and C is projectively equivalent to the sextic*

$$10X^3Y^3 + 9X^5Z + 9Y^5Z - 45X^2Y^2Z^2 - 135XYZ^4 + 27Z^6 = 0.$$

In this case $\mathrm{Aut}(C)$ is equal to A_6, a simple group of order 360.

Furthermore, for any $d \neq 6$, the equality $|\mathrm{Aut}(C)| = 6d^2$ holds if and only if C is projectively equivalent to Fermat curve $F_d : X^d + Y^d + Z^d = 0$, in which case $\mathrm{Aut}(C)$ is a semidirect product of S_3 acting on $\mathbb{Z}_d^2$. In particular, for each $d \geq 4$, there exists a unique smooth plane curve with the full group of automorphisms of maximum order up to projective equivalence.

Remark 2.2. (1) It is classically known that Klein quartic has the Klein group $\mathrm{PSL}(2, \mathbb{F}_7)$ as its group of automorphisms. For the sextic in the above theorem, Wiman [W] proved that its group of automorphisms is isomorphic to A_6. In [DIK] Doi, Idei and Kaneta called this curve *Wiman sextic* and showed that it is the only smooth plane sextic whose full automorphism group has the maximum order 360. We shall give a simpler proof on the uniqueness of Klein quartic (resp. Wiman sextic) as a smooth plane curve of degree four (resp. six) with the group of automorphisms of maximum order (see the proof of Proposition 5.1).
(2) When $d = 6$, the smooth plane sextic defined by the equation

$$X^6 + Y^6 + Z^6 - 10(X^3Y^3 + Y^3Z^3 + Z^3X^3) = 0$$

satisfies $|\mathrm{Aut}(C)| = 216 = 6^3$. In this case $\mathrm{Aut}(C)$ is equal to the Hessian group of order 216, therefore this curve is not a descendant of Fermat curve.
(3) In fact, a more general theorem holds true for linear automorphism groups of irreducible plane curves (see Theorem 6.1).

As another by-product of Main Theorem 1, we also give a stronger upper bound for the order of automorphism groups of smooth plane curves and classify the exceptional cases when $d \geq 60$:

Main Theorem 3. *Let C be a smooth plane curve of degree $d \geq 60$. Then $|\mathrm{Aut}(C)| \leq d^2$ unless C is projectively equivalent to one of the following curves:*

(i) *Fermat curve $F_d : X^d + Y^d + Z^d = 0$ ($|\mathrm{Aut}(F_d)| = 6d^2$).*

(ii) *Klein curve $K_d : XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$ ($|\mathrm{Aut}(K_d)| = 3(d^2 - 3d + 3)$).*

(iii) *The smooth plane curve defined by the equation*

$$Z^d + XY(X^{d-2} + Y^{d-2}) = 0,$$

in which case $|\mathrm{Aut}(C)| = 2d(d-2)$.

(iv) *The descendant of Fermat curve defined by the equation*

$$X^{3m} + Y^{3m} + Z^{3m} - 3\lambda X^m Y^m Z^m = 0,$$

where $d = 3m$ *and* λ *is a non-zero number with* $\lambda^3 \neq 1$. *In this case* $|\mathrm{Aut}(C)| = 2d^2$.

(v) *The descendant of Fermat curve defined by the equation*

$$X^{2m} + Y^{2m} + Z^{2m} + \lambda(X^m Y^m + Y^m Z^m + Z^m X^m) = 0,$$

where $d = 2m$ *and* $\lambda \neq 0, -1, \pm 2$. *In this case* $|\mathrm{Aut}(C)| = 6m^2 = (3/2)d^2$.

3 Preliminary results

Notation and Conventions

In this article we say that a group G acting on a set Ω *fixes* a subset $S \subset \Omega$ if $GS = S$.

We identify a regular matrix with the projective transformation represented by it if no confusion occurs. When a planar projective transformation fixes a smooth plane curve, it is also identified with the automorphism obtained by its restriction to the curve.

We denote by $[H_1, H_2, H_3]$ a planar projective transformation defined by $(X : Y : Z) \mapsto (H_1(X,Y,Z) : H_2(X,Y,Z) : H_3(X,Y,Z))$ for homogeneous linear polynomials H_1, H_2 and H_3.

