On automorphism groups of smooth plane curves*

Takeshi Harui

January 31, 2014

This article is a summary of several results of the author on automorphism groups of smooth plane curves.

1 Introduction

The group of automorphisms of an algebraic curve defined over the complex number field is an old subject of research in algebraic geometry. We consider the following problem in this article:

Problem. Classify automorphism groups of smooth plane curves.

The answer is classically known for the cases of degree at most three. Therefore we argue about smooth plane curves of degree at least four.

We shall give a complete answer for this problem in Main Theorem 1. Roughly speaking, smooth plane curves are divided into five kinds by their full automorphism group. Curves of the first kind is smooth plane curves whose full automorphism group is cyclic. The second kind consists of curves whose full automorphism group is the central extension of a finite subgroup of Möbius group $\mathrm{PGL}(2, \mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1)$ by a cyclic group. Curves of the third (resp. the fourth) kind are descendants of Fermat (resp. Klein) curves (see Section 2 for the definition of this concept). For curves of the fifth kind, their full automorphism group is isomorphic to a primitive subgroup of $\mathrm{PGL}(3, \mathbb{C})$. It seems surprising that a smooth plane curve is a descendant of Fermat curve or Klein curve unless its full automorphism group is primitive or has a fixed point in the plane.

There are several by-products of Main Theorem 1 on automorphism groups of smooth plane curves. We obtain, for example, a sharp upper bound of the order of such groups in Main Theorem 2. For the order of automorphism groups of smooth plane curves, it is natural to expect that there exists a stronger upper bound than Hurwitz's one. Indeed, we show that the order of the full automorphism group of a smooth plane curve $d \neq 4, 6$ is at most $6d^2$, which is attained by Fermat curve.

*arXiv:math.AG/1306.5842

Moreover, a smooth plane curve with the full automorphism group of maximum order is unique for each degree up to projective equivalence.

We remark that Main Theorem 2 has been shown in several special cases: $d = 4$ (classical), $d = 6$ ([DIK]) and d is a prime at most 20 ([KMP]). Furthermore, Pambianco states the same theorem for $d \geq 8$ in his preprint [P], though his proof seems incomplete.

Our third result (Main Theorem 3) is a classification of smooth plane curves with automorphism groups of large order in terms of defining equations.

2 Main results

First of all, we note a simple fact on automorphism groups of smooth plane curves and introduce several concepts. Let G be a group of automorphisms of a smooth plane curve of degree at least four. Then it is naturally considered as a subgroup of $\mathrm{PGL}(3,\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^2)$.

Let F_d be Fermat curve $X^d + Y^d + Z^d = 0$ of degree d. In this article we denote by K_d a smooth plane curve defined by the equation $XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$, which is called *Klein curve* of degree d.

For a non-zero monomial $cX^iY^jZ^k$ we define its *exponent* as $\max\{i, j, k\}$. For a homogeneous polynomial F, the *core* of F is defined as the sum of all terms of F with the greatest exponent. A term of F is said to be *low* if it does not belong to the core of F.

Let C_0 be a smooth plane curve of degree at least four with a defining polynomial F_0. Then a pair (C, G) of a smooth plane curve C and a subgroup $G \subset \mathrm{Aut}(C)$ is said to be a *descendant* of C_0 if C is defined by a homogeneous polynomial whose core coincides with F_0 and G acts on C_0 under a suitable coordinate system. We simply call C a descendant of C_0 if $(C, \mathrm{Aut}(C))$ is a descendant of C_0.

In this article, we denote by $\mathrm{PBD}(2,1)$ the subgroup of $\mathrm{PGL}(3,\mathbb{C})$ that consists of all elements representable by a 3×3 complex matrix A of the form

$$\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{A'}} & 0 \\ & & 0 \\ 0 & 0 & \alpha \end{pmatrix} \ (A' \text{ is a regular } 2 \times 2 \text{ matrix}, \alpha \in \mathbb{C}^*).$$

There exists a natural group homomorphism $\rho : \mathrm{PBD}(2,1) \to \mathrm{PGL}(2,\mathbb{C})$ $([A] \mapsto [A'])$, where $[M]$ denotes the equivalence class of a matrix M. Using these concepts we state our first main result as follows:

Main Theorem 1. *Let C be a smooth plane curve of degree $d \geq 4$, G a subgroup of $\mathrm{Aut}(C)$. Then one of the following holds:*

(a-i) *G fixes a point on C and G is a cyclic group whose order is at most $d(d-1)$. Furthermore, if $d \geq 5$ and $|G| = d(d-1)$, then C is projectively equivalent to the curve $YZ^{d-1} + X^d + Y^d = 0$.*

