SMOOTH PLANE CURVES WHOSE AUTOMORPHISM GROUP IS PRIMITIVE

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This article is based on my talk on December 14, 2018. In this article we show that, for a given finite primitive subgroup G of $PGL(3,\mathbb{C})$, there exists a smooth plane curve such that $\mathrm{Aut}(C) = G$. We also determine the number of quasi-Galois points for smooth plane curves whose automorphism group is primitive. These are joint work with A. Ohbuchi.

1. Introduction

Throughout this article C denotes a smooth plane curve of degree $d \geq 4$ defined over the complex number field $\mathbb{C}$. Note that $\mathrm{Aut}(C)$ is considered as a finite subgroup of $PGL(3,\mathbb{C})$.

In Theorem 1 we use the following notation:

$$\begin{aligned}\mathrm{PBD}(2,1) &:= \left\{ \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{A}} & 0 \\ & & 0 \\ 0 & 0 & \alpha \end{pmatrix} \;\middle|\; A \in GL(2,\mathbb{C}), \alpha \in \mathbb{C}^\times \right\} \Big/ \mathbb{C}^\times \\ &= \left\{ \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{A}} & 0 \\ & & 0 \\ 0 & 0 & \alpha \end{pmatrix} \;\middle|\; A \in SL(2,\mathbb{C}), \alpha \in \mathbb{C}^\times \right\} \Big/ \{\pm E_3\} \\ &\subset PGL(3,\mathbb{C}).\end{aligned}$$

Let $\rho\colon \mathrm{PBD}(2,1) \to PGL(2,\mathbb{C})$ be the natural homomorphism.

First we recall a classification of automorphism groups of smooth plane curves (cf. [H, Theorem 2.3]):

Theorem 1. *Let G be a subgroup of $\mathrm{Aut}(C)$. Then one of the following holds:*

(a-i) *G fixes a point on C and G is a cyclic group whose order is at most $d(d-1)$. Furthermore, if $d \geq 5$ and $|G| = d(d-1)$, then C is projectively equivalent to the curve $YZ^{d-1} + X^d + Y^d = 0$.*

(a-ii) *G fixes a point not lying on C and there exists a commutative diagram*

$$\begin{array}{ccccccccc} 1 \to \mathbb{C}^* & \to & \mathrm{PBD}(2,1) & \xrightarrow{\rho} & PGL(2,\mathbb{C}) & \to 1 & \text{(exact)} \\ \uparrow & & \uparrow & & \uparrow & & \\ 1 \to \mathbb{Z}_n & \longrightarrow & G & \longrightarrow & G' & \to 1 & \text{(exact),} \end{array}$$

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where n is a factor of d and G' is conjugate to $\mathbb{Z}_m$, D_{2m} ($m \leq d-1$), A_4, S_4 or A_5. Furthermore, if $n \neq 1$ and $G' \simeq D_{2m}$ then $m \mid d-2$. In particular $|G| \leq \max\{2d(d-2), 60d\}$.

(b-i) *G is a subgroup of the automorphism group of the Fermat curve $F_d\colon X^d + Y^d + Z^d = 0$. In particular, $|G| \leq 6d^2$.*

(b-ii) *G is a subgroup of the automorphism group of the Klein curve $K_d\colon XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$. In particular, $|G| \leq 3(d^2 - 3d + 3)$ if $d \geq 5$. On the other hand, $|G| \leq 168$ if $d = 4$.*

(c) *G is a primitive subgroup of $PGL(3, \mathbb{C})$. In this case G is conjugate to one of the following subgroups of $PGL(3, \mathbb{C})$: the alternating group A_5 or A_6, the Klein group $K_{168} = PSL(2, 7)$, the Hessian group H_{216} of order 216 or its subgroup of order 36 or 72. In particular $|G| \leq 360$.*

In this article we consider the case (c) in Theorem 1 (see Definition 2 for the definition of primitive subgroups) and discuss the following problems:

Problem. (A) For a given finite primitive subgroup $G \subset PGL(3, \mathbb{C})$, does there exist a smooth plane curve C such that $\mathrm{Aut}(C) = G$?