A projective transformation of finite order is classically called a *homology* if it is defined by $[X, Y, \zeta Z]$ under a suitable coordinate system, where ζ is a root of unity. A nontrivial homology fixes a unique line pointwise and a unique point not lying the line.

A *triangle* means a set of three non-concurrent lines. Each line is called an *edge* of the triangle.

The line defined by the equation $X = 0$ (resp. $Y = 0$, $Z = 0$) will be denoted by L_1 (resp. L_2, L_3). We also denote by P_1 (resp. P_2 and P_3) the point $(1 : 0 : 0)$ (resp. $(0 : 1 : 0)$ and $(0 : 0 : 1)$).

For a positive integer m, we denote by $\mathbb{Z}_m$ (resp. $\mathbb{Z}_m^r$) a cyclic group of order m (resp. the direct product of r copies of $\mathbb{Z}_m$).

In this section C denotes a smooth irreducible projective curve of genus $g \geq 2$ defined over the field of complex numbers. Then the full group of its automorphisms is a finite group and we have a famous upper bound of its order, which is known as *Hurwitz bound*:

Theorem 3.1. (Hurwitz) *Let G be a subgroup of* $\mathrm{Aut}(C)$. *Then* $|G| \leq 84(g-1)$. *More precisely,*

$$\frac{|G|}{g-1} = 84,\ 48,\ 40,\ 36,\ 30\ \text{ or }\ \frac{132}{5} \quad \text{or} \quad \frac{|G|}{g-1} \leq 24.$$

Oikawa [O] and Arakawa [A] gave possibly stronger upper bounds under the assumption that G fixes finite subsets of C:

Theorem 3.2. ([O, Theorem 1], [A, Theorem 3]) *Let G be a subgroup of* $\mathrm{Aut}(C)$.

(1) (*Oikawa's inequality*) *If G fixes a finite subset S of C with $|S| = k \geq 1$, then $|G| \leq 12(g-1) + 6k$.*

(2) (*Arakawa's inequality*) *If G fixes three finite disjoint subsets S_i $(i = 1, 2, 3)$ of C with $|S_i| = k_i \geq 1$, then $|G| \leq 2(g-1) + k_1 + k_2 + k_3$.*

As an application of the former inequality, we can describe the structure of the full automorphism groups of Fermat curves and Klein curves, though the former was determined in a different way and the latter is also probably known.

Proposition 3.3. *Let d be an integer with $d \geq 4$. Then the full group of automorphisms of Fermat curve F_d is generated by the four transformations $[\zeta X, Y, Z]$, $[X, \zeta Y, Z]$, $[Y, Z, X]$ and $[X, Z, Y]$, where ζ is a primitive d-th root of unity. It is isomorphic to a semidirect product of S_3 acting on $\mathbb{Z}_d^2$, in other words, there exists a split short exact sequence of groups*

$$1 \to \mathbb{Z}_d^2 \to \mathrm{Aut}(F_d) \to S_3 \to 1.$$

In particular $|\mathrm{Aut}(F_d)| = 6d^2$.

Proposition 3.4. *If $d \geq 5$ then the full group of automorphisms of Klein curve*

$$K_d : XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$$

is generated by the two transformations $[\xi^{-(d-2)}X, \xi Y, Z]$ and $[Y, Z, X]$, where ξ is a primitive $(d^2 - 3d + 3)$-rd root of unity. It is isomorphic to a semidirect product of $\mathbb{Z}_3$ acting on $\mathbb{Z}_{d^2-3d+3}$, in other words, there exists a split short exact sequence of groups

$$1 \to \mathbb{Z}_{d^2-3d+3} \to \mathrm{Aut}(K_d) \to \mathbb{Z}_3 \to 1.$$

In particular $|\mathrm{Aut}(K_d)| = 3(d^2 - 3d + 3)$. *On the other hand,* $|\mathrm{Aut}(K_4)| = 168$.

In the end of this section, we refer to a theorem on finite groups of planar projective transformations. It is a basic tool to prove Main Theorem 1.

