## Introduction

- Polyhedron : a finite intersection of half-spaces

$$
P:=\left\{x \in \mathbf{R}^{d}: \tilde{a}_{1} x+\tilde{b}_{1} \geq 0, \ldots, \tilde{a}_{n} x+\tilde{b}_{n} \geq 0\right\} \quad\left(\tilde{a}:=\left(\tilde{a}_{1}, \ldots, \tilde{a}_{n}\right) \in \mathbf{R}^{d \times n}, \tilde{b} \in \mathbf{R}^{n}\right)
$$



Figure: Examples of polyhedra

- We call the probability content of a convex polyhedron with a multivariate normal distribution the normal probabilities of polyhedra. D. Q. Naiman and H. P. Wynn (1997); and S. Kuriki, T. Miwa, and A. J. Hayter (2012) studied about methods to evaluate the probabilities, which is important in applications of statistics.
- The normal probability of polyhedra is a function of the parameters $a \in \mathbf{R}^{d \times n}$ and $b \in \mathbf{R}^{n}$ which determine the polyhedron

$$
\varphi(a, b)=\int_{\sum_{i=1}^{d} a_{i j} x_{i}+b_{j} \geq 0,1 \leq j \leq n} \frac{1}{(2 \pi)^{d / 2}} e^{-\frac{1}{2} \sum_{i=1}^{d} x_{i}^{2}} d x_{1} \cdots d x_{d}
$$

This function is a generalization of the Schäfli function which studied by Aomoto (1977)

- By the inclusion-exclusion identity, we have the following: When the polyhedron $P$ in general position, the equation

$$
\varphi(a, b)=\sum_{J \in \mathcal{F}} \int_{\mathbf{R}^{d}} e^{-\frac{1}{2} \sum_{i=1}^{d} x_{i}^{2}} \frac{1}{(2 \pi)^{d / 2}} \prod_{j \in J}\left(H\left(\sum_{i=1}^{d} a_{i j} x_{i}+b_{j}\right)-1\right) d x
$$

holds on a neighborhood of ( $\tilde{a}, \tilde{b}$ ) which defines $P$, where $H(x)$ is the Heaviside function

- We use this decomposition to obtain a holonomic system for $\varphi(a, b)$. The rank of the holonomic module is equal to the number of nonempty faces of the convex polyhedron $P$.


## A Polyhedron in General Position

Definition:

- $P \subset \mathbf{R}^{d}$ : a polyhedron
- $F_{1}, \ldots, F_{n}$ : the all facets of $P$
- $\mathcal{H}=\left\{H_{1}, \ldots, H_{n}\right\}$ : the family of the bounding half-spaces for the polyhedron $P$,
i.e., for $j \in[n], H_{j} \subset \mathbf{R}^{d}$ is the half-space which is valid for $P$ and satisfies $\left(\partial H_{j}\right) \cap P=F_{j}$.
- The homogenization $\hat{H}$ of a half-space $H=\left\{x \in \mathbf{R}^{d} \mid \sum a_{i} x_{i}+a_{0} \geq 0\right\}$ is defined as

$$
\hat{H}=\left\{\left(x_{0}, \ldots, x_{d}\right) \in \mathbf{R}^{d+1} \mid \sum_{i=0}^{d} a_{i} x_{i} \geq 0\right\} .
$$

- For a family of half-spaces $\mathcal{H}=\left\{H_{1}, \ldots, H_{n}\right\}$, we call $\hat{\mathcal{H}}=\left\{\hat{H}_{0}, \hat{H}_{1}, \ldots, \hat{H}_{n}\right\}$ the homogenization of $\mathcal{H}$. Here, we put $\hat{H}_{0}=\left\{x_{0} \geq 0\right\}$.
- We call a family of half-spaces $\mathcal{H}=\left\{H_{1}, \ldots, H_{n}\right\}$ (or it's homogenization $\hat{\mathcal{H}}=\left\{\hat{H}_{0}, \ldots, \hat{H}_{n}\right\}$ ) is in general position when for $J \subset[n+1]$,

$$
\hat{F}_{J}:=\left(\bigcap_{j \in J} \partial \hat{H}_{j}\right) \cap\left(\bigcap_{j=0}^{n} \hat{H}_{j}\right)
$$

is a $d+1-|J|$-dimensional cone (i.e., the affine hull of the cone is $d+1-|J|$-dimensional affine space) or $\{0\}$. Here, we denote by $[n+1]$ the set $\{0,1, \ldots, n\}$.