(B) Assume that $\mathrm{Aut}(C)$ is primitive. How many quasi-Galois points does C have?

The notion of quasi-Galois points is introduced as follows (cf. [FMT, Definition 1.1], see also [Y]):

Definition 1. Put $G[P] := \{\sigma \in \mathrm{Aut}(C) \mid \pi_P \circ \sigma = \pi_P\}$ for $P \in \mathbb{P}^2$, where $\pi_P\colon C \to \mathbb{P}^1$ is the projection with center P. For $k \geq 2$,

$$P \text{ is a } \textit{quasi-Galois point} \text{ for } C \text{ of order } k \quad \underset{\text{def}}{\iff} \quad |G[P]| = k.$$

If $P \in C$ (resp. $P \notin C$) then it is said to be *inner* (resp. *outer*). If $|G[P]| = \deg \pi_P$ then P is called a *Galois point* for C.

Remark 1. Since C is smooth, the following hold:

(1) $G[P]$ is a cyclic group of order k.

(2) $k \mid d-1$ (resp. $k \mid d$) if P is an inner (resp. outer) quasi-Galois point of order k.

For Problem (A) we have the following result:

Theorem A. *Let G be a finite primitive subgroup of $PGL(3, \mathbb{C})$. If $G \neq K_{168}$ then there exists a smooth plane sextic C such that $\mathrm{Aut}(C) = G$.*

Remark 2. The Klein group K_{168} is the full automorphism group of the Klein quartic

$$K\colon XY^3 + YZ^3 + ZX^3 = 0,$$

i.e., $\mathrm{Aut}(K) = K_{168}$. It has 21 outer quasi-Galois points of order two (see [FMT, Theorem 6.9, Remark 6.10]).

For Problem (B) we determine the number of quasi-Galois points when $\mathrm{Aut}(C)$ is primitive. Set

$$S_k = S_k(C) := \{\, P \in \mathbb{P}^2 \mid P \text{ is a quasi-Galois point for } C \text{ of order } k \,\}.$$

Theorem B. *Assume that $G = \mathrm{Aut}(C)$ is primitive. Then the set S_k is empty for $k \neq 2, 3$ and the number of quasi-Galois points of order two or three is determined by Table 1.*

G	$\lvert S_2\rvert$	$\lvert S_3\rvert$
L	15	0
K_{168}	21	0
H_{216}	9	12
H_{36}	9	0
H_{72}	9	0
V	45	0

Table 1 quasi-Galois points

See Proposition 2 for the notation in this table.

2. Primitive subgroups of $PGL(3, \mathbb{C})$ and primitive reflection subgroups of $GL(3, \mathbb{C})$

In this section we recall basic facts on primitive subgroups of $PGL(3, \mathbb{C})$ and primitive reflection subgroups of $GL(3, \mathbb{C})$. See, for example, [B] and [LT] for the contents of this section.

Let $\pi\colon GL(n, \mathbb{C}) \to PGL(n, \mathbb{C})$ be the natural projection and $[A] := \pi(A)$ for $A \in GL(n, \mathbb{C})$.

Definition 2. Let $\widetilde{G} \subset GL(n, \mathbb{C})$ be a subgroup of $GL(n, \mathbb{C})$.

(1) $\widetilde{G}$ is *irreducible* $\underset{\text{def}}{\iff}$ $\mathbb{C}^n$ has no proper $\widetilde{G}$-invariant subspaces.

