Natural Deduction for Quantum Logic

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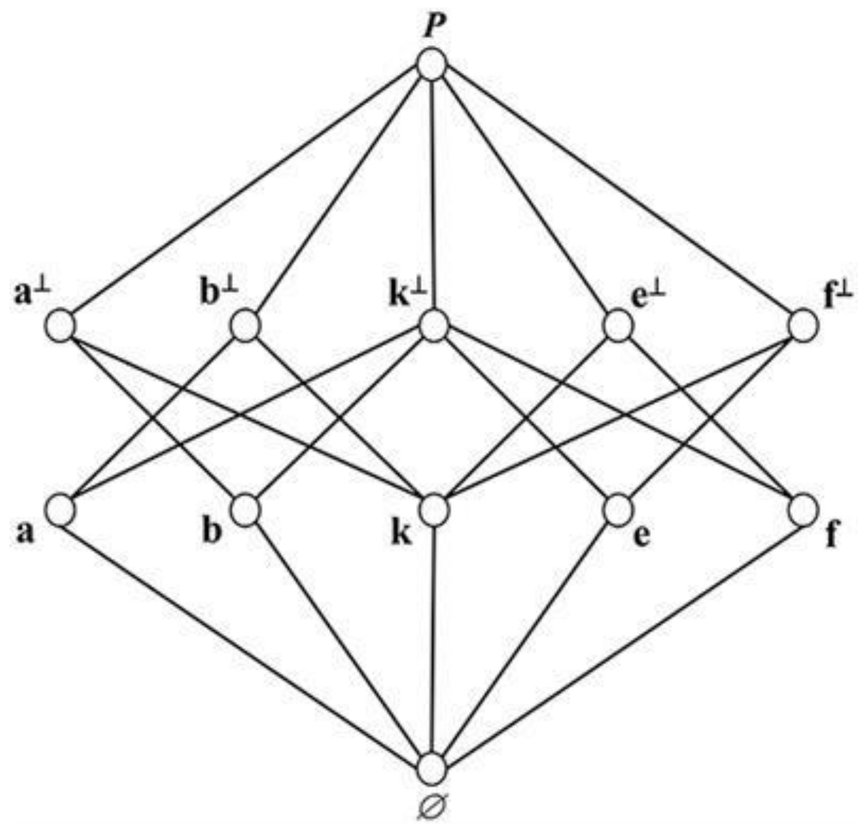

Outline (1/1)

- This paper presents a natural deduction system for ortho-modular quantum logic.
- The system is shown to be provably equivalent to Nishimura's quantum sequent calculus.
- Through the Curry–Howard isomorphism, quantum λ-calculus is also introduced for which strong normalization property is established.

Main contributions (1/2)

- This paper presents a natural deduction system for ortho-modular quantum logic.
- Thanks to its intrinsic and straightforward appearance, we can understand the meaning of inference in quantum logic deeply by comparing the system with those for other logics (e.g. intuitionistic logic or classical logic).

Main contributions (2/2)

- We can also introduce the corresponding quantum λ-calculus, which allows us to further investigate computational theories based on quantum logic, via the Curry–Howard isomorphism.
- We can establish a desirable property regarding normalization of proofs, or equivalently, termination of computation.

Related Studies (1/3)

- As an earlier study for quantum natural deduction, we need to mention Delmas-Rigoutsos's *double deduction system* [5], which incorporates a concept of compatibility into a natural deduction system for classical logic.
- Differently from this approach, we define a natural deduction system that directly corresponds to **GOM**, Nishimura's quantum sequent calculus [13].

Related Studies (2/3)

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- Besides **GOM**, a few systems for quantum sequent calculus have been proposed by Cutland and Gibbins [3] and Nishimura [14].
- Furthermore, an extended logical system containing quantum logic called *basic logic* along with its sequent calculus has been studied by Sambin et al. [19], Faggian and Sambin [7], and Dalla Chiara and Giuntini [4].
- These systems, however, are all inadequate for being translated into natural deduction forms due to their complex treatment of negation ($\neg$) and cut.

Related Studies (3/3)

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- The quantum λ-calculus introduced in this paper is based on orthomodular quantum logic, while several other systems based on intuitionistic linear logic have also been studied under the name *quantum λ-calculus* [20,21].
- The proof of the strong normalization property presented in this paper follows that of Girard et al. [8].

Problems and solutions (1/14)

- It is known that quantum logic has no satisfactory implication operation ($\rightarrow$) as in the case of intuitionistic logic or classical logic. Indeed, Nishimura's **GOM** only adopts conjunction ($\land$) and negation ($\neg$) as the basic set of operations.
- On the other hand, it is almost inevitable to include implication in the basic set of operations for the purpose of developing a natural deduction system and the corresponding λ-calculus.

