Lectures on Ordinal Analysis *

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The lecture rely on the followings, especially on starred ones.

- [Buchholz75] W. Buchholz, Normalfunktionen und konstruktive Systeme von Ordinalzahlen. In: Diller, J., Müller, G. H. (eds.) Proof Theory Symposion Keil 1974, Lect. Notes Math. vol. 500, pp. 4-25, Springer (1975)
- [Buchholz92]* W. Buchholz, A simplified version of local predicativity, in *Proof Theory*, eds. P. H. G. Aczel, H. Simmons and S. S. Wainer (Cambridge UP,1992), pp. 115–147.
- [Buchholz00]* W. Buchholz, Review of the paper: A. Setzer, Well-ordering proofs for Martin-Löf type theory, Bulletin of Symbolic Logic 6 (2000) 478-479.
- [Jäger82]* G. Jäger, Zur Beweistheorie der Kripke-Platek Mengenlehre über den natürlichen Zahlen, Archiv f. math. Logik u. Grundl., 22(1982), 121-139.
- [Jäger83]* G. Jäger, A well-ordering proof for Feferman's theory T_0, Archiv f. math. Logik u. Grundl., 23(1983), 65-77.
- [Rathjen94]* M. Rathjen, Proof theory of reflection, Ann. Pure Appl. Logic 68 (1994) 181–224.
- [Rathjen05b] M. Rathjen, An ordinal analysis of parameter free Π^1_2-comprehension, Arch. Math. Logic 44 (2005) 263-362.
- (An ordinal analysis of set theory) [Jäger82]*.
- (Operator controlled derivations) A streamlined technique introduced in [Buchholz92]*, and its extension in [Rathjen94]*.
- (Shrewd cardinals) [Rathjen05b]
- (Well-foundedness proofs) Distinguished classes are introduced in [Buchholz75]. I have learnt it in [Jäger83]* and its improved version in [Buchholz00]*.

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Plan

1. $\mathsf{KP}\omega$
2. Rathjen's analysis of Π_3-reflection
   Well-foundedness proof in $\mathsf{KP}\Pi_3$ (skipped)
3. First-order reflection
4. First-order reflection (contd.)
5. Π^1_1-reflection
6. Π^1_1-reflection (contd.)
7. Π^1_1-reflection (contd.)
8. Π_1-collection
9. Π_1-collection (contd.)

An ordinal α is said to be *recursive* iff there exists a recursive (computable) well ordering on ω of type α. ω_1^{CK} (*Church-Kleene* ω_1) denotes the least non-recursive ordinal.

Definition 0.1 1. $Prg[\prec, U] :\Leftrightarrow \forall x[\forall y \prec x(y \in U) \to x \in U]$
(U is *progressive* with respect to $\prec$).

2. $\mathrm{TI}[\prec, A] :\Leftrightarrow Prg[\prec, A] \to \forall x\, A(x)$ for formulas $A(x)$, and
$\mathrm{TI}[\prec, U] \Leftrightarrow Prg[\prec, U] \to \forall x\, U(x)$ (transfinite induction on $\prec$).

3. Let $\prec$ be a computable strict partial order on ω. If $\prec$ is well-founded, then let $|n|_\prec := \sup\{|m|_\prec + 1 : m \prec n\}$, and $|\prec| := \sup\{|n|_\prec + 1 : n \in \omega\}$ (the order type of $\prec$). Otherwise let $|\prec| := \omega_1^{CK}$.

Definition 0.2 For a theory T comprising elementary recursive arithmetic EA the *proof-theoretic ordinal* $|T|$ of T is defined by

$$|T| := \sup\{|\prec| : T \vdash \mathrm{TI}[\prec, U] \text{ for some recursive well order } \prec\} \tag{1}$$

where U is a fresh predicate constant.

Now, most brutally speaking, the aim of the ordinal analysis is to compute and/or describe the proof-theoretic ordinals of natural theories, thereby measuring the proof-theoretic strengths of theories with respect to Π^1_1-consequences.

1 Ordinal analysis of $\mathsf{KP}\omega$

1.1 Kripke-Platek set theory

A fragment KP of Zermelo-Fraenkel set theory ZF, Kripke-Platek set theory, is introduced Let $\mathcal{L}_{set} = \{\in, =\}$ be the set-theoretic language. In this section we deal only with set-theoretic models $\langle X; \in\restriction(X \times X)\rangle$, and the model is identified with the sets X.

Definition 1.1 $(\Delta_0, \Sigma_1, \Pi_2, \Sigma)$

1. A set-theoretic formula is said to be a Δ_0-*formula* if every quantifier occurring in it is *bounded* by a set. *Bounded quantifiers* is of the form $\forall x \in u, \exists x \in u$.

2. A formula of the form $\exists xA$ with a Δ_0-matrix A is a Σ_1-*formula*.

   Its dual $\forall xA$ is a Π_1-*formula*.

3. The set of Σ-formulas [Π-formulas] is the smallest class including Δ_0-formulas, closed under positive operations $\wedge, \vee$, bounded quantifications $\forall x \in u, \exists x \in u$, and existential (unbounded) quantification $\exists x$ [universal (unbounded) quantification $\forall x$], resp.

   For example $\forall x \in u \exists y A\,(A \in \Delta_0)$ is a Σ-formula but not a Σ_1-formula.

4. A formula of the form $\forall xA$ with a Σ_1-matrix A is a Π_2-*formula*.

We see easily that Δ_0-formulas are *absolute* in the sense that for any transitive sets $X \subset Y$ (X is transitive iff $\forall y \in X \forall x \in y(x \in X)$), $X \models A[\bar{x}] \Leftrightarrow Y \models A[\bar{x}]$ for any Δ_0-formula A and $\bar{x} = x_1, \ldots, x_n$ with $x_i \in X$.

Definition 1.2 Axioms of KP are **Extensionality** $\forall a, b[\forall x \in a(x \in b) \wedge \forall x \in b(x \in a) \to a = b]$, **Null set**(the empty set $\emptyset$ exists), **Pair** $\forall x, y \exists a(x \in a \wedge y \in a)$, **Union** $\forall a \exists b \forall x \in a \forall y \in x(y \in b)$, and the following three schemata.

Δ_0-Separation For any set a and any Δ_0-*formula* A, the set $b = \{x \in a : A(x)\}$ exists. Namely $\exists b \forall x[x \in b \leftrightarrow x \in a \wedge A(x)]$.

Δ_0-Collection $\forall x \in a \exists y\, A(x, y) \to \exists b \forall x \in a \exists y \in b\, A(x, y)$ for Δ_0-*formulas* A.

Foundation or $\in$-Induction $\forall x[\forall y \in xF(y) \to F(x)] \to \forall xF(x)$ for *arbitrary formula* F.

$\mathsf{KP}\omega$ denotes KP *plus* Axiom of **Infinity** $\exists x \neq \emptyset \forall y \in x[y \cup \{y\} \in x]$.

1.2 Constructible hierarchy and admissible sets

The constructible hierarchy $\{L_\alpha : \alpha \in ON\}$.

1. $L_0 := \emptyset$.
2. $L_{\alpha+1}$ is the collection of all definable sets in $(L_\alpha, \in)$.
3. $L_\lambda := \bigcup_{\alpha<\lambda} L_\alpha$ for limits λ.
4. $L := \bigcup_{\alpha \in ON} L_\alpha$.

Note that $L_{\omega\alpha} \models \mathsf{KP} - (\Delta_0\textbf{-Collection})$ for $\alpha > 0$, and $\omega \in L_{\omega\alpha}$ if $\alpha > 1$.

Definition 1.3 1. A transitive set A is *admissible* if $(A; \in) \models \mathsf{KP}$.

2. An ordinal α is *admissible* if L_α is admissible.
3. A relation R on an admissible set A is *A-recursive* [*A-recursively enumerable, A-r.e.*] (*A-finite*) if R is Δ_1 [Σ_1] ($R \in A$), resp.
4. A function on an admissible set A is *A-recursive* if its graph is A-r.e.
5. An ordinal α is *recursively regular* iff $L_\alpha \models \mathsf{KP}\omega$.

Observe that an ordinal α is recursively regular iff α is a multiplicative principal number$> \omega$, and for any L_α-recursive function $f : \beta \to \alpha$ with a $\beta < \alpha$, $\sup\{f(\gamma) : \gamma < \beta\} < \alpha$ holds.

Theorem 1.4 (Π_2**-Reflection on** L)
For any Σ-predicate A

$$\mathsf{KP}\omega \vdash \forall x \in L \exists y \in L\, A(x, y) \to \exists z \in L \forall x \in z \exists y \in z\, A(x, y).$$

In particular for recursively regular ordinals Ω,

$$\forall \alpha < \Omega \exists \beta < \Omega\, A(\alpha, \beta) \to \exists \gamma < \Omega \forall \alpha < \gamma \exists \beta < \gamma\, A(\alpha, \beta).$$

Lemma 1.5 $|\mathsf{KP}\omega| \leq |\mathsf{KP}\omega|_\Sigma := \min\{\alpha : \forall A \in \Sigma(\mathsf{KP}\omega \vdash A \Rightarrow L_\alpha \models A)\}$.

Proof. Suppose $\mathsf{KP}\omega$ proves $\mathrm{TI}[\prec, U]$ for a computable order $\prec$ on ω, where a unary predicate U may occur in Foundation schema, but not in Δ_0-Separation nor Δ_0-Collection. Then $\forall n \in \omega \exists \alpha(\alpha = |n|_\prec = \sup\{|m|_\prec + 1 : m \prec n\})$ is provable in $\mathsf{KP}\omega$. Therefore $|\mathsf{KP}\omega| \leq |\mathsf{KP}\omega|_\Sigma$. □

The *Mostowski collapsing* clpse(b) of a set b is defined by $C_b(x) = \{C_b(y) : y \in x \cap b\}$ and $\mathrm{clpse}(b) := C_b(b) = \{C_b(x) : x \in b\}$.

Definition 1.6 We say that a class $\mathcal{C}$ is Π_n*-classes* for $n \geq 2$ if there exists a set-theoretic Π_n-formula $F(\bar{a})$ with parameters $\bar{a}$ such that for any transitive set P with $\bar{a} \subset P$, $P \in \mathcal{C} \Leftrightarrow P \models F(\bar{a})$ holds. For a whole universe L, $L \in \mathcal{C}$ denotes the formula $F(\bar{a})$. By a Π^1_0*-class* we mean a Π_n-class for some $n \geq 2$.

1.3 Buchholz' ψ-functions

In this section we work in $\mathsf{KP}\omega$.

We are in a position to introduce a collapsing function $\psi_\sigma(\alpha) < \sigma$ (even if $\alpha \geq \sigma$). The following definition is due to [Buchholz86].

Definition 1.7 Let $\Omega = \omega_1$ or $\Omega = \omega_1^{CK}$. Define simultaneously by recursion on ordinals $\alpha < \Gamma_{\Omega+1}$ the classes $\mathcal{H}_\alpha(X)\,(X \subset \Omega)$ and the ordinals $\psi_\Omega(\alpha)$ as follows.

$\mathcal{H}_\alpha(X)$ is the Skolem hull of $\{0, \Omega\} \cup X$ under the functions $+, \varphi$, and $\beta \mapsto \psi_\Omega(\beta)\,(\beta < \alpha)$.

Let

$$\psi_\Omega(\alpha) = \min(\{\Omega\} \cup \{\beta < \Omega : \mathcal{H}_\alpha(\beta) \cap \Omega \subset \beta\}) \tag{2}$$

Let us interpret $\Omega = \omega_1$. Then we see readily that $\mathcal{H}_\alpha(X)$ is countable for any countable X.

To see that the ordinal $\psi_\Omega(\alpha)$ could be defined, it suffices to show the existence of an ordinal $\beta < \Omega$ such that $\mathcal{H}_\alpha(\beta) \cap \Omega \subset \beta$: let $\beta = \sup\{\beta_n : n \in \omega\}$ with $\beta_{n+1} = \min\{\beta < \Omega : \mathcal{H}_\alpha(\beta_n) \cap \Omega \subset \beta\}$ and $\beta_0 = 0 < \Omega$. Then $\mathcal{H}_\alpha(\beta) \cap \Omega \subset \beta$ since $\mathcal{H}_\alpha(\beta) = \bigcup_n \mathcal{H}_\alpha(\beta_n)$, and $\beta < \Omega$ since $\Omega > \omega$ is regular.

The ordinal $\psi_{\Omega_1}(\varepsilon_{\Omega_1+1})$ is called the *Bachmann-Howard ordinal*.

Proposition 1.8 *1.* $\alpha_0 \leq \alpha_1 \wedge X_0 \subset X_1 \Rightarrow \mathcal{H}_{\alpha_0}(X_0) \subset \mathcal{H}_{\alpha_1}(X_1)$.

2. $\mathcal{H}_\alpha(\psi_\Omega(\alpha)) \cap \Omega = \psi_\Omega(\alpha)$ *and* $\psi_\Omega(\alpha) \notin \mathcal{H}_\alpha(\psi_\Omega(\alpha))$.

3. $\alpha_0 \leq \alpha \Rightarrow \psi_\Omega(\alpha_0) \leq \psi_\Omega(\alpha) \wedge \mathcal{H}_{\alpha_0}(\psi_\Omega(\alpha_0)) \subset \mathcal{H}_\alpha(\psi_\Omega(\alpha))$.

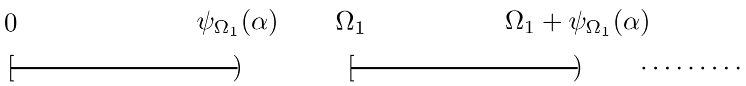
4. $\alpha_0 \in \mathcal{H}_\alpha(\psi_\Omega(\alpha)) \cap \alpha \Rightarrow \psi_\Omega(\alpha_0) < \psi_\Omega(\alpha)$. *Therefore* $\alpha_0 \in \mathcal{H}_{\alpha_0}(\psi_\Omega(\alpha_0)) \wedge \alpha \in \mathcal{H}_\alpha(\psi_\Omega(\alpha)) \Rightarrow (\alpha_0 < \alpha \leftrightarrow \psi_\Omega(\alpha_0) < \psi_\Omega(\alpha))$.

5. $\psi_\Omega(\alpha)$ *is a strongly critical number such that* $\psi_\Omega(\alpha) < \Omega$.

6. $\gamma \in \mathcal{H}_\alpha(\beta) \Leftrightarrow \mathsf{SC}(\gamma) \subset \mathcal{H}_\alpha(\beta)$, *where* $\mathsf{SC}(0) = \mathsf{SC}(\Omega) = \emptyset$, $\mathsf{SC}(\gamma) = \{\gamma\}$ *if* $\gamma \neq \Omega$ *is strongly critical, and* $\mathsf{SC}(\varphi\gamma\delta) = \mathsf{SC}(\gamma + \delta) = \mathsf{SC}(\gamma) \cup \mathsf{SC}(\delta)$.

7. $\mathcal{H}_\alpha(\psi_\Omega(\alpha)) = \mathcal{H}_\alpha(0)$ *and* $\psi_\Omega(\alpha) = \min\{\xi : \xi \notin \mathcal{H}_\alpha(0) \cap \Omega\}$.

Proposition 1.8.7 means that $\psi_\Omega(\alpha)$ is the Mostowski's collapse of the point Ω in the iterated Skolem hull $\mathcal{H}_\alpha(0)$ of ordinals $\{0, \Omega\}$ under addition $+$ and the binary Veblen function φ. This suggests us that the ordinal $\psi_\Omega(\alpha)$ could be a substitute for Ω in a restricted situation.



1.4 Computable notation system $OT(\Omega)$ of ordinals

By Proposition 1.8.7 we have $\mathcal{H}_{\varepsilon_{\Omega+1}}(0) = \mathcal{H}_{\varepsilon_{\Omega+1}}(0) = \mathcal{H}_{\varepsilon_{\Omega+1}}(\psi_\Omega(\varepsilon_{\Omega+1}))$, and hence each ordinal below $\psi_\Omega(\varepsilon_{\Omega+1})$ can be denoted by terms built up from $0, \Omega, +, \varphi, \psi$. Although the representation is not uniquely determined from ordinals, e.g., $\psi_\Omega(\psi_\Omega(\Omega)) = \psi_\Omega(\Omega)$, α can be determined from the ordinal $\psi_\Omega(\alpha)$ if $\alpha \in \mathcal{H}_\alpha(0)$, cf. Propositions 1.8.4 and 1.8.7. We can devise a recursive notation system $OT(\Omega)$ of ordinals with this restriction in such a way that the following holds

Proposition 1.9 EA *proves that* $(OT(\Omega), <)$ *is a linear order.*

1.5 Ramified set theory

Definition 1.10 *RS-terms* t and their *levels* $|t|$ are defined recursively as follows.

1. For each ordinal $\alpha \in OT(\Omega) \cap (\Omega + 1)$, L_α is an RS-term of level $|\mathsf{L}_\alpha| = \alpha$.
2. Let $\theta(x, y_1, \ldots, y_n)$ be a formula in the set-theoretic language, and $s_1, \ldots, s_n$ be RS-terms such that $\max\{|s_i| : 1 \leq i \leq n\} < \alpha$. Then the formal expression $[x \in \mathsf{L}_\alpha : \theta^{\mathsf{L}_\alpha}(x, s_1, \ldots, s_n)]$ is an RS-term of level $|[x \in \mathsf{L}_\alpha : \theta^{\mathsf{L}_\alpha}(x, s_1, \ldots, s_n)]| = \alpha$.

RS denotes the set of all RS-terms.

Let $\theta(x_1, \ldots, x_n)$ be a formula such that each quantifier is bounded by a variable y, $Qx \in y$, all free variables occurring in θ are among the list $x_1, \ldots, x_n$, and each x_i occurs freely in θ. An *RS-formula* is obtained from such a formula $\theta(x_1, \ldots, x_n)$ by substituting RS-terms t_i for each x_i.

Let $\mathsf{k}(\mathsf{L}_\alpha) := \{\alpha\}$, $\mathsf{k}([x \in \mathsf{L}_\alpha : \theta^{\mathsf{L}_\alpha}(x, s_1, \ldots, s_n)]) = \{\alpha\} \cup \bigcup_{i \leq n} \mathsf{k}(s_i)$ and

$$\mathsf{k}(\theta(t_1, \ldots, t_n)) := \bigcup_{i \leq n} \mathsf{k}(t_i),\ |\theta(t_1, \ldots, t_n)| := \max\{|t_1|, \ldots, |t_n|, 0\}.$$

The bound L_Ω in $\exists x \in \mathsf{L}_\Omega$ and $\forall x \in \mathsf{L}_\Omega$ is the replacements of the unbounded quantifiers $\exists$ and $\forall$, resp.

Definition 1.11 Let s, t be RS-terms with $|s| < |t|$.

$$(s \dot{\in} t) :\equiv \begin{cases} B(s) & t \equiv [x \in \mathsf{L}_\alpha : B(x)] \\ \top & t \equiv \mathsf{L}_\alpha \end{cases}$$

where $\top$ denotes a true literal, e.g., $\emptyset \not\in \emptyset$.

We assign disjunctions or conjunctions to sentences as follows. When a disjunction $\bigvee(A_i)_{i \in J}$ [a conjunction $\bigwedge(A_i)_{i \in J}$] is assigned to A, we denote $A \simeq \bigvee(A_i)_{i \in J}$ [$A \simeq \bigwedge(A_i)_{i \in J}$], resp.

Definition 1.12 *1.* $(A_0 \lor A_1) :\simeq \bigvee(A_i)_{i\in J}$ and $(A_0 \land A_1) :\simeq \bigwedge(A_i)_{i\in J}$ with $J := 2$.

2. $(a \in b) :\simeq \bigvee(t\dot{\in}b \land t = a)_{t\in J}$ and $(a \not\in b) :\simeq \bigwedge(t\dot{\in}b \to t \neq a)_{t\in J}$ with $J := Tm(|b|) := \{t \in RS : |t| < |b|\}$.

3. Let a, b be set terms.
$(a \neq b) :\simeq \bigvee(\neg A_i)_{i\in J}$ and $(a = b) :\simeq \bigwedge(A_i)_{i\in J}$ with $J := 2$ and $A_0 :\equiv (\forall x \in a(x \in b))$, $A_1 :\equiv (\forall x \in b(x \in a))$.

4. $\exists x \in b\, A(x) :\simeq \bigvee(t\dot{\in}b \land A(t))_{t\in J}$ and $\forall x \in b\, A(x) :\simeq \bigwedge(t\dot{\in}b \to A(t))_{t\in J}$ with $J := Tm(|b|)$.

Lemma 1.13 $\forall i \in J\, (\mathsf{k}(i) \subset \mathsf{k}(A_i) \subset \mathsf{k}(A) \cup \mathsf{k}(i))$ *for* $A \simeq \bigvee(A_i)_{i\in J}$, *where* $\mathsf{k}(0) = \mathsf{k}(1) = \emptyset$.

The *rank* $\mathrm{rk}(A), \mathrm{rk}(a) < \Omega+\omega$ of RS-formulas A and RS-terms a are defined so that the followings hold for any formula A.

Proposition 1.14 *1.* $\mathrm{rk}(\mathcal{A}) \in \{\omega|\mathcal{A}| + n : n \in \omega\}$ *for RS-terms and RS-formulas* $\mathcal{A}$.

2. $\mathrm{rk}(B(t)) \in \{\omega|t| + n : n \in \omega\} \cup \{\mathrm{rk}(B(\mathsf{L}_0))\}$.

3. Let $A \simeq \bigvee(A_i)_{i\in J}$. *Then* $\forall i \in J(\mathrm{rk}(A_i) < \mathrm{rk}(A))$.

Definition 1.15 1. Let $B(x_1, \ldots, x_n)$ be a Δ_0-formula, and $a_1, \ldots, a_n \in RS$ be $|a_i| < \Omega$. Then $B(a_1, \ldots, a_n)$ is a $\Delta(\Omega)$*-formula*.

2. Let $A(x_1, \ldots, x_n)$ be a Σ-formula, and $a_1, \ldots, a_n \in RS$ be $|a_i| < \Omega$. Then $A^{(\mathsf{L}_\Omega)}(a_1, \ldots, a_n)$ is a $\Sigma(\Omega)$*-formula*, where for RS-terms c, $A^{(c)}$ denotes the result of replacing unbounded existential quantifiers $\exists x(\cdots)$ by $\exists x \in c(\cdots)$.

3. Let $B \equiv A^{(\mathsf{L}_\Omega)}$ be a $\Sigma(\Omega)$-formula, and $\alpha \in OT(\Omega) \cap \Omega$. Then $B^{(\alpha,\Omega)} \equiv A^{(\mathsf{L}_\alpha)}$. For $\Gamma \subset \Sigma(\Omega)$, $\Gamma^{(\alpha,\Omega)} := \{B^{(\alpha,\Omega)} : B \in \Gamma\}$.

Let us define a derivability relation $\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma$ for finite sets Θ of ordinals, $\gamma, a < \varepsilon_{\Omega+1}$, $b < \Omega + \omega$ and RS-sequents, i.e., finite sets of RS-formulas Γ.

Definition 1.16 $\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma$ holds if

$$\{\gamma, a, b\} \cup \mathsf{k}(\Gamma) \subset \mathcal{H}_\gamma[\Theta] \tag{3}$$

and one of the following cases holds:

($\bigvee$) There are $A \in \Gamma$ such that $A \simeq \bigvee(A_i)_{i\in J}$, an $i \in J$ with

$$|i| < a \tag{4}$$

and an $a(i) < a$ for which $\mathcal{H}_\gamma[\Theta] \vdash^{a(i)}_b \Gamma, A_i$ holds.

$$\frac{\mathcal{H}_\gamma[\Theta] \vdash^{a(i)}_b \Gamma, A_i}{\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma}\ (\bigvee)\ (|i| < a)$$

($\bigwedge$) There is an $A \in \Gamma$ such that $A \simeq \bigwedge(A_i)_{i\in J}$, and for each $i \in J$, there is an $a(i)$ such that $a(i) < a$ for which $\mathcal{H}_\gamma[\Theta \cup \mathsf{k}(i)] \vdash^{a(i)}_b \Gamma, A_i$ holds.

$$\frac{\{\mathcal{H}_\gamma[\Theta \cup \mathsf{k}(i)] \vdash^{a(i)}_b \Gamma, A_i\}_{i\in J}}{\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma}\ (\bigwedge)$$

(cut) There are C and $a_0 < a$ such that $\mathrm{rk}(C) < b$, $\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_b \Gamma, \neg C$ and $\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_b C, \Gamma$.

$$\frac{\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_b \Gamma, \neg C \quad \mathcal{H}_\gamma[\Theta] \vdash^{a_0}_b C, \Gamma\ (\mathrm{rk}(C) < b)}{\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma}\ (cut)$$

($\Delta_0(\Omega)$-Coll) $b \geq \Omega$, and there are a formula $C \in \Sigma(\Omega)$ and an $a_0 < a$ such that $\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_b \Gamma, C$ and $\mathcal{H}_\gamma[\Theta \cup \{\alpha\}] \vdash^{a_0}_b \Gamma, \neg C^{(\alpha,\Omega)}$ for every $\alpha < \Omega$.

$$\frac{\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_b \Gamma, C \quad \{\mathcal{H}_\gamma[\Theta \cup \{\alpha\}] \vdash^{a_0}_b \neg C^{(\alpha,\Omega)}, \Gamma\}_{\alpha<\Omega}}{\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma}\ (\Delta_0(\Omega)\text{-Coll})$$

Lemma 1.17 (Tautology) $\mathcal{H}_0[\mathsf{k}(A)] \vdash^{2d}_0 \neg A, A$ *with* $d = \mathrm{rk}(A)$.

Lemma 1.18 (Inversion)
$\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma, A \Rightarrow \forall i \in J(\mathcal{H}_\gamma[\Theta \cup \mathsf{k}(i)] \vdash^a_b \Gamma, A_i)$ *for* $A \simeq \bigwedge(A_i)_{i\in J}$.

Lemma 1.19 (Boundedness) *Let* $a \leq \beta \in \mathcal{H}_\gamma[\Theta] \cap \Omega$ *and* $\Lambda \subset \Sigma(\Omega)$. *Then* $\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma, \Lambda \Rightarrow \mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma, \Lambda^{(\beta,\Omega)}$.

Lemma 1.20 (Embedding)
Let $\Gamma[\vec{x} := \vec{a}]\,(\vec{a} \subset RS)$ *denote a closed instance of a sequent* Γ *with restriction of unbounded quantifiers to* L_Ω. *Assume* $\mathsf{KP}\omega \vdash \Gamma$. *Then*

$$\exists m, l < \omega \forall \vec{a} \subset RS[\mathcal{H}_0[\mathsf{k}(\vec{a})] \vdash^{\Omega+l}_{\Omega+m} \Gamma[\vec{x} := \vec{a}]]$$

where $\mathsf{k}(\vec{a}) = \mathsf{k}(a_1) \cup \cdots \mathsf{k}(a_n)$ *for* $\vec{a} = (a_1, \ldots, a_n)$.

Let $\theta_c(a)$ be the c-th iterate of $\theta_1(a) = \omega^a$. $\theta_0(a) = a$, $\theta_{c\dot{+}d}(a) = \theta_c(\theta_d(a))$, and $\theta_{\omega^c}(a) = \varphi_c(a)$.

Lemma 1.21 (Predicative Cut-elimination)
$\mathcal{H}_\gamma[\Theta] \vdash^a_{b+c} \Gamma \Rightarrow \mathcal{H}_\gamma[\Theta] \vdash^{\theta_c(a)}_b \Gamma$ *if* $\neg(b < \Omega \leq b + c)$.

Theorem 1.22 (Collapsing)
Suppose

$$\Theta \subset \mathcal{H}_\gamma(\psi_\Omega(\gamma)) \tag{5}$$

for a finite set Θ *of ordinals, and* $\Gamma \subset \Sigma(\Omega)$. *Then for* $\hat{a} = \gamma + \omega^a$ *and* $\beta = \psi_\Omega(\hat{a})$

$$\mathcal{H}_\gamma[\Theta] \vdash^a_\Omega \Gamma \Rightarrow \mathcal{H}_{\hat{a}+1}[\Theta] \vdash^\beta_\beta \Gamma.$$

Proof. This is seen by induction on a. Observe that $\mathsf{k}(\Gamma)\cup\{\beta\}\subset\mathcal{H}_{\hat{\alpha}+1}[\Theta]$ by $\gamma<\hat{\alpha}+1$ and (3).

Case 1. The last inference is a $(\bigvee)$.
Let $A\in\Gamma$ be such that $A\simeq\bigvee(A_i)_{i\in J}$, and for an $i\in J$ and an $a(i)<a$

$$\frac{\mathcal{H}_\gamma[\Theta]\vdash^{a(i)}_{\Omega}\Gamma,A_i}{\mathcal{H}_\gamma[\Theta]\vdash^{a}_{\Omega}\Gamma}\ (\bigvee)$$

By IH it suffices to show $|i|<\psi_\Omega(\hat{a})$ for (4). We can assume $\mathsf{k}(i)\subset\mathsf{k}(A_i)$. Then $|i|\in\mathsf{k}(A_i)\subset\mathcal{H}_\gamma[\Theta]\subset\mathcal{H}_\gamma(\psi_\Omega(\gamma))$ by (3) and the assumption (5). On the other hand we have $|i|<\Omega$. Hence $|i|\in\mathcal{H}_{\hat{a}}(\psi_\Omega(\hat{a}))\cap\Omega=\psi_\Omega(\hat{a})$.

Case 2. The last inference is a $(\bigwedge)$.
Let $A\in\Gamma$ be such that $A\simeq\bigwedge(A_i)_{i\in J}$, and for each $i\in J$, there are $a(i)<a$ such that

$$\frac{\{\mathcal{H}_\gamma[\Theta\cup\mathsf{k}(i)]\vdash^{a(i)}_{\Omega}\Gamma,A_i\}_{i\in J}}{\mathcal{H}_\gamma[\Theta]\vdash^{a}_{\Omega}\Gamma}\ (\bigwedge)$$

By IH it suffices to show that $\forall i\in J(\mathsf{k}(i)\subset\mathcal{H}_\gamma(\psi_\Omega(\gamma)))$. For example consider the case when $A\equiv(\forall x\in u\,B(x))$ for a set term u. Then $J=\{t\in RS:|t|<|u|\}$. Since A is a $\Sigma(\Omega)$-sentence, we have $|a|<\Omega$. On the other hand we have $|u|\in\mathcal{H}_\gamma[\Theta]$ for $|u|=\max\mathsf{k}(u)$, and hence $\mathsf{k}(i)\subset|u|\in\mathcal{H}_\gamma(\psi_\Omega(\gamma))\cap\Omega=\psi_\Omega(\gamma)$ for any $i\in J$.