(a-ii) *G fixes a point not lying on C and there exists a commutative diagram*

$$\begin{array}{ccccccccc} 1 \to \mathbb{C}^* & \to & \mathrm{PBD}(2,1) & \xrightarrow{\rho} & \mathrm{PGL}(2,\mathbb{C}) & \to 1 & \text{(exact)} \\ \uparrow & & \uparrow & & \uparrow & & \\ 1 \to N & \longrightarrow & G & \longrightarrow & G' & \to 1 & \text{(exact)}, \end{array}$$

where N is a cyclic group whose order is a factor of d and G' is conjugate to a cyclic group $\mathbb{Z}_m$, a dihedral group D_{2m}, the tetrahedral group A_4, the octahedral group S_4 or the icosahedral group A_5, where m is an integer at most $d-1$. Moreover, if $G' \simeq D_{2m}$, then $m \mid d-2$ or N is trivial. In particular $|G| \leq \max\{2d(d-2), 60d\}$.

(b-i) *(C,G) is a descendant of Fermat curve $F_d : X^d + Y^d + Z^d = 0$. In this case $|G| \leq 6d^2$.*

(b-ii) *(C,G) is a descendant of Klein curve $K_d : XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$. In this case $|G| \leq 3(d^2 - 3d + 3)$ if $d \geq 5$. On the other hand, $|G| \leq 168$ if $d = 4$.*

(c) *G is conjugate to a finite primitive subgroup of* $\mathrm{PGL}(3,\mathbb{C})$, *namely, the icosahedral group A_5, the Klein group* $\mathrm{PSL}(2,\mathbb{F}_7)$, *the alternating group A_6, the Hessian group H_{216} of order 216 or its subgroup of order 36 or 72. In particular $|G| \leq 360$.*

We make some remarks on this theorem.

Remark 2.1. (1) In the case (a-ii), if $|N| = d$ then the fixed point of G is an outer Galois point of C.
(2) The Klein group in the case (c) is the full automorphism group of Klein quartic and the alternating group A_6 is that of Wiman sextic (see Main Theorem 2). The Hessian group of order 216 is generated by the four elements h_i $(i = 1,2,3,4)$ represented by the following matrices:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix},$$

where ω is a primitive third root of unity. Its primitive subgroups of order 36 and 72 are respectively equal to $\langle h_1, h_2, h_3 \rangle$ and $\langle h_1, h_2, h_3, u \rangle$, where $u = h_1^{-1} h_4^2 h_1$.

As a corollary of Main Theorem 1, we obtain a sharp upper bound for the order of automorphism groups of smooth plane curves and classify the extremal cases.

Main Theorem 2. *Let C be a smooth plane curve of degree $d \geq 4$. Then $|\mathrm{Aut}(C)| \leq 6d^2$ except the following cases:*

(i) *$d = 4$ and C is projectively equivalent to Klein quartic $XY^3 + YZ^3 + ZX^3 = 0$. In this case $\mathrm{Aut}(C)$ is the Klein group $\mathrm{PSL}(2,\mathbb{F}_7)$, which is of order 168.*

(ii) $d = 6$ *and* C *is projectively equivalent to the sextic*

$$10X^3Y^3 + 9X^5Z + 9Y^5Z - 45X^2Y^2Z^2 - 135XYZ^4 + 27Z^6 = 0.$$

In this case $\mathrm{Aut}(C)$ *is equal to* A_6*, a simple group of order* 360.

Furthermore, for any $d \neq 6$*, the equality* $|\mathrm{Aut}(C)| = 6d^2$ *holds if and only if* C *is projectively equivalent to Fermat curve* $F_d : X^d + Y^d + Z^d = 0$*, in which case* $\mathrm{Aut}(C)$ *is a semidirect product of* S_3 *acting on* $\mathbb{Z}_d^2$*. In particular, for each* $d \geq 4$*, there exists a unique smooth plane curve with the full group of automorphisms of maximum order up to projective equivalence.*

Remark 2.2. (1) It is classically known that Klein quartic has the Klein group of order 168 as its group of automorphisms (see [Bl]). For the sextic in the above theorem, Wiman [W] proved that its group of automorphisms is isomorphic to A_6. In [DIK] Doi, Idei and Kaneta called this curve *Wiman sextic* and showed that it is the only smooth plane sextic whose full automorphism group has the maximum order 360. We shall give a simpler proof on the uniqueness of Klein quartic (resp. Wiman sextic) as a smooth plane curve of degree four (resp. six) with the group of automorphisms of maximum order.
(2) When $d = 6$, the smooth plane sextic defined by the equation

$$X^6 + Y^6 + Z^6 - 10(X^3Y^3 + Y^3Z^3 + Z^3X^3) = 0$$

also satisfies $|\mathrm{Aut}(C)| = 216 = 6^3$. In this case $\mathrm{Aut}(C)$ is equal to the Hessian group of order 216, therefore this curve is not a descendant of Fermat curve.