Theorem 3.5. ([M, Section 1-10], [DI, Theorem 4.8]) *Let G be a finite subgroup of* $\mathrm{PGL}(3, \mathbb{C})$. *Then one of the following holds:*

(a) *G fixes a line and a point not lying on the line;*

(b) *G fixes a triangle; or*

(c) *G is primitive and conjugate to the icosahedral group A_5, the Klein group* $\mathrm{PSL}(2, \mathbb{F}_7)$ (*of order* 168), *the alternating group A_6, the Hessian group H_{216} of order* 216 *or its subgroup of order* 36 *or* 72.

4 An outline of our proof of Main Theorem 1

In the following sections C denotes a smooth plane curve of degree $d \geq 4$ defined by a homogeneous polynomial F and let G be a subgroup of $\mathrm{Aut}(C)$, which is also considered as a subgroup of $\mathrm{PGL}(3, \mathbb{C})$. We identify an element σ of G with the corresponding planar projective transformation, which is also denoted by σ.

This section is wholly devoted to give a sketch of our proof of Main Theorem 1. From Theorem 3.5 there are three cases:

(A) G fixes a line and a point not lying on the line.

(B) G fixes a triangle and there exists neither a line nor a point fixed by G.

(C) G is primitive and conjugate to a group described in Theorem 3.5.

Note that the last case leads us to the statement (c) in Main Theorem 1. We argue the other cases one by one.

Case (A): G fixes a line L and a point P not lying on L.

We prove that the statement (a-i) (resp. (a-ii)) in Main Theorem 1 holds if $P \in C$ (resp. $P \notin C$). For the sake of simplicity, we omit to estimate the order of G.

If C passes through P, then G is cyclic. In this case (a-i) in Main Theorem 1 holds.

Assume that C does not pass through P. We may assume that L is defined by $Z = 0$ and $P = (0 : 0 : 1)$. Then G is a subgroup of $\mathrm{PBD}(2, 1)$. Hence there exists a short exact sequence of groups

$$1 \to N \to G \xrightarrow{\rho} G' \to 1,$$

where $\rho : \mathrm{PBD}(2, 1) \to \mathrm{PGL}(2, \mathbb{C})$ is a natural homomorphism, $N = \mathrm{Ker}\rho$ and $G' = \mathrm{Im}\rho$.

We show that N is a cyclic group. For each element η of N, there exists a unique diagonal matrix of the form $\mathrm{diag}(1, 1, \zeta)$ that represents η. Hence we have an injective homomorphism $\varphi : N \to \mathbb{C}^*$ $(\eta \mapsto \zeta)$, which implies that N is isomorphic to a finite subgroup of $\mathbb{C}^*$. It follows that N is cyclic.

On the other hand, it is well known that G', a finite subgroup of $\mathrm{PGL}(2, \mathbb{C})$, is isomorphic to $\mathbb{Z}_m$, D_{2m}, A_4, S_4 or A_5. Thus (a-ii) in Main Theorem 1 holds.

Next we show the statement (b-i) or (b-ii) in Main Theorem 1 holds in Case (B).

Case (B): G fixes a triangle Δ and there exists neither a line nor a point fixed by G.

We may assume that Δ consists of three lines $L_1 : X = 0$, $L_2 : Y = 0$ and $L_3 : Z = 0$. Let V be the set of vertices of Δ, i.e., $V = \{P_1, P_2, P_3\}$. Note that G acts on V transitively. Indeed, otherwise G fixes a line or a point, which conflicts with our assumption. It follows that either C and V are disjoint or C contains V.

We note a trivial but useful observation:

Observation. *Each element of G gives a permutation of the set $\{X, Y, Z\}$ of the coordinate functions up to constants.*

If C contains V, we denote by T_i the tangent line to C at P_i $(i = 1, 2, 3)$. Note that these lines are distinct and not concurrent by our assumption. Furthermore, G fixes the set $\{T_1, T_2, T_3\}$ and acts on it transitively. Thus Case (B) is divided into three subcases:

(B-1) C and V are disjoint.
(B-2) C contains V and each of T_i's $(i = 1, 2, 3)$ is an edge of Δ.
(B-3) C contains V and none of T_i's $(i = 1, 2, 3)$ is an edge of Δ.

Subcase (B-1): C and V are disjoint.