Case 3. The last inference is a $(\Delta_0(\Omega)$-Coll$)$.
There are a sentence $C\in\Sigma(\Omega)$ and an $a_0<a$ such that

$$\frac{\mathcal{H}_\gamma[\Theta]\vdash^{a_0}_{\Omega}\Gamma,C\quad\{\mathcal{H}_\gamma[\Theta\cup\{\alpha\}]\vdash^{a_0}_{\Omega}\neg C^{(\alpha,\Omega)},\Gamma\}_{\alpha<\Omega}}{\mathcal{H}_\gamma[\Theta]\vdash^{a}_{\Omega}\Gamma}\ (\Delta_0(\Omega)\text{-Coll})$$

Let $\widehat{a_0}=\gamma+\omega^{a_0}$ and $\beta_0=\psi_\Omega(\widehat{a_0})$. IH yields $\mathcal{H}_{\hat{a}+1}[\Theta]\vdash^{\beta_0}_{\beta_0}\Gamma,C$. Boundedness 1.19 yields $\mathcal{H}_{\widehat{a_0}+1}[\Theta]\vdash^{\beta_0}_{\beta_0}\Gamma,C^{(\beta_0,\Omega)}$, where $\beta_0\in\mathcal{H}_{\widehat{a_0}+1}[\Theta]$. On the other hand we have $\mathcal{H}_\gamma[\Theta\cup\{\beta_0\}]\vdash^{a_0}_{\Omega}\neg C^{(\beta_0,\Omega)},\Gamma$, and $\mathcal{H}_{\widehat{a_0}+1}[\Theta]\vdash^{a_0}_{\Omega}\neg C^{(\beta_0,\Omega)},\Gamma$. IH yields $\mathcal{H}_{\widehat{a_0}+\omega^{a_0}+1}[\Theta]\vdash^{\beta_1}_{\beta}\neg C^{(\beta_0,\Omega)},\Gamma$, where $\beta_1=\psi_\Omega(\widehat{a_0}+\omega^{a_0})$ with $\widehat{a_0}+\omega^{a_0}=\gamma+\omega^{a_0}+\omega^{a_0}<\hat{a}$. A (cut) with $\mathrm{rk}(C^{(\beta_0,\Omega)})<\beta$ yields $\mathcal{H}_{\hat{a}+1}[\Theta]\vdash^{\beta}_{\beta}\Gamma$.

Case 4. The last inference is a (cut).

$$\frac{\mathcal{H}_\gamma[\Theta]\vdash^{a_0}_{\Omega}\Gamma,\neg C\quad\mathcal{H}_\gamma[\Theta]\vdash^{a_0}_{\Omega}C,\Gamma}{\mathcal{H}_\gamma[\Theta]\vdash^{a}_{\Omega}\Gamma}\ (cut)$$

We obtain $\mathrm{rk}(C)<\Omega$, and $\mathrm{rk}(C)\in\mathcal{H}_\gamma[\Theta]\cap\Omega\subset\psi_\Omega(\gamma)\le\beta$. IH followed by a (cut) yields the lemma. □

Lemma 1.23 (Truth)
If $\mathcal{H}_\gamma[\Theta]\vdash^{\alpha}_{\Omega}\Gamma$ *with* $\Gamma\subset\Delta(\Omega)$, *then* $L_\Omega\models\Gamma$.

Theorem 1.24 $\mathsf{KP}\omega \vdash \Gamma$ and $\Gamma \subset \Sigma(\Omega_1) \Rightarrow \exists m < \omega \left[L_\Omega \models \Gamma^{(\psi_\Omega(\omega_m(\Omega+1)),\Omega)} \right]$.

Proof. Let $\mathsf{KP}\omega \vdash \Gamma$ for a set Γ of Σ-sentences. By Embedding 1.20 pick an $m < \omega$ such that $\mathcal{H}_0[\emptyset] \vdash^{\Omega+m}_{\Omega+m} \Gamma$. Predicative Cut-elimination 1.21 yields $\mathcal{H}_0[\emptyset] \vdash^{a}_{\Omega} \Gamma$ for $a = \omega_m(\Omega + m)$. Let $\beta = \psi_\Omega(\hat{a})$ with $\hat{a} = \omega^a = \omega_{m+1}(\Omega + m)$. We then obtain $\mathcal{H}_{\hat{a}+1}[\emptyset] \vdash^{\beta}_{\beta} \Gamma$ by Collapsing 1.22, and $\mathcal{H}_{\hat{a}+1}[\emptyset] \vdash^{\beta}_{\beta} \Gamma^{(\beta,\Omega)}$ by Boundedness 1.19. We see $L_\Omega \models \Gamma^{(\beta,\Omega)}$ from Truth 1.23. From $\beta < \psi_\Omega(\omega_{m+2}(\Omega + 1))$ and the persistency of Σ-formulas, we conclude $L_\Omega \models \Gamma^{(\psi_\Omega(\omega_{m+2}(\Omega+1)),\Omega)}$. □

1.6 Well-foundedness proof in $\mathsf{KP}\omega$

In this subsection $\alpha, \beta, \gamma, \delta, \ldots$ range over ordinal terms in $OT(\Omega)$, and $<$ denotes the relation between ordinal terms defined in Definition **??**. An ordinal term α is identified with the set $\{\beta \in OT(\Omega) : \beta < \alpha\}$. For ordinal terms α, β, ordinal terms $\alpha + \beta$ and ω^α are defined trivially.

In this subsection we show that the theory ID for non-iterated positive elementary inductive definitions on $\mathbb{N}$ proves the fact that the relation $<$ on $OT(\Omega)$ is well-founded up to each $\alpha < \psi_\Omega(\varepsilon_{\Omega+1})$.

Theorem 1.25 *For* each $n < \omega$

$$\mathsf{ID} \vdash \mathrm{TI}[<\restriction \psi_\Omega(\omega_n(\Omega + 1)), B]$$

for any formula B in the language $\mathcal{L}(\mathsf{ID})$.

Acc denotes the accessible part of $<$ in $OT(\Omega)$, which is defined in ID as the least fixed point $P_\mathcal{A}$ of the operator $\mathcal{A}(X, \alpha) :\Leftrightarrow \alpha \subset X \Leftrightarrow (\forall\beta < \alpha(\beta \in X))$. It suffices to show the following, which is equivalent to Theorem 1.25.

Theorem 1.26 *For each $\alpha < \psi_\Omega(\varepsilon_{\Omega+1})$, $\mathsf{ID} \vdash \alpha \in Acc$.*

The least fixed point Acc enjoys $\forall\alpha(\alpha \subset Acc \to \alpha \in Acc)$, and $\forall\alpha(\alpha \subset F \to \alpha \in F) \to Acc \subset F$. From these we see easily that Acc is closed under $+, \varphi$ besides $0 \in Acc$. Hence we obtain $\Gamma_0 = \psi_\Omega(0) \in Acc$. Likewise $\Gamma_1 = \psi_\Omega(1) \in Acc$ follows. To prove $\psi_\Omega(\Omega) \in Acc$, we need to show $\psi_\Omega(\alpha) \in Acc$ for any $\alpha < \Omega$ such that $\psi_\Omega(\alpha)$ is an ordinal term, i.e., $G(\alpha) < \alpha$. This means that when $\psi_\Omega(\beta)$ occurs in α, then $\beta < \alpha$ holds. Thus we have a chance to prove inductively that $\psi_\Omega(\alpha) \in Acc$. The ordinal term α is built from 0, Ω and some ordinal terms $\psi_\Omega(\beta)$ with $\beta < \alpha$ by $+, \varphi$. Let us assume that each of ordinals $\psi_\Omega(\beta) < \Omega$ occurring in α is in $W_0 = Acc \cap \Omega$, and denote the set of such ordinals α by M_1. Though we don't have $\Omega \in Acc$ in hand (since this means that $OT(\Omega) \cap \Omega$ is well-founded, which is the fact we are going to prove), Ω is in the accessible part W_1 of the set M_1. It turns out that W_1 is progressive on M_1, and $\Omega \in W_1$. Moreover $\omega^{\Omega+1} \in W_1$ is seen as for the jump set for epsilon numbers. In this way we see that $\alpha \in W_1$, i.e., $\psi_\Omega(\alpha) \in W_0$ for *each* $\alpha < \varepsilon_{\Omega+1}$.

Let $\mathsf{SC}(\alpha)$ denote the set of strongly critical parts of α defined in Proposition 1.8.6, and let $\mathsf{SC}_\Omega(\alpha) = \mathsf{SC}(\alpha) \cap \Omega$.

Definition 1.27 $M_1 = \{\alpha \in OT(\Omega) : \mathsf{SC}_\Omega(\alpha) \subset W_0\}$.

Proposition 1.28 *$G(\beta) < \alpha \Rightarrow \mathsf{SC}_\Omega(\beta) < \psi_\Omega(\alpha)$ for $\psi_\Omega(\alpha) \in OT(\Omega)$.*

Proof. By induction on the length of ordinal terms β. Assume $G(\beta) < \alpha$. By IH we can assume $\beta = \psi_\Omega(\gamma)$. Then $\gamma \in G(\beta)$ and $\mathsf{SC}_\Omega(\beta) = \{\beta\}$. Hence $\gamma < \alpha$ and $\beta < \psi_\Omega(\alpha)$. □

In what follows we work in ID except otherwise stated.

Lemma 1.29 $M_1 \cap \Omega = W_0$.

$$\mathcal{A}(X) := \{\alpha \in M_1 : M_1 \cap \alpha \subset X\}.$$

Proposition 1.30 *For each formula F, $\mathcal{A}(F) \subset F \to \Omega \in F$.*

Proof. Assuming $\mathcal{A}(F) \subset F$, we see $\alpha \in W_0 \Rightarrow \alpha \in F$ by induction on $\alpha \in W_0$. □

Lemma 1.31 *For each formula F, $\mathcal{A}(F) \subset F \to \mathcal{A}(\mathsf{j}[F]) \subset \mathsf{j}[F]$, where $\mathsf{j}[F] := \{\beta \in OT(\Omega) : \forall\alpha(M_1 \cap \alpha \subset F \to M_1 \cap (\alpha + \omega^\beta) \subset F)\}$.*

Lemma 1.32 *For each formula F and* each $n < \omega$, *$\mathcal{A}(F) \subset F \to \omega_n(\Omega + 1) \in F$.*

$$\alpha \in W :\Leftrightarrow (\psi_\Omega(\alpha) \in OT(\Omega) \to \psi_\Omega(\alpha) \in W_0).$$

Lemma 1.33 $\mathcal{A}(W) \subset W$.

Proof. Assume $\alpha \in \mathcal{A}(W)$ and $\psi_\Omega(\alpha) \in OT(\Omega)$. Then $\alpha \in M_1$ and $M_1 \cap \alpha \subset W$. We show

$$\gamma < \psi_\Omega(\alpha) \to \gamma \in W_0$$

by induction on the length of ordinal terms γ. We can assume that $\gamma = \psi_\Omega(\beta)$. Then $\beta < \alpha$. We see $\beta \in M_1$ from IH. Therefore $\beta \in M_1 \cap \alpha \subset W$, which yields $\gamma = \psi_\Omega(\beta) \in W_0$. Therefore $\psi_\Omega(\alpha) \subset W_0$. □

Let us show Theorem 1.26. We show that ID proves $\psi_\Omega(\omega_n(\Omega + 1)) \in W_0$ for each $n < \omega$. By Lemmas 1.32 and 1.33 we obtain $\omega_n(\Omega + 1) \in W$. Thus $\psi_\Omega(\omega_n(\Omega + 1)) \in W_0$ by the definition of W.

2 Rathjen's analysis of Π_3-reflection

Given an analysis of $\mathsf{KP}\omega$ for a single recursively regular ordinal, it is not hard to extend it to an analysis of theories of recursively regular ordinals of a given order type, e.g., to $\mathsf{KP}\ell$, or equivalently to Π^1_1-CA+BI. Or to an iteration of recursively regularities in another manner. Specifically an ordinal analysis of $\mathsf{KP}M$ for recursively Mahlo ordinals is not an obstacle.

Let us introduce a Π_i-recursively Mahlo operation RM_i and its iterations. A Π_i-recursively Mahlo operation RM_i for $2 \leq i < \omega$, is defined through a universal Π_i-formula $\Pi_i(a)$ such that for each Π_i-formula $\varphi(x)$ there exists a natural number n such that $\mathsf{KP} \vdash \forall x[\varphi(x) \leftrightarrow \Pi_i(\langle n, x\rangle)]$. Let $\mathcal{X}$ be a collection of sets.

$$P \in RM_i(\mathcal{X}) \quad :\Leftrightarrow \quad \forall b \in P\,[P \models \Pi_i(b) \to \exists Q \in \mathcal{X} \cap P(b \in Q \models \Pi_i(b))]$$
$$\text{(read: } P \text{ is } \Pi_i\text{-reflecting on } \mathcal{X}.)$$

Let $RM_i = RM_i(V)$, and V is Π_i-*reflecting* if $V \in RM_i$. Under the axiom $V = L$ of constructibility, $V \in RM_2$ iff $V \models \mathsf{KP}\omega$, and $V \in RM_2(RM_2)$ iff V is *recursively Mahlo* universe. When $V = L_\sigma$, the ordinal σ is recursively Mahlo ordinal.

Let $\mathsf{KP}M$ denote a set theory for recursively Mahlo universes. For an ordinal analysis of $\mathsf{KP}M$, it suffices for us to have two step collapsings $\alpha \mapsto \sigma = \psi_M(\alpha) \in RM_2$ and $(\sigma, \beta) \mapsto \psi_\sigma(\beta)$.

Assume that $P \in \mathcal{X}$ is given by a Δ_0-formula. Then there exists a Π_{i+1}-formula rm_i such that for any non-empty transitive sets $P \in V \cup \{V\}$, $P \in RM_i(\mathcal{X}) \leftrightarrow rm_i^P$, where rm_i^P denotes the result of restricting unbounded quantifiers in rm_i to P.

An iteration of RM_i along a definable relation $\prec$ is defined as follows.

$$P \in RM_i(a; \prec) :\Leftrightarrow a \in P \in \bigcap\{RM_i(RM_i(b; \prec)) : b \in P \models b \prec a\}.$$

Assume that $b \prec a$ is given by a Σ_1-formula. Then there exists a Π_{i+1}-formula $rm_i(a, \prec)$ such that for any non-empty transitive sets $P \in V \cup \{V\}$ and $a \in P$, $P \in RM_i(a; \prec) \leftrightarrow rm_i^P(a, \prec)$.

For $2 \leq N < \omega$, $\mathsf{KP}\Pi_N$ denotes a set theory for Π_N-reflecting universes V, which is obtained from $\mathsf{KP}\omega$ by adding an axiom $V \in RM_N$ (the axiom for Π_N-reflection) stating that its universe is Π_N-reflecting. This means that for each Π_N-formula φ, $\varphi(a) \to \exists c[ad_N^c \wedge a \in c \wedge \varphi^c(a)]$ is an axiom, where $ad_2^c :\equiv (\forall x \in c \forall y \in x(y \in c))$, i.e., c is transitive, and for $N > 2$, $ad \equiv ad_N$ denotes a Π_3-sentence such that $P \models ad \Leftrightarrow P \models \mathsf{KP}\omega$ for any transitive and well-founded sets P. $\mathsf{KP}\Pi_2$ is a subtheory of $\mathsf{KP}\omega + (V = L)$, which is interpreted in $\mathsf{KP}\omega$: $\mathsf{KP}\omega + (V = L) \vdash \varphi \Rightarrow \mathsf{KP}\omega \vdash \varphi^L$, cf. Theorem 1.4.

$\mathsf{KP}\Pi_{N+1}$ is much stronger than $\mathsf{KP}\Pi_N$ since Π_N-recursively Mahlo operation RM_N can be iterated in $\mathsf{KP}\Pi_{N+1}$. For example, $\mathsf{KP}\Pi_{N+1}$ proves $\forall \alpha \in ON[V \in RM_N(\alpha; <)]$ by induction on ordinals α. Suppose $\forall \beta < \alpha[V \in RM_N(\beta; <)]$. Let φ be a Π_N-formula such that $V \models \varphi$, and $\beta < \alpha$. We can reflect a Π_{N+1}-formula $V \in RM_N(\beta; <) \wedge \varphi$, and obtain a set P such that $P \in RM_N(\beta; <) \wedge P \models \varphi$. Hence $V \in RM_N(\alpha; <)$. This means that V is in the diagonal intersection $\triangle_\alpha RM_N(\alpha; <)$, i.e., $V \in \bigcap\{RM_N(\alpha; <) : \alpha \in ON \cap V\}$. Since this is a Π_{N+1}-formula, the Π_{N+1}-reflecting universe V reflects it: there exists a set $P \in V$ such that P is in the diagonal intersection, i.e., $P \in \bigcap\{RM_N(\alpha; <) : \alpha \in ON \cap P\}$, and so forth.

Let $ON \subset V$ denote the class of ordinals, $ON^\varepsilon \subset V$ and $<^\varepsilon$ be Δ-predicates such that for any transitive and well-founded model V of $\mathsf{KP}\omega$, $<^\varepsilon$ is a well order of type $\varepsilon_{\mathbb{K}+1}$ on ON^{ε} for the order type $\mathbb{K}$ of the class ON in V. $\lceil\omega_n(\mathbb{K}+1)\rceil \in ON^{\varepsilon}$ denotes the code of the 'ordinal' $\omega_n(\mathbb{K}+1)$, which is assumed to be a closed 'term' built from the code $\lceil\mathbb{K}\rceil$ and n, e.g., $\lceil\alpha\rceil = \langle 0, \alpha\rangle$ for $\alpha \in ON$, $\lceil\mathbb{K}\rceil = \langle 1, 0\rangle$ and $\lceil\omega_n(\mathbb{K}+1)\rceil = \langle 2, \langle 2, \cdots \langle 2, \langle 3, \lceil\mathbb{K}\rceil, \langle 0, 1\rangle\rangle\rangle \cdots\rangle\rangle$.

$<^{\varepsilon}$ is assumed to be a standard epsilon order with base $\mathbb{K}$ (not on $\mathbb{N}$, but on V) such that $\mathsf{KP}\omega$ proves the fact that $<^{\varepsilon}$ is a linear ordering, and for any formula φ and each $n < \omega$,

$$\mathsf{KP}\omega \vdash \forall x(\forall y <^{\varepsilon} x\, \varphi(y) \to \varphi(x)) \to \forall x <^{\varepsilon} \lceil\omega_n(\mathbb{K}+1)\rceil \varphi(x) \tag{6}$$

Theorem 2.1 ([A14a])
For each $N \geq 2$, $\mathsf{KP}\Pi_{N+1}$ is Π_{N+1}-conservative over the theory

$$\mathsf{KP}\omega + \{V \in RM_N(\lceil\omega_n(\mathbb{K}+1)\rceil; <^{\varepsilon}) : n \in \omega\}.$$

From (6) we see that $\mathsf{KP}\Pi_{N+1}$ proves $V \in RM_N(\lceil\omega_n(\mathbb{K}+1)\rceil; <^{\varepsilon})$ for each $n \in \omega$.

Let us consider the simplest case $N = 3$, i.e., an ordinal analysis of set theory $\mathsf{KP}\Pi_3$ for Π_3-reflecting universe. It turns out that $\mathsf{KP}\Pi_3$ is proof-theoretically reducible to iterations of recursively Mahlo operations $V \in RM_2(\lceil\omega_n(\mathbb{K}+1)\rceil; <^{\varepsilon})\,(n \in \omega)$, but how to analyze it proof-theoretically? Here we need a breakthrough done by [Rathjen94].

2.1 Ordinals for $\mathsf{KP}\Pi_3$

In this subsection we define collapsing functions $\psi_{\sigma}^{\xi}(a)$ for $\mathsf{KP}\Pi_3$. It is much easier for us to justify the definitions with an existence of a small large cardinal. Let $\mathbb{K}$ be the least weakly compact cardinal, i.e., Π^1_1-indescribable cardinal, and $\Omega = \omega_1$. In general for $n \geq 0$, $A \subset ON$ is Π^1_n-*indescribable* in an ordinal π iff for every $\Pi^1_n(P)$-formula $\varphi(P)$ with a predicate P and $C \subset \pi$, if $(L_{\pi}, C) \models \varphi(P)$, then $(L_{\alpha}, C \cap \alpha) \models \varphi(P)$ for an $\alpha \in A \cap \pi$. First let us introduce the Mahlo operation. Let $A \subset \mathbb{K}$ be a set, and $\alpha \leq \mathbb{K}$ a limit ordinal. $\alpha \in M_2(A)$ iff $A \cap \alpha$ is Π^1_0-indescribable in α.

As in Definition 1.7 we define the Skolem hull $\mathcal{H}_a(X)$ and simultaneously classes $Mh^a_2(\xi)$ as follows.

Definition 2.2 Define simultaneously by recursion on ordinals $a < \varepsilon_{\mathbb{K}+1}$ the classes $\mathcal{H}_a(X)\,(X \subset \Gamma_{\mathbb{K}+1})$, $Mh^a_2(\xi)\,(\xi < \varepsilon_{\mathbb{K}+1})$ and the ordinals $\psi_{\sigma}^{\xi}(a)$ as follows.

1. $\mathcal{H}_a(X)$ denotes the Skolem hull of $\{0, \Omega, \mathbb{K}\} \cup X$ under the functions $+, \varphi$, and $(\sigma, \nu, b) \mapsto \psi_{\sigma}^{\nu}(b)\,(b < a)$.

2. Let for $\xi > 0$,

$$\pi \in Mh^a_2(\xi) :\Leftrightarrow \{a, \xi\} \subset \mathcal{H}_a(\pi)\,\&\, \forall \nu \in \mathcal{H}_a(\pi) \cap \xi\,(\pi \in M_2(Mh^a_2(\nu))) \tag{7}$$

$\pi \in Mh^a_2(0)$ iff π is a limit ordinal.

3. For $0 \leq \xi < \varepsilon_{\mathbb{K}+1}$,

$$\psi^{\xi}_{\pi}(a) = \min\left(\{\pi\} \cup \{\kappa \in Mh^a_2(\xi) : \{\xi, \pi, a\} \subset \mathcal{H}_a(\kappa) \,\&\, \mathcal{H}_a(\kappa) \cap \pi \subset \kappa\}\right) \tag{8}$$

and $\psi_{\Omega}(\alpha) = \min\{\beta < \Omega : \mathcal{H}_{\alpha}(\beta) \cap \Omega \subset \beta\}$.

We see that each of $x = \mathcal{H}_a(y)$, $x = \psi^{\xi}_{\kappa} a$ and $x \in Mh^a_2(\xi)$, is a Σ_1-predicate as fixed points in ZFL

Since the cardinality of the set $\mathcal{H}_{\varepsilon_{\mathbb{K}+1}}(\pi)$ is π for any infinite cardinal $\pi \leq \mathbb{K}$, pick an injection $f : \mathcal{H}_{\varepsilon_{\mathbb{K}+1}}(\mathbb{K}) \to \mathbb{K}$ so that $f"\mathcal{H}_{\varepsilon_{\mathbb{K}+1}}(\pi) \subset \pi$ for any weakly inaccessibles $\pi \leq \mathbb{K}$.

Lemma 2.3 (Cf. Theorem 4.12 in [Rathjen94].)

1. *There exists a Π^1_1-formula $mh^a_2(x)$ such that $\pi \in Mh^a_2(\xi)$ iff $L_{\pi} \models mh^a_2(\xi)$ for any weakly inaccessible cardinals $\pi \leq \mathbb{K}$ with $f"(\{a, \xi\}) \subset L_{\pi}$.*

2. *$\mathbb{K} \in Mh^a_2(\varepsilon_{\mathbb{K}+1}) \cap M_2(Mh^a_2(\varepsilon_{\mathbb{K}+1}))$ for every $a < \varepsilon_{\mathbb{K}+1}$.*

Proof. 2.3.1. Let π be a weakly inaccessible cardinal and f an injection such that $f"\mathcal{H}_{\varepsilon_{\mathbb{K}+1}}(\pi) \subset L_{\pi}$. Assume that $f"(\{a, \xi\}) \subset L_{\pi}$. Then for $f(\xi) \in f"\mathcal{H}_a(\pi)$, $\pi \in Mh^a_2(\xi)$ iff for any $f(\nu) \in L_{\pi}$, if $f(\nu) \in f"\mathcal{H}_a(\pi)$ and $\nu < \xi$, then $\pi \in M_2(Mh^a_2(\nu))$, where $f"\mathcal{H}_a(\pi) \subset L_{\pi}$ is a class in L_{π}.

2.3.2. We show the following $B(\xi)$ is progressive in $\xi < \varepsilon_{\mathbb{K}+1}$:

$$B(\xi) :\Leftrightarrow \mathbb{K} \in Mh^a_2(\xi) \cap M_2(Mh^a_2(\xi))$$

Note that $\xi \in \mathcal{H}_a(\mathbb{K})$ holds for any $\xi < \varepsilon_{\mathbb{K}+1}$.

Suppose $\forall \nu < \xi\, B(\nu)$. We have to show that $Mh^a_2(\xi)$ is Π^1_0-indescribable in $\mathbb{K}$. It is easy to see that if $\pi \in M_2(Mh^a_2(\xi))$, then $\pi \in Mh^a_2(\xi)$ by induction on π. Let $\theta(P)$ be a first-order formula with a predicate P such that $(L_{\mathbb{K}}, C) \models \theta(P)$ for $C \subset \mathbb{K}$.

By IH we have $\forall \nu < \xi[\mathbb{K} \in M_2(Mh^a_2(\nu))]$. In other words, $\mathbb{K} \in Mh^a_2(\xi)$, i.e., $(L_{\mathbb{K}}, C) \models mh^a_2(\xi) \wedge \theta(P)$. Since the universe $L_{\mathbb{K}}$ is Π^1_1-indescribable, pick a $\pi < \mathbb{K}$ such that $(L_{\pi}, C \cap \pi)$ enjoys the Π^1_1-sentence $mh^a_2(\xi) \wedge \theta(P)$, and $\{f(a), f(\xi)\} \subset L_{\pi}$. Therefore $\pi \in Mh^a_2(\xi)$. Thus $\mathbb{K} \in M_2(Mh^a_2(\xi))$. □

Lemma 2.4 *For every $\{a, \xi\} \subset \varepsilon_{\mathbb{K}+1}$, $\psi^{\xi}_{\mathbb{K}}(a) < \mathbb{K}$ for the Π^1_1-indescribable cardinal $\mathbb{K}$.*

Proof. Let $\{a, \xi\} \subset \varepsilon_{\mathbb{K}+1}$. By Lemma 2.3.2 we obtain $\mathbb{K} \in M_2(Mh^a_2(\xi))$. On the other, $\{\kappa < \mathbb{K} : \{\xi, a\} \subset \mathcal{H}_a(\kappa), \mathcal{H}_a(\kappa) \cap \mathbb{K} \subset \kappa\}$ is a club subset of $\mathbb{K}$. Hence $\psi^{\xi}_{\mathbb{K}}(a) < \mathbb{K}$ by the definition (8). □

From the definition (8) we see

$$\pi \in Mh^a_2(\mu) \cap \mathcal{H}_a(\pi) \,\&\, \xi \in \mathcal{H}_a(\pi) \cap \mu \Rightarrow \pi \in M_2(Mh^a_2(\xi)) \,\&\, \psi^{\xi}_{\pi}(a) < \pi$$

In what follows M_2 denote the Π_2-recursively Mahlo operation RM_2.

2.2 Operator controlled derivations for $\mathsf{KP}\Pi_3$

$OT(\Pi_3)$ denotes a computable notation system of ordinals with collapsing functions $\psi^{\nu}_{\sigma}(b)$. $\kappa=\psi^{\nu}_{\sigma}(b)\in OT(\Pi_3)$ if $\{\sigma,\nu,b\}\subset OT(\Pi_3)\cap\mathcal{H}_b(\kappa)$, $\nu=m_2(\kappa)<m_2(\sigma)$ and

$$SC_{\mathbb{K}}(\nu)\subset\kappa\,\&\,\nu\leq b \tag{9}$$

where $m_2(\Omega)=1$ and $m_2(\mathbb{K})=\varepsilon_{\mathbb{K}+1}$. We need the condition (9) in our well-foundedness proof of $OT(\Pi_3)$, cf. Proposition 3.30 and Lemma 3.38.

Operator controlled derivations for $\mathsf{KP}\Pi_3$ are defined as in Definition 1.16 for $\mathsf{KP}\omega$ together with the following inference rules. For ordinals $\pi=\psi^{\xi}_{\sigma}(a)$, let $m_2(\pi)=\xi$.

($\mathrm{rfl}_{\Pi_3}(\mathbb{K})$) $b\geq\mathbb{K}$. There exist an ordinal $a_0\in\mathcal{H}_\gamma[\Theta]\cap a$, and a $\Sigma_3(\mathbb{K})$-sentence A enjoying the following conditions:

$$\frac{\mathcal{H}_\gamma[\Theta]\vdash^{a_0}_{b}\Gamma,\neg A\quad\{\mathcal{H}_\gamma[\Theta\cup\{\rho\}]\vdash^{a_0}_{b}\Gamma,A^{(\rho,\mathbb{K})}:\rho<\mathbb{K}\}}{\mathcal{H}_\gamma[\Theta]\vdash^{a}_{b}\Gamma}\ (\mathrm{rfl}_{\Pi_3}(\mathbb{K}))$$

The inference says that $\mathbb{K}\in RM_3$.

($\mathrm{rfl}_{\Pi_2}(\alpha,\pi,\nu)$) There exist ordinals $\alpha<\pi\leq b<\mathbb{K}$, $\nu<m_2(\pi)$ such that $SC_{\mathbb{K}}(\nu)\subset\pi$ and $\nu\leq\gamma$, cf. (9), $a_0<a$, and a finite set Δ of $\Sigma_2(\pi)$-sentences enjoying the following conditions:

1. $\{\alpha,\pi,\nu\}\subset\mathcal{H}_\gamma[\Theta]$.
2. For each $\delta\in\Delta$, $\mathcal{H}_\gamma[\Theta]\vdash^{a_0}_{b}\Gamma,\neg\delta$.
3. For each $\alpha<\rho\in Mh_2(\nu)\cap\pi$, $\mathcal{H}_\gamma[\Theta\cup\{\rho\}]\vdash^{a_0}_{b}\Gamma,\Delta^{(\rho,\pi)}$ holds. By $\rho\in Mh_2(\nu)$ we mean $\nu\leq m_2(\rho)$.

$$\frac{\{\mathcal{H}_\gamma[\Theta]\vdash^{a_0}_{b}\Gamma,\neg\delta\}_{\delta\in\Delta}\quad\{\mathcal{H}_\gamma[\Theta\cup\{\rho\}]\vdash^{a_0}_{b}\Gamma,\Delta^{(\rho,\pi)}:\alpha<\rho\in Mh_2(\nu)\cap\pi\}}{\mathcal{H}_\gamma[\Theta]\vdash^{a}_{b}\Gamma}\ (\mathrm{rfl}_{\Pi_2}(\alpha,\pi,\nu))$$

The inference says that $\pi\in M_2(Mh^{\gamma}_{2}(\nu))$ provided that $\{m_2(\pi),\gamma,\nu\}\subset\mathcal{H}_\gamma(\pi)$.