(2) Assume that $\widetilde{G}$ is irreducible. It is said to be *imprimitive* if there exists a direct sum decomposition

$$\mathbb{C}^n = W_1 \oplus W_2 \oplus \cdots \oplus W_r$$

such that

$$\forall \sigma \in \widetilde{G},\ \forall i,\ \exists j,\ \sigma(W_i) = W_j.$$

Otherwise $\widetilde{G}$ is said to be *primitive.*

(3) A subgroup $G \subset PGL(n, \mathbb{C})$ is said to be *irreducible* (resp. imprimitive, primitive) if $\pi^{-1}(G) \subset GL(n, \mathbb{C})$ is irreducible (resp. imprimitive, primitive).

Proposition 2. *Any finite primitive subgroup of $PGL(3, \mathbb{C})$ is conjugate to one of the following:*

(i) $L \simeq A_5 \simeq PSL(2, 5)$ (*alternating group of degree five*).
(ii) *The Klein group* $K_{168} \simeq PSL(2, 7)$.
(iii) *The Hessian group* H_{216}*, or its subgroup* H_{36} *or* H_{72}.
(iv) $V \simeq A_6$ (*alternating group of degree six*).

Remark 3. (1) The Hessian group H_{216} is generated by the four elements h_i $(i = 1, 2, 3, 4)$ represented by the following matrices:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix},$$

where ω is a primitive third root of unity. This group is the full automorphism group of the smooth plane sextic

$$H\colon X^6 + Y^6 + Z^6 - 10(X^3Y^3 + Y^3Z^3 + Z^3X^3) = 0,$$

i.e., $\mathrm{Aut}(H) = H_{216}$. This curve has 12 outer quasi-Galois points of order three (cf. [FMT, Theorem 4.12]).

The subgroups H_{36} and H_{72} are respectively equal to $\langle h_1, h_2, h_3 \rangle$ and $\langle h_1, h_2, h_3, u \rangle$, where $u = h_1^{-1} h_4^2 h_1$.

(2) The group V is the full automorphism group of the Wiman sextic

$$W\colon 10X^3Y^3 + 9(X^5 + Y^5)Z - 45X^2Y^2Z^2 - 135XYZ^4 + 27Z^6 = 0,$$

i.e., $\mathrm{Aut}(W) = V \simeq A_6$. It has 45 outer quasi-Galois points of order two.

Definition 3. (1) A matrix $A \in GL(n, \mathbb{C})$ is called a *reflection* $\underset{\text{def}}{\Longleftrightarrow}$ $\mathrm{rank}(A - E) = 1$ (i.e., A has a unique eigenvalue $\neq 1$).

(2) An element $\sigma \in PGL(n, \mathbb{C})$ is called a *projective reflection* $\underset{\text{def}}{\Longleftrightarrow}$ there exists a reflection $A \in GL(n, \mathbb{C})$ such that $\sigma = [A]$.

(3) A subgroup of $GL(n, \mathbb{C})$ (resp. $PGL(n, \mathbb{C})$) is called a *reflection group* (resp. *projective reflection group*) if it is generated by reflections (resp. projective reflections).

Note that a projective reflection can be represented by a non-reflection.

Example 1. *Let ω be a primitive third root of unity. A reflection $A = \mathrm{diag}(1, 1, \omega)$ and another matrix $A' = \mathrm{diag}(-1, -1, -\omega)$ give the same projective reflection $\sigma = [A] = [A']$, but A' is not a reflection.*

The following classification of three-dimensional finite primitive reflection groups is well-known. We use the notation in [LT].

Proposition 3. *Any finite primitive reflection subgroup of $GL(3, \mathbb{C})$ is conjugate to one of the following:*

(i) G_{23} $(\pi(G_{23}) = L,\ |G_{23}| = 120)$
(ii) G_{24} $(\pi(G_{24}) = K_{168},\ |G_{24}| = 336)$
(iii) G_{25} $(\pi(G_{25}) = H_{216},\ |G_{25}| = 648)$
(iii') $G_{26} = G_{25} \times \{\pm E_3\}$ $(\pi(G_{26}) = H_{216},\ |G_{26}| = 1296)$
(iv) G_{27} $(\pi(G_{27}) = V,\ |G_{27}| = 2160)$

In particular, any finite primitive subgroup of $PGL(3, \mathbb{C})$ is a projective reflection group or its subgroup.