We show that (C, G) is a descendant of Fermat curve $F_d : X^d + Y^d + Z^d = 0$ in this subcase. By our assumption the defining polynomial F of C is of the form

$$F = aX^d + bY^d + cZ^d + (\text{low terms}) \quad (a, b, c \neq 0).$$

Furthermore we may assume that $a = b = c = 1$ after a suitable coordinate change if necessary. Then the core of F is $X^d + Y^d + Z^d$, which is fixed by G up to a constant from the above observation. It follows that G also acts on Fermat curve F_d, in other words, G is a subgroup of $\mathrm{Aut}(F_d)$. Thus we conclude that (C, G) is a descendant of F_d.

Subcase (B-2): C contains V and each T_i $(i = 1, 2, 3)$ is an edge of Δ.

We show that (C, G) is a descendant of Klein curve $K_d : XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$ in this subcase. Without loss of generality we may assume that $T_1 = L_3$, $T_2 = L_1$ and $T_3 = L_2$. Then the defining polynomial F of C is of the form

$$F = aXY^{d-1} + bYZ^{d-1} + cZX^{d-1} + (\text{low terms}) \quad (a, b, c \neq 0).$$

Again we may assume that $a = b = c = 1$ after a suitable coordinate change if necessary. Then the core of F is $XY^{d-1} + YZ^{d-1} + ZX^{d-1}$, which is fixed by G up to a constant from the above observation. Hence G also acts on Klein curve K_d, that is to say, G is a subgroup of $\mathrm{Aut}(K_d)$. Thus (C, G) is a descendant of K_d.

Subcase (B-3): C contains V and no T_i $(i = 1, 2, 3)$ is an edge of Δ.

We exclude this subcase. Let $V' = \{P'_1, P'_2, P'_3\}$ be the set of the intersection points of T_1, T_2 and T_3, where P'_i is the intersection point of T_j and T_k with $\{i, j, k\} = \{1, 2, 3\}$. These points are pairwise distinct. Indeed, otherwise T_1, T_2 and T_3 are concurrent and G fixes the intersection point of them, which conflicts with our assumption. Thus T_1, T_2 and T_3 constitute a triangle Δ', which is fixed by G and V' is the set of its vertices. Furthermore, V and V' are disjoint by our assumption.

Every element $\sigma \in G$ can be written in the form $\sigma = [\alpha X_i, \beta X_j, \gamma X_k]$ for some constants α, β and γ, where $\{i, j, k\} = \{1, 2, 3\}$, $X_1 = X$, $X_2 = Y$ and $X_3 = Z$. Hence we have a natural homomorphism $\rho : G \to S_3$ defined by

$$\rho(\sigma) = \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}.$$

Then $\mathrm{Im}\rho$ is isomorphic to $\mathbb{Z}_3$ or S_3, since there exists neither a line nor a point fixed by G. Furthermore, $\mathrm{Ker}\rho$ is trivial. Indeed, any element of $\mathrm{Ker}\rho$ can be written in the form $[\alpha X, \beta Y, Z]$ ($\alpha, \beta \neq 0$). Hence it fixes V pointwise, which implies that it fixes V' also pointwise. It is easy to show that such a planar projective transformation is trivial. Thus $G \simeq \mathrm{Im}\rho \simeq \mathbb{Z}_3$ or S_3.

If G is isomorphic to $\mathbb{Z}_3$, then G fixes a line, which contradicts our assumption. Thus G is isomorphic to S_3. Hence G is generated by $\eta = [Y, Z, X]$ and another element τ of order two with $\tau\eta\tau = \eta^{-1}$ after a suitable coordinate change if necessary. Then we may assume that $\tau = [\omega Y, \omega^{-1} X, Z]$ ($\omega^3 = 1$). Both η and τ fixes the same point $(1 : \omega^2 : \omega)$. Therefore G also fixes this point, which is a contradiction. It follows that this subcase is excluded.

Thus we complete the proof of Main Theorem 1 thoroughly.

5 An outline of our proof of Main Theorem 2 and 3

In this section we describe an outline of our proof of Main Theorem 2 and Main Theorem 3. First we consider primitive groups acting on smooth plane curves.