- The polyhedron $P$ is in general position when the family $\mathcal{H}$ of the bounding half-spaces for $P$ is in general position
Example: In the following two cases, neither polyhedron is in general position.
- There exists a vertex like the apex of the pyramid (a).
- There are two facet which intersect at points at infinity (b)

(a)

(b)

Figure: Polyhedra not in general position

## The Inclusion-Exclusion Identity

The Abstract Simplicial Complex: The nerve of $\left\{F_{1}, \ldots, F_{n}\right\}$ is the abstract simplicial complex defined by

$$
\mathcal{F}=\left\{J \subset[n]: F_{J} \neq \emptyset\right\}, \quad\left(F_{J}:=\bigcap_{j \in J} F_{j}\right)
$$

We also call $\mathcal{F}$ the abstract simplicial complex associated with the polyhedron $P$.

Example. The abstract simplicial complex associated with the polyhedron in the figure on the right is

$$
\mathcal{F}=\{\emptyset, 1,2,3,12,23,31\} .
$$

Note that each element of $\mathcal{F}$ corresponds to a non-empty face of the polyhedron $P$


Edelsbrunner (1995) showed the inclusion-exclusion identity:

$$
\prod_{j=1}^{n} H\left(\sum_{i=1}^{d} \tilde{a}_{i j} x_{i}+\tilde{b}_{j}\right)=\sum_{J \in \mathcal{F}} \prod_{j \in J}\left(H\left(\sum_{i=1}^{d} \tilde{a}_{i j} x_{i}+\tilde{b}_{j}\right)-1\right)
$$

## where $H(x)$ is the Heaviside function.

Example: In the case where the polyhedron $P$ is the 2-simplex, the inclusion-exclusion identity is

$$
\begin{aligned}
\prod_{j=1}^{3} \mathbf{1}_{H_{j}}= & 1+\left(\mathbf{1}_{H_{1}}-1\right)+\left(\mathbf{1}_{H_{2}}-1\right)+\left(\mathbf{1}_{H_{3}}-1\right) \\
& +\left(\mathbf{1}_{H_{1}}-1\right)\left(\mathbf{1}_{H_{2}}-1\right)+\left(\mathbf{1}_{H_{2}}-1\right)\left(\mathbf{1}_{H_{3}}-1\right)+\left(\mathbf{1}_{H_{3}}-1\right)\left(\mathbf{1}_{H_{1}}-1\right)
\end{aligned}
$$

where $\left\{H_{1}, H_{2}, H_{3}\right\}$ is the family of bounding half-spaces for $P$, and $\mathbf{1}_{E}$ denotes the indicator
function of the domain $E$. Each of the term in the right-hand side corresponds to the cones in the following figure.


Under the general position assumption, this identity can be generalized as follows.

## Theorem

If the polyhedron $P$ is in general position, then there exists a neighborhood $U$ of $(\tilde{a}, \tilde{b})$ such that the equation

$$
\prod_{j=1}^{n} H\left(\sum_{i=1}^{d} a_{i j} x_{i}+b_{j}\right)=\sum_{J \in \mathcal{F}} \prod_{j \in J}\left(H\left(\sum_{i=1}^{d} a_{i j} x_{i}+b_{j}\right)-1\right)
$$

holds for all $(a, b, x) \in U \times \mathbf{R}^{d}$.
Remark: When a polyhedron $P$ is in general position, the abstract simplicial complex associated with $P$ is stable under perturbations of the parameters.

Example: The abstract simplicial complexes associated with the tow polyhedra in the figure on the right are equivalent to the following abstract simplicial complex:

$$
\mathcal{F}=\{\emptyset, 1,2,3,12,23,31\}
$$

## Complex Integrals

By the inclusion-exclusion identity with parameters, the normal probability of the polyhedron can be decomposed as follows:

$$
\varphi(a, b)=\sum_{F \in \mathcal{F}} \varphi_{F}(a, b)
$$

$$
\varphi_{F}(a, b)=\int_{\mathbf{R}^{d}} \frac{1}{(2 \pi)^{d / 2}} \exp \left(-\frac{1}{2} \sum_{i=1}^{d} x_{i}^{2}\right) \prod_{j \in F}\left(H\left(f_{j}(a, b, x)\right)-1\right) d x
$$

for $F \in \mathcal{F}$. The function $\varphi_{F}(a, b)$ can be written in terms of complex integral. In fact, we have

$$
\begin{equation*}
\int_{\gamma} \frac{1}{(2 \pi)^{d / 2}} \exp \left(-\frac{1}{2} \sum_{i=1}^{d} z_{i}^{2}\right) \prod_{j \in F}\left(-\frac{\log \left(f_{j}(a, b, z)\right)}{2 \pi \sqrt{-1}}\right) d z \tag{1}
\end{equation*}
$$

on a connected open neighborhood of $(\tilde{a}, \tilde{b})$ in $U_{F}:=\left\{(a, b): \operatorname{det} \alpha_{F}(a) \neq 0\right\}$. Here, $\gamma$ is a $d$-simplex in $\mathbf{C}^{d}$, and we suppose the multivalued function $\log$ satisfies $\log (1)=0$ and the branch cut is $\{z \in \mathbf{C}: \Re(z) \leq 0\}$. Consequently, we have the following theorem.