Problems and solutions (2/14)

- To handle such a contradiction, we employ the *Sasaki hook*, a kind of quasi-implication, as one of the basic operations of our system.
- Although it fails to satisfy the deduction theorem, the Sasaki hook still enjoys some expected properties of implication such as *modus ponens*.

Problems and solutions (3/14)

- Another problem that we encounter when associating **GOM** with a natural deduction system is how to treat assumptions in a deduction process.
- In the usual natural deduction system for intuitionistic logic or classical logic, assumptions that are not used in the application of a rule may be omitted.

Problems and solutions (4/14)

- That is, for example:

$$\cfrac{\begin{matrix}[\alpha] \\ \vdots \\ \beta\end{matrix}}{\alpha \to \beta}$$

- In this case, it is legitimate that assumptions other than α are not explicitly stated even if they exist.

Problems and solutions (5/14)

- The point becomes clear when we express this situation in the sequent calculus form:

$$\frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \to \beta}$$

- Here, all (undischarged) assumptions other than α are explicitly written as Γ, a (possibly empty) set of formulas.

Problems and solutions (6/14)

- In quantum logic, however, the introduction rule of implication (the Sasaki hook) is subject to a restriction due to the failure of the deduction theorem.
- That is, the introduction rule of implication can only be applied if there exist no assumptions other than α:

$$\frac{\alpha \vdash \beta}{\vdash \alpha \to \beta}$$

Problems and solutions (7/14)

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Intuitionistic / Classical

$$\frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \to \beta}$$

Quantum

$$\frac{\alpha \vdash \beta}{\vdash \alpha \to \beta}$$

Problems and solutions (8/14)

- Taking this restriction into account, we will use the following convention in defining and applying rules of our natural deduction system: *the assumptions that are not explicitly stated must not exist.*

Problems and solutions (9/14)

- When written in this style, the introduction rule of implication in intuitionistic logic or classical logic would become as follows:

$$\dfrac{\begin{array}{c}[\alpha], \Gamma \\ \vdots \\ \beta\end{array}}{\alpha \to \beta}$$

Problems and solutions (10/14)

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Intuitionistic / Classical (usual style)

$$\dfrac{\begin{array}{c}[\alpha] \\ \vdots \\ \beta\end{array}}{\alpha \rightarrow \beta}$$

Intuitionistic / Classical (our style)

$$\dfrac{\begin{array}{c}[\alpha], \Gamma \\ \vdots \\ \beta\end{array}}{\alpha \rightarrow \beta}$$

Problems and solutions (11/14)

- Example: A legitimate proof in intuitionistic or classical logic.

$$\dfrac{\dfrac{\dfrac{\dfrac{[\mathrm{b} : \beta \to \gamma] \qquad \dfrac{[\mathrm{c} : \alpha \to \beta] \qquad [\mathrm{a} : \alpha]}{\beta}}{\gamma}}{\alpha \to \gamma}\;\mathrm{(a)}}{(\beta \to \gamma) \to (\alpha \to \gamma)}\;\mathrm{(b)}}{(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))}\;\mathrm{(c)}$$

Problems and solutions (12/14)

- However, the proof is illegitimate in quantum logic.

$$\dfrac{\dfrac{\dfrac{\dfrac{[\mathrm{b}:\beta\to\gamma]\qquad\dfrac{[\mathrm{c}:\alpha\to\beta]\qquad[\mathrm{a}:\alpha]}{\beta}}{\gamma}}{\alpha\to\gamma}\,(\mathrm{a})}{(\beta\to\gamma)\to(\alpha\to\gamma)}\,(\mathrm{b})}{(\alpha\to\beta)\to((\beta\to\gamma)\to(\alpha\to\gamma))}\,(\mathrm{c})$$

Problems and solutions (13/14)

- Indeed, $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$ is not a theorem of quantum logic.

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\mathrm{b} : \beta \to \gamma] \qquad \dfrac{[\mathrm{c} : \alpha \to \beta] \qquad [\mathrm{a} : \alpha]}{\beta}}{\gamma}}{\alpha \to \gamma}\,(\mathrm{a})}{(\beta \to \gamma) \to (\alpha \to \gamma)}\,(\mathrm{b})}{(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))}\,(\mathrm{c})
$$

Problems and solutions (14/14)

- Once a natural deduction system for quantum logic is obtained, the corresponding *quantum λ-calculus* can be introduced via the Curry–Howard isomorphism: the proofs of the natural deduction system can be reversibly translated into the terms of the λ-calculus, respectively.
- Finally, we will prove the *strong normalization property* for the quantum λ-calculus, which claims that any computation in the quantum λ-calculus eventually terminates.