Proposition 5.1. *Let C be a smooth plane curve of degree $d \geq 4$, G a finite subgroup of* $\mathrm{Aut}(C)$*. If G is primitive, then $|G| \leq 6d^2$ except the following cases:*

(i) *$d = 4$ and C is projectively equivalent to Klein quartic $XY^3 + YZ^3 + ZX^3 = 0$ and $G \simeq \mathrm{Aut}(K_4) \simeq \mathrm{PSL}(2, \mathbb{F}_7)$.*

(ii) *$d = 6$ and C is projectively equivalent to Wiman sextic W_6, which is defined by*

$$10X^3Y^3 + 9ZX^5 + 9Y^5Z - 45X^2Y^2Z^2 - 135XYZ^4 + 27Z^6 = 0$$

and $G \simeq \mathrm{Aut}(W_6) \simeq A_6$.

Proof. First note that $\mathrm{Aut}(C)$ is also primitive, which implies that $|G| \leq |\mathrm{Aut}(C)| \leq 360$ by Theorem 3.5. Hence $|G| < 6d^2$ if $d \geq 8$.

Assume that $d \leq 7$. If $d = 5$ or 7, then we have the inequality $|G| < 6d^2$ except for $(d, |G|) = (5, 168)$, $(5, 216)$, $(5, 360)$ or $(7, 360)$ again by Theorem 3.5. It is easy to check by Theorem 3.1 that these four exceptional cases do not occur.

Assume that $d = 4$. We then have the inequality $|G| \leq 168$ by Hurwitz's theorem. If $|G| < 168$, then $|G| \leq 72 < 6d^2$ holds by Theorem 3.5. Suppose that $|G| = 168$

and C is not projectively equivalent to Klein quartic K_4. Then G is conjugate to the Klein group $\mathrm{PSL}(2, \mathbb{F}_7)$. Hence we may assume that G acts on both C and K_4. In particular $C \cap K_4$ is fixed by G. This is a non-empty subset of C of order at most $4^2 = 16$ by virtue of Bézout's theorem. It follows from Oikawa's inequality that $168 = |G| \leq 12 \cdot 2 + 6 \cdot 16 = 120$, a contradiction.

Next assume that $d = 6$. If $|G| < 360$, then $|G| \leq 216 = 6d^2$ by Theorem 3.5. Suppose that $|G| = 360$ and C is not projectively equivalent to Wiman sextic W_6. Then G is conjugate to A_6. Hence we may assume that G acts on both C and W_6. It follows from Bézout's theorem again that $C \cap W_6$ is a non-empty subset of C of order at most $6^2 = 36$, which is fixed by G. Again applying Oikawa's inequality we come to the conclusion that $360 = |G| \leq 12 \cdot 9 + 6 \cdot 36 = 324$, a contradiction. □

We show Main Theorem 2 by using the above proposition, Theorem 3.5 and Oikawa's inequality.

Proof of Main Theorem 2. We may assume that $\mathrm{Aut}(C)$ is not primitive by virtue of Proposition 5.1. Then it follows from Theorem 3.5 that $\mathrm{Aut}(C)$ fixes a line or a triangle. First suppose that $\mathrm{Aut}(C)$ fixes a line L. Then $S := C \cap L$ is a non-empty set of order at most d, which is also fixed by $\mathrm{Aut}(C)$. Hence Oikawa's inequality implies that

$$|\mathrm{Aut}(C)| \leq 12(g-1) + 6|S| \leq 6d(d-3) + 6d = 6d(d-2) < 6d^2.$$

Next suppose that $\mathrm{Aut}(C)$ fixes a triangle Δ. Then $C \cap \Delta$ is a non-empty set of order at most $3d$, which is also fixed by $\mathrm{Aut}(C)$. Thus we have the inequality $|\mathrm{Aut}(C)| \leq 6d^2$ by the same argument as above.