## Theorem

The function $\varphi(a, b)$ is a real analytic function, and it has an analytic continuation along every path in $\bigcap_{F \in \mathcal{F}} U_{F}$.

## Holonomic System and Pfaffian equation

We consider a holonomic system for the function

$$
\begin{equation*}
\chi_{P}(a, b, x)=\sum_{F \in \mathcal{F}} \chi_{F}(a, b, x) . \tag{2}
\end{equation*}
$$

For $J \subset[n]$, we define a hyperfunction $\chi^{J}$ by $\chi^{J}=\partial_{b}^{J} \chi_{P} \quad\left(\partial_{b}^{J}:=\prod_{i \in J} \partial_{b_{i}}\right)$. Regarding the Heaviside function as a hyper function, we have

$$
\partial_{x_{i}} \chi^{J}=\sum_{j=1}^{n} a_{i j} \partial_{b_{j}} \chi^{J}, \quad \partial_{a_{i j}} \chi^{J}=x_{i} \partial_{b_{j}} \chi^{J}, \quad \partial_{b_{\ell}} \chi^{J}=\chi^{J \cup\{\ell\}}, \quad f_{k} \chi^{J}=0
$$

for $J \in \mathcal{F}, 1 \leq i \leq d, 1 \leq j \leq n, k \in J$, and $\ell \in[n] \backslash J$. Here, we put $g^{J \cup\{j\}}=0$ for $J \cup\{j\} \notin \mathcal{F}$. These differential equations define a holonomic module $M_{\chi}$
We can obtain a holonomic module $M_{q}$ for $\exp \left(-\frac{1}{2} \sum_{i=1}^{d} x_{i}^{2}\right) \chi_{P}$ from $M_{\chi}$. Calculating the integration module of $M_{q}$ with respect to $x$, we have the following

## Theorem

The function $\varphi(a, b)$ satisfies the following linear partial equations:

$$
\begin{aligned}
\left(\begin{array}{rl}
\left.\partial_{a_{i j}}-\sum_{k=1}^{n} a_{i k} \partial_{b_{k}} \partial_{b_{j}}\right) g^{J}=0 & (1 \leq i \leq d, 1 \leq j \leq n, J \in \mathcal{F}), \\
\partial_{b_{j}} g^{J}-g^{J \cup\{j\}}=0 & \left(j \in J^{c}, J \in \mathcal{F}\right), \\
\left(b_{j}+\sum_{k=1}^{n} \sum_{i=1}^{d} a_{i j} a_{i k} \partial_{b_{k}}\right) g^{J} & =0 \\
g^{J} & =0 \\
(j \in J, J \in \mathcal{F}), & (J \notin \mathcal{F})
\end{array}\right.
\end{aligned}
$$

where $g^{J}:=\left(\prod_{j \in J} \partial_{b_{j}}\right) \varphi$ for $J \subset[n]$. Moreover, the above equations define a holonomic module $M$.
Theorem
The holonomic rank of $M$ is equal to the number of non-empty faces of $P$, i.e.,

$$
\operatorname{rank}(M)=|\mathcal{F}|
$$

## Reference

- K. Aomoto. Analytic structure of Schläfli function. Nagoya Math.J., 68:1-16, 1977.
- T. Koyama, Holonomic modules associated with multivariate normal probabilities of polyhedra
http://arxiv.org/abs/1311.6905, 2013, to appear in FUNKCIALAJ EKVACIOJ.
Daniel Q. Naiman and Henry P. Wynn. Abstract tubes, improved inclusion-exclusion identities and inequalities and importance sampling. The Annals of Statistics 25(5):1954-1983, 1997.
H. Nakayama, K. Nishiyama, M. Noro, K. Ohara, T. Sei, N. Takayama, and A. Takemura. Holonomic gradient desce and its application to the Fisher-Bingham integral. Advances in Applied Mathematics, 47:639-658, 2011