Organization of the paper (1/1)

- Section 1 **Introduction**
- Section 2 **Formal Definition of NQ**
- Section 3 **Equivalence Between NQ and GOM**
- Section 4 **Quantum λ-Calculus**
- Section 5 **Strong Normalization**
- Section 6 **Conclusion**

2 **Formal Definition of NQ** (1/9)

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- We employ $\wedge$ (conjunction) and $\rightarrow$ (implication) as the basic connectives, and introduce the following as abbreviations: $\alpha \vee \beta \equiv \neg(\neg\alpha \wedge \neg\beta)$ (disjunction), $\neg\alpha \equiv \alpha \rightarrow \bot$ (negation)

2 **Formal Definition of NQ** (2/9)

- Proof of **NQ**

<u>Assumption rule</u>

$$\mathrm{a} : \alpha$$

<u>⊥-rule</u>

$$\begin{array}{c} \Gamma \\ \vdots \\ \perp \end{array} \quad \Rightarrow \quad \dfrac{\begin{array}{c} \Gamma \\ \vdots \\ \perp \end{array}}{\alpha}$$

2 **Formal Definition of NQ** (3/9)

<u>∧El-rule</u>

$$\begin{array}{c}\Gamma\\ \vdots\\ \alpha \wedge \beta\end{array} \quad \Rightarrow \quad \begin{array}{c}\Gamma\\ \vdots\\ \dfrac{\alpha \wedge \beta}{\alpha}\end{array}$$

<u>∧Er-rule</u>

$$\begin{array}{c}\Gamma\\ \vdots\\ \alpha \wedge \beta\end{array} \quad \Rightarrow \quad \begin{array}{c}\Gamma\\ \vdots\\ \dfrac{\alpha \wedge \beta}{\beta}\end{array}$$

2 **Formal Definition of NQ** (4/9)

∧I-rule

$$\begin{array}{c} \Gamma_1 \\ \vdots \\ \alpha \end{array} \qquad \begin{array}{c} \Gamma_2 \\ \vdots \\ \beta \end{array} \quad \Rightarrow \quad \dfrac{\begin{array}{c} \Gamma_1 \\ \vdots \\ \alpha \end{array} \qquad \begin{array}{c} \Gamma_2 \\ \vdots \\ \beta \end{array}}{\alpha \land \beta}$$

2 **Formal Definition of NQ** (5/9)

→E-rule (Modus Ponens)

$$
\begin{array}{c}
\Gamma_1 \\
\vdots \\
\alpha \to \beta
\end{array}
\qquad\qquad
\begin{array}{c}
\Gamma_2 \\
\vdots \\
\alpha
\end{array}
\qquad \Longrightarrow \qquad
\dfrac{\begin{array}{c}\Gamma_1 \\ \vdots \\ \alpha \to \beta\end{array} \qquad \begin{array}{c}\Gamma_2 \\ \vdots \\ \alpha\end{array}}{\beta}
$$

2 **Formal Definition of NQ** (6/9)

→I-rule

$$\begin{array}{c} \alpha \\ \vdots \\ \beta \end{array} \qquad \Rightarrow \qquad \dfrac{\begin{array}{c} [\mathrm{a} : \alpha] \\ \vdots \\ \beta \end{array}}{\alpha \to \beta}$$

No assumptions other than a : α are allowed to be made when this rule is applied.

2 **Formal Definition of NQ** (7/9)

MT (Modus Tollens)

$$
\begin{array}{c}\emptyset\\ \vdots\\ \alpha \to \beta\end{array}
\qquad
\begin{array}{c}\Gamma\\ \vdots\\ \beta \to \bot\end{array}
\qquad \Rightarrow \qquad
\dfrac{\begin{array}{c}\emptyset\\ \vdots\\ \alpha \to \beta\end{array}\qquad\begin{array}{c}\Gamma\\ \vdots\\ \beta \to \bot\end{array}}{\alpha \to \bot}
$$

2 **Formal Definition of NQ** (8/9)

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¬¬E-rule

$$\begin{array}{c}\Gamma\\ \vdots\\ (\alpha \to \bot) \to \bot\end{array} \quad \Rightarrow \quad \begin{array}{c}\Gamma\\ \vdots\\ \dfrac{(\alpha \to \bot) \to \bot}{\alpha}\end{array}$$

2 **Formal Definition of NQ** (9/9)