Finally assume that $|\mathrm{Aut}(C)| = 6d^2$ and $d \neq 6$. From Proposition 5.1 and the above argument $\mathrm{Aut}(C)$ fixes a triangle and does not fix a line. Then C is a descendant of Fermat curve F_d by virtue of Main Theorem 1. Comparing the order of two groups we know that $G = \mathrm{Aut}(F_d)$. Let $cX^iY^jZ^k$ ($c \neq 0, i+j+k = d$) be a term of F. Note that $[\zeta X, Y, Z]$ and $[X, \zeta Y, Z]$ (ζ is a primitive d-th root of unity), which are elements of G, preserve F. Hence they also preserve the monomial $cX^iY^jZ^k$. Then $\zeta^i = \zeta^j = 1$ holds, which implies that $(i,j,k) = (d,0,0)$, $(0,d,0)$ or $(0,0,d)$. It follows that $F = X^d + Y^d + Z^d$. □

In the rest of this section we give a sketch of our proof of Main Theorem 3. First we describe the full automorphism groups of curves in three exceptional cases (iii), (iv) and (v) in the theorem without proof.

Proposition 5.2. *Assume that $d \geq 4$ and C is the smooth plane curve defined by the equation $Z^d + XY(X^{d-2} + Y^{d-2}) = 0$.*

(i) *If $d \neq 4, 6$, then $\mathrm{Aut}(C)$ is a central extension of $D_{2(d-2)}$ by $\mathbb{Z}_d$. In particular $|\mathrm{Aut}(C)| = 2d(d-2)$.*
(ii) *If $d = 6$, $\mathrm{Aut}(C)$ is a central extension of S_4 by $\mathbb{Z}_6$. In particular $|\mathrm{Aut}(C)| = 144$.*

(iii) *If* $d = 4$*, then* C *is isomorphic to Fermat quartic* F_4*. In particular* $\mathrm{Aut}(C) \simeq \mathbb{Z}_4^2 \rtimes S_3$ $(|\mathrm{Aut}(C)| = 96)$*.*

Proposition 5.3. *For a positive integer* $d = 3m \geq 6$*, let* F_d' *be the smooth plane curve defined by*

$$X^{3m} + Y^{3m} + Z^{3m} - 3\lambda X^m Y^m Z^m = 0,$$

where λ *is a non-zero number with* $\lambda^3 \neq 1$*. It is a descendant of Fermat curve* F_d *and* $\mathrm{Aut}(F_d')$ *is generated by the five transformations* $[\zeta^3 X, Y, Z]$*,* $[X, \zeta^3 Y, Z]$*,* $[\zeta X, \zeta^{-1} Y, Z]$*,* $[Y, Z, X]$ *and* $[Y, X, Z]$*, where* ζ *is a primitive* d*-th root of unity. In this case* $|\mathrm{Aut}(C)| = 2d^2$*.*

Proposition 5.4. *For a positive even integer* $d = 2m \geq 8$*, let* F_d'' *be the smooth plane curve defined by*

$$X^{2m} + Y^{2m} + Z^{2m} + \lambda(X^m Y^m + Y^m Z^m + Z^m X^m) = 0,$$

where $\lambda \neq 0, -1, \pm 2$*. It is a descendant of Fermat curve* F_d *and* $\mathrm{Aut}(F_d'')$ *is generated by the four transformations* $[\zeta^2 X, Y, Z]$*,* $[X, \zeta^2 Y, Z]$*,* $[Y, Z, X]$ *and* $[Y, X, Z]$*, where* ζ *is a primitive* d*-th root of unity. It is isomorphic to a semidirect product of* S_3 *acting on* $\mathbb{Z}_m^2$*, in other words, there exists a split short exact sequence of groups*

$$1 \to \mathbb{Z}_m^2 \to \mathrm{Aut}(F_d'') \to S_3 \to 1.$$

In particular $|\mathrm{Aut}(F_d')| = 6m^2 = (3/2)d^2$*.*

Furthermore we obtain a characterization of these descendants of Fermat curve.

Lemma 5.5. *Two curves* F_d' *and* F_d'' *are the only descendants of Fermat curve* F_d *whose group of automorphisms has order greater than* d^2 *up to projective equivalence, except* F_d *itself.*

We also need the uniqueness of smooth plane curves of degree d whose full automorphism group is of order $3(d^2 - 3d + 3)$.

Proposition 5.6. *Let* C *be a smooth plane curve of degree* $d \geq 5$*,* G *a subgroup of* $\mathrm{Aut}(C)$*. Assume that* $|G| = 3(d^2 - 3d + 3)$*. Then* C *is projectively equivalent to Klein curve* K_d *and* $G = \mathrm{Aut}(K_d)$*.*

By using these facts we can prove Main Theorem 3.