Remark 4. Let $\widetilde{G_1}$ and $\widetilde{G_2}$ be finite reflection groups contained in $GL(3, \mathbb{C})$. If $\widetilde{G_1}$ is primitive and $\widetilde{G_1} \subset \widetilde{G_2}$, then $\widetilde{G_2}$ is also primitive. Hence $(\widetilde{G_1}, \widetilde{G_2}) = (G_{23}, G_{27})$ or (G_{25}, G_{26}).

3. Outline of proof of Theorem A

For Theorem A, we recall that

$$\mathrm{Aut}(K) = K_{168}, \quad \mathrm{Aut}(H) = H_{216} \quad \text{and} \quad \mathrm{Aut}(W) = V \simeq A_6,$$

where

$$\begin{aligned} K &= XY^3 + YZ^3 + ZX^3, \\ H &= X^6 + Y^6 + Z^6 - 10(Y^3Z^3 + Z^3X^3 + X^3Y^3), \\ W &= 10X^3Y^3 + 9(X^5 + Y^5)Z - 45X^2Y^2Z^2 - 135XYZ^4 + 27Z^6. \end{aligned}$$

Thus we only have to give examples whose automorphism group is L, H_{36} or H_{72}.

The homogeneous part of degree six of the invariant ring of G_{23} is

$$\mathbb{C}[X,Y]_6^{G_{23}} = \mathbb{C}W \oplus \mathbb{C}U.$$

The polynomials U is defined by

$$U := 7(X^6+Y^6+Z^6) + \alpha(X^4Y^2+Y^4Z^2+Z^4X^2) \\ + \beta(X^2Y^4+Y^2Z^4+Z^2X^4) + 54X^2Y^2Z^2 = 0,$$

where

$$\alpha = -21(\zeta_5+\zeta_5^4) - 18(\zeta_5^2+\zeta_5^3), \quad \beta = -18(\zeta_5+\zeta_5^4) - 21(\zeta_5^2+\zeta_5^3)$$

(ζ_5 is a primitive fifth root of unity).

Proposition 4. Let C_λ ($\lambda \in \mathbb{C}$) be the plane sextic defined by $U + \lambda W = 0$. If λ is general then $\mathrm{Aut}(C_\lambda) = L \simeq A_5$.

Set $\widetilde{H}_{36} = G_{26} \cap \pi^{-1}(H_{36})$. Then

$$\mathbb{C}[X,Y]_6^{\widetilde{H}_{36}} = \mathbb{C}H \oplus \mathbb{C}F.$$

The polynomials F is defined by

$$F := 4(X^6+Y^6+Z^6) - 15\omega XYZ(X^3+Y^3+Z^3) \\ - 10(Y^3Z^3+Z^3X^3+X^3Y^3) - 45\omega^2X^2Y^2Z^2,$$

where ω is a primitive third root of unity.

Proposition 5. Let C'_λ ($\lambda \in \mathbb{C}$) be the plane sextic defined by $F + \lambda H = 0$. If λ is general then $\mathrm{Aut}(C'_\lambda) = H_{36}$. On the other hand, $\mathrm{Aut}(C'_{-3/2}) = H_{72}$.

Thus we obtain Theorem A.