As another by-product of Main Theorem 1, we also give a stronger upper bound for the order of automorphism groups of smooth plane curves and classify the exceptional cases when $d \geq 60$:

Main Theorem 3. *Let* C *be a smooth plane curve of degree* $d \geq 60$*. Then* $|\mathrm{Aut}(C)| \leq d^2$ *unless* C *is projectively equivalent to one of the following curves:*

(i) *Fermat curve* $F_d : X^d + Y^d + Z^d = 0$ ($|\mathrm{Aut}(F_d)| = 6d^2$).

(ii) *Klein curve* $K_d : XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$ ($|\mathrm{Aut}(K_d)| = 3(d^2 - 3d + 3)$).

(iii) *a smooth plane curve defined by the equation*

$$Z^d + XY(X^{d-2} + Y^{d-2}) = 0,$$

in which case $|\mathrm{Aut}(C)| = 2d(d-2)$.

(iv) *a descendant of Fermat curve defined by the equation*

$$X^{3m} + Y^{3m} + Z^{3m} - 3\lambda X^mY^mZ^m = 0,$$

where $d = 3m$ *and* λ *is a non-zero number with* $\lambda^3 \neq 1$*. In this case* $|\mathrm{Aut}(C)| = 2d^2$.

(v) *a descendant of Fermat curve defined by the equation*

$$X^{2m} + Y^{2m} + Z^{2m} + \lambda(X^m Y^m + Y^m Z^m + Z^m X^m) = 0,$$

where $d = 2m$ *and* $\lambda \neq 0, -1, \pm 2$. *In this case* $|\mathrm{Aut}(C)| = 6m^2 = (3/2)d^2$.

3 Preliminary results

Notation and Conventions

We identify a non-zero matrix with the projective transformation represented by the matrix if no confusion occurs. When a planar projective transformation preserves a smooth plane curve, it is also identified with the automorphism obtained by its restriction to the curve.

We denote by $[H_1(X,Y,Z), H_2(X,Y,Z), H_3(X,Y,Z)]$ a planar projective transformation defined by $(X : Y : Z) \mapsto (H_1(X,Y,Z) : H_2(X,Y,Z) : H_3(X,Y,Z))$, where H_1, H_2 and H_3 are homogeneous linear polynomials.

A projective transformation of finite order is classically called a *homology* if it is defined by $[X, Y, \zeta Z]$ under a suitable coordinate system, where ζ is a root of unity. A non-trivial homology fixes a unique line pointwise and a unique point not lying the line. They are respectively called its *axis* and *center*.

A *triangle* means a set of three non-concurrent lines. Each line is called an *edge* of the triangle.

The line defined by the equation $X = 0$ (resp. $Y = 0$, $Z = 0$) will be denoted by L_1 (resp. L_2, L_3). We also denote by P_1 (resp. P_2 and P_3) the point $(1 : 0 : 0)$ (resp. $(0 : 1 : 0)$ and $(0 : 0 : 1)$).

For a positive integer m, we denote by $\mathbb{Z}_m$ (resp. $\mathbb{Z}_m^r$) a cyclic group of order m (resp. the direct product of r copies of $\mathbb{Z}_m$).

In this section C denotes a smooth irreducible projective curve of genus $g \geq 2$ defined over the field of complex numbers. Then the full group of its automorphisms is a finite group and we have a famous upper bound of its order, which is known as *Hurwitz bound*:

Theorem 3.1. (Hurwitz) *Let G be a subgroup of* $\mathrm{Aut}(C)$. *Then* $|G| \leq 84(g-1)$. *More precisely,*

$$\frac{|G|}{g-1} = 84,\ 48,\ 40,\ 36,\ 30 \ \text{ or } \ \frac{132}{5} \quad \text{or} \quad \frac{|G|}{g-1} \leq 24.$$

Oikawa [O] and Arakawa [A] gave possibly stronger upper bounds under the assumption that G fixes finite subsets of C. The following theorem is an application of Riemann-Hurwitz formula:

Theorem 3.2. ([O, Theorem 1], [A, Theorem 3]) *Let G be a subgroup of* $\mathrm{Aut}(C)$.

(1) (*Oikawa's inequality*) *If G fixes a finite subset S of C, i.e., $GS = S$, with $|S| = k \geq 1$, then $|G| \leq 12(g-1) + 6k$.*

(2) (*Arakawa's inequality*) *If G fixes three finite disjoint subsets S_i $(i = 1, 2, 3)$ of C with $|S_i| = k_i \geq 1$, then $|G| \leq 2(g-1) + k_1 + k_2 + k_3$.*

As an application of the former inequality, we can describe the structure of the full automorphism groups of Fermat curves and Klein curves, though the former was determined by a different way and the latter is also probably known.