The axiom for Π_3-reflection follows from the inference ($\mathrm{rfl}_{\Pi_3}(\mathbb{K})$) as follows. Let $A\in\Sigma_3(\mathbb{K})$ with $d=\mathrm{rk}(A)<\mathbb{K}+\omega$, and $d_\rho=\mathrm{rk}(A^{(\rho,\mathbb{K})})$ for $\rho<\mathbb{K}$.

$$\frac{\mathcal{H}_0[\mathsf{k}(A)]\vdash^{2d}_{0}A,\neg A\quad\dfrac{\mathcal{H}_0[\mathsf{k}(A)\cup\{\rho\}]\vdash^{2d_\rho}_{0}A^{(\rho,\mathbb{K})},\neg A^{(\rho,\mathbb{K})}}{\mathcal{H}_0[\mathsf{k}(A)\cup\{\rho\}]\vdash^{\mathbb{K}}_{0}\exists z\,A^{(z,\mathbb{K})},\neg A^{(\rho,\mathbb{K})}}}{\mathcal{H}_0[\mathsf{k}(A)]\vdash^{\mathbb{K}+\omega}_{\mathbb{K}}\neg A,\exists z\,A^{(z,\mathbb{K})}}\ (\mathrm{rfl}_{\Pi_3}(\mathbb{K}))$$

An appropriate name for this collapsing technique would be stationary collapsing since in order for this procedure to work, a single derivation has to be collapsed into a "stationary" family of derivations. [Rathjen94]

We see from the following proof that $\alpha=\psi_{\mathbb{K}}(\gamma+\mathbb{K})$ holds in every inference $(\mathrm{rfl}_{\Pi_2}(\alpha,\kappa,a_0))$ occurring in a witnessed derivation of $\mathcal{H}_{\hat{a}+1}[\Theta\cup\{\kappa\}]\vdash^{\beta}_{\beta}\Gamma^{(\kappa,\mathbb{K})}$. Let us call the unique ordinal α a *base*.

Lemma 2.5 *Assume* $\Gamma\subset\Sigma_2(\mathbb{K})$, $\Theta\subset\mathcal{H}_\gamma(\psi_{\mathbb{K}}(\gamma))$, *and* $\mathcal{H}_\gamma[\Theta]\vdash^{a}_{\mathbb{K}}\Gamma$ *with* $a\leq\gamma$. *Then* $\mathcal{H}_{\hat{a}+1}[\Theta\cup\{\kappa\}]\vdash^{\beta}_{\beta}\Gamma^{(\kappa,\mathbb{K})}$ *holds for any* $\kappa\in Mh_2(a)\cap\psi_{\mathbb{K}}(\gamma+\mathbb{K}\cdot\omega)$ *such that* $\psi_{\mathbb{K}}(\gamma+\mathbb{K})<\kappa$, *where* $\hat{a}=\gamma+\omega^{\mathbb{K}+a}$ *and* $\beta=\psi_{\mathbb{K}}(\hat{a})$.

Proof. By induction on a. Note that there exists a $\kappa\in OT(\Pi_3)$ such that $\psi_{\mathbb{K}}(\gamma+\mathbb{K})<\kappa\in Mh_2(a)\cap\psi_{\mathbb{K}}(\gamma+\mathbb{K}\cdot\omega)$. F.e. $\kappa=\psi^{a}_{\mathbb{K}}(\gamma+\mathbb{K}+1)$.
Case 1. Consider the case when the last inference is a $(\mathrm{rfl}_{\Pi_3}(\mathbb{K}))$. For $\Sigma_3\ni A\simeq\bigvee(A_i)_{i\in J}$,

$$\dfrac{\mathcal{H}_\gamma[\Theta]\vdash^{a_0}_{\mathbb{K}}\Gamma,\neg A\quad\{\mathcal{H}_\gamma[\Theta\cup\{\rho\}]\vdash^{a_0}_{\mathbb{K}}\Gamma,A^{(\rho,\mathbb{K})}:\rho<\mathbb{K}\}}{\mathcal{H}_\gamma[\Theta]\vdash^{a}_{\mathbb{K}}\Gamma}\ (\mathrm{rfl}_{\Pi_3}(\mathbb{K}))$$

Let
$$\psi_{\mathbb{K}}(\gamma+\mathbb{K})\leq\sigma\in Mh_2(a_0)\cap\kappa.$$
Let $i\in Tm(\sigma)$, i.e., $\mathsf{k}(i)\subset\sigma$. For each $i\in Tm(\sigma)$ Inversion yields $\mathcal{H}_{\gamma+|i|}[\Theta\cup\mathsf{k}(i)]\vdash^{a_0}_{\mathbb{K}}\Gamma,\neg A_i$ with $\mathsf{k}(i)<\psi_{\mathbb{K}}(\gamma+|i|)$. By IH we obtain $\mathcal{H}_{\hat{a}+1}[\Theta\cup\{\sigma\}\cup\mathsf{k}(i)]\vdash^{\beta_0}_{\beta}\Gamma^{(\sigma,\mathbb{K})},\neg A_i^{(\sigma,\mathbb{K})}$ for every $i\in Tm(\sigma)$, where $\beta_0=\psi_{\mathbb{K}}(\widehat{a_0})$ with $\widehat{a_0}=\gamma+\omega^{\mathbb{K}+a_0}=\gamma+|i|+\omega^{\mathbb{K}+a_0}$. A $(\bigwedge)$ yields

$$\mathcal{H}_{\hat{a}+1}[\Theta\cup\{\sigma\}]\vdash^{\beta_0+1}_{\beta}\Gamma^{(\sigma,\mathbb{K})},\neg A^{(\sigma,\mathbb{K})}$$

On the other hand we have $\mathcal{H}_{\gamma+\sigma}[\Theta\cup\{\sigma\}]\vdash^{a_0}_{\mathbb{K}}\Gamma,A^{(\sigma,\mathbb{K})}$ with $\sigma\in\mathcal{H}_{\gamma+\sigma}(\psi_{\mathbb{K}}(\gamma+\sigma))$, but $\sigma\not\in\mathcal{H}_\gamma(\psi_{\mathbb{K}}(\gamma+\mathbb{K}))$. We obtain $\kappa\in Mh_2(a_0)$ by $a_0<a$, and $\gamma+\sigma+\mathbb{K}=\gamma+\mathbb{K}$. IH yields

$$\mathcal{H}_{\hat{a}+1}[\Theta\cup\{\kappa,\sigma\}]\vdash^{\beta_0}_{\beta}\Gamma^{(\kappa,\mathbb{K})},A^{(\sigma,\mathbb{K})}$$

A (cut) of the cut formula $A^{(\sigma,\mathbb{K})}$ with $\mathrm{rk}(A^{(\sigma,\mathbb{K})})<\kappa<\psi_{\mathbb{K}}(\gamma+\mathbb{K}\cdot\omega)\leq\beta$ yields

$$\mathcal{H}_{\hat{a}+1}[\Theta\cup\{\kappa,\sigma\}]\vdash^{\beta_0+2}_{\beta}\Gamma^{(\kappa,\mathbb{K})},\Gamma^{(\sigma,\mathbb{K})}$$

On the other side

$$\mathcal{H}_\gamma[\Theta\cup\{\kappa\}]\vdash^{2d}_{0}\neg\theta^{(\kappa,\mathbb{K})},\Gamma^{(\kappa,\mathbb{K})}$$

holds for each $\theta\in\Gamma\subset\Sigma_2(\mathbb{K})$, where $d=\max\{\mathrm{rk}(\theta^{(\kappa,\mathbb{K})}):\theta\in\Gamma\}<\kappa+\omega<\beta$.

Moreover we have $a_0<\hat{a}$, $SC_{\mathbb{K}}(a_0)\subset\mathcal{H}_\gamma[\Theta]\cap\mathbb{K}\subset\mathcal{H}_\gamma(\psi_{\mathbb{K}}(\gamma))\cap\mathbb{K}\subset\kappa$. A $(\mathrm{rfl}_{\Pi_2}(\delta,\kappa,a_0))$ with $\delta=\psi_{\mathbb{K}}(\gamma+\mathbb{K})$, $\{\delta,\kappa,a_0\}\subset\mathcal{H}_{\hat{a}+1}[\Theta\cup\{\kappa\}]$ yields $\mathcal{H}_{\hat{a}+1}[\Theta\cup\{\kappa\}]\vdash^{\beta}_{\beta}\Gamma^{(\kappa,\mathbb{K})}$.

$$\dfrac{\{\mathcal{H}_\gamma[\Theta\cup\{\kappa\}]\vdash^{2d}_{0}\neg\theta^{(\kappa,\mathbb{K})},\Gamma^{(\kappa,\mathbb{K})}\}_{\theta\in\Gamma}\quad\dfrac{\mathcal{H}_{\hat{a}+1}[\Theta\cup\{\sigma\}]\vdash^{\beta_0+1}_{\beta}\Gamma^{(\sigma,\mathbb{K})},\neg A^{(\sigma,\mathbb{K})}\quad\mathcal{H}_{\hat{a}+1}[\Theta\cup\{\kappa,\sigma\}]\vdash^{\beta_0}_{\beta}\Gamma^{(\kappa,\mathbb{K})},A^{(\sigma,\mathbb{K})}}{\{\mathcal{H}_{\hat{a}+1}[\Theta\cup\{\kappa,\sigma\}]\vdash^{\beta_0+2}_{\beta}\Gamma^{(\kappa,\mathbb{K})},\Gamma^{(\sigma,\mathbb{K})}\}_{\delta<\sigma\in Mh_2(a_0)\cap\kappa}}}{\mathcal{H}_{\hat{a}+1}[\Theta\cup\{\kappa\}]\vdash^{\beta}_{\beta}\Gamma^{(\kappa,\mathbb{K})}}\ (\mathrm{rfl}_{\Pi_2}(\delta,\kappa,a_0))$$

Case 2. The last inference is a (*cut*) of a cut formula C with $\mathrm{rk}(C) < \mathbb{K}$. Then $\mathrm{rk}(C) \in \mathcal{H}_\gamma[\Theta] \cap \mathbb{K} \subset \psi_\mathbb{K}(\gamma) < \beta$ by (3), Proposition 3.1 and the assumption $\Theta \subset \mathcal{H}_\gamma(\psi_\mathbb{K}(\gamma))$.
Case 3. The last inference is a $(\bigwedge)$ with a main formula $\Pi_1(\mathbb{K}) \ni A \simeq \bigwedge(A_\iota)_{\iota \in J}$. We may assume $J = Tm(\mathbb{K})$. Then $A^{(\kappa,\mathbb{K})} \simeq \bigwedge(A_\iota)_{\iota \in Tm(\kappa)}$, and we obtain the lemma by pruning the branches for $\iota \notin Tm(\kappa)$.
Case 4. The last inference is a $(\bigvee)$ with a main formula $\Sigma_2(\mathbb{K}) \ni A \simeq \bigvee(A_\iota)_{\iota \in J}$. We may assume $J = Tm(\mathbb{K})$. Then $A^{(\kappa,\mathbb{K})} \simeq \bigvee(A_\iota)_{\iota \in Tm(\kappa)}$.

$$\frac{\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_{\mathbb{K}} \Gamma, A_\iota}{\mathcal{H}_\gamma[\Theta] \vdash^{a}_{\mathbb{K}} \Gamma}\ (\bigvee)$$

We may assume that $\mathsf{k}(\iota) \subset \mathsf{k}(A_\iota)$. Then by (3) and $\Theta \subset \mathcal{H}_\gamma(\psi_\mathbb{K}(\gamma))$ we obtain $\mathsf{k}(\iota) \subset \mathcal{H}_\gamma[\Theta] \cap \mathbb{K} \subset \mathcal{H}_\gamma(\psi_\mathbb{K}(\gamma)) \cap \mathbb{K} \subset \kappa$, and $\iota \in Tm(\kappa)$. □

An ordinal term α in $OT(\Pi_3)$ is said to be *regular* if either $\alpha \in \{\Omega, \mathbb{K}\}$ or $\alpha = \psi^\nu_\sigma(a)$ for some σ, a and $\nu > 0$.

Lemma 2.6 *Let λ be regular, $\Gamma \subset \Sigma_1(\lambda)$ and $\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma$, where $a < \mathbb{K}$, $\mathcal{H}_\gamma[\Theta] \ni \lambda \leq b < \mathbb{K}$, and $\forall \kappa \in [\lambda, b)(\Theta \subset \mathcal{H}_\gamma(\psi_\kappa(\gamma)))$. Let $\hat{a} = \gamma + \theta_b(a)$ and $\beta = \psi^\eta_\lambda(\hat{a})$ such that $0 \leq \eta \in \mathcal{H}_\gamma[\Theta]$, $\eta < m_2(\lambda)$, $SC_\mathbb{K}(\eta) \subset \beta$ and $\eta \leq \gamma$. Then $\mathcal{H}_{\hat{a}+1}[\Theta] \vdash^\beta_\beta \Gamma$ holds.*

Proof. By main induction on b with subsidiary induction on a as in Theorem 1.22.
Case 1. Consider first the case when the last inference is a $(\mathrm{rfl}_{\Pi_2}(\alpha, \sigma, \nu))$ with $b \geq \sigma > \alpha$.

$$\frac{\{\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_b \Gamma, \neg\delta\}_{\delta \in \Delta} \quad \{\mathcal{H}_\gamma[\Theta \cup \{\rho\}] \vdash^{a_0}_b \Gamma, \Delta^{(\rho,\sigma)} : \alpha < \rho \in Mh_2(\nu) \cap \sigma\}}{\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma}\ (\mathrm{rfl}_{\Pi_2}(\alpha, \sigma, \nu))$$

where $\Delta \subset \Sigma_2(\sigma)$, $\{\alpha, \sigma, \nu\} \subset \mathcal{H}_\gamma[\Theta]$, $\nu < m_2(\sigma)$, $\nu \leq \gamma$ and $SC_\mathbb{K}(\nu) \subset \sigma$.
Case 1.1. $\sigma < \lambda$: Then $\{\neg\delta\} \cup \Delta^{(\rho,\sigma)} \subset \Delta_0(\lambda)$ for each $\delta \in \Delta$. For any $\lambda \leq \kappa < b$, we obtain $\rho < \sigma \in \mathcal{H}_\gamma[\Theta] \cap \kappa \subset \psi_\kappa(\gamma)$. SIH yields the lemma.
Case 1.2. $\sigma \geq \lambda$: For each $\delta \in \Delta$, let $\delta \simeq \bigvee(\delta_i)_{i \in J}$. We may assume $J = Tm(\sigma)$. Inversion yields $\mathcal{H}_{\gamma+|i|}[\Theta \cup \mathsf{k}(i)] \vdash^{a_0}_b \Gamma, \neg\delta_i$. Let $\widehat{a_0} = \gamma + \theta_b(a_0)$ and $\rho = \psi^\nu_\sigma(\widehat{a_0} + \alpha)$, where $\Theta \subset \mathcal{H}_\gamma(\rho)$ by the assumption, $\{\alpha, \sigma, \nu, \widehat{a_0}\} \subset \mathcal{H}_\gamma[\Theta]$ with $\nu < m_2(\sigma)$. Hence $\{\alpha, \sigma, \nu, \widehat{a_0}\} \subset \mathcal{H}_\gamma(\rho)$ and $\alpha < \rho$ by $\alpha < \sigma$. Therefore, cf. (9), $SC_\mathbb{K}(\nu) \subset \rho \in Mh_2(\nu) \cap \sigma \cap \mathcal{H}_{\widehat{a_0}+\alpha+1}[\Theta]$.

For each $\mathsf{k}(i) \subset \rho$ and $\neg\delta_i \in \Sigma_1(\sigma)$, we obtain $\gamma + |i| + \theta_b(a_0) = \widehat{a_0}$ by $|i| < \rho < \sigma \leq b$, and $\mathcal{H}_{\widehat{a_0}+1}[\Theta \cup \mathsf{k}(i)] \vdash^{\rho_0}_{\rho_0} \Gamma, \neg\delta_i$ by SIH for $\rho_0 = \psi_{|sig}(\widehat{a_0}) \leq \rho$. Hence $\mathcal{H}_{\widehat{a_0}+\alpha+1}[\Theta \cup \mathsf{k}(i)] \vdash^\rho_\rho \Gamma, \neg\delta_i$ By Boundedness we obtain $\mathcal{H}_{\widehat{a_0}+\alpha+1}[\Theta \cup \mathsf{k}(i)] \vdash^\rho_\rho \Gamma, \neg\delta_i^{(\rho,\sigma)}$. A $(\bigwedge)$ yields

$$\mathcal{H}_{\widehat{a_0}+\alpha+1}[\Theta] \vdash^{\rho+1}_\rho \Gamma, \neg\delta^{(\rho,\sigma)}.$$

On the other hand we have $\mathcal{H}_\gamma[\Theta\cup\{\rho\}]\vdash^{a_0}_b\Gamma,\Delta^{(\rho,\sigma)}$, and $\mathcal{H}_{\widehat{a_0}+\alpha+1}[\Theta]\vdash^{a_0}_b\Gamma,\Delta^{(\rho,\sigma)}$. By SIH we obtain

$$\mathcal{H}_{\widehat{a_1}+1}[\Theta]\vdash^{\beta_1}_{\beta_1}\Gamma,\Delta^{(\rho,\sigma)}$$

for $\beta_1=\psi_\sigma(\widehat{a_1})>\rho$, with $\widehat{a_1}=\widehat{a_0}+\alpha+\theta_b(a_0)\leq\gamma+\theta_b(a_0)\cdot 3<\hat{a}$. Therefore we obtain $\mathcal{H}_{\widehat{a_1}+1}[\Theta]\vdash^{\beta_1+\omega}_{\beta_1}\Gamma$ by several (cut)'s of $\mathrm{rk}(\delta^{(\rho,\sigma)})<\rho+\omega<\beta_1$.

If $\sigma=\lambda$, then we are done. Let $\lambda<\sigma\leq b$. Then $\lambda\in\mathcal{H}_\gamma[\Theta]\cap\sigma\subset\beta_1$. MIH yields $\mathcal{H}_{\widehat{a_2}+1}[\Theta]\vdash^{\beta_2}_{\beta_2}\Gamma$, where $\widehat{a_2}=\widehat{a_1}+\theta_{\beta_1}(\beta_1+\omega)<\hat{a}$ by $\beta_1<\sigma\leq b$, and $\beta_2=\psi_\lambda(\widehat{a_2})<\psi_\lambda(\hat{a})\leq\beta$.

Case 2. Next the last inference is a (cut) of a cut formula C with $d=\mathrm{rk}(C)<b$.

$$\frac{\mathcal{H}_\gamma[\Theta]\vdash^{a_0}_b\Gamma,\neg C\quad \mathcal{H}_\gamma[\Theta]\vdash^{a_0}_b\Gamma,C}{\mathcal{H}_\gamma[\Theta]\vdash^{a}_b\Gamma}\,(cut)$$

If $d<\lambda$, then SIH yields the lemma. Let $\lambda\leq d$ and $\widehat{a_0}=\gamma+\theta_b(a_0)$.

Case 2.1. There exists a regular $\sigma\in\mathcal{H}_\gamma[\Theta]$ such that $d<\sigma\leq b$: For $\{\neg C,C\}\subset\Delta_0(\sigma)$, we obtain $\mathcal{H}_{\widehat{a_0}+1}[\Theta]\vdash^{\beta_0}_{\beta_0}\Gamma,C$ and $\mathcal{H}_{\widehat{a_0}+1}[\Theta]\vdash^{\beta_0}_{\beta_0}\Gamma,\neg C$ for $\beta_0=\psi_\sigma(\widehat{a_0})$ by SIH. A (cut) yields $\mathcal{H}_{\widehat{a_0}+1}[\Theta]\vdash^{\beta_0+1}_{\beta_0}\Gamma$. MIH yields $\mathcal{H}_{\widehat{a_1}+1}[\Theta]\vdash^{\beta_1}_{\beta_1}\Gamma$, where $\widehat{a_1}=\widehat{a_0}+\theta_{\beta_0}(\beta_0+1)<\hat{a}$ and $\beta_1=\psi_\lambda(\widehat{a_1})<\psi_\lambda(\hat{a})\leq\beta$.

Case 2.2. Otherwise: Then there is no regular $\sigma\in\mathcal{H}_\gamma[\Theta]$ such that $d<\sigma\leq b$. Let $d+c=b$. Then by Cut-elimination we obtain $\mathcal{H}_\gamma[\Theta]\vdash^{\theta_c(a)}_d\Gamma$. MIH yields $\mathcal{H}_{\hat{a}+1}[\Theta]\vdash^{\psi_\lambda(\hat{a})}_{\psi_\lambda(\hat{a})}\Gamma$, where $\gamma+\theta_d(\theta_c(a))=\gamma+\theta_b(a)=\hat{a}$. □

Theorem 2.7 *Assume* $\mathsf{KP}\Pi_3\vdash\theta^{L_\Omega}$ *for* $\theta\in\Sigma$. *Then there exists an* $n<\omega$ *such that* $L_\alpha\models\theta$ *for* $\alpha=\psi_\Omega(\omega_n(\mathbb{K}+1))$ *in* $OT(\Pi_3)$.

Proof. By Embedding there exists an $m>0$ such that $\mathcal{H}_0[\emptyset]\vdash^{\mathbb{K}+m}_{\mathbb{K}+m}\theta^{L_\Omega}$. By Cut-elimination, $\mathcal{H}_0[\emptyset]\vdash^a_{\mathbb{K}}\theta^{L_\Omega}$ and $\mathcal{H}_a[\emptyset]\vdash^a_{\mathbb{K}}\theta^{L_\Omega}$ for $a=\omega_m(\mathbb{K}+m)$. By Lemma 2.5 we obtain $\mathcal{H}_{\omega^a+1}[\{\kappa\}]\vdash^\beta_\beta\theta^{L_\Omega}$, where $\beta=\psi_{\mathbb{K}}(\omega^a)$, $a+\omega^{\mathbb{K}+a}=\omega^a$, $(\theta^{L_\Omega})^{(\kappa,\mathbb{K})}\equiv\theta^{L_\Omega}$ and $\psi_{\mathbb{K}}(a+\mathbb{K})<\kappa\in Mh_2(a)\cap\psi_{\mathbb{K}}(a+\mathbb{K}\cdot\omega)$. F.e. $\kappa=\psi^a_{\mathbb{K}}(a+\mathbb{K}+1)\in\mathcal{H}_{a+\mathbb{K}+2}[\emptyset]$. Hence $\mathcal{H}_{\omega^a+\mathbb{K}+2}[\emptyset]\vdash^\beta_\beta\theta^{L_\Omega}$. Lemma 2.6 then yields $\mathcal{H}_{\gamma+1}[\emptyset]\vdash^{\beta_1}_{\beta_1}\theta^{L_\Omega}$ for $\gamma=\omega^a+\mathbb{K}+\theta_\beta(\beta)$ and $\beta_1=\psi_\Omega(\gamma)<\psi_\Omega(\omega^a+\mathbb{K}\cdot 2)<\psi_\Omega(\omega_{m+2}(\mathbb{K}+1))=\alpha$. Therefore $L_\alpha\models\theta$. □

3 Well-foundedness proof in $\mathsf{KP}\Pi_3$

$OT(\Pi_3)$ denotes the computable notation system in section 2. $\kappa=\psi^\nu_\sigma(b)\in OT(\Pi_3)$ only if $\nu=m_2(\kappa)<m_2(\sigma)$, $SC_{\mathbb{K}}(\nu)\subset\kappa$ and $\nu\leq b$, cf. (9). In this section we show the

Theorem 3.1 $\mathsf{KP}\Pi_3$ *proves the well-foundedness of* $OT(\Pi_3)$ *up to each* $\alpha<\psi_\Omega(\varepsilon_{\mathbb{K}+1})$.

We assume a standard encoding $OT(\Pi_3)\ni\alpha\mapsto\lceil\alpha\rceil\in\omega$, and identify ordinal terms α with its code $\lceil\alpha\rceil$.

3.1 Distinguished sets

In this subsection we work in $\mathsf{KP}\ell$.

Definition 3.2 [Buchholz00].
For $\alpha \in OT(\Pi_3), X \subset OT(\Pi_3)$, let

$$\begin{aligned}\mathcal{C}^\alpha(X) \ &:= \ \text{closure of } \{0, \Omega, \mathbb{K}\} \cup (X \cap \alpha) \text{ under } +, \varphi \\ &\quad\ \text{and } (\sigma, \alpha, \nu) \mapsto \psi_\sigma^\nu(\alpha) \text{ for } \sigma > \alpha \text{ in } OT(\Pi_3)\end{aligned} \tag{10}$$

$\alpha^+ = \Omega_{a+1}$ denotes the least regular term above α if such a term exists. Otherwise $\alpha^+ := \infty$.

Proposition 3.3 *Assume* $\forall \gamma \in X[\gamma \in \mathcal{C}^\gamma(X)]$ *for a set* $X \subset OT(\Pi_3)$.

1. $\alpha \leq \beta \Rightarrow \mathcal{C}^\beta(X) \subset \mathcal{C}^\alpha(X)$.
2. $\alpha < \beta < \alpha^+ \Rightarrow \mathcal{C}^\beta(X) = \mathcal{C}^\alpha(X)$.

Proof. 3.3.1. We see by induction on $\ell\gamma\, (\gamma \in OT(\Pi_3))$ that

$$\forall \beta \geq \alpha[\gamma \in \mathcal{C}^\beta(X) \Rightarrow \gamma \in \mathcal{C}^\alpha(X) \cup (X \cap \beta)] \tag{11}$$

For example, if $\psi_\pi^\nu(\delta) \in \mathcal{C}^\beta(X)$ with $\pi > \beta \geq \alpha$ and $\{\pi, \delta, \nu\} \subset \mathcal{C}^\alpha(X) \cup (X \cap \beta)$, then $\pi \in \mathcal{C}^\alpha(X)$, and for any $\gamma \in \{\delta, \nu\}$, either $\gamma \in \mathcal{C}^\alpha(X)$ or $\gamma \in X \cap \beta$. If $\gamma < \alpha$, then $\gamma \in X \cap \alpha \subset \mathcal{C}^\alpha(X)$. If $\alpha \leq \gamma \in X \cap \beta$, then $\gamma \in \mathcal{C}^\gamma(X)$ by the assumption, and by IH we have $\gamma \in \mathcal{C}^\alpha(X) \cup (X \cap \gamma)$, i.e., $\gamma \in \mathcal{C}^\alpha(X)$. Therefore $\{\pi, \delta, \nu\} \subset \mathcal{C}^\alpha(X)$, and $\psi_\pi^\nu(\delta) \in \mathcal{C}^\alpha(X)$.

Using (11) we see from the assumption that $\forall \beta \geq \alpha[\gamma \in \mathcal{C}^\beta(X) \Rightarrow \gamma \in \mathcal{C}^\alpha(X)]$.
3.3.2. Assume $\alpha < \beta < \alpha^+$. Then by Proposition 3.3.1 we have $\mathcal{C}^\beta(X) \subset \mathcal{C}^\alpha(X)$. $\mathcal{C}^\alpha(X) \subset \mathcal{C}^\beta(X)$ is easily seen from $\beta < \alpha^+$. □

Definition 3.4 1. $Prg[X, Y] :\Leftrightarrow \forall \alpha \in X(X \cap \alpha \subset Y \to \alpha \in Y)$.

2. For a definable class $\mathcal{X}$, $TI[\mathcal{X}]$ denotes the schema:
   $TI[\mathcal{X}] :\Leftrightarrow Prg[\mathcal{X}, \mathcal{Y}] \to \mathcal{X} \subset \mathcal{Y}$ holds for any definable classes $\mathcal{Y}$.
3. For $X \subset OT(\Pi_3)$, $W(X)$ denotes the *well-founded part* of X.
4. $Wo[X] :\Leftrightarrow X \subset W(X)$.

Note that for $\alpha \in OT(\Pi_3)$, $W(X) \cap \alpha = W(X \cap \alpha)$.

Definition 3.5 For $X \subset OT(\Pi_3)$ and $\alpha \in OT(\Pi_3)$,

1. $$D[X] :\Leftrightarrow \forall \alpha(\alpha \leq X \to W(\mathcal{C}^\alpha(X)) \cap \alpha^+ = X \cap \alpha^+) \tag{12}$$

   A set X is said to be a *distinguished set* if $D[X]$.

2. $\mathcal{W} := \bigcup\{X : D[X]\}$.

Let $\alpha \in X$ for a distinguished set X. Then $W(\mathcal{C}^\alpha(X)) \cap \alpha^+ = X \cap \alpha^+$. Hence X is a well order.

Proposition 3.6 *Let X be a distinguished set. Then $\alpha \in X \Rightarrow \forall\beta[\alpha \in \mathcal{C}^\beta(X)]$.*

Proof. Let $D[X]$ and $\alpha \in X$. Then $\alpha \in X \cap \alpha^+ = W(\mathcal{C}^\alpha(X)) \cap \alpha^+ \subset \mathcal{C}^\alpha(X)$. Hence $\forall\gamma \in X(\gamma \in \mathcal{C}^\gamma(X))$, and $\alpha \in \mathcal{C}^\beta(X)$ for any $\beta \leq \alpha$ by Proposition 3.3.1. Moreover for $\beta > \alpha$ we have $\alpha \in X \cap \beta \subset \mathcal{C}^\beta(X)$. □

Proposition 3.7 *$X \cap \alpha = Y \cap \alpha \Rightarrow \forall\beta < \alpha^+ \left[\mathcal{C}^\beta(X) = \mathcal{C}^\beta(Y)\right]$ if $\forall\gamma \in X(\gamma \in \mathcal{C}^\gamma(X))$ and $\forall\gamma \in Y(\gamma \in \mathcal{C}^\gamma(Y))$.*

Proof. Assume that $X \cap \alpha = Y \cap \alpha$ and $\alpha \leq \beta < \alpha^+$. We obtain $\mathcal{C}^\alpha(X) = \mathcal{C}^\alpha(Y)$. On the other hand we have $\mathcal{C}^\beta(X) = \mathcal{C}^\alpha(X)$ and similarly for $\mathcal{C}^\beta(Y)$ by Proposition 3.3.2. Hence $\mathcal{C}^\beta(X) = \mathcal{C}^\beta(Y)$. □

Proposition 3.8 *$\alpha \leq X \,\&\, \alpha \leq Y \Rightarrow X \cap \alpha^+ = Y \cap \alpha^+$ if $D[X]$ and $D[Y]$.*

Proof. For distinguished set X, $\alpha \leq X \Rightarrow X \cap \alpha^+ = W(\mathcal{C}^\alpha(X)) \cap \alpha^+$. Hence the proposition follows from Propositions 3.6 and 3.7. □

Proposition 3.9 *$\mathcal{W}$ is the maximal distinguished class.*

Proof. First we show $\forall\gamma \in \mathcal{W}(\gamma \in \mathcal{C}^\gamma(\mathcal{W}))$. Let $\gamma \in \mathcal{W}$, and pick a distinguished set X such that $\gamma \in X$. Then $\gamma \in \mathcal{C}^\gamma(X) \subset \mathcal{C}^\gamma(\mathcal{W})$ by $X \subset \mathcal{W}$.