4. Smooth plane curves whose automorphism group is primitive

In what follows G denotes $\mathrm{Aut}(C)$ and assume that it is primitive. We define

$$\widetilde{G} := \begin{cases} G_{23} & \text{if } G = L \\ G_{24} & \text{if } G = K_{168} \\ G_{26} & \text{if } G = H_{216} \\ \widetilde{H}_{36} = G_{26} \cap \pi^{-1}(H_{36}) & \text{if } G = H_{36} \\ \widetilde{H}_{72} = G_{26} \cap \pi^{-1}(H_{72}) & \text{if } G = H_{72} \\ G_{27} & \text{if } G = V. \end{cases}$$

Recall

$$S_k = S_k(C) = \{\, P \in \mathbb{P}^2 \mid P \text{ is a quasi-Galois point for } C \text{ of order } k \,\}$$

and set

$R_k = R_k(G) := \{ \sigma \in G \mid \sigma \text{ is a projective reflection of order } k \}, \quad R := \bigcup R_k,$

$R^{(k)} = R^{(k)}(G) := \{ \langle \sigma \rangle \subset G \mid \sigma \in R_k \},$

$\widetilde{R}_k = \widetilde{R}_k(\widetilde{G}) := \{ A \in \widetilde{G} \mid A \text{ is a reflection of order } k \}, \quad \widetilde{R} := \bigcup \widetilde{R}_k,$

$\widetilde{R}^{(k)} = \widetilde{R}^{(k)}(\widetilde{G}) := \{ \langle A \rangle \subset \widetilde{G} \mid A \in \widetilde{R}_k \}.$

Note that $G[P] \in R^{(k)}$ if $P \in S_k$. Furthermore

Lemma 6. *The map $S_k \to R^{(k)}$ $(P \mapsto G[P])$ is bijective.*

By a classification of reflection groups we see that $\widetilde{R}^{(k)} = \emptyset$ for $k \neq 2, 3$ and obtain the following table.

G	$\widetilde{G}$	$\lvert\widetilde{R}^{(2)}\rvert$	$\lvert\widetilde{R}^{(3)}\rvert$
L	G_{23}	15	0
K_{168}	G_{24}	21	0
H_{216}	G_{26}	9	12
H_{36}	$\widetilde{H}_{36}$	9	0
H_{72}	$\widetilde{H}_{72}$	9	0
V	G_{27}	45	0

Table 2 Reflections

Remark 5. (1) $|\widetilde{R}_k| = \varphi(k)|\widetilde{R}^{(k)}|$, $|R_k| = \varphi(k)|R^{(k)}|$ (φ is Euler's totient function).

(2) Note that $H_{216} = \pi(G_{25})$ has 9 projective reflections of order 2, e.g.,

$$h_3^2 = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right],$$

but G_{25} has no reflections of order 2.

The following holds:

Proposition 7. *$\pi|_{\widetilde{R}} \colon \widetilde{R} \to R$ $(A \mapsto [A])$ is bijection and $\pi(\widetilde{R}_k) = R_k$. In particular $|S_k| = |R^{(k)}| = |\widetilde{R}^{(k)}|$.*

Thus we give a solution to Problem (B), i.e., we can determine the number of quasi-Galois points for C by counting reflections in $\widetilde{G}$.

G	$\lvert S_2\rvert$	$\lvert S_3\rvert$
L	15	0
K_{168}	21	0
H_{216}	9	12
H_{36}	9	0
H_{72}	9	0
V	45	0

Table 3 quasi-Galois points

5. Outline of proof of Theorem B

We show some lemmas to prove Proposition 7.

Lemma 8. *Let $\sigma \in PGL(n, \mathbb{C})$ be a projective reflection of finite order and $A \in GL(n, \mathbb{C})$ a reflection such that $\sigma = [A]$. Then A is of finite order and $\operatorname{ord} A = \operatorname{ord} \sigma$.*

Proof. It suffices to show that $\operatorname{ord} A \mid \operatorname{ord} \sigma$. There exists a matrix $A_0 \in GL(n, \mathbb{C})$ of finite order such that $\sigma = [A_0]$ (For example, if we take $A_0 \in SL(n, \mathbb{C})$ then $\operatorname{ord} A_0 \leq n \operatorname{ord} \sigma$). Then $A = cA_0$ for some $c \in \mathbb{C}^\times$ and A_0 is diagonalizable (since it is of finite order). Hence A is also diagonalizable, i.e.,

$$PAP^{-1} = \operatorname{diag}(1, \dots, 1, \zeta) \quad (\zeta \text{ is a root of unity})$$

for some $P \in GL(n, \mathbb{C})$.