¬¬I-rule

$$\begin{matrix}\Gamma\\ \vdots\\ \alpha\end{matrix} \quad \Rightarrow \quad \dfrac{\begin{matrix}\Gamma\\ \vdots\\ \alpha\end{matrix}}{(\alpha \to \bot) \to \bot}$$

3 **Equivalence Between NQ and GOM** (1/5)

- Nishimura's **GOM**

Axioms

$$\alpha \vdash \alpha$$

Rules

$$\frac{\Gamma \vdash \Delta}{\Pi, \Gamma \vdash \Delta, \Sigma}\,(\text{extension}) \qquad \frac{\Gamma_1 \vdash \Delta_1, \alpha \qquad \alpha, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,(\text{cut})$$

3 Equivalence Between NQ and GOM (2/5)

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$$\frac{\alpha, \Gamma \vdash \Delta}{\alpha \wedge \beta, \Gamma \vdash \Delta}\ (\wedge \mathrm{l}_1) \qquad \frac{\beta, \Gamma \vdash \Delta}{\alpha \wedge \beta, \Gamma \vdash \Delta}\ (\wedge \mathrm{l}_2)$$

$$\frac{\Gamma \vdash \Delta, \alpha \qquad \Gamma \vdash \Delta, \beta}{\Gamma \vdash \Delta, \alpha \wedge \beta}\ (\wedge \mathrm{r})$$

$$\frac{\Gamma \vdash \Delta, \alpha}{\neg\alpha, \Gamma \vdash \Delta}\ (\neg \mathrm{l}) \qquad \frac{\alpha \vdash \Delta}{\neg\Delta \vdash \neg\alpha}\ (\neg \mathrm{r})$$

3 Equivalence Between NQ and GOM (3/5)

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$$\frac{\alpha, \Gamma \vdash \Delta}{\neg\neg\alpha, \Gamma \vdash \Delta}\ (\neg\neg\mathrm{l}) \qquad \frac{\Gamma \vdash \Delta, \alpha}{\Gamma \vdash \Delta, \neg\neg\alpha}\ (\neg\neg\mathrm{r})$$

$$\frac{\neg\beta \vdash \neg\alpha \qquad \neg\alpha, \beta \vdash}{\neg\alpha \vdash \neg\beta}\ (\text{O-modular})$$

3 **Equivalence Between NQ and GOM** (4/5)

- In **GOM**, we regard $\alpha \rightarrow \beta$ as a shorthand for $\neg(\alpha \land \neg(\alpha \land \beta))$ (the Sasaki hook). Additionally, we regard $\bot$ as a shorthand for $\alpha \land \neg\alpha$.
- **Theorem 3.5** (Admissibility of the rules of **NQ** in **GOM**). *The rules of* **NQ** *are provable in* **GOM**.
- **Theorem 3.6** (Admissibility of the axioms and rules of **GOM** in **NQ**). *The axioms and the rules of* **GOM** *are provable in* **NQ**.

3 **Equivalence Between NQ and GOM** (5/5)

- Here we have used the following fact.
- **Proposition 3.2.** *O-modular can be replaced by the following MP rule.*

$$\frac{\Gamma \vdash \Delta, \alpha \to \beta \qquad \Gamma \vdash \Delta, \alpha}{\Gamma \vdash \Delta, \beta}\ (\mathrm{MP})$$

4 Quantum $\boldsymbol{\lambda}$-Calculus (1/8)

- **Definition 4.1** (*Type*). *Types* of λ-terms are inductively defined as follows:
  - α is a type if α is a type variable.
  - α is a type if α is $\perp$.
  - $(\alpha \times \beta)$ is a type if α and β are types.
  - $(\alpha \rightarrow \beta)$ is a type if α and β are types.

4 Quantum λ-Calculus (2/8)

- **Definition 4.4** (*Typed λ-Term*). *Typed λ-terms* (or simply *λ-terms*), are inductively defined as follows:
  - $x : \alpha$ is a λ-term under Γ if $x : \alpha \in \Gamma$.
  - $\varepsilon(M) : \alpha$ is a λ-term under Γ if $M : \bot$ is a λ-term under.
  - $\pi_1(M) : \alpha$ and $\pi_2(M) : \beta$ are λ-terms under Γ if $M : \alpha \times \beta$ is a λ-term under Γ.
  - $\langle M, N\rangle$: $\alpha \times \beta$ is a λ-term under $\Gamma_1 \cup \Gamma_2$ if $M : \alpha$ is a λ-term under Γ_1 and $N : \beta$ is a λ-term under Γ_2.