Proof of Main Theorem 3. Let C be a smooth plane curve of degree $d \geq 60$, F a defining homogeneous polynomial of C. Assume that a subgroup G of $\mathrm{Aut}(C)$ is of order greater than d^2.

Since $d \geq 60$, we have the inequalities $|G| > 60d$ and $|G| > 360$. Then there are only three possibilities from Main Theorem 1:

(i) G fixes a point P not lying on C and G is isomorphic to a central extension of $D_{2(d-2)}$ by $\mathbb{Z}_d$.
(ii) (C, G) is a descendant of Fermat curve $F_d : X^d + Y^d + Z^d = 0$.
(iii) (C, G) is a descendant of Klein curve $K_d : XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$.

Case (i) In this case G also fixes a line L not containing P. We may assume that $P = (0 : 0 : 1)$ and L is defined by $Z = 0$. Then G is generated by the three elements $\eta = [X, Y, \zeta Z]$, $\sigma = [X, \omega Y, \omega' Z]$ and $\tau = [\gamma Y, \gamma X, Z]$, where ζ, ω, ω' and γ are roots of unity and the order of ζ (resp. ω) is d (resp. $d - 2$). Since η preserves F up to a constant, F is written as $F = Z^d + \hat{F}(X, Y)$, where $\hat{F}(X, Y)$ is a homogeneous polynomial of X and Y without multiple factors. Furthermore, we can show that C intersects L transversally at $P_1 = (1 : 0 : 0)$ and $P_2 = (0 : 1 : 0)$, respectively. It follows that $\hat{F}(X, Y)$ has a factor of the form $X - cY$ $(c \neq 0)$. Since σ preserves $\hat{F}(X, Y)$ up to a constant, we conclude that $\hat{F}(X, Y) = \lambda XY\Pi_{k=0}^{d-3}(X - \omega^k cY) = \lambda XY(X^{d-2} - c^{d-2}Y^{d-2})$ $(\lambda \in \mathbb{C}^*)$. Thus it is clear that C is projectively equivalent to the curve defined by $Z^d + XY(X^{d-2} + Y^{d-2}) = 0$.

Case (ii) From Lemma 5.5 we see that C is projectively equivalent to F_d, F'_d or F''_d in this case.

Case (iii) In this case G is a subgroup of $\mathrm{Aut}(K_d)$. Since $\mathrm{Aut}(K_d)$ has an odd order $3(d^2 - 3d + 3)$, we see that $G = \mathrm{Aut}(K_d)$ by our assumption that $|G| > d^2$. It follows from Proposition 5.6 that C is projectively equivalent to Klein curve K_d. □

6 On linear automorphism groups of plane curves

Let X be a Riemann surface of genus at least two, Γ a plane model of X, i.e., an irreducible plane curve birationally equivalent to X. Then $d := \deg\Gamma \geq 4$.

We denote by $\mathrm{Lin}(\Gamma)$ the group of linear automorphisms of Γ, that is to say, the subgroup of $\mathrm{PGL}(3, \mathbb{C})$ consisting of projective transformations preserving Γ. It is naturally considered as a subgroup of $\mathrm{Aut}(X)$. In particular it is a finite group by our assumption. We close this article by proposing a recent result on the order of $\mathrm{Lin}(\Gamma)$ (cf. [H2]) without proof:

Theorem 6.1. *The order of the group* $\mathrm{Lin}(\Gamma)$ *is at most* $6d^2$ *unless* Γ *is projectively equivalent to Klein quartic* K_4 $(d = 4)$ *or Wiman sextic* W_6 $(d = 6)$*. Furthermore,* $|\mathrm{Lin}(\Gamma)| = 6d^2$ *if and only if* Γ *is projectively equivalent to Fermat curve* F_d *or Hessian sextic* $H_6 : X^6 + Y^6 + Z^6 - 10(X^3Y^3 + Y^3Z^3 + Z^3X^3) = 0$ $(d = 6)$*.*

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Takeshi Harui
Academic Support Center, Kogakuin University
Nakano, Hachioji, Tokyo 192-0015, Japan
e-mail address: takeshi@cwo.zaq.ne.jp
kt13459@ns.kogakuin.ac.jp