Proposition 3.3. *Let d be an integer with $d \geq 4$. Then the full group of automorphisms of Fermat curve F_d is generated by four transformations $[\zeta X, Y, Z]$, $[X, \zeta Y, Z]$, $[Y, Z, X]$ and $[X, Z, Y]$, where ζ is a primitive d-th root of unity. It is isomorphic to a semidirect product of S_3 acting on $\mathbb{Z}_d^2$, in other words, there exists a split short exact sequence of groups*

$$1 \to \mathbb{Z}_d^2 \to \mathrm{Aut}(F_d) \to S_3 \to 1.$$

In particular $|\mathrm{Aut}(F_d)| = 6d^2$.

Proposition 3.4. *If $d \geq 5$ then the full group of automorphisms of Klein curve*

$$K_d : XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$$

is generated by two transformations $[\xi^{-(d-2)}X, \xi Y, Z]$ and $[Y, Z, X]$, where ξ is a primitive $(d^2 - 3d + 3)$-rd root of unity. It is isomorphic to a semidirect product of $\mathbb{Z}_3$ acting on $\mathbb{Z}_{d^2-3d+3}$, in other words, there exists a split short exact sequence of groups

$$1 \to \mathbb{Z}_{d^2-3d+3} \to \mathrm{Aut}(K_d) \to \mathbb{Z}_3 \to 1.$$

In particular $|\mathrm{Aut}(K_d)| = 3(d^2 - 3d + 3)$. On the other hand, $|\mathrm{Aut}(K_4)| = 168$.

The following is a well-known classical result:

Proposition 3.5. *If a subgroup G of $\mathrm{Aut}(C)$ fixes a point on C, then G is cyclic.*

For cyclic groups of automorphisms of smooth plane curves, we have the following lemma.

Lemma 3.6. *Let C be a smooth plane curve of degree d, G a cyclic subgroup of $\mathrm{Aut}(C)$. Then $|G| \leq d^2$ holds. Furthermore, if G is generated by a homology, then $|G|$ is a factor of $d-1$ or d. The equality $|G| = d-1$ (resp. $|G| = d$) holds if and only if C has an inner (resp. outer) Galois point and G is the Galois group at the point.*

Combining Proposition 3.5 and Lemma 3.6, we can determine the group of automorphisms of the curve $YZ^{d-1} + X^d + Y^d = 0$.

Proposition 3.7. *For $d \geq 5$, let $F_{d,d-1}$ be the smooth plane curve defined by the equation $YZ^{d-1} + X^d + Y^d = 0$. Then $\mathrm{Aut}(F_{d,d-1})$ is isomorphic to a cyclic group of order $d(d-1)$.*

In the end of this section, we refer to a theorem on finite groups of planar projective transformations. It is a basic tool to prove Main Theorem 1.

Theorem 3.8. ([M, Section 1-10], [DI, Theorem 4.8]) *Let G be a finite subgroup of $\mathrm{PGL}(3, \mathbb{C})$. Then one of the following holds:*

(a) *G fixes a line and a point not lying on the line;*

(b) *G fixes a triangle; or*

(c) *G is primitive and conjugate to the icosahedral group A_5, the Klein group $\mathrm{PSL}(2, \mathbb{F}_7)$ (of order 168), the alternating group A_6, the Hessian group H_{216} of order 216 or its subgroup of order 36 or 72.*

Remark 3.9. To be precise, Mitchell [M] proved that G fixes a point, a line or a triangle unless G is primitive and isomorphic to a group as in the case (c). In fact, the first two cases are equivalent. Indeed, if G fixes a point (resp. a line) then G also fixes a line not passing through the point (resp. a point not lying the line). It is a direct consequence of Maschke's theorem in group representation theory. Combining this fact with Mitchell's result we obtain the above theorem.

4 Outline of proof of Main Theorem 1

In the following sections C denotes a smooth plane curve of degree $d \geq 4$ defined by a homogeneous polynomial F and let G be a finite subgroup of $\mathrm{Aut}(C)$, which is also considered as a subgroup of $\mathrm{PGL}(3, \mathbb{C})$. We identify an element σ of G with the corresponding planar projective transformation, which is also denoted by σ.

This section is wholly devoted to prove Main Theorem 1. From Theorem 3.8 there are three cases:

(A) G fixes a line and a point not lying on the line.

(B) G fixes a triangle and there exist neither a line nor a point fixed by G.

(C) G is primitive and conjugate to a group described in Theorem 3.8.

Note that the last case leads us to the statement (c) in Main Theorem 1. We argue the other cases one by one.

Case (A): G fixes a line L and a point P not lying on L.

We prove that the statement (a-i) (resp. (a-ii)) in Main Theorem 1 holds if $P \in C$ (resp. $P \notin C$). For the sake of simplicity, we omit to estimate the order of G.

If C passes through P, then G is cyclic by virtue of Proposition 3.5. Hence (a-i) in Main Theorem 1 holds.