Let $\alpha \leq \mathcal{W}$. Pick a distinguished set X such that $\alpha \leq X$. We claim that $\mathcal{W} \cap \alpha^+ = X \cap \alpha^+$. Let Y be a distinguished set and $\beta \in Y \cap \alpha^+$. Then $\beta \in Y \cap \beta^+ = X \cap \beta^+$ by Proposition 3.8. The claim yields $W(\mathcal{C}^\alpha(\mathcal{W})) \cap \alpha^+ = W(\mathcal{C}^\alpha(X)) \cap \alpha^+ = X \cap \alpha^+ = \mathcal{W} \cap \alpha^+$. Hence $D[\mathcal{W}]$. □

Definition 3.10 $\mathcal{G}(X) := \{\alpha \in OT(\Pi_3) : \alpha \in \mathcal{C}^\alpha(X) \,\&\, \mathcal{C}^\alpha(X) \cap \alpha \subset X\}$.

Lemma 3.11 *For $D[X]$, $X \subset \mathcal{G}(X)$.*

Proof. Let $\gamma \in X$. We have $\gamma \in W(\mathcal{C}^\gamma(X)) \cap \gamma^+ = X \cap \gamma^+$. Hence $\gamma \in \mathcal{C}^\gamma(X)$. Assume $\alpha \in \mathcal{C}^\gamma(X) \cap \gamma$. Then $\alpha \in W(\mathcal{C}^\gamma(X)) \cap \gamma^+ \subset X$. Therefore $\mathcal{C}^\gamma(X) \cap \gamma \subset X$. □

Definition 3.12 For ordinal terms $\alpha, \delta \in OT(\Pi_3)$, finite sets $G_\delta(\alpha) \subset OT(\Pi_3)$ are defined recursively as follows.

1. $G_\delta(\alpha) = \emptyset$ for $\alpha \in \{0, \Omega, \mathbb{K}\}$. $G_\delta(\alpha_m + \cdots + \alpha_0) = \bigcup_{i \leq m} G_\delta(\alpha_i)$. $G_\delta(\varphi\beta\gamma) = G_\delta(\beta) \cup G_\delta(\gamma)$.

2. $G_\delta(\psi_\pi^\nu(a)) = \begin{cases} G_\delta(\{\pi, a, \nu\}) & \delta < \pi \\ \{\psi_\pi^\nu(a)\} & \pi \leq \delta \end{cases}$.

Proposition 3.13 *For $\{\alpha, \delta, a, b, \rho\} \subset OT(\Pi_3)$,*

1. $G_\delta(\alpha) \leq \alpha$.

2. $\alpha \in \mathcal{H}_a(b) \Rightarrow G_\delta(\alpha) \subset \mathcal{H}_a(b)$.

Proof. These are shown simultaneously by induction on the lengths $\ell\alpha$ of ordinal terms α. It is easy to see that

$$G_\delta(\alpha) \ni \beta \Rightarrow \beta < \delta \,\&\, \ell\beta \leq \ell\alpha \tag{13}$$

3.13.1. Consider the case $\alpha = \psi_\pi^\nu(a)$ with $\delta < \pi$. Then $G_\delta(\alpha) = G_\delta(\{\pi, a, \nu\})$. On the other hand we have $\{\pi, a, \nu\} \subset \mathcal{H}_a(\alpha)$. Proposition 3.13.2 with (13) yields $G_\delta(\{\pi, a, \nu\}) \subset \mathcal{H}_a(\alpha) \cap \pi \subset \alpha$. Hence $G_\delta(\alpha) < \alpha$.
3.13.2. Since $G_\delta(\alpha) \leq \alpha$ by Proposition 3.13.1, we can assume $\alpha \geq b$.

Consider the case $\alpha = \psi_\pi^\nu(a)$ with $\delta < \pi$. Then $\{\pi, a, \nu\} \subset \mathcal{H}_a(b)$ and $G_\delta(\alpha) = G_\delta(\{\pi, a, \nu\})$. IH yields the proposition. □

Proposition 3.14 *Let $\gamma < \beta$. Assume $\alpha \in \mathcal{C}^\gamma(X)$ and $\forall\kappa \leq \beta[G_\kappa(\alpha) < \gamma]$. Moreover assume $\forall\delta[\ell\delta \leq \ell\alpha \,\&\, \delta \in \mathcal{C}^\gamma(X) \cap \gamma \Rightarrow \delta \in \mathcal{C}^\beta(X)]$. Then $\alpha \in \mathcal{C}^\beta(X)$.*

Proof. By induction on $\ell\alpha$. If $\alpha < \gamma$, then $\alpha \in \mathcal{C}^\gamma(X) \cap \gamma$. The third assumption yields $\alpha \in \mathcal{C}^\beta(X)$. Assume $\alpha \geq \gamma$. Except the case $\alpha = \psi_\pi^\nu(a)$ for some π, a, ν, IH yields $\alpha \in \mathcal{C}^\beta(X)$. Suppose $\alpha = \psi_\pi^\nu(a)$ for some $\{\pi, a, \nu\} \subset \mathcal{C}^\gamma(X)$ and $\pi > \gamma$. If $\pi \leq \beta$, then $\{\alpha\} = G_\pi(\alpha) < \gamma$ by the second assumption. Hence this is not the case, and we obtain $\pi > \beta$. Then $G_\kappa(\{\pi, a, \nu\}) = G_\kappa(\alpha) < \gamma$ for any $\kappa \leq \beta < \pi$. IH yields $\{\pi, a, \nu\} \subset \mathcal{C}^\beta(X)$. We conclude $\alpha \in \mathcal{C}^\beta(X)$ from $\pi > \beta$. □

Lemma 3.15 *Suppose $D[Y]$ and $\alpha \in \mathcal{G}(Y)$. Let $X = W(\mathcal{C}^\alpha(Y)) \cap \alpha^+$. Assume that the following condition (71) is fulfilled. Then $\alpha \in X$ and $D[X]$.*

$$\forall\beta \left(Y \cap \alpha^+ < \beta \,\&\, \beta^+ < \alpha^+ \to W(\mathcal{C}^\beta(Y)) \cap \beta^+ \subset Y\right) \tag{14}$$

Proof. Let $\alpha \in \mathcal{G}(Y)$. By $\mathcal{C}^\alpha(Y) \cap \alpha \subset Y$ and $Wo[Y]$ we obtain by Proposition 3.6

$$X \cap \alpha = Y \cap \alpha = \mathcal{C}^\alpha(Y) \cap \alpha \tag{15}$$

Hence $\alpha \in X$.

Claim 3.16 $\alpha^+ = \gamma^+ \,\&\, \gamma \in X \Rightarrow \gamma \in \mathcal{C}^\gamma(X)$.

Proof of Claim 3.16. Let $\alpha^+ = \gamma^+$ and $\gamma \in X = W(\mathcal{C}^\alpha(Y)) \cap \alpha^+$. We obtain $\gamma \in \mathcal{C}^\alpha(Y) = \mathcal{C}^\gamma(Y)$ by Propositions 3.6 and 3.3. Hence $Y \cap \gamma \subset \mathcal{C}^\gamma(Y) \cap \gamma = \mathcal{C}^\alpha(Y) \cap \gamma$. $\gamma \in W(\mathcal{C}^\alpha(Y))$ yields $Y \cap \gamma \subset X$. Therefore we obtain $\gamma \in \mathcal{C}^\gamma(Y) \subset \mathcal{C}^\gamma(X)$. □ of Claim 3.16.

Claim 3.17 $D[X]$.

Proof of Claim 3.17. We have $X \cap \alpha = Y \cap \alpha$ by (15). Let $\beta \leq X$. We show $W(\mathcal{C}^{\beta}(X)) \cap \beta^{+} = X \cap \beta^{+}$.
Case 1. $\beta^{+} = \alpha^{+}$: We obtain $\mathcal{C}^{\beta}(X) = \mathcal{C}^{\alpha}(X) = \mathcal{C}^{\alpha}(Y)$ by Proposition 3.3, Claim 3.16 and (15).
Case 2. $\beta^{+} < \alpha^{+}$: Then $\beta^{+} \leq \alpha$.

First let $Y \cap \alpha^{+} < \beta$. Then the assumption (71) yields $W(\mathcal{C}^{\beta}(Y)) \cap \beta^{+} \subset Y$. We obtain $W(\mathcal{C}^{\beta}(X)) \cap \beta^{+} \subset Y \cap \beta^{+} = X \cap \beta^{+}$ by (15). It remains to show $Y \cap \beta^{+} \subset W(\mathcal{C}^{\beta}(Y))$. Let $\gamma \in Y \cap \beta^{+}$. We obtain $\gamma \in W(\mathcal{C}^{\gamma}(Y))$ by $D[Y]$. On the other hand we have $\mathcal{C}^{\beta}(Y) \subset \mathcal{C}^{\gamma}(Y)$ by Proposition 3.3. Moreover Proposition 3.6 yields $\gamma \in \mathcal{C}^{\beta}(Y)$. Hence $\gamma \in W(\mathcal{C}^{\beta}(Y))$.

Next let $\beta \leq Y \cap \alpha^{+}$. We obtain $Y \cap \beta^{+} = \mathcal{W}(\mathcal{C}^{\beta}(Y)) \cap \beta^{+}$, and $X \cap \beta^{+} = \mathcal{W}(\mathcal{C}^{\beta}(X)) \cap \beta^{+}$ by (15). $\Box$ of Claim 3.17.

This completes a proof of Lemma 3.15. $\Box$

Proposition 3.18 *Let* $D[X]$.

1. *Let* $\{\alpha, \beta\} \subset X$ *with* $\alpha + \beta = \alpha \# \beta$ *and* $\alpha > 0$. *Then* $\gamma = \alpha + \beta \in X$.
2. *If* $\{\alpha, \beta\} \subset X$, *then* $\varphi_{\alpha}(\beta) \in X$.

Proof. Proposition 3.18.2 is seen by main induction on $\alpha \in X$ with subsidiary induction on $\beta \in X$ using Proposition 3.18.1. We show Proposition 3.18.1. We obtain $\alpha \in X \cap \gamma^{+} = W(\mathcal{C}^{\gamma}(X)) \cap \gamma^{+}$ with $\gamma^{+} = \alpha^{+}$. We see that $\alpha + \beta \in W(\mathcal{C}^{\gamma}(X))$ by induction on $\beta \in X \cap \alpha \subset \mathcal{C}^{\gamma}(X)$. $\Box$

Proposition 3.19 *Let* $X_0 = W(\mathcal{C}^{0}(\emptyset)) \cap 0^{+}$ *with* $0^{+} = \Omega$, *and* $X_1 = W(\mathcal{C}^{\Omega}(X_0)) \cap \Omega^{+}$. *Then* $0 \in X_0$, $\Omega \in X_1$ *and* $D[X_i]$ *for* $i = 0, 1$.

Proof. For each $\alpha \in \{0, \Omega\}$ and any set $Y \subset OT(\Pi_3)$ we have $\alpha \in \mathcal{C}^{\alpha}(Y)$. First we obtain $0 \in \mathcal{G}(\emptyset)$ and $D[\emptyset]$. Also there is no β such that $\beta^{+} < 0^{+}$. Hence the condition (71) is fulfilled, and we obtain $0 \in X_0$ and $D[X_0]$ by Lemma 3.15.

Next let $\gamma \in \mathcal{C}^{\Omega}(X_0) \cap \Omega$. We show $\gamma \in X_0$ by induction on the lengths $\ell\gamma$ of ordinal terms γ as follows. We see that each strongly critical number $\gamma \in \mathcal{C}^{\Omega}(X_0) \cap \Omega$ is in X_0 since if $\psi_{\sigma}^{\nu}(\beta) < \Omega$, then $\sigma = \Omega$. Otherwise $\gamma \in X_0$ is seen from IH using Proposition 3.18 and $0 \in X_0$. Therefore we obtain $\alpha \in \mathcal{G}(X_0)$. Let $\beta^{+} < \alpha^{+}$. Then $\beta^{+} = \Omega$ and $\beta < \Omega$. Then $W(\mathcal{C}^{\beta}(X_0)) \cap \Omega = W(\mathcal{C}^{0}(X_0)) \cap \Omega = X_0$ by Proposition 3.3. Hence the condition (71) is fulfilled, and we obtain $\Omega \in X_1$ and $D[X_1]$ by Lemma 3.15. $\Box$

Definition 3.20 $\beta \prec \alpha$ iff there exists a sequence $\{\sigma_i\}_{i \leq n} (n > 0)$ such that $\alpha = \sigma_0$, $\beta = \sigma_n$ and for each $i < n$, there are some ν_i, a_i such that $\sigma_{i+1} = \psi_{\sigma_i}^{\nu_i}(a_i)$.

Note that $\beta \prec \alpha \Rightarrow m_2(\beta) < m_2(\alpha)$.

Lemma 3.21 *Suppose* $D[Y]$ *with* $\{0, \Omega\} \subset Y$, *and for* $\eta \in OT(\Pi_3)$

$$\eta \in \mathcal{G}(Y) \tag{16}$$

and

$$\forall\gamma\prec\eta(\gamma\in\mathcal{G}(Y)\Rightarrow\gamma\in Y) \tag{17}$$

Let $X=W(\mathcal{C}^\eta(Y))\cap\eta^+$. *Then* $\eta\in X$ *and* $D[X]$.

Proof. By Lemma 3.15 and the hypothesis (16) it suffices to show (71), i.e.,

$$\forall\beta\left(Y\cap\eta^+<\beta\,\&\,\beta^+<\eta^+\to W(\mathcal{C}^\beta(Y))\cap\beta^+\subset Y\right).$$

Assume $Y\cap\eta^+<\beta$ and $\beta^+<\eta^+$. We have to show $W(\mathcal{C}^\beta(Y))\cap\beta^+\subset Y$. We prove this by induction on $\gamma\in W(\mathcal{C}^\beta(Y))\cap\beta^+$. Suppose $\gamma\in\mathcal{C}^\beta(Y)\cap\beta^+$ and

$$\text{MIH}:\mathcal{C}^\beta(Y)\cap\gamma\subset Y.$$

We show $\gamma\in Y$. We can assume that

$$Y\cap\eta^+<\gamma \tag{18}$$

since if $\gamma\leq\delta$ for some $\delta\in Y\cap\eta^+$, then by $Y\cap\eta^+<\beta$ and $\gamma\in\mathcal{C}^\beta(Y)$ we obtain $\delta<\beta$, $\gamma\in\mathcal{C}^\delta(Y)$ and $\delta\in W(\mathcal{C}^\delta(Y))\cap\delta^+=Y\cap\delta^+$. Hence $\gamma\in W(\mathcal{C}^\delta(Y))\cap\delta^+\subset Y$.

We show first

$$\gamma\in\mathcal{G}(Y) \tag{19}$$

First $\gamma\in\mathcal{C}^\gamma(Y)$ by $\gamma\in\mathcal{C}^\beta(Y)\cap\beta^+$ and Proposition 3.3. Second we show the following claim by induction on $\ell\alpha$:

$$\alpha\in\mathcal{C}^\gamma(Y)\cap\gamma\Rightarrow\alpha\in Y \tag{20}$$

Proof of (20). Assume $\alpha\in\mathcal{C}^\gamma(Y)$. We can assume $\gamma^+\leq\beta$ for otherwise we have $\alpha\in\mathcal{C}^\gamma(Y)\cap\gamma=\mathcal{C}^\beta(Y)\cap\gamma\subset Y$ by MIH.

By induction hypothesis on lengths, $\alpha<\gamma<\beta^+<\eta^+$, Proposition 3.18, and $\{0,\Omega\}\subset Y$, we can assume that $\alpha=\psi_\pi^\nu(a)$ for some $\pi>\gamma$ such that $\{\pi,a,\nu\}\subset\mathcal{C}^\gamma(Y)$.

Case 1. $\beta<\pi$: Then $G_\beta(\{\pi,a,\nu\})=G_\beta(\alpha)<\alpha<\gamma$ by Proposition 3.13.1. Proposition 3.14 with induction hypothesis on lengths yields $\{\pi,a,\nu\}\subset\mathcal{C}^\beta(Y)$. Hence $\alpha\in\mathcal{C}^\beta(Y)\cap\gamma$ by $\pi>\beta$. MIH yields $\alpha\in Y$.

Case 2. $\beta\geq\pi$: We have $\alpha<\gamma<\pi\leq\beta$. It suffices to show that $\alpha\leq Y\cap\eta^+$. Then by (18) we have $\alpha\leq\delta\in Y\cap\eta^+$ for some $\delta<\gamma$. $\mathcal{C}^\delta(Y)\ni\alpha\leq\delta\in Y\cap\delta^+=W(\mathcal{C}^\delta(Y))\cap\delta^+$ yields $\alpha\in W(\mathcal{C}^\delta(Y))\cap\delta^+\subset Y$.

Assume first that γ is not a strongly critical number. By $\alpha=\psi_\pi^\nu(a)<\gamma$, we can assume that $\gamma\neq 0$. Let δ denote the largest immediate subterm of γ. We obtain $\delta\in\mathcal{C}^\beta(Y)\cap\gamma$ by (18), $Y\cap\eta^+<\gamma\in\mathcal{C}^\beta(Y)$. Hence $\delta\in Y$ by MIH. Also by $\alpha<\gamma$, we obtain $\alpha\leq\delta$, i.e., $\alpha\leq Y$, and we are done.

Next let $\gamma=\psi_\kappa^\xi(b)$ for some b,ξ and $\kappa>\beta$ by (18) and $\gamma\in\mathcal{C}^\beta(Y)$. We have $\alpha<\gamma<\pi\leq\beta<\kappa$. We obtain $\pi\notin\mathcal{H}_b(\gamma)$ since otherwise by $\pi<\kappa$ we would have $\pi<\gamma$. Therefore $\alpha=\psi_\pi^\nu(a)<\psi_\kappa^\xi(b)=\gamma<\pi<\kappa$ with $\pi\in\mathcal{H}_a(\alpha)$ and $\pi\notin\mathcal{H}_b(\gamma)$. This yields $a>b$ and $\{\kappa,b,\xi\}\not\subset\mathcal{H}_a(\alpha)$.

On the other hand we have $\{\kappa, b, \xi\} \subset \mathcal{H}_a(\gamma)$. This means that there exists a subterm $\delta < \gamma$ of one of κ, b, ξ such that $\delta \not\in \mathcal{H}_a(\alpha)$. Also we have $\{\kappa, b, \xi\} \subset \mathcal{C}^\beta(Y)$. Then $\delta \in \mathcal{C}^\beta(Y) \cap \gamma$. By MIH we obtain $\alpha \leq \delta \in \mathcal{C}^\beta(Y) \cap \gamma \subset Y$.

□ of (20) and (19).

Hence we obtain $\gamma \in \mathcal{G}(Y)$. We have $\gamma < \beta^+ \leq \eta$ and $\gamma \in \mathcal{C}^\gamma(Y)$. If $\gamma \prec \eta$, then the hypothesis (17) yields $\gamma \in Y$. In what follows assume $\gamma \not\prec \eta$.

If $G_\eta(\gamma) < \gamma$, then Proposition 3.14 yields $\gamma \in \mathcal{C}^\eta(Y) \cap \eta \subset Y$ by $\eta \in \mathcal{G}(Y)$.

Suppose $G_\eta(\gamma) = \{\gamma\}$. This means, by $\gamma \not\prec \eta$, that $\gamma \prec \tau$ for a $\tau < \eta$. Let τ denote the maximal such one. We have $\gamma < \tau < \eta$. From $\gamma \in \mathcal{C}^\gamma(Y)$ we see $\tau \in \mathcal{C}^\gamma(Y)$. Next we show that

$$G_\eta(\tau) < \gamma \tag{21}$$

Let $\tau = \psi^\mu_\kappa(b)$. Then $\eta < \kappa$ by the maximality of τ, and $G_\eta(\tau) = G_\eta(\{\kappa, b, \mu\}) < \tau$ by Proposition 3.13.1. On the other hand we have $\tau \in \mathcal{H}_a(\gamma)$. Proposition 3.13.2 yields $G_\eta(\tau) \subset \mathcal{H}_a(\gamma)$. We see $G_\eta(\tau) < \gamma$ inductively.

Proposition 3.14 with (21) yields $\tau \in \mathcal{C}^\eta(Y)$, and $\tau \in \mathcal{C}^\eta(Y) \cap \eta \subset Y$ by $\eta \in \mathcal{G}(Y)$. Therefore $Y \cap \eta^+ < \gamma < \tau \in Y$. This is not the case by (18). We are done. □

Proposition 3.22 $\alpha \leq \mathcal{W} \cap \beta^+ \,\&\, \alpha \in \mathcal{C}^\beta(\mathcal{W}) \Rightarrow \alpha \in \mathcal{W}$.

Proof. This is seen from Propositions 3.3, 3.6 an 7.39. □

3.2 Mahlo universes

In Proposition 3.9, we saw that $\mathcal{W}$ is the maximal distinguished class, which is Σ^{1-}_2-definable and a proper class in $\mathsf{KP}\Pi_3$. $\mathcal{W}^P$ in Definition 3.25 denotes the maximal distinguished class *inside* a set P. $\mathcal{W}^P$ exists as a set.

Let ad denote a Π^-_3-sentence such that a transitive set z is admissible iff $(z; \in) \models ad$. Let $lmtad :\Leftrightarrow \forall x \exists y (x \in y \wedge ad^y)$. Observe that $lmtad$ is a Π^-_2-sentence.

Definition 3.23 L denotes a whole universe, which is a model of $\mathsf{KP}\Pi_3$.

1. By a *universe* we mean either the whole universe L or a transitive set $Q \in L$ with $\omega \in Q$. Universes are denoted by $P, Q, \ldots$

2. For a universe P and a set-theoretic sentence φ, $P \models \varphi :\Leftrightarrow (P; \in) \models \varphi$.

3. A universe P is said to be a *limit universe* if $lmtad^P$ holds, i.e., P is a limit of admissible sets. The class of limit universes is denoted by $Lmtad$.

Lemma 3.24 *$W(\mathcal{C}^\alpha(X))$ as well as $D[X]$ are absolute for limit universes P.*

Proof. Let P be a limit universe and $X \in \mathcal{P}(\omega) \cap P$. Then $W(X)$ is Δ_1 in P, and so is $W(\mathcal{C}^\alpha(X))$. Hence $W(\mathcal{C}^\alpha(X)) = \{\beta \in OT(\Pi_3) : P \models \beta \in W(\mathcal{C}^\alpha(X))\}$, and $D[X] \Leftrightarrow P \models D[X]$. □

Definition 3.25 For a universe P, let $\mathcal{W}^P := \bigcup\{X \in P : D[X]\}$.

Lemma 3.26 *Let P be a universe closed under finite unions, and $\alpha \in OT(\Pi_3)$.*

1. *There is a finite set $K(\alpha) \subset OT(\Pi_3)$ such that $\forall Y \in P \forall \gamma [K(\alpha) \cap Y = K(\alpha) \cap \mathcal{W}^P \Rightarrow \left(\alpha \in \mathcal{C}^\gamma(\mathcal{W}^P) \Leftrightarrow \alpha \in \mathcal{C}^\gamma(Y)\right)]$.*

2. *There exists a distinguished set $X \in P$ such that $\forall Y \in P \forall \gamma [X \subset Y \,\&\, D[Y] \Rightarrow (\alpha \in \mathcal{C}^\gamma(\mathcal{W}^P) \Leftrightarrow \alpha \in \mathcal{C}^\gamma(Y))]$.*

Proof. 3.26.1. F.e. the set of subterms of α enjoys the condition for $K(\alpha)$.
3.26.2. By $X, Y \in P \Rightarrow X \cup Y \in P$, pick a distinguished set $X \in P$ such that $K(\alpha) \cap \mathcal{W}^P \subset X$. □

Proposition 3.27 *For each limit universe P, $D[\mathcal{W}^P]$ holds, and $\exists X (X = \mathcal{W}^P)$ if P is a set.*

Proof. $D[\mathcal{W}^P]$ is seen as in Proposition 3.9. □

For a universal Π_n-formula $\Pi_n(a)$ $(n > 0)$ uniformly on admissibles, let

$$P \in M_2(\mathcal{C}) :\Leftrightarrow P \in Lmtad \,\&\, \forall b \in P[P \models \Pi_2(b) \to \exists Q \in \mathcal{C} \cap P(Q \models \Pi_2(b))].$$

Lemma 3.28 *Let $\mathcal{C}$ be a Π^1_0-class such that $\mathcal{C} \subset Lmtad$. Suppose $P \in M_2(\mathcal{C})$ and $\alpha \in \mathcal{G}(\mathcal{W}^P)$. Then there exists a universe $Q \in \mathcal{C}$ such that $\alpha \in \mathcal{G}(\mathcal{W}^Q)$.*

Proof. Suppose $P \in M_2(\mathcal{C})$ and $\alpha \in \mathcal{G}(\mathcal{W}^P)$. First by $\alpha \in \mathcal{C}^\alpha(\mathcal{W}^P)$ and Lemma 3.26 pick a distinguished set $X_0 \in P$ such that $\alpha \in \mathcal{C}^\alpha(X_0)$ and $K(\alpha) \cap \mathcal{W}^P \subset X_0$. Next writing $\mathcal{C}^\alpha(\mathcal{W}^P) \cap \alpha \subset \mathcal{W}^P$ analytically we have

$$\forall \beta < \alpha[\beta \in \mathcal{C}^\alpha(\mathcal{W}^P) \Rightarrow \exists Y \in P(D[Y] \,\&\, \beta \in Y)]$$

By Lemma 3.26 we obtain $\beta \in \mathcal{C}^\alpha(\mathcal{W}^P) \Leftrightarrow \exists X \in P\{D[X] \,\&\, K(\beta) \cap \mathcal{W}^P \subset X \,\&\, \beta \in \mathcal{C}^\alpha(X)\}$. Hence for any $\beta < \alpha$ and any distinguished set $X \in P$, there are $\gamma \in K(\beta)$, $Z \in P$ and a distinguished set $Y \in P$ such that if $\gamma \in Z \,\&\, D[Z] \to \gamma \in X$ and $\beta \in \mathcal{C}^\alpha(X)$, then $\beta \in Y$. By Lemma 3.24 $D[X]$ is absolute for limit universes. Hence the following Π_2-predicate holds in the universe $P \in M_2(\mathcal{C})$:

$$\forall \beta < \alpha \forall X \exists \gamma \in K(\beta) \exists Z \exists Y [\{D[X] \,\&\, (\gamma \in Z \,\&\, D[Z] \to \gamma \in X) \,\&\, \beta \in \mathcal{C}^\alpha(X)\} \Rightarrow (D[Y] \,\&\, \beta \in Y)] \tag{22}$$

Now pick a universe $Q \in \mathcal{C} \cap P$ with $X_0 \in Q$ and $Q \models (22)$. Tracing the above argument backwards in the limit universe Q we obtain $\mathcal{C}^\alpha(\mathcal{W}^Q) \cap \alpha \subset \mathcal{W}^Q$ and $X_0 \subset \mathcal{W}^Q = \bigcup\{X \in Q : Q \models D[X]\} \in P$. Thus Lemma 3.26 yields $\alpha \in \mathcal{C}^\alpha(\mathcal{W}^Q)$. We obtain $\alpha \in \mathcal{G}(\mathcal{W}^Q)$. □

Definition 3.29 We define the class $M_2(\alpha)$ of α-recursively Mahlo universes for $\alpha \in OT(\Pi_3)$ as follows:

$$P \in M_2(\alpha) \Leftrightarrow P \in Lmtad \,\&\, \forall \beta \prec \alpha [SC_{\mathbb{K}}(m_2(\beta)) \subset \mathcal{W}^P \Rightarrow P \in M_2(M_2(\beta))] \tag{23}$$

$M_2(\alpha)$ is a Π_3-class.

Proposition 3.30 *If* $\eta \in \mathcal{G}(Y)$*, then* $SC_{\mathbb{K}}(m_2(\eta)) \subset Y$*.*

Proof. Let $\nu = m_2(\eta)$. Then $SC_{\mathbb{K}}(\nu) \subset \eta$ by (9). From $\eta \in \mathcal{C}^{\eta}(Y)$ we see $SC_{\mathbb{K}}(\nu) \subset \mathcal{C}^{\eta}(Y)$. Hence $SC_{\mathbb{K}}(\nu) \subset \mathcal{C}^{\eta}(Y) \cap \eta \subset Y$ by $\eta \in \mathcal{G}(Y)$. □

Lemma 3.31 *If* $\eta \in \mathcal{G}(\mathcal{W}^P)$ *and* $P \in M_2(M_2(\eta))$*, then* $\eta \in \mathcal{W}^P$*.*

Proof. We show this by induction on $\in$. Suppose, as IH, the lemma holds for any $Q \in P$. By Lemma 3.28 pick a $Q \in P$ such that $Q \in M_2(\eta)$, and for $Y = \mathcal{W}^Q \in P$, $\{0, \Omega\} \subset Y$ and

$$\eta \in \mathcal{G}(Y) \tag{16}$$

On the other the definition (23) yields $\forall \gamma \prec \eta[SC_{\mathbb{K}}(m_2(\gamma)) \subset \mathcal{W}^Q \Rightarrow Q \in M_2(M_2(\gamma))]$. Hence by Proposition 3.30 $\forall \gamma \prec \eta[\gamma \in \mathcal{G}(\mathcal{W}^Q) \Rightarrow Q \in M_2(M_2(\gamma))]$.

IH yields with $Y = \mathcal{W}^Q$

$$\forall \gamma \prec \eta(\gamma \in \mathcal{G}(Y) \Rightarrow \gamma \in Y) \tag{17}$$

Therefore by Lemma 3.21 we conclude $\eta \in X$ and $D[X]$ for $X = W(\mathcal{C}^{\eta}(Y)) \cap \eta^+$.