4 **Quantum $\boldsymbol{\lambda}$-Calculus** (3/8)

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- $(\lambda x.M) : \alpha \rightarrow \beta$ is a λ-term under Γ if $M : \beta$ is a λ-term under $\{x : \alpha\}$.
- $(MN) : \beta$ is a λ-term under $\Gamma_1 \cup \Gamma_2$ if $M : \alpha \rightarrow \beta$ is a λ-term under Γ_1 and $N : \alpha$ is a λ-term under Γ_2.
- $\tau(M, N) : \alpha \rightarrow \perp$ is a λ-term under Γ if $M : \alpha \rightarrow \beta$ is a λ-term under $\emptyset$ and $N : \beta \rightarrow \perp$ is a λ-term under Γ.
- $\eta(M) : \alpha$ is a λ-term under Γ if $M : (\alpha \rightarrow \perp) \rightarrow \perp$ is a λ-term under Γ.
- $\theta(M) : (\alpha \rightarrow \perp) \rightarrow \perp$ is a λ-term under Γ if $M : \alpha$ is a λ-term under Γ.

4 **Quantum λ-Calculus** (4/8)

- **Theorem 4.6** (Curry–Howard isomorphism between the formulas and the types). *There exists an isomorphism between the formulas of* **NQ** *and the types of the quantum λ-calculus.*
- **Theorem 4.7** (Curry–Howard isomorphism between the proofs and the terms). *There exists an isomorphism between the proofs of* **NQ** *and the λ-terms of the quantum λ-calculus.*

4 **Quantum λ-Calculus** (5/8)

- **Definition 4.8** (*Conversion*). A λ-term M is said to *be converted* if a subterm of M is substituted with another λ-term in the following way:
  - $\pi_1(N{:}\,\alpha, L{:}\,\beta)$ is substituted with $N{:}\,\alpha$.
    In NQ:

$$\dfrac{\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\\ \alpha\end{array} \qquad \begin{array}{c}\Gamma_2\\ \vdots\\ \beta\end{array}}{\alpha \wedge \beta}}{\alpha} \quad \Rightarrow \quad \begin{array}{c}\Gamma_1\\ \vdots\\ \alpha\end{array}$$

4 **Quantum λ-Calculus** (6/8)

- $\pi_2(N{:}\,\alpha, L{:}\,\beta)$ is substituted with $L{:}\,\beta$.
- $(\lambda x : \alpha . N : \beta)(L : \alpha)$ is converted to $(N[x := L : \alpha]) : \beta$.
  In NQ:

$$\dfrac{\dfrac{\begin{array}{c}[\mathrm{a} : \alpha]\\ \vdots\\ \beta\end{array}}{\alpha \to \beta} \qquad \begin{array}{c}\Gamma\\ \vdots\\ \alpha\end{array}}{\beta} \quad\Rightarrow\quad \begin{array}{c}\Gamma\\ \vdots\\ \alpha\\ \vdots\\ \beta\end{array}$$

4 Quantum **λ**-Calculus (7/8)

- $\eta(\theta(M : \alpha))$ is converted to $M : \alpha$.

$$\dfrac{\dfrac{\begin{matrix}\Gamma\\ \vdots\\ \alpha\end{matrix}}{(\alpha \to \bot) \to \bot}}{\alpha} \quad\Rightarrow\quad \begin{matrix}\Gamma\\ \vdots\\ \alpha\end{matrix}$$

4 **Quantum λ-Calculus** (8/8)

- **Definition 4.9** (*Normal form*). A λ-term (or a proof) is said to be in its *normal form* if it cannot be further converted. A λ-term is said to be *normalizable* if there exists a conversion sequence that starts with itself and ends with its normal form.

5 **Strong Normalization** (1/1)

- **Definition 5.1** (*Strongly normalizing*). A λ-term is said to be *strongly normalizing* if there exists no infinite conversion sequence that starts with itself.
- **Theorem 5.2** (Strong normalization). *The λ-terms of the quantum λ-calculus are strongly normalizing.*

6 **Conclusion** (1/2)

- In this paper, we have presented a natural deduction system for orthomodular quantum logic and the corresponding λ-calculus.
- Proof theory and computational theory for quantum logic have not been thoroughly studied so far. One of the reasons for this is that quantum logic lacks a satisfactory implication operation.

6 **Conclusion** (2/2)

- By treating the Sasaki hook as a quasi-implication and adopting it as a basic operation, we have obtained a straightforward formalization of natural deduction for quantum logic and the corresponding λ-calculus.
- We hope that both systems will contribute to the study of proof theory and computational theory for orthomodular quantum logic.