Assume that C does not pass through P. We may assume that L is defined by $Z = 0$ and $P = (0 : 0 : 1)$. Then G is a subgroup of $\mathrm{PBD}(2, 1)$. Hence there exists a short exact sequence of groups

$$1 \to N \to G \stackrel{\rho}{\to} G' \to 1,$$

where $\rho : \mathrm{PBD}(2, 1) \to \mathrm{PGL}(2, \mathbb{C})$ is a natural map, $N = \mathrm{Ker}\rho$ and $G' = \mathrm{Im}\rho$.

We show that N is a cyclic group. For each element η of N, there exists a unique diagonal matrix of the form $\mathrm{diag}(1, 1, \zeta)$ that represents η. Hence we have an injective homomorphism $\varphi : N \to \mathbb{C}^*$ $(\eta \mapsto \zeta)$, which implies that N is isomorphic to a finite subgroup of $\mathbb{C}^*$. It follows that N is cyclic.

On the other hand, it is well known that G', a finite subgroup of $\mathrm{PGL}(2, \mathbb{C})$, is isomorphic to $\mathbb{Z}_m$, D_{2m}, A_4, S_4 or A_5. Thus (a-ii) in Main Theorem 1 holds.

Next we show the statement (b-i) or (b-ii) in Main Theorem 1 holds in Case (B).

Case (B): G fixes a triangle Δ and there exist neither a line nor a point fixed by G.

We may assume that Δ consists of three lines $L_1 : X = 0$, $L_2 : Y = 0$ and $L_3 : Z = 0$. Let V be the set of vertices of Δ, i.e., $V = \{P_1, P_2, P_3\}$. Then G acts on V transitively because otherwise G fixes a line or a point, which conflicts with our assumption. It follows that either C and V are disjoint or C contains V.

Let F be a defining homogeneous polynomial of C. We note a trivial but useful observation:

Observation. *Each element of G gives a permutation of the set $\{X, Y, Z\}$ of the coordinate functions up to constants.*

If C contains V, we denote by T_i the tangent line to C at P_i $(i = 1, 2, 3)$. Note that these lines are distinct and not concurrent by our assumption. Furthermore, G fixes the set $\{T_1, T_2, T_3\}$ and acts on it transitively. Thus Case (B) is divided into three subcases:

(B-1) C and V are disjoint.

(B-2) C contains V and each of T_i's $(i = 1, 2, 3)$ is an edge of Δ.

(B-3) C contains V and none of T_i's $(i = 1, 2, 3)$ is an edge of Δ.

Subcase (B-1): C and V are disjoint.

We show that (C, G) is a descendant of Fermat curve $F_d : X^d + Y^d + Z^d = 0$ in this subcase. By our assumption the defining polynomial F of C is of the form

$$F = aX^d + bY^d + cZ^d + (\text{low terms}) \quad (a, b, c \neq 0).$$

Furthermore we may assume that $a = b = c = 1$ after a suitable coordinate change if necessary. Then the core of F is $X^d + Y^d + Z^d$, which is fixed by G up to a constant from the above observation. It follows that G also acts on Fermat curve F_d, in other words, G is a subgroup of $\mathrm{Aut}(F_d)$. Thus we conclude that (C, G) is a descendant of F_d.

Subcase (B-2): C contains V and each T_i $(i = 1, 2, 3)$ is an edge of Δ.

We show that (C, G) is a descendant of Klein curve $K_d : XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$ in this subcase. Without loss of generality we may assume that $T_1 = L_3$, $T_2 = L_1$ and $T_3 = L_2$. Then the defining polynomial F of C is of the form

$$F = aXY^{d-1} + bYZ^{d-1} + cZX^{d-1} + (\text{low terms}) \quad (a, b, c \neq 0).$$

Again we may assume that $a = b = c = 1$ after a suitable coordinate change if necessary. Then the core of F is $XY^{d-1} + YZ^{d-1} + ZX^{d-1}$, which is fixed by G up to a constant from the above observation. Hence G also acts on Klein curve K_d, that is to say, G is a subgroup of $\mathrm{Aut}(K_d)$. Thus (C, G) is a descendant of K_d.

Subcase (B-3): C contains V and no T_i $(i = 1, 2, 3)$ is an edge of Δ.

We show that this subcase does not actually occur.

Let $V' = \{P'_1, P'_2, P'_3\}$ be the set of the intersection points of T_1, T_2 and T_3, where P'_i is the intersection point of T_j and T_k with $\{i, j, k\} = \{1, 2, 3\}$. They are pairwise distinct because otherwise T_1, T_2 and T_3 are concurrent and the intersection point of them is fixed by G, which conflicts with our assumption. Thus T_1, T_2 and T_3 constitute a triangle Δ', which is fixed by G and V' is the set of its vertices. Furthermore, V and V' are disjoint by our assumption.