$X \in P$ follows from $Y \in P \in Lmtad$. Consequently $\eta \in \mathcal{W}^P$. □

Lemma 3.32 *1.* $\mathcal{C}^{\mathbb{K}}(\mathcal{W}) \cap \mathbb{K} = \mathcal{W} \cap \mathbb{K}$*.*

2. $\mathbb{K} \in \mathcal{C}^{\mathbb{K}}(\mathcal{W})$*.*

3. For each $n \in \omega$, $TI[\mathcal{C}^{\mathbb{K}}(\mathcal{W}) \cap \omega_n(\mathbb{K}+1)]$*.*

Proof. We show Lemma 3.32.3. It suffices to show $TI[\mathcal{W}]$. Assume $Prg[\mathcal{W}, A]$ for a formula A, and $\alpha \in \mathcal{W}$. Pick a distinguished set X such that $\alpha \in X$. Then $X \cap \alpha^+ = \mathcal{W} \cap \alpha^+$, and hence $Prg[X \cap (\alpha+1), A]$. $Wo[X]$ yields $A(\alpha)$. □

Lemma 3.33 $\forall \eta[m_2(\eta) \in \mathcal{C}^{\mathbb{K}}(\mathcal{W}) \cap \omega_n(\mathbb{K}+1) \Rightarrow L \in M_2(M_2(\eta))]$ *holds for* each $n \in \omega$*.*

Proof. We show the lemma by induction on $\nu = m_2(\eta) \in \mathcal{C}^{\mathbb{K}}(\mathcal{W})$ up to each $\omega_n(\mathbb{K}+1)$. Suppose $\nu \in \mathcal{C}^{\mathbb{K}}(\mathcal{W})$ and $L \models \Pi_2(b)$ for a $b \in L$. We have to find a universe $Q \in L$ such that $b \in Q$, $Q \in M_2(\eta)$ and $Q \models \Pi_2(b)$.

By the definition (23) $L \in M_2(\eta)$ is equivalent to $\forall \gamma \prec \eta[m_2(\gamma) \in \mathcal{C}^{\mathbb{K}}(\mathcal{W}) \Rightarrow L \in M_2(M_2(\gamma))]$. We obtain $\gamma \prec \eta \Rightarrow m_2(\gamma) < m_2(\eta) = \nu$. Thus IH yields $L \in M_2(\eta)$. Let g be a primitive recursive function in the sense of set theory such that $L \in M_2(\eta) \Leftrightarrow P \models \Pi_3(g(\eta))$. Then $L \models \Pi_2(b) \wedge \Pi_3(g(\eta))$. Since this is a Π_3-formula which holds in a Π_3-reflecting universe L, we conclude for some $Q \in L$, $Q \models \Pi_2(b) \wedge \Pi_3(g(\eta))$ and hence $Q \in M_2(\eta)$. We are done. □

Remark 3.34 Only here we need Π_3-reflection. Therefore it sufffices for a whole universe L to admit iterations of Π_2-recursively Mahlo operations along a well founded relation $\prec$ which is Σ on L: $L \in M_2^{\prec}(\mu) = \bigcap\{M_2(M_2^{\prec}(\nu)) : L \models \nu \prec \mu\}$. Hence our wellfoundednes proof is formalizable in a set theory axiomatizing such universes L.

Lemma 3.35 *For each $n \in \omega$, $m_2(\eta) < \omega_n(\mathbb{K}+1) \,\&\, \eta \in \mathcal{G}(\mathcal{W}) \Rightarrow \eta \in \mathcal{W}$.*

Proof. Assume $\nu = m_2(\eta) < \omega_n(\mathbb{K}+1)$ and $\eta \in \mathcal{G}(\mathcal{W})$. By Proposition 3.30 we obtain $\nu \in \mathcal{C}^{\mathbb{K}}(\mathcal{W})$. Lemma 3.33 yields $L \in M_2(M_2(\eta))$. From this we see $L \in M_2(\mathcal{C})$ with $\mathcal{C} = M_2(M_2(\eta))$ as in the proof of Lemma 3.33 using Π_3-reflection of the whole universe L once again. Then by Lemma 3.28 pick a set $P \in L$ such that $\eta \in \mathcal{G}(\mathcal{W}^P)$ and $P \in \mathcal{C} = M_2(M_2(\eta))$. Lemma 3.31 yields $\eta \in \mathcal{W}^P \subset \mathcal{W}$. □

3.3 Well-foundedness proof (concluded)

Definition 3.36 For terms $\alpha, \kappa, \delta \in OT(\Pi_3)$, finite sets $\mathcal{E}(\alpha), K_\delta(\alpha), k_\delta(\alpha) \subset OT(\Pi_3)$ are defined recursively as follows.

1. $\mathcal{E}(\alpha) = \emptyset$ for $\alpha \in \{0, \Omega, \mathbb{K}\}$. $\mathcal{E}(\alpha_m + \cdots + \alpha_0) = \bigcup_{i \leq m} \mathcal{E}(\alpha_i)$. $\mathcal{E}(\varphi\beta\gamma) = \mathcal{E}(\beta) \cup \mathcal{E}(\gamma)$. $\mathcal{E}(\psi_\pi^\nu(a)) = \{\psi_\pi^\nu(a)\}$.
2. $\mathcal{A}(\alpha) = \bigcup\{\mathcal{A}(\beta) : \beta \in \mathcal{E}(\alpha)\}$ for $\mathcal{A} \in \{K_\delta, k_\delta\}$.
3. $K_\delta(\psi_\pi^\nu(a)) = \left\{ \begin{array}{ll} \{a\} \cup K_\delta(\{\pi, a\} \cup SC_{\mathbb{K}}(\nu)) & \psi_\pi^\nu(a) \geq \delta \\ \emptyset & \psi_\pi^\nu(a) < \delta \end{array} \right.$.
4. $k_\delta(\psi_\pi^\nu(a)) = \left\{ \begin{array}{ll} \{\psi_\pi^\nu(a)\} \cup k_\delta(\{\pi, a\} \cup SC_{\mathbb{K}}(\nu)) & \psi_\pi^\nu(a) \geq \delta \\ \emptyset & \psi_\pi^\nu(a) < \delta \end{array} \right.$.

Note that $K_\delta(\alpha) < a \Leftrightarrow \alpha \in \mathcal{H}_a(\delta)$.

Definition 3.37 For $a, \nu \in OT(\Pi_3)$, define:

$$A(a,\nu) \;:\Leftrightarrow\; \forall \sigma \in \mathcal{C}^{\mathbb{K}}(\mathcal{W})[\psi_\sigma^\nu(a) \in OT(\Pi_3) \Rightarrow \psi_\sigma^\nu(a) \in \mathcal{W}]. \tag{24}$$

$$\mathrm{MIH}(a) \;:\Leftrightarrow\; \forall b \in \mathcal{C}^{\mathbb{K}}(\mathcal{W}) \cap a \forall \nu \in \mathcal{C}^{\mathbb{K}}(\mathcal{W})\, A(b,\nu). \tag{25}$$

$$\mathrm{SIH}(a,\nu) \;:\Leftrightarrow\; \forall \xi \in \mathcal{C}^{\mathbb{K}}(\mathcal{W})[\xi < \nu \Rightarrow A(a,\xi)]. \tag{26}$$

Lemma 3.38 *For each n the following holds: Assume $\{a, \nu\} \subset \mathcal{C}^{\mathbb{K}}(\mathcal{W}) \cap \omega_n(\mathbb{K}+1)$, $\mathrm{MIH}(a)$, and $\mathrm{SIH}(a,\nu)$ in Definition 3.37. Then*

$$\forall \kappa \in \mathcal{C}^{\mathbb{K}}(\mathcal{W})[\psi_\kappa^\nu(a) \in OT(\Pi_3) \Rightarrow \psi_\kappa^\nu(a) \in \mathcal{W}].$$

Proof. Let $\alpha_1 = \psi_\kappa^\nu(a) \in OT(\Pi_3)$ with $\{a, \kappa, \nu\} \subset \mathcal{C}^{\mathbb{K}}(\mathcal{W})$ and $\nu \leq a < \omega_n(\mathbb{K}+1)$, cf. (9). By Lemma 3.35 it suffices to show $\alpha_1 \in \mathcal{G}(\mathcal{W})$.

By Proposition 3.6 we have $\{\kappa, a, \nu\} \subset \mathcal{C}^{\alpha_1}(\mathcal{W})$, and hence $\alpha_1 \in \mathcal{C}^{\alpha_1}(\mathcal{W})$. It suffices to show the following claim.

$$\forall \beta_1 \in \mathcal{C}^{\alpha_1}(\mathcal{W}) \cap \alpha_1[\beta_1 \in \mathcal{W}]. \tag{27}$$

Proof of (27) by induction on $\ell\beta_1$. Assume $\beta_1 \in \mathcal{C}^{\alpha_1}(\mathcal{W}) \cap \alpha_1$ and let

$$\mathrm{LIH} :\Leftrightarrow \forall \gamma \in \mathcal{C}^{\alpha_1}(\mathcal{W}) \cap \alpha_1[\ell\gamma < \ell\beta_1 \Rightarrow \gamma \in \mathcal{W}].$$

We show $\beta_1 \in \mathcal{W}$. By Propositions 3.18, 3.19 and LIH, we may assume that $\beta_1 = \psi^{\xi}_{\pi}(b)$ for some π, b, ξ such that $\{\pi, b, \xi\} \subset \mathcal{C}^{\alpha_1}(\mathcal{W})$.

$\beta_1 = \psi^{\xi}_{\pi}(b) < \psi^{\nu}_{\kappa}(a) = \alpha_1$ holds iff one of the following holds: (1) $\pi \leq \alpha_1$. (2) $b < a$, $\beta_1 < \kappa$ and $\{\pi, b, \xi\} \subset \mathcal{H}_a(\alpha_1)$. (3) $b = a$, $\pi = \kappa$, $\xi \in \mathcal{H}_a(\alpha_1)$ and $\xi < \nu$. (4) $a \leq b$ and $\{\kappa, a, \nu\} \not\subset \mathcal{H}_b(\beta_1)$.

Case 1. $\pi \leq \alpha_1$: Then $\beta_1 \in \mathcal{W}$ by $\beta_1 \in \mathcal{C}^{\alpha_1}(\mathcal{W})$.

Case 2. $b < a$, $\beta_1 < \kappa$ and $\{\pi, b, \xi\} \subset \mathcal{H}_a(\alpha_1)$: Let B denote a set of subterms of β_1 defined recursively as follows. First $\{\pi, b\} \cup SC_{\mathbb{K}}(\xi) \subset B$. Let $\alpha_1 \leq \beta \in B$. If $\beta =_{NF} \gamma_m + \cdots + \gamma_0$, then $\{\gamma_i : i \leq m\} \subset B$. If $\beta =_{NF} \varphi\gamma\delta$, then $\{\gamma, \delta\} \subset B$. If $\beta = \psi^{\mu}_{\sigma}(c)$, then $\{\sigma, c\} \cup SC_{\mathbb{K}}(\mu) \subset B$.

Then from $\{\pi, b, \xi\} \subset \mathcal{C}^{\alpha_1}(\mathcal{W})$ we see inductively that $B \subset \mathcal{C}^{\alpha_1}(\mathcal{W})$. Hence by LIH we obtain $B \cap \alpha_1 \subset \mathcal{W}$. Moreover if $\alpha_1 \leq \psi^{\mu}_{\sigma}(c) \in B$, then we see $c < a$ from $\{\pi, b, \xi\} \subset \mathcal{H}_a(\alpha_1)$. We claim that

$$\forall \beta \in B(\beta \in \mathcal{C}^{\mathbb{K}}(\mathcal{W})) \tag{28}$$

Proof of (28) by induction on $\ell\beta$. Let $\beta \in B$. We can assume that $\alpha_1 \leq \beta = \psi^{\mu}_{\sigma}(c)$ by induction hypothesis on the lengths. Then by induction hypothesis we have $\{\sigma, c\} \cup SC_{\mathbb{K}}(\mu) \subset \mathcal{C}^{\mathbb{K}}(\mathcal{W})$. On the other hand we have $\mu \leq c < a$ by (9). MIH(a) yields $\beta \in \mathcal{W}$. Thus (28) is shown. □

In particular we obtain $\{\pi, b\} \cup SC_{\mathbb{K}}(\xi) \subset \mathcal{C}^{\mathbb{K}}(\mathcal{W})$. Moreover we have $\xi \leq b < a$ by (9). Therefore once again MIH(a) yields $\beta_1 \in \mathcal{W}$.

Case 3. $b = a$, $\pi = \kappa$, $\xi \in \mathcal{H}_a(\alpha_1)$ and $\xi < \nu \leq a$: As in (28) we see that $SC_{\mathbb{K}}(\xi) \subset \mathcal{W}$ from MIH(a). SIH(a, ν) yields $\beta_1 \in \mathcal{W}$.

Case 4. $a \leq b$ and $\{\kappa, a, \nu\} \not\subset \mathcal{H}_b(\beta_1)$: It suffices to find a γ such that $\beta_1 \leq \gamma \in \mathcal{W} \cap \alpha_1$. Then $\beta_1 \in \mathcal{W}$ follows from $\beta_1 \in \mathcal{C}^{\alpha_1}(\mathcal{W})$ and Proposition 3.22.

$k_\delta(\alpha)$ denotes the set in Definition 3.36. In general we see that $a \in K_\delta(\alpha)$ iff $\psi^h_\sigma(a) \in k_\delta(\alpha)$ for some σ, h, and for each $\psi^h_\sigma(a) \in k_\delta(\psi^{h_0}_{\sigma_0}(a_0))$ there exists a sequence $\{\alpha_i\}_{i \leq m}$ of subterms of $\alpha_0 = \psi^{h_0}_{\sigma_0}(a_0)$ such that $\alpha_m = \psi^h_\sigma(a)$, $\alpha_i = \psi^{h_i}_{\sigma_i}(a_i)$ for some σ_i, a_i, h_i, and for each $i < m$, $\delta \leq \alpha_{i+1} \in \mathcal{E}(C_i)$ for $C_i = \{\sigma_i, a_i\} \cup SC_{\mathbb{K}}(h_i)$.

Let $\delta \in SC_{\mathbb{K}}(f) \cup \{\kappa, a\}$ such that $b \leq \gamma$ for a $\gamma \in K_{\beta_1}(\delta)$. Pick an $\alpha_2 = \psi^{h_2}_{\sigma_2}(a_2) \in \mathcal{E}(\delta)$ such that $\gamma \in K_{\beta_1}(\alpha_2)$, and an $\alpha_m = \psi^{h_m}_{\sigma_m}(a_m) \in k_{\beta_1}(\alpha_2)$ for some σ_m, h_m and $a_m \geq b \geq a$. We have $\alpha_2 \in \mathcal{W}$ by $\delta \in \mathcal{W}$. If $\alpha_2 < \alpha_1$, then $\beta_1 \leq \alpha_2 \in \mathcal{W} \cap \alpha_1$, and we are done. Assume $\alpha_2 \geq \alpha_1$. Then $a_2 \in K_{\alpha_1}(\alpha_2) < a \leq b$, and $m > 2$.

Let $\{\alpha_i\}_{2 \leq i \leq m}$ be the sequence of subterms of α_2 such that $\alpha_i = \psi^{h_i}_{\sigma_i}(a_i)$ for some σ_i, a_i, h_i, and for each $i < m$, $\beta_1 \leq \alpha_{i+1} \in \mathcal{E}(C_i)$ for $C_i = \{\sigma_i, a_i\} \cup SC_{\mathbb{K}}(h_i)$.

Let $\{n_j\}_{0 \leq j \leq k}$ $(0 < k \leq m-2)$ be the increasing sequence $n_0 < n_1 < \cdots < n_k \leq m$ defined recursively by $n_0 = 2$, and assuming n_j has been defined so that $n_j < m$ and $\alpha_{n_j} \geq \alpha_1$, n_{j+1} is defined by $n_{j+1} = \min(\{i : n_j < i < m, \alpha_i < \alpha_{n_j}\} \cup \{m\})$. If either $n_j = m$ or $\alpha_{n_j} < \alpha_1$, then $k = j$ and n_{j+1} is undefined. Then we claim that

$$\forall j \leq k(\alpha_{n_j} \in \mathcal{W})\,\&\,\alpha_{n_k} < \alpha_1 \tag{29}$$

Proof of (29). By induction on $j \leq k$ we show first that $\forall j \leq k(\alpha_{n_j} \in \mathcal{W})$. We have $\alpha_{n_0} = \alpha_2 \in \mathcal{W}$. Assume $\alpha_{n_j} \in \mathcal{W}$ and $j < k$. Then $n_j < m$, i.e., $\alpha_{n_{j+1}} < \alpha_{n_j}$, and by $\alpha_{n_j} \in \mathcal{C}^{\alpha_{n_j}}(\mathcal{W})$, we have $C_{n_j} \subset \mathcal{C}^{\alpha_{n_j}}(\mathcal{W})$, and hence $\alpha_{n_j+1} \in \mathcal{E}(C_{n_j}) \subset \mathcal{C}^{\alpha_{n_j}}(\mathcal{W})$. We see inductively that $\alpha_i \in \mathcal{C}^{\alpha_{n_j}}(\mathcal{W})$ for any i with $n_j \leq i \leq n_{j+1}$. Therefore $\alpha_{n_{j+1}} \in \mathcal{C}^{\alpha_{n_j}}(\mathcal{W}) \cap \alpha_{n_j} \subset \mathcal{W}$ by Proposition 3.22.

Next we show that $\alpha_{n_k} < \alpha_1$. We can assume that $n_k = m$. This means that $\forall i(n_{k-1} \leq i < m \Rightarrow \alpha_i \geq \alpha_{n_{k-1}})$. We have $\alpha_2 = \alpha_{n_0} > \alpha_{n_1} > \cdots > \alpha_{n_{k-1}} \geq \alpha_1$, and $\forall i < m(\alpha_i \geq \alpha_1)$. Therefore $\alpha_m \in k_{\alpha_1}(\alpha_2) \subset k_{\alpha_1}(\{\kappa, a\} \cup SC_{\mathbb{K}}(h))$, i.e., $a_m \in K_{\alpha_1}(\{\kappa, a\} \cup SC_{\mathbb{K}}(h))$ for $\alpha_m = \psi_{\sigma_m}^{h_m}(a_m)$. On the other hand we have $K_{\alpha_1}(\{\kappa, a\} \cup SC_{\mathbb{K}}(h)) < a$ for $\alpha_1 = \psi_{\sigma}^{h}(a)$. Thus $a \leq a_m < a$, a contradiction.

(29) is shown, and we obtain $\beta_1 \leq \alpha_{n_k} \in \mathcal{W} \cap \alpha_1$.

This completes a proof of (27) and of the lemma. □

Lemma 3.39 *For each* $\alpha \in OT(\Pi_3)$, $\alpha \in \mathcal{C}^{\mathbb{K}}(\mathcal{W})$.

Proof. This is seen by meta-induction on $\ell\alpha$. By Propositions 3.18, 3.19, and Lemma 3.32, we may assume $\alpha = \psi_{\kappa}^{\nu}(a)$. By IH pick an $n < \omega$ such that $\{\kappa, \nu, a\} \subset \mathcal{C}^{\mathbb{K}}(\mathcal{W}) \cap \omega_{n+1}(\mathbb{K} + 1)$. Lemma 3.38 yields $\alpha \in \mathcal{W}$. □

Theorem 3.1 follows from Lemma 3.39 and the fact $\mathcal{W} \cap \Omega = W(\mathcal{C}^0(\emptyset)) \cap \Omega = W(OT(\Pi_3)) \cap \Omega$.

4 Π_4-reflection

> In this paper we focus on the ordinal analysis of Π_3 reflection. This means no genuine loss of generality, as the removal of Π_3 reflection rules in derivations already exhibits the pattern of cut elimination that applies for arbitrary Π_n reflection rules as well. ([Rathjen94])

In this section $\mathbb{K}$ denotes either a Π^1_2-indescribable cardinal or a Π_4-reflecting ordinal. Skolem hull $\mathcal{H}_a(X)$ and a Mahlo class $Mh^a_3(\xi)$ are defined as in Definition 2.2: Let for $\xi > 0$,

$$\pi \in Mh^a_3(\xi) :\Leftrightarrow [\{a, \xi\} \subset \mathcal{H}_a(\pi) \,\&\, \forall \nu \in \mathcal{H}_a(\pi) \cap \xi \, (\pi \in M_3(Mh^a_3(\nu)))]$$

where $\alpha \in M_3(A)$ iff A is Π^1_1-indescribable in α or α is Π_3-reflecting on A.

Then as in (8)

$$\psi^{\xi}_{\pi}(a) = \min(\{\pi\} \cup \{\kappa \in Mh^a_3(\xi) : \{\xi, \pi, a\} \subset \mathcal{H}_a(\kappa) \,\&\, \mathcal{H}_a(\kappa) \cap \pi \subset \kappa\})$$

where $\xi = m_3(\psi^{\xi}_{\pi}(a))$.

As in Lemmas 2.3 and 2.4 we see the following for Π^1_2-indescribable cardinal $\mathbb{K}$.

Lemma 4.1 *Let* $a \in \mathcal{H}_a(\mathbb{K}) \cap \varepsilon_{\mathbb{K}+1}$. *Then* $\mathbb{K} \in M_3(Mh^a_3(\varepsilon_{\mathbb{K}+1}))$. *For every* $\xi \in \mathcal{H}_a(\mathbb{K}) \cap \varepsilon_{\mathbb{K}+1}$, $\psi^{\xi}_{\mathbb{K}}(a) < \mathbb{K}$.

Operator controlled derivations for $\mathsf{KP}\Pi_4$ are closed under the following inference rules. For convenience let us attach an assignment $\bar{m} : \pi \mapsto \bar{m}(\pi) = (\bar{m}_2(\pi), \bar{m}_3(\pi))$ to the derivations, where $\bar{m}_i(\pi) \leq m_i(\pi)$ for $i = 2, 3$. Although our derivability relation should be written as $(\mathcal{H}_\gamma[\Theta], \bar{m}) \vdash^a_b \Gamma$, let us write $\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma$.

$(\mathrm{rfl}_{\Pi_4}(\mathbb{K}))$ $b \geq \mathbb{K}$. There exist an ordinal $a_0 \in \mathcal{H}_\gamma[\Theta] \cap a$, and a $\Sigma_4(\mathbb{K})$-sentence A enjoying the following conditions:

$$\frac{\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_b \Gamma, \neg A \quad \{\mathcal{H}_\gamma[\Theta \cup \{\rho\}] \vdash^{a_0}_b \Gamma, A^{(\rho,\mathbb{K})} : \rho < \mathbb{K}\}}{\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma} \ (\mathrm{rfl}_{\Pi_4}(\mathbb{K}))$$

$(\mathrm{rfl}_{\Pi_3}(\alpha, \pi, \nu))$ There exist ordinals $\alpha < \pi \leq b < \mathbb{K}$, $\nu < \bar{m}_3(\pi) \leq m_3(\pi)$ with $SC_{\mathbb{K}}(\nu) \subset \pi$ and $\nu \leq \gamma$, $a_0 < a$, and a finite set Δ of $\Sigma_3(\pi)$-sentences enjoying the following conditions:

1. $\{\alpha, \pi, \nu\} \cup \bar{m}(\pi) \subset \mathcal{H}_\gamma[\Theta]$.
2. For each $\delta \in \Delta$, $\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_b \Gamma, \neg\delta$.
3. Let
$$\rho \in Mh_3(\nu) :\Leftrightarrow \nu \leq m_3(\rho).$$
Then for each $\alpha < \rho \in Mh_3(\nu) \cap \pi$, $\mathcal{H}_\gamma[\Theta \cup \{\rho\}] \vdash^{a_0}_b \Gamma, \Delta^{(\rho)}$.

$$\frac{\{\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_b \Gamma, \neg\delta\}_{\delta\in\Delta} \quad \{\mathcal{H}_\gamma[\Theta \cup \{\rho\}] \vdash^{a_0}_b \Gamma, \Delta^{(\rho,\pi)}\}_{\alpha<\rho\in Mh_3(\nu)\cap\pi}}{\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma} \ (\mathrm{rfl}_{\Pi_3}(\alpha, \pi, \nu))$$

Finite proofs in $\mathsf{KP}\Pi_4$ are embedded to controlled derivations with inferences $(\mathrm{rfl}_{\Pi_4}(\mathbb{K}))$, and then $(\mathrm{rfl}_{\Pi_4}(\mathbb{K}))$ is replaced by inferences $(\mathrm{rfl}_{\Pi_3}(\alpha, \pi, \nu))$ as in Lemma 2.5.

Lemma 4.2 *Assume $\Gamma \subset \Sigma_3(\mathbb{K})$, $\Theta \subset \mathcal{H}_\gamma(\psi_{\mathbb{K}}(\gamma))$, and $\mathcal{H}_\gamma[\Theta] \vdash^a_{\mathbb{K}} \Gamma$ with $a \leq \gamma$. Then $\mathcal{H}_{\hat{a}+1}[\Theta \cup \{\kappa\}] \vdash^\beta_\beta \Gamma^{(\kappa,\mathbb{K})}$ holds for every $\kappa \in Mh_3(a) \cap \psi_{\mathbb{K}}(\gamma + \mathbb{K} \cdot \omega)$ such that $\psi_{\mathbb{K}}(\gamma + \mathbb{K}) < \kappa$, where $\hat{a} = \gamma + \omega^{\mathbb{K}+a}$ and $\beta = \psi_{\mathbb{K}}(\hat{a})$.*

Let us try to eliminate inferences $(\mathrm{rfl}_{\Pi_3}(\alpha, \pi, \nu))$ from the resulting derivations following the proof of Lemma 2.5. Let $Mh_2(\xi; a)$ be a Mahlo class for which the following holds.

Lemma 4.3 *Let $\Gamma \subset \Sigma_2(\pi)$ with $\xi = m_3(\pi)$, and $\mathcal{H}_\gamma[\Theta] \vdash^a_\pi \Gamma$. Then for any $\kappa \in Mh_2(\xi; a) \cap \pi$, $\mathcal{H}_\gamma[\Theta \cup \{\kappa\}] \vdash^{\kappa+\omega a}_\pi \Gamma^{(\kappa,\pi)}$ holds*[1].

[1]Here we don't need to collapse derivations and cut ranks$< \pi$.

Consider the crucial case. Let $\Delta\subset\Sigma_3(\pi)$, $\pi\in Mh_3(\xi)$ and $\nu<\xi$.

$$\frac{\{\mathcal{H}_\gamma[\Theta]\vdash^{a_0}_{\pi}\Gamma,\neg\delta\}_{\delta\in\Delta}\quad \{\mathcal{H}_\gamma[\Theta\cup\{\rho\}]\vdash^{a_0}_{\pi}\Gamma,\Delta^{(\rho,\pi)}:\alpha<\rho\in Mh_3(\nu)\cap\pi\}}{\mathcal{H}_\gamma[\Theta]\vdash^{a}_{\pi}\Gamma}\,(\mathrm{rfl}_{\Pi_3}(\alpha,\pi,\nu))$$

Let $\sigma\in Mh_2(\xi;a_0)\cap\kappa$. By IH with Inversion we obtain $\mathcal{H}_\gamma[\Theta\cup\{\sigma\}]\vdash^{\kappa+\omega a_0+1}_{\pi}$ $\Gamma^{(\sigma,\pi)},\neg\delta^{(\sigma,\pi)}$ for each $\delta\in\Delta$.

On the other hand we have $\mathcal{H}_\gamma[\Theta\cup\{\sigma\}]\vdash^{a_0}_{\pi}\Gamma,\Delta^{(\sigma,\pi)}$ for $\alpha<\sigma\in Mh_3(\nu)\cap\pi$. Assume $Mh_2(\xi;a)\subset Mh_2(\xi;a_0)$. IH yields $\mathcal{H}_\gamma[\Theta\cup\{\kappa,\sigma\}]\vdash^{\kappa+\omega a_0}_{\pi}\Gamma^{(\kappa,\pi)},\Delta^{(\sigma,\pi)}$.

Let $\alpha<\sigma\in Mh_2(\xi;a_0)\cap Mh_3(\nu)\cap\kappa$. A (cut) of the cut formulas $\delta^{(\sigma,\pi)}$ then yields $\mathcal{H}_\gamma[\Theta\cup\{\kappa,\sigma\}]\vdash^{\kappa+\omega a_0+p}_{\pi}\Gamma^{(\kappa,\pi)},\Gamma^{(\sigma,\pi)}$ for a $p<\omega$.

On the other hand we have $\mathcal{H}_\gamma[\Theta\cup\{\kappa\}]\vdash^{2d}_{0}\neg\theta^{(\kappa,\pi)},\Gamma^{(\kappa,\pi)}$ for each $\theta\in\Gamma\subset\Sigma_2(\pi)$, where $d=\max\{\mathrm{rk}(\theta^{(\kappa,\pi)}):\theta\in\Gamma\}<\kappa+\omega<\pi$.

Now $\kappa\in Mh_2(\xi;a)\cap\pi$ needs to reflect $\Pi_2(\kappa)$-formulas $\neg\theta^{(\kappa,\pi)}$ down to some $\alpha<\sigma\in Mh_2(\xi;a_0)\cap Mh_3(\nu)\cap\kappa$.

$$a_0<a\,\&\,\nu<\xi\Rightarrow Mh_2(\xi;a)\subset M_2(Mh_2(\xi;a_0)\cap Mh_3(\nu))$$

Thus we arrive at the following definition of the Mahlo classes $Mh^\gamma_2(\xi;a)$, which is a Π_3-class in the sense that there is a Π_3-formula $\theta(\gamma,\xi,a)$ such that $\alpha\in Mh^\gamma_2(\xi;a)$ iff $L_\alpha\models\theta(\gamma,\xi,a)$, while $Mh^\gamma_3(\nu)$ is a Π_4-class.

$\pi\in Mh^\gamma_2(\xi;a)$ iff $\{\gamma,\xi,a\}\subset\mathcal{H}_\gamma(\pi)$ and

$$\forall\{\nu,b\}\subset\mathcal{H}_\gamma(\pi)\,[\nu<\xi\,\&\,b<a\Rightarrow\pi\in M_2\left(Mh^\gamma_2(\xi;b)\cap Mh^\gamma_3(\nu)\right)].$$

It turns out that we need Mahlo classes $Mh^\gamma_2(\bar\xi;\bar a)$ for finite sequences $\bar\xi$ and $\bar a$ in our proof-theoretic study, cf. Lemma 4.13. Let us explain the classes intuitively in the next subsection.