Put $k = \operatorname{ord} \sigma$. Then

$$E = [P]\sigma^k[P^{-1}] = [(PAP^{-1})^k] = [\operatorname{diag}(1, \dots, 1, \zeta^k)].$$

Hence $\zeta^k = 1$, which implies that $A^k = E$. Thus $\operatorname{ord} A \mid k$. □

Set $N = \operatorname{lcm}\{\operatorname{ord} g \mid g \in \widetilde{G}\}$.

Lemma 9. *Let $\sigma \in G$ be projective reflection. Take a reflection $A \in GL(3, \mathbb{C})$ and a matrix $A_0 \in \widetilde{G}$ such that $\sigma = [A] = [A_0]$. If $A = cA_0$ ($c \in \mathbb{C}^\times$) then $c^N = 1$.*

Proof. We see that $A^N = c^N E$ since $A_0^N = E$. Then $c^N = 1$ because A is a reflection. □

Set

$$Z_N := \{ aE_3 \mid a^N = 1 \} \subset GL(3, \mathbb{C}), \quad \widehat{G} := \langle \widetilde{G}, Z_N \rangle \quad \text{and}$$
$$\widehat{R} = \{ A \in \widehat{G} \mid A \text{ is a reflection} \}.$$

Then $\widehat{G}$ is a finite group since $Z_N \subset Z(GL(3,\mathbb{C}))$.

Lemma 10. *If* $\widetilde{G} = G_i$ $(i = 23, 24, 26, 27)$ *then* $\widehat{R} = \widetilde{R}$.

Proof. Assume that $\widehat{R} \supsetneq \widetilde{R}$. Take $A \in \widehat{R} \setminus \widetilde{R}$. Note that $\langle \widetilde{G}, A\rangle$ is a finite reflection group strictly containing $\widetilde{G} = G_i$, hence it is primitive. Then $\widetilde{G} = G_{23}$ and $\langle \widetilde{G}, A\rangle = G_{27}$. In particular $N = 30$. However,

$$\widehat{G} = \langle G_{23}, Z_{30}\rangle = G_{23} \times Z_{15},$$

which implies that

$$|\widehat{G}| = 120 \cdot 15 = 1800 < |G_{27}| = 2160.$$

This is a contradiction. □

We prove Proposition 7 by using above lemmas.

Proof of Proposition 7. We may assume that $\widetilde{G} = G_i$ $(i = 23, 24, 26, 27)$. Take any element $\sigma \in R$. By Lemma 9 there exist $A_0 \in \widetilde{G}$ and $c \in Z_N$ such that $\sigma = [A_0]$ and cA_0 is a reflection. Then $cA_0 \in \widehat{R}$ since $cA_0 \in \widehat{G}$. It follows from Lemma 10 that $cA_0 \in \widetilde{R}$. Thus $\pi|_{\widetilde{R}} \colon \widetilde{R} \to R$ is surjective since $[cA_0] = \sigma$.

Suppose that $A, A' \in \widetilde{R}$ and $[A] = [A']$. Then

$$A' = cA \quad (\exists\, c \in \mathbb{C}^\times), \quad A \sim \mathrm{diag}(1,1,\zeta) \quad \text{and} \quad A' \sim \mathrm{diag}(1,1,\zeta') \quad (\zeta^N = \zeta'^N = 1).$$

Hence

$$\mathrm{diag}(c, c, c\zeta) \sim \mathrm{diag}(1,1,\zeta'),$$

which implies that $c = 1$, that is to say, $A = A'$. Thus $\pi|_{\widetilde{R}} \colon \widetilde{R} \to R$ is injective. Furthermore, π preserves the order of elements by Lemma 8. □

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