Any element $\sigma \in G$ can be written in the form $\sigma = [\alpha X_i, \beta X_j, \gamma X_k]$ with some constants α, β and γ, where $\{i, j, k\} = \{1, 2, 3\}$, $X_1 = X$, $X_2 = Y$ and $X_3 = Z$. Hence we have a natural homomorphism $\rho : G \to S_3$ defined by

$$\rho(\sigma) = \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}.$$

Then $\mathrm{Im}\rho$ is isomorphic to $\mathbb{Z}_3$ or S_3, since there exist neither a line nor a point fixed by G. Furthermore, $\mathrm{Ker}\rho$ is trivial. Indeed, any element of $\mathrm{Ker}\rho$ can be written in the form $[\alpha X, \beta Y, Z]$ $(\alpha, \beta \neq 0)$. Hence it fixes V pointwise, which implies that it fixes V' also pointwise. It is easy to show that such a planar projective transformation is trivial. Thus $G \simeq \mathrm{Im}\rho \simeq \mathbb{Z}_3$ or S_3.

If G is isomorphic to $\mathbb{Z}_3$, then G fixes a line, which contradicts our assumption. Thus G is isomorphic to S_3. Hence G is generated by $\eta = [Y, Z, X]$ and another element τ of order two with $\tau\eta\tau = \eta^{-1}$ after a suitable coordinate change if necessary. Then we may assume that $\tau = [\omega Y, \omega^{-1} X, Z]$ $(\omega^3 = 1)$. Both η and τ fixes the same point $(1 : \omega^2 : \omega)$. Therefore G also fixes this point, which conflicts with our assumption. It follows that this subcase is excluded.

Thus we complete the proof of Main Theorem 1 thoroughly.

5 Outline of proof of Main Theorem 2 and 3

In this section we shall prove Main Theorem 2 and Main Theorem 3. First we consider primitive groups acting on smooth plane curves.

Proposition 5.1. *Let C be a smooth plane curve of degree $d \geq 4$, G a finite subgroup of* $\mathrm{Aut}(C)$*. If G is primitive, then $|G| \leq 6d^2$ except the following cases:*

(i) *$d = 4$ and C is projectively equivalent to Klein quartic $XY^3+YZ^3+ZX^3 = 0$ and $G \simeq \mathrm{Aut}(K_4) \simeq \mathrm{PSL}(2, \mathbb{F}_7)$.*

(ii) *$d = 6$ and C is projectively equivalent to Wiman sextic W_6, which is defined by*

$$10X^3Y^3 + 9ZX^5 + 9Y^5Z - 45X^2Y^2Z^2 - 135XYZ^4 + 27Z^6 = 0$$

and $G \simeq \mathrm{Aut}(W_6) \simeq A_6$.

Proof. First note that $\mathrm{Aut}(C)$ is also primitive, which implies that $|G| \leq |\mathrm{Aut}(C)| \leq 360$ by Theorem 3.8. Hence $|G| < 6d^2$ if $d \geq 8$.

Assume that $d \leq 7$. If $d = 5$ or 7, then we have the inequality $|G| < 6d^2$ except for $(d, |G|) = (5, 168)$, $(5, 216)$, $(5, 360)$ or $(7, 360)$ again by Theorem 3.8. It is easy to check by Theorem 3.1 that these four exceptional cases do not occur.

Assume that $d = 4$. We then have the inequality $|G| \leq 168$ by Hurwitz's theorem. If $|G| < 168$, then $|G| \leq 72 < 6d^2$ holds by Theorem 3.8. Suppose that $|G| = 168$ and C is not projectively equivalent to Klein quartic K_4. Then G is conjugate to the Klein group $\mathrm{PSL}(2, \mathbb{F}_7)$. Hence we may assume that G acts on both C and K_4. In particular $C \cap K_4$ is fixed by G. This is a non-empty subset of C of order at most $4^2 = 16$ by virtue of Bézout's theorem. It follows from Oikawa's inequality that $168 = |G| \leq 12 \cdot 2 + 6 \cdot 16 = 120$, a contradiction.

Next assume that $d = 6$. If $|G| < 360$, then $|G| \leq 216 = 6d^2$ by Theorem 3.8. Suppose that $|G| = 360$ and C is not projectively equivalent to Wiman sextic W_6. Then G is conjugate to A_6. Hence we may assume that G acts on both C and W_6. It follows from Bézout's theorem again that $C \cap W_6$ is a non-empty subset of C of order at most $6^2 = 36$, which is fixed by G. Again applying Oikawa's inequality we come to the conclusion that $360 = |G| \leq 12 \cdot 9 + 6 \cdot 36 = 324$, a contradiction. □

We show Main Theorem 2 by using Theorem 3.8 and Oikawa's inequality.