4.1 Mahlo classes

Let $M_i=RM_i$ and $P,Q,\ldots$ denote transitive classes in $L\cup\{L\}$ for a Π_4-reflecting universe L. For classes $\mathcal{X},\mathcal{Y}$ and $i=2,3$ let

$$\mathcal{X}\prec_i\mathcal{Y}:\Leftrightarrow\forall P\in\mathcal{Y}(P\in M_i(\mathcal{X}))$$

Definition 4.4 Let

$$M_2(\xi;a):=\bigcap\{M_2\left(M_2(\xi;b)\cap M_3(\nu)\right):\nu<\xi,b<a\}.$$

In general for classes $\mathcal{Y}$ let

$$M^{\mathcal{Y}}_2(\xi;a):=\mathcal{Y}\cap\bigcap\{M_2\left(M^{\mathcal{Y}}_2(\xi;b)\cap M_3(\nu)\right):\nu<\xi,b<a\}.$$

Proposition 4.5 *For a Π_3-class $\mathcal{Y}$ and $\mu<\xi$, $M^{\mathcal{Y}}_2(\xi;a)\cap M_3(\mu)\prec_2\mathcal{Y}\cap M_3(\xi)$ and $M^{\mathcal{Y}}_2(\xi;a)\supset\mathcal{Y}\cap M_3(\xi)$.*

Proof. By induction on a, we show $P \in \mathcal{Y} \cap M_3(\xi) \Rightarrow P \in M_2^{\mathcal{Y}}(\xi; a)$.

Let $P \in \mathcal{Y} \cap M_3(\xi)$, $\nu < \xi$ and $b < a$. By IH we obtain $P \in M_2^{\mathcal{Y}}(\xi; b)$. Since $M_2^{\mathcal{Y}}(\xi; b)$ is a Π_3-class, we obtain $P \in M_2\left(M_2^{\mathcal{Y}}(\xi; b) \cap M_3(\nu)\right)$ by $P \in M_3(\xi)$. Therefore $P \in M_2^{\mathcal{Y}}(\xi; a)$.

Since $M_2^{\mathcal{Y}}(\xi; a)$ is a Π_3-class and $P \in M_3(\xi) \subset M_3(M_3(\mu))$, we obtain $P \in M_2(M_2^{\mathcal{Y}}(\xi; a) \cap M_3(\mu))$. □

Let $\nu < \mu < \xi$. From Proposition 4.5 we see $M_2(\xi; a) \cap M_3(\mu) \prec_2 M_3(\xi)$, and $M_2^{\mathcal{Y}}(\mu; b) \cap M_3(\nu) \prec_2 \mathcal{Y} \cap M_3(\mu)$ for $\mathcal{Y} = M_2(\xi; a)$.

Let us write $M_2((\xi, \mu); (a, b))$ for $M_2^{\mathcal{Y}}(\mu; b)$, where $\xi > \mu$. Let $\nu < \mu < \xi$. We obtain $M_2((\xi, \mu); (a, b)) \cap M_3(\nu) \prec_2 M_2(\xi; a) \cap M_3(\mu) \prec_2 M_3(\xi)$.

Proposition 4.6 *Let $\xi_1, \zeta < \xi$, $c < b$ and $d < a$. Then $M_2((\xi, \mu); (a, c)) \cap M_3(\nu) \prec_2 M_2((\xi, \mu); (a, b))$ and $M_2((\xi, \xi_1); (d, e)) \cap M_3(\zeta) \prec_2 M_2((\xi, \mu); (a, b))$.*

Proof. Let $\mathcal{Y} = M_2(\xi; a)$. Then $M_2((\xi, \mu); (a, c)) \cap M_3(\nu) = M_2^{\mathcal{Y}}(\mu; c) \cap M_3(\nu) \prec_2 M_2^{\mathcal{Y}}(\mu; b) = M_2((\xi, \mu); (a, b))$ by $c < b$ and $\nu < \mu$.

Next we show $M_2^{\mathcal{X}}(\xi_1; e) \cap M_3(\zeta) \prec_2 \mathcal{Y} \supset M_2^{\mathcal{Y}}(\mu; b)$, where $\mathcal{X} = M_2(\xi; d)$ and $M_2((\xi, \xi_1); (d, e)) = M_2^{\mathcal{X}}(\xi_1; e)$. We have $\mathcal{X} \cap M_3(\xi_1) \cap M_3(\zeta) = M_2(\xi; d) \cap M_3(\xi_1) \cap M_3(\zeta) \prec_2 M_2(\xi; a) = \mathcal{Y}$ by $d < a$ and $\xi_1, \zeta < \xi$. On the other hand we have $M_2^{\mathcal{X}}(\xi_1; e) \supset \mathcal{X} \cap M_3(\xi_1)$ by Proposition 4.5. Hence $M_2^{\mathcal{X}}(\xi_1; e) \cap M_3(\zeta) \prec_2 \mathcal{Y}$. □

The same argument applies not only to pairs $(\xi > \mu)$, (a, b), but also to triples, and so forth.

Let $\bar{\xi} = (\xi_0 > \xi_1 > \cdots > \xi_n)$ and $\bar{a} = (a_0, a_1, \ldots, a_n)$ be sequences in the same lengths. By iterating the process $\mathcal{Y} \mapsto \{M_2^{\mathcal{Y}}(\xi; a)\}_a$ with $M_3(\xi)$, we now define classes $M_2(\bar{\xi}; \bar{a})$ by induction on the length n of the sequences $\bar{\xi}, \bar{a}$ as follows.

$M_2(\langle\rangle; \langle\rangle)$ denotes the class of transitive sets in $\mathrm{L} \cup \{\mathrm{L}\}$.

For $\bar{\xi} * (\xi) = (\xi_0 > \cdots > \xi_n > \xi)$ and $\bar{a} * (a) = (a_0, \ldots, a_n, a)$ define for the Π_3-class $\mathcal{Y} = M_2(\bar{\xi}; \bar{a})$

$$M_2(\bar{\xi} * (\xi); \bar{a} * (a)) = M_2^{\mathcal{Y}}(\xi; a)$$

Namely

$$M_2(\bar{\xi} * (\xi); \bar{a} * (a)) = M_2(\bar{\xi}; \bar{a}) \cap \bigcap \{M_2\left(M_2(\bar{\xi} * (\xi); \bar{a} * (b)) \cap M_3(\nu)\right) : \nu < \xi, b < a\}$$

Proposition 4.6 is extended to finite sequences. To state an extension, let us redefine classes $M_2(\bar{\xi}; \bar{a})$ through ordinals $\alpha = \Lambda^{\xi_0} a_0 + \cdots + \Lambda^{\xi_n} a_n$ as follows, where Λ is a big enough ordinal such that $\Lambda > a_0$.

Let $\alpha = \Lambda^{\xi_0} a_0 + \cdots + \Lambda^{\xi_n} a_n$, where $\xi_0 > \cdots > \xi_n$ and $a_0, \ldots, a_n \neq 0$.

$$M_2(\alpha) := \bigcap \{M_2(M_2(\beta) \cap M_3(\nu)) : (\beta, \nu) < \alpha\}$$

where for segments $\alpha_i = \Lambda^{\xi_0} a_0 + \cdots + \Lambda^{\xi_i} a_i$ of $\alpha = \Lambda^{\xi_0} a_0 + \cdots + \Lambda^{\xi_n} a_n$

$$(\beta, \nu) < \alpha :\Leftrightarrow \exists i \leq n\, [\beta < \alpha_i \,\&\, \nu < \xi_i] .$$

F.e. in Proposition 4.6 we have $(\Lambda^{\xi}a+\Lambda^{\mu}c,\nu) < \Lambda^{\xi}a+\Lambda^{\mu}b$ and $(\Lambda^{\xi}d+\Lambda^{\xi_1}e,\zeta) < \Lambda^{\xi}a+\Lambda^{\mu}b$, but $(\Lambda^{\xi}a+\Lambda^{\mu}c,\mu) \not< \Lambda^{\xi}a+\Lambda^{\mu}b$, where $\nu<\mu<\xi$, $\xi_1,\zeta<\xi$, $c<b$ and $d<a$.

Proposition 4.7 $(\beta,\nu)<\alpha<\gamma \Rightarrow (\beta,\nu)<\gamma$.

$\alpha\dot{+}\beta$ designates that $\alpha+\beta=\alpha\#\beta$.

Lemma 4.8 (Cf. Lemma 3.2 in [A09].)
If $\xi>0$ and $\beta<\Lambda^{\xi+1}$, then $M_2(\alpha\dot{+}\beta)\prec_2 M_2(\alpha,\xi):=M_2(\alpha)\cap M_3(\xi)$.

Proof. Suppose $P\in M_2(\alpha,\xi)=M_2(\alpha)\cap M_3(\xi)$ and $\beta<\Lambda^{\xi+1}$.

We show $P\in M_2(\alpha\dot{+}\beta)$ by induction on ordinals β. Let $(\gamma,\nu)<\alpha\dot{+}\beta$. We need to show that $P\in M_2(M_2(\gamma,\nu))$.

Let δ be a segment of $\alpha\dot{+}\beta$ such that $\gamma<\delta$ and $\nu<\mu$ where $\delta=\cdots+\Lambda^{\mu}b$. If δ is a segment of α, then $P\in M_2(M_2(\gamma,\nu))$ by $P\in M_2(\alpha)$.

Let $\delta=\alpha\dot{+}\beta_0$, where β_0 is a segment of β. Then $\nu<\mu\leq\xi$. We claim that $P\in M_2(\gamma)$. If $\gamma<\alpha$, then Proposition 4.7 yields $P\in M_2(\alpha)\subset M_2(\gamma)$. Let $\gamma=\alpha\dot{+}\gamma_0<\alpha\dot{+}\beta_0$. IH yields $P\in M_2(\gamma)$. Thus the claim is shown. On the other hand we have $P\in M_3(\xi)$ and $\nu<\xi$. Since $M_2(\gamma)$ is a Π_3-class, we obtain $P\in M_3(M_2(\gamma,\nu))\subset M_2(M_2(\gamma,\nu))$. $P\in M_2(\alpha\dot{+}\beta)$ is shown.

By $P\in M_2(\alpha\dot{+}\beta)$ and $P\in M_3(\xi)\subset M_3$ with $\xi>0$, we obtain $P\in M_3(M_2(\alpha\dot{+}\beta))\subset M_2(M_2(\alpha\dot{+}\beta))$. □

4.2 Skolem hulls and collapsing functions

We can assume $\xi<\varepsilon_{\mathbb{K}+1}$ and $a<\Lambda=\mathbb{K}$. For $\alpha<\Lambda^{\varepsilon_{\mathbb{K}+1}}$, let us define $Mh_2^{\gamma}(\alpha)$ as follows. (β,ν) denotes pairs of ordinals $\beta<\Lambda^{\varepsilon_{\mathbb{K}+1}}$ and $\nu<\varepsilon_{\mathbb{K}+1}$ such that $\beta+\Lambda^{\nu}=\beta\#\Lambda^{\nu}$. Let $\alpha=\Lambda^{\beta_0}a_0+\cdots+\Lambda^{\beta_n}a_n$, where $\varepsilon_{\mathbb{K}+1}>\beta_0>\cdots>\beta_n$ and $0<a_0,\ldots,a_n<\Lambda$. Then $\pi\in Mh_2^{\gamma}(\alpha)$ iff $\{\gamma,\alpha\}\subset\mathcal{H}_{\gamma}(\pi)$ and

$$\forall\{\nu,\beta\}\subset\mathcal{H}_{\gamma}(\pi)\,[(\beta,\nu)<\alpha\Rightarrow\pi\in M_2\left(Mh_2^{\gamma}(\beta)\cap Mh_3^{\gamma}(\nu)\right)]$$

where for segments $\alpha_i=\Lambda^{\beta_0}a_0+\cdots+\Lambda^{\beta_i}a_i$ of $\alpha=\Lambda^{\beta_0}a_0+\cdots+\Lambda^{\beta_n}a_n$

$$(\beta,\nu)<\alpha:\Leftrightarrow\exists i\leq n\,[\beta<\alpha_i\,\&\,\nu<\beta_i]\,.$$

For example, if $\nu<\xi$ and $a_0<a$, then $(\Lambda^{\xi}a_0,\nu)<\Lambda^{\xi}a$. The exponents β_i of α designate 'Π_3-Mahlo degrees'.

Proposition 4.9 $(\beta,\nu)<\alpha<\gamma\Rightarrow(\beta,\nu)<\gamma$.

Definition 4.10 Define simultaneously by recursion on ordinals $a<\varepsilon_{\mathbb{K}+1}$ the classes $\mathcal{H}_a(X)\,(X\subset\Gamma_{\mathbb{K}+1})$, $Mh_2^a(\alpha)\,(\xi<\varepsilon_{\mathbb{K}+1})$, the ordinals $\psi_{\sigma}^{(\alpha,\xi)}(a)$ as follows.

1. $\mathcal{H}_a(X)$ denotes the Skolem hull of $\{0,\Omega,\mathbb{K}\}\cup X$ under the functions $+,\varphi$, and the following.

   Let $\{\sigma,b,\alpha,\xi\}\subset\mathcal{H}_a(X)$, $\alpha\in\{0\}\cup[\Lambda,\Lambda^{\varepsilon_{\mathbb{K}+1}})$, $\xi\in[0,\varepsilon_{\mathbb{K}+1})$ and $b<a$. Then $\psi_{\sigma}^{(\alpha,\xi)}(b)\in\mathcal{H}_a(X)$.

2. $\pi \in Mh_3^a(\xi) :\Leftrightarrow \{a,\xi\} \subset \mathcal{H}_a(\pi) \,\&\, \forall \nu \in \mathcal{H}_a(\pi) \cap \xi\, (\pi \in M_3(Mh_3^a(\nu)))$, where $\alpha \in Mh_3^a(0)$ iff α is a limit ordinal.

3. For $\alpha < \Lambda^{\varepsilon_{\mathbb{K}+1}}$ and $a < \varepsilon_{\mathbb{K}+1}$, $\pi \in Mh_2^a(\alpha)$ iff $\{a,\alpha\} \subset \mathcal{H}_a(\pi)$ and

$$\forall\{\beta,\nu\} \subset \mathcal{H}_a(\pi)\, [(\beta,\nu) < \alpha \to \pi \in M_2\,(Mh_2^a(\beta,\nu))]$$

where

$$Mh_2^a(\beta,\nu) = Mh_2^a(\beta) \cap Mh_3^a(\nu)$$

and $\alpha \in Mh_2^a(0)$ iff α is a limit ordinal. Note that $Mh_2^a(\alpha)$ is a Π_3-class.

4. Let $m_2(\mathbb{K}) = 0$, $m_3(\mathbb{K}) = \varepsilon_{\mathbb{K}+1}$, $m_2(\Omega) = 1$ and $m_3(\Omega) = 0$.

   (a) For $\{\xi,a\} \subset \mathcal{H}_a(\mathbb{K}) \cap \varepsilon_{\mathbb{K}+1}$ with $0 < \xi \leq a$, let
   $\psi_{\mathbb{K}}^{(0,\xi)}(a) = \min\,(\{\mathbb{K}\} \cup \{\kappa \in Mh_3^a(\xi) : \{\xi,a\} \subset \mathcal{H}_a(\kappa) \,\&\, \mathcal{H}_a(\kappa) \cap \mathbb{K} \subset \kappa\})$.
   $m_2(\psi_{\mathbb{K}}^{(0,\xi)}(a)) = 0$ and $m_3(\psi_{\mathbb{K}}^{(0,\xi)}(a)) = \xi$.

   (b) Let $0 \leq \alpha < \Lambda^{\varepsilon_{\mathbb{K}+1}}$ and $0 < \xi < \varepsilon_{\mathbb{K}+1}$ be ordinals, $0 < c \leq a < \Lambda = \mathbb{K}$ with $c \in \mathcal{H}_a(\sigma)$ and $\sigma \in Mh_2^a(\alpha,\xi)$. Then for $\beta = \alpha \dot{+} \Lambda^{\xi} c$

   $$\psi_{\sigma}^{(\beta,0)}(a) = \min\,(\{\sigma\} \cup \{\kappa \in Mh_2^a(\beta) : \{\sigma,\alpha,\xi,c,a\} \subset \mathcal{H}_a(\kappa) \,\&\, \mathcal{H}_a(\kappa) \cap \sigma \subset \kappa\})\,.$$

   $m_2(\psi_{\sigma}^{(\beta,0)}(a)) = \beta$ and $m_3(\psi_{\sigma}^{(\beta,0)}(a)) = 0$.

   (c) Let $0 < \beta, \alpha < \Lambda^{\varepsilon_{\mathbb{K}+1}}$ and $0 < \nu < \varepsilon_{\mathbb{K}+1}$ be such that $\{\beta,\nu\} \subset \mathcal{H}_a(\sigma)$, $SC_{\mathbb{K}}(\beta,\nu) \subset (a+1) < \mathbb{K}$ and $(\beta,\nu) < \alpha$. Then for $\sigma \in Mh_2^a(\alpha)$ with $m_3(\sigma) = 0$

   $$\psi_{\sigma}^{(\beta,\nu)}(a) = \min\,(\{\sigma\} \cup \{\kappa \in Mh_2^a(\beta,\nu) : \{\sigma,\beta,\nu,a\} \subset \mathcal{H}_a(\kappa) \,\&\, \mathcal{H}_a(\kappa) \cap \sigma \subset \kappa\})\,.$$

   $m_2(\psi_{\sigma}^{(\beta,\nu)}(a)) = \beta$ and $m_3(\psi_{\sigma}^{(\beta,\nu)}(a)) = \nu$.

   (d)

   $$\psi_{\sigma}(a) = \min\{\kappa \leq \sigma : \{\sigma,a\} \subset \mathcal{H}_a(\kappa) \,\&\, \mathcal{H}_a(\kappa) \cap \sigma \subset \kappa\}.$$

   We write $\psi_{\sigma}(a)$ for $\psi_{\sigma}^{(0,0)}(a)$.

Let $\mathbb{K}$ be a Π_2^1-indescribable cardinal. As in Lemmas 2.3 and 2.4 we see that $\psi_{\mathbb{K}}^{(0,\xi)}(a) < \mathbb{K}$ for every $\{a,\xi\} \subset \mathcal{H}_a(\mathbb{K}) \cap \varepsilon_{\mathbb{K}+1}$.

It is easy to see that $\psi_{\sigma}^{(\beta,\nu)}(a) < \sigma$ if $(\beta,\nu) < \alpha$, $\sigma \in Mh_2^a(\alpha)$ and $\{\beta,\nu\} \subset \mathcal{H}_a(\sigma)$.

Lemma 4.11 (Cf. Lemma 3.2 in [A09].) *Assume $\mathbb{K} \geq \sigma \in Mh_2^a(\alpha,\xi)$ with $0 < \xi < \varepsilon_{\mathbb{K}+1}$, $\beta < \Lambda^{\xi+1}$ and $\beta \in \mathcal{H}_a(\sigma)$. Then $\sigma \in M_3(Mh_2^a(\alpha \dot{+} \beta))$ holds, a fortiori $\sigma \in M_2(Mh_2^a(\alpha \dot{+} \beta))$.*

Proof. Suppose $\sigma \in Mh_2^a(\alpha, \xi) = Mh_2^a(\alpha) \cap Mh_3^a(\xi)$ and $\beta \in \mathcal{H}_a(\sigma)$ with $\beta < \Lambda^{\xi+1}$. We show $\sigma \in Mh_2^a(\alpha \dot{+} \beta)$ by induction on ordinals β. Let $\{\gamma, \nu\} \subset \mathcal{H}_a(\sigma)$ and $(\gamma, \nu) < \alpha \dot{+} \beta$. We need to show that $\sigma \in M_2(Mh_2^a(\gamma, \nu))$.

Let δ be a segment of $\alpha \dot{+} \beta$ such that $\gamma < \delta$ and $\nu < \mu$ where $\delta = \cdots + \Lambda^\mu b$. If δ is a segment of α, then $\sigma \in M_2(Mh_2^a(\gamma, \nu))$ by $\sigma \in Mh_2^a(\alpha)$.

Let $\delta = \alpha \dot{+} \beta_0$, where β_0 is a segment of β. Then $\nu < \mu \leq \xi$. We claim that $\sigma \in Mh_2^a(\gamma)$. If $\gamma < \alpha$, then Proposition 4.9 with $\gamma \in \mathcal{H}_a(\sigma)$ yields $\sigma \in Mh_2^a(\alpha) \subset Mh_2^a(\gamma)$. Let $\gamma = \alpha \dot{+} \gamma_0 < \alpha \dot{+} \beta_0$ with $\gamma_0 \in \mathcal{H}_a(\sigma)$. IH yields $\sigma \in Mh_2^a(\gamma)$. Thus the claim is shown. On the other hand we have $\sigma \in Mh_3^a(\xi)$ and $\nu \in \mathcal{H}_a(\sigma) \cap \xi$. Since $Mh_2^a(\gamma)$ is a Π_3-class, we obtain $\sigma \in M_3(Mh_2^a(\gamma, \nu)) \subset M_2(Mh_2^a(\gamma, \nu))$ with $Mh_2^a(\gamma, \nu) = Mh_2^a(\gamma) \cap Mh_3^a(\nu)$. $\sigma \in Mh_2^a(\alpha \dot{+} \beta)$ is shown.

By $\sigma \in Mh_2^a(\alpha \dot{+} \beta)$ and $\sigma \in Mh_3^a(\xi) \subset M_3$ with $\xi > 0$, we obtain $\sigma \in M_3(Mh_2^a(\alpha \dot{+} \beta))$. □

Corollary 4.12 *If $\sigma \in Mh_2^a(\alpha, \xi)$ and $c \in \mathcal{H}_a(\sigma) \cap \Lambda$ with $\xi > 0$, then $\psi_\sigma^{(\beta,0)}(a) < \sigma$ for $\beta = \alpha \dot{+} \Lambda^\xi c$.*

Proof. We obtain $\sigma \in M_2(Mh_2^a(\beta))$ by Lemma 4.11. Since $\{\kappa < \sigma : \{\beta, a, \sigma\} \subset \mathcal{H}_a(\kappa), \mathcal{H}_a(\kappa) \cap \sigma \subset \kappa\}$ is a club subset of σ, we obtain $\psi_\sigma^{(\beta,0)}(a) < \sigma$. □

$OT(\Pi_4)$ denotes a computable notation system of ordinals with collapsing functions $\psi_\sigma^{(\alpha,\xi)}(a)$. Although in our well-foundedness proof in $\mathsf{KP}\Pi_4$, ordinal terms $\psi_\sigma^{(\beta,\nu)}(a)$ has to obey some restrictions such as (9) for $OT(\Pi_3)$, it is cumbersome to verify the conditions, and let us skip it.

Operator controlled derivations for $\mathsf{KP}\Pi_4$ are closed under the inference rules $(\mathrm{rfl}_{\Pi_4}(\mathbb{K}))$, $(\mathrm{rfl}_{\Pi_3}(\alpha, \pi, \nu))$ and the following.

$(\mathrm{rfl}_{\Pi_2}(\alpha, \pi, \beta, \nu))$ There exist ordinals $\alpha < \pi \leq b < \mathbb{K}$, $(\beta, \nu) < \bar{m}_2(\pi) \leq m_2(\pi)$, $a_0 < a$, and a finite set Δ of $\Sigma_2(\pi)$-sentences enjoying the following conditions:

1. $\{\alpha, \pi, \beta, \nu\} \cup \bar{m}(\pi) \subset \mathcal{H}_\gamma[\Theta]$.
2. For each $\delta \in \Delta$, $\mathcal{H}_\gamma[\Theta] \vdash_b^{a_0} \Gamma, \neg\delta$.
3. For each $\alpha < \rho \in Mh_2(\beta, \nu) \cap \pi$, $\mathcal{H}_\gamma[\Theta \cup \{\rho\}] \vdash_b^{a_0} \Gamma, \Delta^{(\rho,\pi)}$.

$$\frac{\{\mathcal{H}_\gamma[\Theta] \vdash_b^{a_0} \Gamma, \neg\delta\}_{\delta \in \Delta} \quad \{\mathcal{H}_\gamma[\Theta \cup \{\rho\}] \vdash_b^{a_0} \Gamma, \Delta^{(\rho,\pi)}\}_{\alpha < \rho \in Mh_2(\beta,\nu) \cap \pi}}{\mathcal{H}_\gamma[\Theta] \vdash_b^a \Gamma} \ (\mathrm{rfl}_{\Pi_2}(\alpha, \pi, \beta, \nu))$$

This inference says that $\pi \in M_2(Mh_2^\gamma(\beta) \cap Mh_3^\gamma(\nu))$.

Lemma 4.13 *Let $\Gamma \subset \Sigma_2(\pi)$. Assume $\mathcal{H}_\gamma[\Theta] \vdash_\pi^a \Gamma$ for a $\pi < \mathbb{K}$, and $\{\xi, \alpha\} \subset \mathcal{H}_\gamma[\Theta]$ for $\alpha = \bar{m}_2(\pi)$, $\xi = \bar{m}_3(\pi)$. Let η be the base for $(\mathrm{rfl}_{\Pi_3}(\eta, \pi, \nu))$ in $\mathcal{H}_\gamma[\Theta] \vdash_\pi^a \Gamma$. Then for any $\eta < \kappa \in Mh_2(\alpha \dot{+} \Lambda^\xi(1+a)) \cap \pi$, $\mathcal{H}_\gamma[\Theta \cup \{\kappa\}] \vdash_\pi^{\kappa+\omega a} \Gamma^{(\kappa,\pi)}$ holds, where $\alpha \dot{+} \Lambda^\xi(1+a) \leq \bar{m}_2(\kappa) \in \mathcal{H}_\gamma[\Theta]$. Moreover when $\Theta \subset \mathcal{H}_\gamma(\kappa)$, $\mathcal{H}_\gamma[\Theta \cup \{\kappa\}] \vdash_\kappa^{\kappa+\omega a} \Gamma^{(\kappa,\pi)}$ holds.*

Proof. By induction on a. Let $\pi' = \kappa$ if $\Theta \subset \mathcal{H}_\gamma(\kappa)$. Otherwise $\pi' = \pi$. Note that there exists a κ such that $\kappa \in Mh_2(\alpha \dot{+} \Lambda^\xi(1+a)) \cap \pi$ if $\Theta \cup \{\pi\} \subset \mathcal{H}_\gamma(\pi)$. F.e. $\kappa = \psi_\pi^{(\alpha+\Lambda^\xi(1+a),0)}(\gamma + \max\Theta)$.

Let η be the base for $(\text{rfl}_{\Pi_3}(\eta,\pi,\nu))$ in $\mathcal{H}_\gamma[\Theta] \vdash^a_\pi \Gamma$.

Case 1. $(\text{rfl}_{\Pi_3}(\eta,\pi,\nu))$: Then $\eta < \pi$, $\{\eta,\pi,\nu\} \cup \bar{m}(\pi) \subset \mathcal{H}_\gamma[\Theta]$, $SC_{\mathbb{K}}(\nu) \subset \pi$, and $\nu < \bar{m}_3(\pi) \leq m_3(\pi)$. Let $\Delta \subset \Sigma_3(\pi)$.

$$\frac{\{\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_\pi \Gamma, \neg\delta\}_{\delta\in\Delta} \quad \{\mathcal{H}_\gamma[\Theta\cup\{\rho\}] \vdash^{a_0}_\pi \Gamma, \Delta^{(\rho,\pi)}\}_{\eta<\rho\in Mh_3(\nu)\cap\pi}}{\mathcal{H}_\gamma[\Theta] \vdash^a_\pi \Gamma}\ (\text{rfl}_{\Pi_3}(\eta,\pi,\nu))$$

Let $\alpha_0 = \alpha \dot{+} \Lambda^\xi(1+a_0)$. Then $(\alpha_0,\nu) < \alpha_1 = \alpha \dot{+} \Lambda^\xi(1+a)$. We obtain $\{\kappa,\alpha_1,\nu,\alpha_0\} \subset \mathcal{H}_\gamma[\Theta\cup\{\kappa\}]$. In the following derivation $\alpha_1 \leq \bar{m}_2(\kappa)$ with $\bar{m}(\kappa) \subset \mathcal{H}_\gamma[\Theta]$.

$$\frac{\{\mathcal{H}_\gamma[\Theta\cup\{\kappa\}] \vdash^{2d}_0 \neg\theta^{(\kappa,\pi)}, \Gamma^{(\kappa,\pi)}\}_{\theta\in\Gamma} \quad \dfrac{\{\mathcal{H}_\gamma[\Theta\cup\{\sigma\}] \vdash^{\sigma+\omega a_0+1}_{\pi'} \Gamma^{(\sigma,\pi)}, \neg\delta^{(\sigma,\pi)}\}_{\delta\in\Delta} \quad \mathcal{H}_\gamma[\Theta\cup\{\kappa,\sigma\}] \vdash^{\kappa+\omega a_0}_{\pi'} \Gamma^{(\kappa,\pi)}, \Delta^{(\sigma,\pi)}}{\{\mathcal{H}_\gamma[\Theta\cup\{\kappa,\sigma\}] \vdash^{\kappa+\omega a_0+p}_{\pi'} \Gamma^{(\kappa,\pi)}, \Gamma^{(\sigma,\pi)}\}_{\eta<\sigma\in Mh_2(\alpha_0,\nu)\cap\pi}}}{\mathcal{H}_\gamma[\Theta\cup\{\kappa\}] \vdash^{\kappa+\omega a}_{\pi'} \Gamma^{(\kappa,\pi)}}\ (\text{rfl}_{\Pi_2}(\eta,\kappa,\alpha_0,\nu))$$

Case 2. $(\text{rfl}_{\Pi_2}(\mu,\pi,\beta,\nu))$: $(\beta,\nu) < \alpha = \bar{m}_2(\pi) \leq m_2(\pi)$, $\mu < \pi$, $\{\mu,\pi,\alpha,\beta,\nu\} \subset \mathcal{H}_\gamma[\Theta]$ and $\Delta \subset \Sigma_2(\pi)$.

$$\frac{\{\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_\pi \Gamma, \neg\delta\}_{\delta\in\Delta} \quad \{\mathcal{H}_\gamma[\Theta\cup\{\rho\}] \vdash^{a_0}_\pi \Gamma, \Delta^{(\rho,\pi)}\}_{\mu<\rho\in Mh_2(\beta,\nu)\cap\pi}}{\mathcal{H}_\gamma[\Theta] \vdash^a_\pi \Gamma}\ (\text{rfl}_{\Pi_2}(\pi,\beta,\nu))$$

Then $(\beta,\nu) < \alpha_1 = \alpha \dot{+} \Lambda^\xi(1+a) \leq \bar{m}_2(\kappa)$ with the segment α of $\alpha \dot{+} \Lambda^\xi(1+a)$. We have $\Delta^{(\rho,\pi)} = \left(\Delta^{(\kappa,\pi)}\right)^{(\rho,\kappa)}$ and $\{\kappa,\alpha_1,\beta,\nu\} \subset \mathcal{H}_\gamma[\Theta\cup\{\kappa\}]$.

$$\frac{\{\mathcal{H}_\gamma[\Theta\cup\{\kappa\}] \vdash^{\kappa+\omega a_0+1}_{\pi'} \Gamma^{(\kappa,\pi)}, \neg\delta^{(\kappa,\pi)}\}_{\delta\in\Delta} \quad \{\mathcal{H}_\gamma[\Theta\cup\{\kappa,\rho\}] \vdash^{\kappa+\omega a_0}_{\pi'} \Gamma^{(\kappa,\pi)}, \Delta^{(\rho,\pi)}\}_{\mu<\rho\in Mh_2(\beta,\nu)\cap\kappa}}{\mathcal{H}_\gamma[\Theta\cup\{\kappa\}] \vdash^{\kappa+\omega a}_{\pi'} \Gamma^{(\kappa,\pi)}}\ (\text{rfl}_{\Pi_2}(\mu,\kappa,\beta,\nu))$$

Case 3. The last inference is a (cut) of a cut formula C: Then $\text{rk}(C) \in \mathcal{H}_\gamma[\Theta] \cap \pi$ and $C \in \Delta_0(\pi)$. If $\Theta \subset \mathcal{H}_\gamma(\kappa)$, then $\text{rk}(C) < \kappa$.

Case 4. The last inference is either a $(\text{rfl}_{\Pi_3}(\sigma,\nu))$ or a $(\text{rfl}_{\Pi_2}(\sigma,\delta,\nu))$ with $\sigma \in \mathcal{H}_\gamma[\Theta] \cap \pi$: IH yields the lemma. If $\Theta \subset \mathcal{H}_\gamma(\kappa)$, then $\sigma < \kappa$. □

We see from the above proof, if there is a base η for inferences $(\text{rfl}_{\Pi_3}(\mu_3,\sigma,\nu))$ and simultaneously for $(\text{rfl}_{\Pi_2}(\mu_2,\sigma,\delta,\nu))$ in $\mathcal{H}_\gamma[\Theta] \vdash^a_\pi \Gamma$ (in the sense that $\eta = \mu_3 = \mu_2$), then the same η is a base for inferences $(\text{rfl}_{\Pi_3}(\mu_3,\sigma,\nu))$ and simultaneously for $(\text{rfl}_{\Pi_2}(\mu_2,\sigma,\delta,\nu))$ in $\mathcal{H}_\gamma[\Theta\cup\{\kappa\}] \vdash^{\kappa+\omega a}_{\pi'} \Gamma^{(\kappa,\pi)}$.

Lemma 4.14 *Let $\Gamma \subset \Sigma_1(\lambda)$ and $\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma$ with $a < \Lambda = \mathbb{K}$, $\mathcal{H}_\gamma[\Theta] \ni \lambda \leq b < \mathbb{K}$ and λ regular, and assume $\forall\kappa \in [\lambda, b)(\Theta \subset \mathcal{H}_\gamma(\psi_\kappa(\gamma)))$.*

Let $\hat{a} = \gamma + \theta_b(a)$ and $\delta = \psi_\lambda^{(\beta,\nu)}(\hat{a})$ when $\lambda \in Mh_2^\gamma(\alpha)$, $m_3(\lambda) = 0$ and $(\beta,\nu) < \alpha$ with $\{\beta,\nu\} \subset \mathcal{H}_\gamma[\Theta]$. Then $\mathcal{H}_{\hat{a}+1}[\Theta] \vdash^\delta_\delta \Gamma$ holds.

Proof. By main induction on b with subsidiary induction on a as in Lemma 2.6. Let η be a base for reflection inferences in $\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma$.

Case 1. Consider the case when the last inference is a $(\mathrm{rfl}_{\Pi_3}(\eta,\sigma,\nu))$ with $b \geq \sigma$.

$$\frac{\{\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_b \Gamma, \neg\delta\}_{\delta\in\Delta} \quad \{\mathcal{H}_\gamma[\Theta\cup\{\rho\}] \vdash^{a_0}_b \Gamma, \Delta^{(\rho,\sigma)}\}_{\eta<\rho\in Mh_3(\nu)\cap\sigma}}{\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma} \ (\mathrm{rfl}_{\Pi_3}(\eta,\sigma,\nu))$$

where $\Delta \subset \Sigma_3(\sigma)$, $SC_{\mathbb{K}}(\nu) \subset \sigma$, $\nu < \xi = \bar{m}_3(\sigma) \leq m_3(\sigma)$, $\alpha = \bar{m}_2(\sigma) \leq m_2(\sigma)$, $\eta < \sigma$ and $\{\eta,\sigma,\xi,\alpha,\nu\} \subset \mathcal{H}_\gamma[\Theta]$. We may assume that $\sigma \geq \lambda$.

Case 1.1. There exists a regular $\pi \in \mathcal{H}_\gamma[\Theta]$ such that $\sigma < \pi \leq b$: Then $\Delta \subset \Delta_0(\pi)$ and $\sigma < b_0 = \psi_\pi(\widehat{a_0})$ for $\widehat{a_0} = \gamma + \theta_b(a_0)$. SIH yields $\mathcal{H}_{\widehat{a_0}+1}[\Theta] \vdash^{b_0}_{b_0} \Gamma, \neg\delta$ for each $\delta \in \Delta$, and $\mathcal{H}_{\widehat{a_0}+1}[\Theta\cup\{\rho\}] \vdash^{b_0}_{b_0} \Gamma, \Delta^{(\rho,\sigma)}$ for each $\eta < \rho \in Mh_3(\nu)\cap\sigma$. A $(\mathrm{rfl}_{\Pi_3}(\eta,\sigma,\nu))$ yields $\mathcal{H}_{\widehat{a_0}+1}[\Theta] \vdash^{b_0+1}_{b_0} \Gamma$, where $b_0 < b$. Let $\delta_0 = \psi_\lambda(\widehat{a_1})$ with $\widehat{a_1} = \widehat{a_0} + \theta_{b_0}(b_0+1) = \gamma + \theta_b(a_0) + \theta_{b_0}(b_0+1) < \gamma + \theta_b(a) = \hat{a}$. We obtain $\mathcal{H}_{\widehat{a_1}+1}[\Theta] \vdash^{\delta_0}_{\delta_0} \Gamma$ by MIH, and the lemma follows.

Case 1.2. Otherwise: By Cut-elimination we obtain $\mathcal{H}_\gamma[\Theta] \vdash^{\theta_b(a_0)}_\sigma \Gamma, \neg\delta$ for each $\delta \in \Delta$, and $\mathcal{H}_\gamma[\Theta\cup\{\rho\}] \vdash^{\theta_b(a_0)}_\sigma \Gamma, \Delta^{(\rho,\sigma)}$ for each $\eta < \rho \in Mh_3(\nu)\cap\sigma$. A $(\mathrm{rfl}_{\Pi_3}(\eta,\sigma,\nu))$ yields $\mathcal{H}_\gamma[\Theta] \vdash^{a_1}_\sigma \Gamma$ for $a_1 = \theta_b(a_0)+1$. Let $\beta = \alpha + \Lambda^\xi(1+a_1)$ for $\alpha = \bar{m}_2(\sigma) \leq m_2(\sigma)$ and $\xi = \bar{m}_3(\pi) \leq m_3(\sigma)$, and $\kappa = \psi^{(\beta,0)}_\sigma(\gamma)$. We obtain $\Theta \subset \mathcal{H}_\gamma(\kappa)$ by the assumption. Hence $\{\gamma,\sigma,\beta\} \subset \mathcal{H}_\gamma(\kappa)$, and $\eta < \kappa \in Mh_2(\beta)\cap\sigma$, cf. Corollary 4.12. Moreover we have $\kappa \in \mathcal{H}_{\gamma+1}[\Theta]$.

Lemma 4.13 yields $\mathcal{H}_\gamma[\Theta\cup\{\kappa\}] \vdash^{\kappa+\omega a_1}_\kappa \Gamma^{(\kappa,\sigma)}$ and $\mathcal{H}_{\gamma+1}[\Theta] \vdash^{\kappa+\omega a_1}_\kappa \Gamma^{(\kappa,\sigma)}$, where $\beta \leq \bar{m}_2(\kappa)$ with $\bar{m}(\kappa) \subset \mathcal{H}_\gamma[\Theta]$, and $\Gamma^{(\kappa,\sigma)} = \Gamma$ if $\lambda < \sigma$, and $\Gamma^{(\kappa,\sigma)} = \Gamma^{(\kappa,\lambda)}$ otherwise. In each case we obtain $\mathcal{H}_{\gamma+1}[\Theta] \vdash^{\kappa+\omega a_1}_\kappa \Gamma$. MIH then yields $\mathcal{H}_{\widehat{a_1}+1}[\Theta] \vdash^{\delta_1}_{\delta_1} \Gamma$, where $\delta_1 = \psi_\lambda(\widehat{a_1})$ with $\widehat{a_1} = \gamma + \theta_\kappa(\kappa+\omega a_1) < \gamma + \theta_b(a) = \hat{a}$ by $\kappa < \sigma \leq b$ and $a_1 < \theta_b(a)$.

Case 2. Consider the case when the last inference is a $(\mathrm{rfl}_{\Pi_2}(\eta,\sigma,\beta,\nu))$ with $b \geq \sigma$.

$$\frac{\{\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_b \Gamma, \neg\delta\}_{\delta\in\Delta} \quad \{\mathcal{H}_\gamma[\Theta\cup\{\rho\}] \vdash^{a_0}_b \Gamma, \Delta^{(\rho,\sigma)}\}_{\eta<\rho\in Mh_2(\beta,\nu)\cap\sigma}}{\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma} \ (\mathrm{rfl}_{\Pi_2}(\eta,\sigma,\beta,\nu))$$

where $\Delta \subset \Sigma_2(\sigma)$, $(\beta,\nu) < \alpha = \bar{m}_2(\sigma) \leq m_2(\sigma)$, $\xi = \bar{m}_3(\sigma) \leq m_3(\sigma)$, $\eta < \sigma$ and $\{\eta,\sigma,\alpha,\xi,\beta,\nu\} \subset \mathcal{H}_\gamma[\Theta]$.

We may assume that $\sigma \geq \lambda$. For each $\delta \in \Delta$, let $\delta \simeq \bigvee(\delta_i)_{i\in J}$. We may assume $J = Tm(\sigma)$. Inversion yields $\mathcal{H}_{\gamma+|i|}[\Theta\cup\mathsf{k}(i)] \vdash^{a_0}_b \Gamma, \neg\delta_i$, where $\Gamma\cup\{\neg\delta_i\} \subset \Sigma_1(\sigma)$. Let $\widehat{a_0} = \gamma + \theta_b(a_0)$ and $\rho = \psi^{(\beta,\nu)}_\sigma(\widehat{a_0})$, where $\Theta \subset \mathcal{H}_\gamma(\rho)$ by the assumption, $\{\eta,\sigma,\beta,\nu,\widehat{a_0}\} \subset \mathcal{H}_\gamma[\Theta]$ with $(\beta,\nu) < m_2(\sigma)$. Hence $\{\eta,\sigma,\beta,\nu,\widehat{a_0}\} \subset \mathcal{H}_\gamma(\rho)$ and $\mathcal{H}_\gamma(\rho)\cap\sigma \subset \rho$. Therefore $< \eta < \rho \in Mh_2(\beta,\nu)\cap\sigma\cap\mathcal{H}_{\widehat{a_0}+1}[\Theta]$.

We see the lemma as in Lemma 2.6 by Inversion, picking the ρ-th branch from the right upper seqeunts, and then introducing several (cut)'s instead of $(\mathrm{rfl}_{\Pi_2}(\eta,\sigma,\beta,\nu))$. Use MIH when $\lambda < \sigma$.

Case 3. As in Lemma 2.6 we see the case when the last inference is a (cut) of a cut formula C with $d = \mathrm{rk}(C) < b$. □

Theorem 4.15 *Assume* $\mathsf{KP}\Pi_4 \vdash \theta^{L_\Omega}$ *for* $\theta \in \Sigma$. *Then there exists an* $n < \omega$ *such that* $L_\alpha \models \theta$ *for* $\alpha = \psi_\Omega(\omega_n(\mathbb{K}+1))$ *in* $OT(\Pi_4)$.

Proof. By Embedding there exists an $m > 0$ such that $\mathcal{H}_0[\emptyset] \vdash^{\mathbb{K}+m}_{\mathbb{K}+m} \theta^{L_\Omega}$. By Cut-elimination, $\mathcal{H}_0[\emptyset] \vdash^{a}_{\mathbb{K}} \theta^{L_\Omega}$ for $a = \omega_m(\mathbb{K}+m)$. By Lemma 4.2 we obtain $\mathcal{H}_{\omega^a+1}[\{\kappa\}] \vdash^{\beta}_{\beta} \theta^{L_\Omega}$, where $\beta = \psi_{\mathbb{K}}(\omega^a)$, $\mathbb{K} + a = a$, $(\theta^{L_\Omega})^{(\kappa,\mathbb{K})} \equiv \theta^{L_\Omega}$ and $\kappa \in Mh_2(a) \cap \psi_{\mathbb{K}}(\mathbb{K})$. F.e. $\kappa = \psi^{(0,a)}_{\mathbb{K}}(0) \in \mathcal{H}_1[\emptyset]$. Hence $\mathcal{H}_{\omega^a+1}[\emptyset] \vdash^{\beta}_{\beta} \theta^{L_\Omega}$. Lemma 4.14 then yields $\mathcal{H}_{\gamma+1}[\emptyset] \vdash^{\beta_1}_{\beta_1} \theta^{L_\Omega}$ for $\gamma = \omega^a + \theta_\beta(\beta)$ and $\beta_1 = \psi_\Omega(\gamma) < \psi_\Omega(\omega^a + \mathbb{K}) < \psi_\Omega(\omega_{m+2}(\mathbb{K}+1)) = \alpha$. Therefore $L_\alpha \models \theta$. □

5 First order reflection

Having established an ordinal analysis for Π_4-reflection in section 4, it is not hard to extend it to first-order reflection. As expected, an exponential ordinal structure emerges in resolving higher Mahlo classes.

Let $\mathbb{K} = \Lambda$ be either a Π^1_{N-2}-indescribable cardinal or a Π_N-reflecting ordinal for an integer $N \geq 3$. Let for $k > 0$, $\alpha \in M_{k+2}(A)$ iff A is Π^1_k-indescribable in α or α is Π_{k+2}-reflecting on A. Let $(\nu_k, \nu_{k+1}, \ldots, \nu_{N-1})$ be a sequence of ordinals $\nu_i < \varepsilon_{\Lambda+1}$, and $\varepsilon_{\Lambda+1} > \alpha = \Lambda^{\beta_0}a_0 + \cdots + \Lambda^{\beta_n}a_n$ with $\beta_0 > \cdots > \beta_n$ and $0 < a_0, \ldots, a_n < \Lambda$. Then $(\nu_k, \nu_{k+1}, \ldots, \nu_{N-1}) < \alpha$ iff there exists a segment $\alpha_i = \Lambda^{\beta_0}a_0 + \cdots + \Lambda^{\beta_i}a_i$ of α such that $\nu_k < \alpha_i$ and $(\nu_{k+1}, \ldots, \nu_{N-1}) < \beta_i$.

Proposition 5.1 *$\vec{\nu} < \alpha < \gamma \Rightarrow \vec{\nu} < \gamma$.*

5.1 Mahlo classes for Π_N-reflection

As in subsection 4.1 $P \in M_i(\mathcal{X})$ designates that P is Π_i-reflecting on $\mathcal{X}$. Let

$$M_k(\alpha) := \bigcap\{M_k(M_k(\bar{\nu})) : \bar{\nu} = (\nu_k, \nu_{k+1}, \ldots, \nu_{N-1}) < \alpha\}$$

where

$$M_k((\nu_k, \nu_{k+1}, \ldots, \nu_{N-1})) := \bigcap_{i \geq k} M_i(\nu_i).$$

By Proposition 5.1 we obtain $\alpha_0 > \alpha \Rightarrow M_k(\alpha_0) \subset M_k(\alpha)$. Hence for $(\max\{\bar{\nu}, \bar{\mu}\})_i = \max\{\nu_i, \mu_i\}$, cf. **Case 1** in Lemma 5.8,

$$M_2(\bar{\nu}) \cap M_2(\bar{\mu}) = M_2(\max\{\bar{\nu}, \bar{\mu}\}).$$

Let $\bar{\nu} = (\nu_2, \ldots, \nu_{N-1})$ and $\bar{\mu} = (\mu_2, \ldots, \mu_{N-1})$. Then let

$$\bar{\nu} \prec_k \bar{\mu} :\Leftrightarrow M_2(\bar{\nu}) \prec_k M_2(\bar{\mu}).$$

Proposition 5.2 *Let $\bar{\mu} = (\mu_2, \ldots, \mu_{k-1})$, $\bar{\nu} = (\nu_{k+1}, \ldots, \nu_{N-1})$, and $\bar{\xi} = (\xi_{k+1}, \ldots, \xi_{N-1})$.*

1. *If $(\nu_k) * \bar{\nu} < \xi_k$, then $\bar{\mu} * (\nu_k) * \bar{\nu} \prec_k \bar{\mu} * (\xi_k) * \bar{\xi}$.*
2. *(Cf. Lemma 4.8) If $\xi_{k+1}, a > 0$, then $\bar{\mu} * (\xi_k \dot{+} \Lambda^{\xi_{k+1}}a) * \bar{0} \prec_k \bar{\mu} * (\xi_k) * \bar{\xi}$.*

Proof. 5.2.1. Let $P \in M_2(\bar{\mu} * (\xi_k) * \bar{\xi}) \subset M_2(\bar{\mu} * \bar{0}) \cap M_k(\xi_k)$. By $(\nu_k) * \bar{\nu} < \xi_k$ we obtain $P \in M_k(M_k((\nu_k) * \bar{\nu}))$. Since $P \in M_2(\bar{\mu} * \bar{0})$ is Π_k on P, we conclude $P \in M_k(M_2(\bar{\mu} * \bar{0}) \cap M_k((\nu_k) * \bar{\nu})) = M_k(M_k(\bar{\mu} * (\nu_k) * \bar{\nu}))$.
5.2.2. It suffices to show that $M_k(\xi_k \dot{+} \Lambda^{\xi_{k+1}} a) \prec_k M_k(\xi_k) \cap M_{k+1}(\xi_{k+1})$, and this follows from $M_k(\xi_k) \cap M_{k+1}(\xi_{k+1}) \subset M_k(\xi_k \dot{+} \Lambda^{\xi_{k+1}} a)$. The latter is shown by induction on a as in Lemma 4.8 using the fact that $P \in M_k(\gamma) \cap M_{k+1}(\xi_{k+1}) \Rightarrow P \in M_k(M_k(\gamma) \cap M_{k+1}(\nu))$ for $\nu < \xi_{k+1}$. □

5.2 Ordinals for first order reflection

Definition 5.3 Define simultaneously by recursion on ordinals $a < \varepsilon_{\mathbb{K}+1}$ the classes $\mathcal{H}_a(X)\,(X \subset \Gamma_{\mathbb{K}+1})$, $Mh^a_k(\vec{\nu})\,(lh(\vec{\nu}) = N - k)$, the ordinals $\psi^{\vec{\nu}}_\sigma(a)$ as follows.

1. $\mathcal{H}_a(X)$ denotes the Skolem hull of $\{0, \Omega, \mathbb{K}\} \cup X$ under the functions $+, \varphi$, and the following.

   Let $\vec{\nu} = (\nu_2, \ldots, \nu_{N-1})$, $\{\sigma, b\} \cup \vec{\nu} \subset \mathcal{H}_a(X)$ and $b < a$. Then $\psi^{\vec{\nu}}_\sigma(b) \in \mathcal{H}_a(X)$.

2. For $2 \leq k < N$, $\pi \in Mh^a_k(\alpha)$ iff $\{a, \alpha\} \subset \mathcal{H}_a(\pi)$ and

   $$\forall \vec{\nu} = (\nu_k, \ldots, \nu_{N-1}) \subset \mathcal{H}_a(\pi)\,[\vec{\nu} < \alpha \to \pi \in M_k\,(Mh^a_k(\vec{\nu}))]$$

   where
   $$Mh^a_k(\vec{\nu}) = \bigcap_{i \geq k} Mh^a_i(\nu_i).$$

   Note that $Mh^a_k(\alpha)$ is a Π_{k+1}-class.

3. $\psi_\sigma(a) = \min(\{\sigma\} \cup \{\kappa < \sigma : \{a, \sigma\} \subset \mathcal{H}_a(\kappa)\,\&\,\mathcal{H}_a(\kappa) \cap \sigma \subset \kappa\})$.

   $m_i(\psi_\sigma(a)) = 0$ for $i < N$.

4. Let $\sigma \in Mh^a_2(\vec{\xi})$ for $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1})$ with $\xi_{k+1} > 0$, and $0 < c < \Lambda = \mathbb{K}$ with $c \in \mathcal{H}_a(\sigma)$. Let $\vec{\nu} = (\xi_2, \ldots, \xi_{k-1}, \xi_k \dot{+} \Lambda^{\xi_{k+1}} c, 0, \ldots, 0)$. Then

   $$\psi^{\vec{\nu}}_\sigma(a) = \min\left(\{\sigma\} \cup \{\kappa \in Mh^a_2(\vec{\nu}) \cap \sigma : \{a\} \cup \vec{\nu} \subset \mathcal{H}_a(\kappa)\,\&\,\mathcal{H}_a(\kappa) \cap \sigma \subset \kappa\}\right).$$

   $m_i(\psi^{\vec{\nu}}_\sigma(a)) = \nu_i$ for $i < N$, cf. Proposition 5.2.2.

5. Let $\sigma \in Mh^a_2(\vec{\mu} * \vec{\xi})$ with $\vec{\mu} = (\mu_2, \ldots, \mu_{k-1})$ and $\vec{\xi} = (\xi_k, \ldots, \xi_{N-1})$, and $\vec{\nu} = (\nu_k, \ldots, \nu_{N-1}) < \xi_k$, cf. Proposition 5.2.1.

   $$\psi^{\vec{\mu} * \vec{\nu}}_\sigma(a) = \min\left(\{\sigma\} \cup \{\kappa \in Mh^a_2(\vec{\mu} * \vec{\nu}) \cap \sigma : \{a\} \cup \vec{\mu} \cup \vec{\nu} \subset \mathcal{H}_a(\kappa)\,\&\,\mathcal{H}_a(\kappa) \cap \sigma \subset \kappa\}\right).$$

   $m_i(\psi^{\vec{\mu} * \vec{\nu}}_\sigma(a)) = \mu_i$ for $i < k$, and $m_i(\psi^{\vec{\mu} * \vec{\nu}}_\sigma(a)) = \nu_i$ for $i \geq k$.

As in section 4 for Π_4-reflection we see the following lemmas for Π^1_{N-2}-indescribable cardinal $\mathbb{K}$.

Lemma 5.4 *Let* $a \in \mathcal{H}_a(\mathbb{K}) \cap \varepsilon_{\mathbb{K}+1}$*. Then* $\mathbb{K} \in M_{N-1}(Mh^a_{N-1}(\varepsilon_{\mathbb{K}+1}))$*, where* $\varepsilon_{\mathbb{K}+1}$ *denotes the sequence* $\vec{\nu} = \vec{0} * (\nu_{N-1})$ *with* $\nu_{N-1} = \varepsilon_{\mathbb{K}+1}$*. For every* $\xi \in \mathcal{H}_a(\mathbb{K}) \cap \varepsilon_{\mathbb{K}+1}$*,* $\psi_{\mathbb{K}}^{\vec{0}*(\xi)}(a) < \mathbb{K}$*.*

Lemma 5.5 *Let* $\vec{\nu} = (\xi_2, \ldots, \xi_{k-1}, \xi_k \dot{+} \Lambda^{\xi_{k+1}} c, 0, \ldots, 0)$*, where* $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1})$ *with* $\xi_{k+1} > 0$*, and* $0 < c < \Lambda$ *with* $c \in \mathcal{H}_a(\sigma)$*.*

Assume $\sigma \in Mh^a_2(\vec{\xi})$*. Then* $\sigma \in M_2(Mh^a_2(\vec{\nu}))$ *and* $\psi^{\vec{\nu}}_\sigma(a) < \sigma$*, cf. Proposition 5.2.2.*

Lemma 5.6 *Let* $\vec{\mu} = (\mu_2, \ldots, \mu_{k-1})$ *and* $\vec{\nu} = (\nu_k, \ldots, \nu_{N-1}) < \xi$*. Assume* $\vec{\nu} \subset \mathcal{H}_a(\sigma)$ *and* $\sigma \in Mh^a_2(\vec{\mu} * (\xi))$*. Then* $\psi^{\vec{\mu}*\vec{\nu}}_\sigma(a) < \sigma$*, cf. Proposition 5.2.1.*

5.3 Operator controlled derivations for first order reflection

Operator controlled derivations for $\mathsf{KP}\Pi_N$ are closed under the following inference rules. $\bar{m} : \pi \mapsto \bar{m}(\pi) = (\bar{m}_2(\pi), \ldots, \bar{m}_{N-1}(\pi))$ is an additional data for the derivations, where $\bar{m}_i(\pi) \le m_i(\pi)$ for $2 \le i \le N-1$.

$(\mathrm{rfl}_{\Pi_N}(\mathbb{K}))$ $b \ge \mathbb{K}$. There exist an ordinal $a_0 \in \mathcal{H}_\gamma[\Theta] \cap a$, and a $\Sigma_N(\mathbb{K})$-sentence A enjoying the following conditions:

$$\frac{\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_b \Gamma, \neg A \quad \{\mathcal{H}_\gamma[\Theta \cup \{\rho\}] \vdash^{a_0}_b \Gamma, A^{(\rho,\mathbb{K})} : \rho < \mathbb{K}\}}{\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma} \ (\mathrm{rfl}_{\Pi_N}(\mathbb{K}))$$

$(\mathrm{rfl}_{\Pi_k}(\eta, \pi, \vec{\nu}))$ for each $2 \le k \le N-1$, cf. Proposition 5.2.1.

There exist ordinals $\eta < \pi \le b < \mathbb{K}$, $\vec{\nu} = (\nu_k, \ldots, \nu_{N-1}) < \bar{m}_k(\pi) \le m_k(\pi)$, $a_0 < a$, and a finite set Δ of $\Sigma_k(\pi)$-sentences enjoying the following conditions:

1. $\{\eta, \pi\} \cup \vec{\nu} \cup \bar{m}(\pi) \subset \mathcal{H}_\gamma[\Theta]$.
2. For each $\delta \in \Delta$, $\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_b \Gamma, \neg\delta$.
3. For any $\eta < \rho \in Mh_2(\bar{m}_{<k}(\pi) * \vec{\nu}) \cap \pi$, $\mathcal{H}_\gamma[\Theta \cup \{\rho\}] \vdash^{a_0}_b \Gamma, \Delta^{(\rho,\pi)}$, where $\bar{m}_{<k}(\pi) = (\bar{m}_2(\pi), \ldots, \bar{m}_{k-1}(\pi))$ and $\rho \in Mh_k(\vec{\nu})$ iff $\nu_i \le m_i(\rho)$ for every $k \le i \le N-1$.

$$\frac{\{\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_b \Gamma, \neg\delta\}_{\delta\in\Delta} \quad \{\mathcal{H}_\gamma[\Theta \cup \{\rho\}] \vdash^{a_0}_b \Gamma, \Delta^{(\rho,\pi)}\}_{\eta<\rho\in Mh_2(\bar{m}_{<k}(\pi)*\vec{\nu})\cap\pi}}{\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma} \ (\mathrm{rfl}_{\Pi_k}(\eta, \pi, \vec{\nu}))$$

Lemma 5.7 *Assume* $\Gamma \subset \Sigma_{N-1}(\mathbb{K})$*,* $\Theta \subset \mathcal{H}_\gamma(\psi_{\mathbb{K}}(\gamma))$*, and* $\mathcal{H}_\gamma[\Theta] \vdash^a_{\mathbb{K}} \Gamma$*. Then* $\mathcal{H}_{\hat{a}+1}[\Theta \cup \{\kappa\}] \vdash^\beta_\beta \Gamma^{(\kappa,\mathbb{K})}$ *holds for any* $\eta = \psi_{\mathbb{K}}(\gamma + \mathbb{K}) < \kappa \in Mh_{N-1}(a) \cap \psi_{\mathbb{K}}(\gamma + \mathbb{K} \cdot \omega)$*, where* $\hat{a} = \gamma + \omega^{\mathbb{K}+a}$ *and* $\beta = \psi_{\mathbb{K}}(\hat{a})$*.*

Lemma 5.8 *Assume $\bar{m}(\pi) \subset \mathcal{H}_\gamma[\Theta]$, and there exists a $2 \leq k < N-1$ such that $\bar{m}_{k+1}(\pi) > 0$, and let $k = \max\{k : \bar{m}_{k+1}(\pi) > 0\}$ and $\alpha = \bar{m}_k(\pi)$, $\xi = \bar{m}_{k+1}(\pi)$. Moreover assume $\mathcal{H}_\gamma[\Theta] \vdash^a_\pi \Gamma$ for $a, \pi < \mathbb{K}$ and $\Gamma \subset \Sigma_k(\pi)$.*

Then for any $\eta < \kappa \in Mh_2(\bar{m}_{<k}(\pi)) \cap Mh_k(\alpha \dot{+} \Lambda^\xi(1+a)) \cap \pi$, $\mathcal{H}_\gamma[\Theta \cup \{\kappa\}] \vdash^{\kappa+\omega a}_\pi \Gamma^{(\kappa,\pi)}$ holds, where η is a base, $\alpha \dot{+} \Lambda^\xi(1+a) \leq \bar{m}_k(\kappa) \in \mathcal{H}_\gamma[\Theta]$ and $\bar{m}_{<k}(\kappa) = \bar{m}_{<k}(\pi)$. Moreover when $\Theta \subset \mathcal{H}_\gamma(\kappa)$, $\mathcal{H}_\gamma[\Theta \cup \{\kappa\}] \vdash^{\kappa+\omega a}_\kappa \Gamma^{(\kappa,\pi)}$ holds.