Proof of Main Theorem 2. We may assume that $\mathrm{Aut}(C)$ is not primitive by virtue of Proposition 5.1. Then it follows from Theorem 3.8 that $\mathrm{Aut}(C)$ fixes a line or a triangle. First suppose that $\mathrm{Aut}(C)$ fixes a line L. Then $S := C \cap L$ is a non-empty set of order at most d, which is also fixed by $\mathrm{Aut}(C)$. Applying Theorem 3.2 (1) we obtain the inequality

$$|\mathrm{Aut}(C)| \leq 12(g-1) + 6|S| \leq 6d(d-3) + 6d = 6d(d-2) < 6d^2.$$

Next suppose that $\mathrm{Aut}(C)$ fixes a triangle Δ. Then $C \cap \Delta$ is a non-empty set of order at most $3d$, which is also fixed by $\mathrm{Aut}(C)$. Thus we have the inequality $|\mathrm{Aut}(C)| \leq 6d^2$ by the same argument as above.

Finally assume that $|\mathrm{Aut}(C)| = 6d^2$ and $d \neq 6$. From Proposition 5.1 and the above argument $\mathrm{Aut}(C)$ fixes a triangle and does not fix a line. Then C is a descendant of Fermat curve F_d by virtue of Main Theorem 1. Comparing the order of two groups we know that $G = \mathrm{Aut}(F_d)$. Let $cX^iY^jZ^k$ $(c \neq 0, i+j+k = d)$ be a term of F. Note that $[\zeta X, Y, Z]$ and $[X, \zeta Y, Z]$ (ζ is a primitive d-th root of unity), which are elements of G, preserve F. Hence they also preserve the monomial $cX^iY^jZ^k$. Then $\zeta^i = \zeta^j = 1$ holds, which implies that $(i, j, k) = (d, 0, 0)$, $(0, d, 0)$ or $(0, 0, d)$. It follows that $F = X^d + Y^d + Z^d$. □

In the rest of this section we give a sketch of our proof of Main Theorem 3. First we describe the full automorphism groups of curves in three exceptional cases (iii), (iv) and (v) in the theorem without proof.

Proposition 5.2. *Assume that $d \geq 4$ and C is a smooth plane curve defined by the equation $Z^d + XY(X^{d-2} + Y^{d-2}) = 0$.*

(i) *If $d \neq 4, 6$, then $\mathrm{Aut}(C)$ is a central extension of $D_{2(d-2)}$ by $\mathbb{Z}_d$. In particular $|\mathrm{Aut}(C)| = 2d(d-2)$.*

(ii) *If $d = 6$, $\mathrm{Aut}(C)$ is a central extension of S_4 by $\mathbb{Z}_6$. In particular $|\mathrm{Aut}(C)| = 144$.*

(iii) *If $d = 4$, then C is isomorphic to Fermat quartic F_4. In particular $\mathrm{Aut}(C) \simeq \mathbb{Z}_4^2 \rtimes S_3$ ($|\mathrm{Aut}(C)| = 96$).*

Proposition 5.3. *For a positive integer $d = 3m$, let F_d' be the smooth plane curve defined by*

$$X^{3m} + Y^{3m} + Z^{3m} - 3\lambda X^m Y^m Z^m = 0,$$

where λ is a non-zero number with $\lambda^3 \neq 1$. It is a descendant of Fermat curve F_d and $\mathrm{Aut}(F_d')$ is generated by five transformations $[\zeta^3 X, Y, Z]$, $[X, \zeta^3 Y, Z]$, $[\zeta X, \zeta^{-1} Y, Z]$, $[Y, Z, X]$ and $[Y, X, Z]$, where ζ is a primitive d-th root of unity. In this case $|\mathrm{Aut}(C)| = 2d^2$.

Proposition 5.4. *For a positive even integer $d = 2m \geq 8$, let F_d'' be the smooth plane curve defined by*

$$X^{2m} + Y^{2m} + Z^{2m} + \lambda(X^m Y^m + Y^m Z^m + Z^m X^m) = 0,$$

where $\lambda \neq 0, -1, \pm 2$. It is a descendant of Fermat curve F_d and $\mathrm{Aut}(F_d'')$ is generated by four transformations $[\zeta^2 X, Y, Z]$, $[X, \zeta^2 Y, Z]$, $[Y, Z, X]$ and $[Y, X, Z]$, where ζ is a primitive d-th root of unity. It is isomorphic to a semidirect product of S_3 acting on $\mathbb{Z}_m^2$, in other words, there exists a split short exact sequence of groups

$$1 \to \mathbb{Z}_m^2 \to \mathrm{Aut}(F_d'') \to S_3 \to 1.$$

In particular $|\mathrm{Aut}(F_d')| = 6m^2 = (3/2)d^2$.

Furthermore we obtain a characterization of these descendants of Fermat curve.

Lemma 5.5. *Two curves F_d' and F_d'' are the only descendants of Fermat curve F_d whose group of automorphisms has order greater than d^2 up to projective equivalence, except F_d itself.*

We also need the uniqueness of smooth plane curve of degree d whose full automorphism group is of order $3(d^2 - 3d + 3)$.

Proposition 5.6. *Let C be a smooth plane curve of degree $d \geq 5$, G a subgroup of* $\mathrm{Aut}(C)$*. Assume that $|G| = 3(d^2 - 3d + 3)$. Then C is projectively equivalent to Klein curve K_d and $G = \mathrm{Aut}(K_d)$.*

By using these facts we can prove Main Theorem 3.

Proof of Main Theorem 3. Let C be a smooth plane curve of degree $d \geq 60$, F a defining homogeneous polynomial of C. Assume that a subgroup G of $\mathrm{Aut}(C)$ is of order greater than d^2.

Since $d \geq 60$, we have the inequalities $|G| > 60d$ and $|G| > 360$. Then there are only three possibilities from Main Theorem 1:

(i) G fixes a point P not lying on C and G is isomorphic to a central extension of $D_{2(d-2)}$ by $\mathbb{Z}_d$.

(ii) (C, G) is a descendant of Fermat curve $F_d : X^d + Y^d + Z^d = 0$.

(iii) (C, G) is a descendant of Klein curve $K_d : XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$.

Case (i) In this case G also fixes a line L not containing P. We may assume that $P = (0 : 0 : 1)$ and L is defined by $Z = 0$. Then G is generated by three elements $\eta = [X, Y, \zeta Z]$, $\sigma = [X, \omega Y, \omega' Z]$ and $\tau = [\gamma Y, \gamma X, Z]$, where ζ, ω, ω' and γ are certain roots of unity and the order of ζ (resp. ω) is d (resp. $d-2$). Since η preserves F up to a constant, F is written as $F = Z^d + \hat{F}(X, Y)$, where $\hat{F}(X, Y)$ is a homogeneous polynomial of X and Y without multiple factors. Furthermore, we can show that C intersects L transversally at $P_1 = (1 : 0 : 0)$ and $P_2 = (0 : 1 : 0)$ respectively. It follows that $\hat{F}(X, Y)$ has a factor of the form $X - cY$ $(c \neq 0)$. Since σ preserves $\hat{F}(X, Y)$ up to a constant, we conclude that $\hat{F}(X, Y) = \lambda XY \Pi_{k=0}^{d-3}(X - \omega^k cY) = \lambda XY(X^{d-2} - c^{d-2}Y^{d-2})$ $(\lambda \in \mathbb{C}^*)$. Thus it is clear that C is projectively equivalent to the curve defined by $Z^d + XY(X^{d-2} + Y^{d-2}) = 0$.

Case (ii) From Lemma 5.5 we know that C is projectively equivalent to F_d, F_d' or F_d'' in this case.

Case (iii) In this case G is a subgroup of $\mathrm{Aut}(K_d)$. Since $\mathrm{Aut}(K_d)$ has an odd order $3(d^2 - 3d + 3)$, we know that $G = \mathrm{Aut}(K_d)$ by our assumption that $|G| > d^2$. It follows from Proposition 5.6 that C is projectively equivalent to Klein curve K_d. □

Acknowledgments. The author expresses his sincere gratitude to the organizers of the conference for giving him an opportunity to talk and holding a nice conference.

References

[A] T. Arakawa, Automorphism groups of compact Riemann surfaces with invariant subsets, Osaka J. Math. **37**, No. 4 (2000), 823–846.

[Bl] H. Blichfeldt, Finite Collineation Groups: With an Introduction to the Theory of Groups of Operators and Substitution Groups, Univ. of Chicago Press, Chicago (1917).

[DI] I. Dolgachev and V. Iskovskikh, Finite subgroups of the plane Cremona group, Algebra, Arithmetic, and Geometry, Progress in Mathematics Volume **269** (2009), 443–548.

[DIK] H. Doi, K. Idei and H. Kaneta, Uniqueness of the most symmetric nonsingular plane sextics. Osaka J. Math. **37** no. 3 (2000), 667–687.

[KMP] H. Kaneta, S. Marcugini and F. Pambianco, The most symmetric nonsingular plane curves of degree $n \leq 20$, I, Geom. Dedicata **85** (2001), 317–334.

[M] H. H. Mitchell, Determination of the ordinary and modular ternary linear groups, Trans. Amer. Math. Soc. **12**, no. 2 (1911), 207–242.

[O] K. Oikawa, Notes on conformal mappings of a Riemann surface onto itself, Kodai Math. Sem. Rep. **8**, no. 1 (1956), 23–30.

[P] F. Pambianco, The Fermat curve $x^n + y^n + z^n$: the most symmetric non-singular algebraic plane curve, preprint.

[W] A. Wiman, Ueber eine einfache Gruppe von 360 ebenen Collineationen, Math. Ann. **47** no. 4 (1896), 531–556.

Takeshi Harui
Academic Support Center, Kogakuin University,
Nakano, Hachioji, Tokyo 192-0015, Japan
e-mail address: takeshi@cwo.zaq.ne.jp
kt13459@ns.kogakuin.ac.jp