Proof. This is seen as in Lemma 4.13 by induction on a. Let $\pi' = \kappa$ if $\Theta \subset \mathcal{H}_\gamma(\kappa)$. Otherwise $\pi' = \pi$. Consider the cases when the last inference is a $(\mathrm{rfl}_{\Pi_n}(\eta, \pi, \vec{\nu}))$. We have $n \leq k+1$, $\eta < \pi$, $\{\eta, \pi\} \cup \vec{\nu} \cup \bar{m}(\pi) \subset \mathcal{H}_\gamma[\Theta]$, $\vec{\nu} = (\nu_n, \ldots, \nu_{N-1}) < \bar{m}_n(\pi) \leq m_n(\pi)$ and $\Delta \subset \Sigma_n(\pi)$.

$$\frac{\{\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_\pi \Gamma, \neg\delta\}_{\delta\in\Delta} \quad \{\mathcal{H}_\gamma[\Theta \cup \{\rho\}] \vdash^{a_0}_\pi \Gamma, \Delta^{(\rho,\pi)}\}_{\eta<\rho\in Mh_2(\bar{m}_{<n}(\pi)*\vec{\nu})\cap\pi}}{\mathcal{H}_\gamma[\Theta] \vdash^a_\pi \Gamma} \ (\mathrm{rfl}_{\Pi_n}(\eta, \pi, \vec{\nu}))$$

Case 1. $n = k+1$: Let $\alpha_0 = \alpha \dot{+} \Lambda^\xi(1+a_0)$. Then $\vec{\mu} = (\alpha_0) * \vec{\nu} < \alpha_1 = \alpha \dot{+} \Lambda^\xi(1+a)$ by $\vec{\nu} < \xi = \bar{m}_{k+1}(\pi)$. We obtain $\eta < \kappa$, $\{\eta, \kappa, \alpha_0\} \cup \bar{m}(\kappa) \cup \vec{\nu} \subset \mathcal{H}_\gamma[\Theta \cup \{\kappa\}]$. In the following derivation $\alpha_1 \leq \bar{m}_k(\kappa)$ with $\bar{m}(\kappa) \subset \mathcal{H}_\gamma[\Theta]$. Note that $\bar{m}_{<k}(\kappa) * \vec{\mu} = \bar{m}_{<k}(\pi) * (\alpha_0) * \vec{\nu} = \max\{(\bar{m}_{<k}(\pi) * (\alpha_0) * \bar{0}), (\bar{m}_{<k}(\pi) * (\alpha) * \vec{\nu})\}$.

$$\frac{\{\mathcal{H}_\gamma[\Theta \cup \{\kappa\}] \vdash^{2d}_0 \neg\theta^{(\kappa,\pi)}, \Gamma^{(\kappa,\pi)}\}_{\theta\in\Gamma} \ \{\mathcal{H}_\gamma[\Theta \cup \{\kappa, \sigma\}] \vdash^{\kappa+\omega a_0+p}_{\pi'} \Gamma^{(\kappa,\pi)}, \Gamma^{(\sigma,\pi)}\}_{\eta<\sigma\in Mh_2(\bar{m}_{<k}(\kappa)*\vec{\mu})\cap\kappa}}{\mathcal{H}_\gamma[\Theta \cup \{\kappa\}] \vdash^{\kappa+\omega a}_{\pi'} \Gamma^{(\kappa,\pi)}} \ (\mathrm{rfl}_{\Pi_k}(\eta, \kappa, \vec{\mu}))$$

where the premise $\{\mathcal{H}_\gamma[\Theta \cup \{\kappa, \sigma\}] \vdash^{\kappa+\omega a_0+p}_{\pi'} \Gamma^{(\kappa,\pi)}, \Gamma^{(\sigma,\pi)}\}$ is derived by

$$\frac{\{\mathcal{H}_\gamma[\Theta \cup \{\sigma\}] \vdash^{\sigma+\omega a_0+1}_{\pi'} \Gamma^{(\sigma,\pi)}, \neg\delta^{(\sigma,\pi)}\}_{\delta\in\Delta} \quad \mathcal{H}_\gamma[\Theta \cup \{\kappa, \sigma\}] \vdash^{\kappa+\omega a_0}_{\pi'} \Gamma^{(\kappa,\pi)}, \Delta^{(\sigma,\pi)}}{\mathcal{H}_\gamma[\Theta \cup \{\kappa, \sigma\}] \vdash^{\kappa+\omega a_0+p}_{\pi'} \Gamma^{(\kappa,\pi)}, \Gamma^{(\sigma,\pi)}}$$

Case 2. $n \leq k$: If $n < k$, then $\vec{\nu} < \bar{m}_n(\pi) = \bar{m}_n(\kappa) \leq m_n(\kappa)$. If $n = k$, then $\vec{\nu} < \alpha \dot{+} \Lambda^\xi(1+a) \leq \bar{m}_k(\kappa)$ with the segment α of $\alpha \dot{+} \Lambda^\xi(1+a)$.

$$\frac{\{\mathcal{H}_\gamma[\Theta \cup \{\kappa\}] \vdash^{\kappa+\omega a_0+1}_{\pi'} \Gamma^{(\kappa,\pi)}, \neg\delta^{(\kappa,\pi)}\}_{\delta\in\Delta} \ \{\mathcal{H}_\gamma[\Theta \cup \{\kappa, \rho\}] \vdash^{\kappa+\omega a_0}_{\pi'} \Gamma^{(\kappa,\pi)}, \Delta^{(\rho,\pi)}\}_{\eta<\rho\in Mh_2(\bar{m}_{<n}(\kappa)*\vec{\nu})\cap\kappa}}{\mathcal{H}_\gamma[\Theta \cup \{\kappa\}] \vdash^{\kappa+\omega a}_{\pi'} \Gamma^{(\kappa,\pi)}} \ (\mathrm{rfl}_{\Pi_n}(\eta, \kappa, \vec{\nu}))$$

□

Lemma 5.9 *Let $\Gamma \subset \Sigma_1(\lambda)$ and $\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma$ with $a < \mathbb{K}$, $\mathcal{H}_\gamma[\Theta] \ni \lambda \leq b < \mathbb{K}$ and λ regular. Assume $\forall\kappa \in [\lambda, b)(\Theta \subset \mathcal{H}_\gamma(\psi_\kappa(\gamma)))$.*

Let $\hat{a} = \gamma + \theta_b(a)$ and $\delta = \psi^{\vec{\nu}}_\lambda(\hat{a})$ when $\lambda \in Mh^\gamma_k(\alpha)$ and $\vec{\nu} < \alpha$ with $\vec{\nu} \subset \mathcal{H}_\gamma[\Theta]$. Then $\mathcal{H}_{\hat{a}+1}[\Theta] \vdash^\delta_\delta \Gamma$ holds.

Proof. This is seen as in Lemma 4.14 by main induction on b with subsidiary induction on a. Let η be a base.
Case 1. Consider the case when the last inference is a $(\mathrm{rfl}_{\Pi_{k+1}}(\eta, \sigma, \vec{\nu}))$ with $2 \leq k < N-1$ and $b \geq \sigma$.

$$\frac{\{\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_b \Gamma, \neg\delta\}_{\delta\in\Delta} \quad \{\mathcal{H}_\gamma[\Theta \cup \{\rho\}] \vdash^{a_0}_b \Gamma, \Delta^{(\rho,\sigma)}\}_{\eta<\rho\in Mh_2(\bar{m}_{\leq k}(\sigma)*\vec{\nu})\cap\sigma}}{\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma} \ (\mathrm{rfl}_{\Pi_{k+1}}(\eta, \sigma, \vec{\nu}))$$

where $\Delta \subset \Sigma_{k+1}(\sigma)$, $\vec{\nu} < \xi = \bar{m}_{k+1}(\sigma) \leq m_{k+1}(\sigma)$, $\eta < \sigma$ and $\{\eta, \sigma\} \cup \bar{m}(\sigma) \cup \vec{\nu} \subset \mathcal{H}_\gamma[\Theta]$. We may assume that $\sigma \geq \lambda$ and there is no regular $\pi \in \mathcal{H}_\gamma[\Theta]$ such that $\sigma < \pi \leq b$.

We obtain the lemma by Cut-elimination, Lemma 5.8 for $\kappa=\psi_\sigma^{\bar{m}_{<k}(\sigma)*(\beta)*\vec{0}}(\gamma)$ with $\beta=\bar{m}_k(\sigma)\dot{+}\Lambda^{\bar{m}_{k+1}(\sigma)}(1+a_1)$ and $a_1=\theta_b(a_0)+1$, and MIH.
Case 2. Next consider the case when the last inference is a $(\mathrm{rfl}_{\Pi_2}(\eta,\sigma,\vec{\nu}))$ with $b\geq\sigma$.

$$\frac{\{\mathcal{H}_\gamma[\Theta]\vdash^{a_0}_b \Gamma,\neg\delta\}_{\delta\in\Delta}\quad \{\mathcal{H}_\gamma[\Theta\cup\{\rho\}]\vdash^{a_0}_b \Gamma,\Delta^{(\rho,\sigma)}\}_{\eta<\rho\in Mh_2(\vec{\nu})\cap\sigma}}{\mathcal{H}_\gamma[\Theta]\vdash^a_b\Gamma}\ (\mathrm{rfl}_{\Pi_2}(\eta,\sigma,\vec{\nu}))$$

where $\Delta\subset\Sigma_2(\sigma)$, $\vec{\nu}<\xi=\bar{m}_2(\sigma)\leq m_2(\sigma)$, $\eta<\sigma$ and $\{\eta,\sigma\}\cup\bar{m}(\sigma)\cup\vec{\nu}\subset\mathcal{H}_\gamma[\Theta]$. We may assume that $\sigma\geq\lambda$. Let $\rho=\psi_\sigma^{\vec{\nu}}(\widehat{a_0})$. We see $\eta<\rho\in Mh_2(\vec{\nu})\cap\sigma\cap\mathcal{H}_{\widehat{a_0}+1}[\Theta]$ from the assumption $\Theta\subset\mathcal{H}_\gamma(\rho)$.

We see the lemma as in Lemma 2.6 by Inversion, picking the ρ-th branch from the right upper seqeunts, and then introducing several (cut)'s instead of $(\mathrm{rfl}_{\Pi_2}(\sigma,\vec{\nu}))$. Use MIH when $\lambda<\sigma$. □

$OT(\Pi_N)$ denotes a computable notation system of ordinals with collapsing functions $\psi_\sigma^{\vec{\nu}}(a)$.

Theorem 5.10 *Assume* $\mathsf{KP}\Pi_N\vdash\theta^{L_\Omega}$ *for* $\theta\in\Sigma$. *Then there exists an* $n<\omega$ *such that* $L_\alpha\models\theta$ *for* $\alpha=\psi_\Omega(\omega_n(\mathbb{K}+1))$ *in* $OT(\Pi_N)$.

Proof. This is seen from Lemmas 5.7 and 5.9. □

6 Π^1_1-reflection

Definition 6.1 σ is said to be *α-stable* for $\alpha>\sigma$ if $L_\sigma\prec_{\Sigma_1}L_\alpha$.

It is known that σ is $(\sigma+1)$-stable iff σ is Π^1_0-reflecting, and σ is σ^+-stable iff σ is Π^1_1-reflecting, where σ^+ denotes the next admissible ordinal above σ, cf. [Richter-Aczel74].

Let S_1 denote the theory obtained from $\mathsf{KP}\omega+(V=L)$ by adding the following axioms for an individual constant $\mathbb{S}$: $\mathbb{S}$ is a limit ordinal and

$$L_{\mathbb{S}}\prec_{\Sigma_1}L.$$

The latter denotes a schema

$$\exists x\,B(x,v)\land v\in L_{\mathbb{S}}\to\exists x\in L_{\mathbb{S}}\,B(x,v)$$

for each Δ_0-formula B. Let $L=L_{\mathbb{S}^+}\models S_1$.

An exponential structure emerges in iterating (recursively) Mahlo operations to resolve first-order reflections M_N in terms of Mahlo classes $Mh^a_k(\alpha)$ and $Mh^a_k(\vec{\nu})$. Viewing the vector $\vec{\nu}=(\nu_2,\nu_3,\ldots,\nu_{N-1})$ as a function $\{2,3,\ldots,N-1\}\ni k\mapsto\nu_k$, each k in its domain designates the class of Π_k-formulas or the Mahlo operation M_k, while its value ν_k corresponds to the height of derivations, cf. **Case 1** in the proof of Lemma 5.8.

On the other side, the axiom $L_{\mathbb{S}}\prec_{\Sigma_1}L_{\mathbb{S}^+}$ says that $\mathbb{S}$ 'reflects' $\Pi_{\mathbb{S}^+}$-formulas in transfinite levels. In place of vectors in finite lengths, we need functions

$f : \mathbb{S}^+ \to ON$. Each c in the domain of the function f corresponds to formulas of ranks$< c$ in inference rules for higher reflections. Its support $\mathrm{supp}(f) = \{c < \mathbb{S}^+ : f(c) \neq 0\}$ may be assumed to be *finite*, while its value $f(c) < \varepsilon_{\mathbb{S}^++1}$. A Veblen function $\tilde{\theta}_b(\xi)$ is used to denote ordinals instead of the exponential function $\tilde{\theta}_1(\xi) = (\mathbb{S}^+)^\xi$. The relation $\vec{\nu} < \alpha$ in section 5 is replaced by a relation $f <^c \xi$ for ordinals c, ξ and finite function f. $f <^c \xi$ holds if $f(c) < \mu$ for a segment $\mu = \cdots + \tilde{\theta}_b(\nu)$ of ξ, and $f(c+d) < \tilde{\theta}_{-d}(\tilde{\theta}_b(\nu))$ for $d = \min\{d > 0 : c + d \in \mathrm{supp}(f)\}$, and so forth, where $\tilde{\theta}_{-d}(\xi)$ denotes an inverse of the function $\xi \mapsto \tilde{\theta}_d(\xi)$.

Mahlo classes $Mh^a_c(\xi)$ introduced in (32) reflects every fact $\pi \in Mh^a_0(g_c) = \bigcap\{Mh^a_d(g(d)) : c > d \in \mathrm{supp}(g)\}$ on the ordinals $\pi \in Mh^a_c(\xi)$ in lower level, down to 'smaller' Mahlo classes $Mh^a_c(f) = \bigcap\{Mh^a_d(f(d)) : c \leq d \in \mathrm{supp}(f)\}$, where $f <^c \xi$.

This apparatus would suffice to analyze reflections in transfinite levels. We need another for the axiom $L_{\mathbb{S}} \prec_{\Sigma_1} L_{\mathbb{S}^+}$ of Π^1_1-reflection, i.e., a (formal) *Mostowski collapsing*: Assume that $B(u, v)$ with $v \in L_{\mathbb{S}}$ for a Δ_0-formula B. We need to find a substitute $u' \in L_{\mathbb{S}}$ for $u \in L_{\mathbb{S}^+}$, i.e., $B(u', v)$. For simplicity let us assume that $v = \beta < \mathbb{S}$ and $u = \alpha < \mathbb{S}^+$ are ordinals. We may assume that $\alpha \geq \mathbb{S}$. Let $\rho < \mathbb{S}$ be an ordinal, which is bigger than every ordinal$< \mathbb{S}$ occurring in the 'context' of $B(\alpha, \beta)$. This means that if an ordinal $\delta < \mathbb{S}$ occurs in a 'relevant' branch of a derivation of $B(\alpha, \beta)$, $\delta < \rho$ holds. Then we can define a Mostwosiki collapsing $\alpha \mapsto \alpha[\rho/\mathbb{S}]$ for ordinal terms α such that $\beta[\rho/\mathbb{S}] = \beta$ for each relevant $\beta < \mathbb{S}$, $\mathbb{S}[\rho/\mathbb{S}] = \rho$ and $\alpha[\rho/\mathbb{S}] < (\mathbb{S}^+)[\rho/\mathbb{S}] = \rho^+ < \mathbb{S}$, cf. Definition 6.22. Then we see that $B(\alpha[\rho/\mathbb{S}], \beta)$ holds.

Although the above scheme would seem to work, how to implement the plan? Let $E^{\mathbb{S}}_\rho$ denote the set of ordinal terms α such that every subterm $\beta < \mathbb{S}$ of α is smaller than ρ. It turns out that $\mathcal{H}_\gamma(E^{\mathbb{S}}_\rho) \subset E^{\mathbb{S}}_\rho$ if $\mathcal{H}_\gamma(\rho) \cap \mathbb{S} \subset \rho$. Let $\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma$, and assume that (3), $\{\gamma, a, b\} \cup \mathsf{k}(\Gamma) \subset \mathcal{H}_\gamma[\Theta]$ holds in Definition 1.16. Moreover let us assume that $\Theta \subset E^{\mathbb{S}}_\rho$ holds. Then we obtain $\{\gamma, a, b\} \cup \mathsf{k}(\Gamma) \subset \mathcal{H}_\gamma[\Theta] \subset \mathcal{H}_\gamma(E^{\mathbb{S}}_\rho) \subset E^{\mathbb{S}}_\rho$. This means that $\mathsf{k}(\Gamma) \subset E^{\mathbb{S}}_\rho$ holds as long as $\Theta \subset E^{\mathbb{S}}_\rho$ holds, i.e., as long as we are concerned with branches for $\mathsf{k}(\iota) \subset E^{\mathbb{S}}_\rho$ in, e.g., inferences $(\bigwedge)$: $A \simeq \bigwedge(A_\iota)_{\iota \in J}$

$$\frac{\{\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_b \Gamma, A, A_\iota\}_{\iota \in J}}{\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma, A}\ (\bigwedge) \rightsquigarrow \frac{\{\mathcal{H}_\gamma[\Theta] \vdash^{a_0}_b \Gamma, A, A_\iota\}_{\iota \in J, \mathsf{k}(\iota) \subset E^{\mathbb{S}}_\rho}}{\mathcal{H}_\gamma[\Theta] \vdash^a_b \Gamma, A}\ (\bigwedge)$$

and dually $\mathsf{k}(\iota) \subset E^{\mathbb{S}}_\rho$ for a minor formula A_ι of a $(\bigvee)$ with the main formula $A \simeq \bigvee(A_\iota)_{\iota \in J}$, *provided that* $\mathcal{H}_\gamma(\rho) \cap \mathbb{S} \subset \rho$. The proviso means that $\gamma_1 \geq \gamma$ when $\rho = \psi^f_{\mathbb{S}}(\gamma_1)$. Such a $\rho \in \mathcal{H}_\gamma[\Theta]$ only when $\rho \in \Theta$. Let us try to replace the inferences for the stability of $\mathbb{S}$

$$\frac{(\mathcal{H}_\gamma, \Theta) \vdash \Gamma, B(u) \quad \{(\mathcal{H}_\gamma, \Theta \cup \{\sigma\}) \vdash \Gamma, \neg B(u)^{[\sigma/\mathbb{S}]}\}_{\Theta \subset E^{\mathbb{S}}_\sigma}}{(\mathcal{H}_\gamma, \Theta) \vdash \Gamma}\ (\mathrm{stbl})$$

by inferences for reflection of ρ with $\Theta \subset E^{\mathbb{S}}_\rho$: If $B(u)^{[\rho/\mathbb{S}]}$ holds, then $B(u)^{[\sigma/\mathbb{S}]}$

holds for some $\sigma < \rho$.

$$\frac{(\mathcal{H}_\gamma, \Theta \cup \{\rho\}) \vdash \Gamma^{[\rho/\mathbb{S}]}, B(u)^{[\rho/\mathbb{S}]} \quad \{(\mathcal{H}_\gamma, \Theta \cup \{\rho, \sigma\}) \vdash \Gamma^{[\rho/\mathbb{S}]}, \neg B(u)^{[\sigma/\mathbb{S}]}\}_{\Theta \subset E^{\mathbb{S}}_\sigma, \sigma < \rho}}{(\mathcal{H}_\gamma, \Theta \cup \{\rho\}) \vdash \Gamma^{[\rho/\mathbb{S}]}} \text{ (rfl)}$$

However we need to eliminate the inferences for reflections in transfinite levels. In view of analysis in section 5 for first-order reflection, $\Gamma^{[\rho/\mathbb{S}]}, B(u)^{[\rho/\mathbb{S}]}$ is replaced by $\Gamma^{[\sigma/\mathbb{S}]}, B(u)^{[\sigma/\mathbb{S}]}$, and $\Gamma^{[\rho/\mathbb{S}]}, \neg B(u)^{[\sigma/\mathbb{S}]}$ by $\Gamma^{[\kappa/\mathbb{S}]}, \neg B(u)^{[\sigma/\mathbb{S}]}$ with $\sigma < \kappa < \rho$.

$$\frac{\{(\mathcal{H}_\gamma, \Theta \cup \{\kappa\}) \vdash \Gamma^{[\kappa/\mathbb{S}]}, \neg\theta^{[\kappa/\mathbb{S}]}\}_{\theta \in \Gamma} \quad \dfrac{(\mathcal{H}_\gamma, \Theta \cup \{\rho, \sigma\}) \vdash \Gamma^{[\sigma/\mathbb{S}]}, B(u)^{[\sigma/\mathbb{S}]} \quad (\mathcal{H}_\gamma, \Theta \cup \{\kappa, \rho, \sigma\}) \vdash \Gamma^{[\kappa/\mathbb{S}]}, \neg B(u)^{[\sigma/\mathbb{S}]}}{\{(\mathcal{H}_\gamma, \Theta \cup \{\kappa, \rho, \sigma\}) \vdash \Gamma^{[\kappa/\mathbb{S}]}, \Gamma^{[\sigma/\mathbb{S}]}\}_\sigma} (cut)}{(\mathcal{H}_\gamma, \Theta \cup \{\kappa, \rho\}) \vdash \Gamma^{[\kappa/\mathbb{S}]}} \text{ (rfl)}$$

We are replacing formulas $\Gamma^{[\rho/\mathbb{S}]}$ by $\Gamma^{[\sigma/\mathbb{S}]}$ or by $\Gamma^{[\kappa/\mathbb{S}]}$. This means that $\alpha[\sigma/\mathbb{S}]$ is substituted for each $\alpha[\rho/\mathbb{S}]$. Namely a composition of uncollapsing and collapsing $\alpha[\rho/\mathbb{S}] \mapsto \alpha \mapsto \alpha[\sigma/\mathbb{S}]$ arises. Hence we need $\alpha \in E^{\mathbb{S}}_\sigma \subsetneq E^{\mathbb{S}}_\rho$ for $\sigma < \rho$. However we have $\Theta \cup \{\rho\} \not\subset E^{\mathbb{S}}_\sigma$, and the schema seems to be broken. Moreover the finite sets $\Theta \cup \{\rho\}$ becomes bigger to $\Theta \cup \{\kappa, \rho\}$. Is it remain finite in eliminating inferences of reflections in transfinite level?

Looking back at the proof of Lemma 4.13, for $\Gamma \subset \Sigma_2$ and $\Delta \subset \Pi_2$

$$\frac{\{\mathcal{H}_\gamma[\Theta] \vdash \Gamma^{(\pi,\mathbb{K})}, \neg\delta^{(\pi,\mathbb{K})}\}_{\delta \in \Delta} \quad \{\mathcal{H}_\gamma[\Theta \cup \{\rho\}] \vdash \Gamma^{(\pi,\mathbb{K})}, \Delta^{(\rho,\mathbb{K})}\}_\rho}{\mathcal{H}_\gamma[\Theta] \vdash \Gamma^{(\pi,\mathbb{K})}} \text{ (rfl}_{\Pi_3})$$

is rewritten to

$$\frac{\{\mathcal{H}_\gamma[\Theta \cup \{\kappa\}] \vdash \neg\theta^{(\kappa,\mathbb{K})}, \Gamma^{(\kappa,\mathbb{K})}\}_{\theta \in \Gamma} \quad \dfrac{\{\mathcal{H}_\gamma[\Theta \cup \{\sigma\}] \vdash \Gamma^{(\sigma,\mathbb{K})}, \neg\delta^{(\sigma,\mathbb{K})}\}_{\delta \in \Delta} \quad \mathcal{H}_\gamma[\Theta \cup \{\kappa, \sigma\}] \vdash \Gamma^{(\kappa,\mathbb{K})}, \Delta^{(\sigma,\mathbb{K})}}{\{\mathcal{H}_\gamma[\Theta \cup \{\kappa, \sigma\}] \vdash \Gamma^{(\kappa,\mathbb{K})}, \Gamma^{(\sigma,\mathbb{K})}\}_\sigma}}{\mathcal{H}_\gamma[\Theta \cup \{\kappa\}] \vdash \Gamma^{(\kappa,\mathbb{K})}} \text{ (rfl}_{\Pi_2})$$

This is done by replacing the restriction $^{(\pi,\mathbb{K})}$ by $^{(\sigma,\mathbb{K})}$ or $^{(\kappa,\mathbb{K})}$, and ordinals π, σ, κ enter derivations, but do we need to control these ordinals? Instead of the restriction $^{(\pi,\mathbb{K})}$, formulas could put on *caps* π, σ, κ in such a way that $\mathsf{k}(A^{(\sigma)}) = \mathsf{k}(A)$. This means that the cap σ does not 'occur' in a capped formula $A^{(\sigma)}$. If we choose an ordinal γ_0 big enough (depending on a given finite proof figure), every ordinal 'occurring' in derivations (including the subscript $\gamma \leq \gamma_0$ in the operators $\mathcal{H}_\gamma$) is in $\mathcal{H}_{\gamma_0} = \mathcal{H}_{\gamma_0}(\emptyset)$ for the ordinal γ_0, while each cap ρ exceeds the *threshold* γ_0 in the sense that $\rho \notin \mathcal{H}_{\gamma_0}(\rho) \cap \mathbb{S} \subset \rho$. Then every ordinal 'occurring' in derivations is in the domain $E^{\mathbb{S}}_\rho$ of the Mostowski collapsing $\alpha \mapsto \alpha[\rho/\mathbb{S}]$. Now details follow.

6.1 Ordinals for one stable ordinal

For a while, $\mathbb{S}$ denotes a weakly inaccessible cardinal.

Definition 6.2 Let $\Lambda = \omega_{\mathbb{S}+1}$ or $\Lambda = \mathbb{S}^+$. $\varphi_b(\xi)$ denotes the binary Veblen function on Λ^+ with $\varphi_0(\xi) = \omega^\xi$, and $\tilde{\varphi}_b(\xi) := \varphi_b(\Lambda \cdot \xi)$ for the epsilon number Λ.

Let $b, \xi < \Lambda^+$. $\theta_b(\xi)$ $[\tilde{\theta}_b(\xi)]$ denotes a b-th iterate of $\varphi_0(\xi) = \omega^\xi$ [of $\tilde{\varphi}_0(\xi) = \Lambda^\xi$], resp.

Definition 6.3 Let $\xi < \varphi_\Lambda(0)$ be a non-zero ordinal with its normal form:

$$\xi = \sum_{i \leq m} \tilde{\theta}_{b_i}(\xi_i) \cdot a_i =_{NF} \tilde{\theta}_{b_m}(\xi_m) \cdot a_m + \cdots + \tilde{\theta}_{b_0}(\xi_0) \cdot a_0 \tag{30}$$

where $\tilde{\theta}_{b_i}(\xi_i) > \xi_i$, $\tilde{\theta}_{b_m}(\xi_m) > \cdots > \tilde{\theta}_{b_0}(\xi_0)$, $b_i = \omega^{c_i} < \Lambda$, and $0 < a_0, \ldots, a_m < \Lambda$. $SC_\Lambda(\xi) = \bigcup_{i \leq m}(\{a_i\} \cup SC_\Lambda(\xi_i))$.

$\tilde{\theta}_{b_0}(\xi_0)$ is said to be the *tail* of ξ, denoted $\tilde{\theta}_{b_0}(\xi_0) = tl(\xi)$, and $\tilde{\theta}_{b_m}(\xi_m)$ the *head* of ξ, denoted $\tilde{\theta}_{b_m}(\xi_m) = hd(\xi)$.

1. ζ is a *segment* of ξ iff there exists an $n \, (0 \leq n \leq m+1)$ such that $\zeta =_{NF} \sum_{i \geq n} \tilde{\theta}_{b_i}(\xi_i) \cdot a_i = \tilde{\theta}_{b_m}(\xi_m) \cdot a_m + \cdots + \tilde{\theta}_{b_n}(\xi_n) \cdot a_n$ for ξ in (30).

2. Let $\zeta =_{NF} \tilde{\theta}_b(\xi)$ with $\tilde{\theta}_b(\xi) > \xi$ and $b = \omega^{b_0}$, and c be ordinals. An ordinal $\tilde{\theta}_{-c}(\zeta)$ is defined recursively as follows. If $b \geq c$, then $\tilde{\theta}_{-c}(\zeta) = \tilde{\theta}_{b-c}(\xi)$. Let $c > b$. If $\xi > 0$, then $\tilde{\theta}_{-c}(\zeta) = \tilde{\theta}_{-(c-b)}(\tilde{\theta}_{b_m}(\xi_m))$ for the head term $hd(\xi) = \tilde{\theta}_{b_m}(\xi_m)$ of ξ in (30). If $\xi = 0$, then let $\tilde{\theta}_{-c}(\zeta) = 0$.

Definition 6.4

1. A function $f : \Lambda \to \varphi_\Lambda(0)$ with a *finite* support $\text{supp}(f) = \{c < \Lambda : f(c) \neq 0\} \subset \Lambda$ is said to be a *finite function* if $\forall i > 0 (a_i = 1)$ and $a_0 = 1$ when $b_0 > 1$ in $f(c) =_{NF} \tilde{\theta}_{b_m}(\xi_m) \cdot a_m + \cdots + \tilde{\theta}_{b_0}(\xi_0) \cdot a_0$ for any $c \in \text{supp}(f)$.

   It is identified with the finite function $f \restriction \text{supp}(f)$. When $c \notin \text{supp}(f)$, let $f(c) := 0$. $SC_\Lambda(f) := \bigcup\{\{c\} \cup SC_\Lambda(f(c))\} : c \in \text{supp}(f)\}$. $f, g, h, \ldots$ range over finite functions.

   For an ordinal c, f_c and f^c are restrictions of f to the domains $\text{supp}(f_c) = \{d \in \text{supp}(f) : d < c\}$ and $\text{supp}(f^c) = \{d \in \text{supp}(f) : d \geq c\}$. $g_c * f^c$ denotes the concatenated function such that $\text{supp}(g_c * f^c) = \text{supp}(g_c) \cup \text{supp}(f^c)$, $(g_c * f^c)(a) = g(a)$ for $a < c$, and $(g_c * f^c)(a) = f(a)$ for $a \geq c$.

2. Let f be a finite function and c, ξ ordinals. A relation $f <^c \xi$ is defined by induction on the cardinality of the finite set $\{d \in \text{supp}(f) : d > c\}$ as follows. If $f^c = \emptyset$, then $f <^c \xi$ holds. For $f^c \neq \emptyset$, $f <^c \xi$ iff there exists a segment μ of ξ such that $f(c) < \mu$ and $f <^{c+d} \tilde{\theta}_{-d}(tl(\mu))$ for $d = \min\{c + d \in \text{supp}(f) : d > 0\}$.

Proposition 6.5 $f <^c \xi \leq \zeta \Rightarrow f <^c \zeta$.

6.2 Mahlo classes for Π^1_1-reflection

In Lemma 4.8 and Proposition 5.2.2, it is crucial the fact that $P \in M_k(\gamma) \Rightarrow P \in M_k(M_k(\gamma) \cap M_{k+1}(\nu))$ if $P \in M_{k+1}(\xi_{k+1})$ and $\nu < \xi_{k+1}$. This means that if P is in a higher Mahlo class, then P reflects a fact on P in lower Mahlo classes.

$P \in M_c(\xi)$ is defined by main induction on c with subsidiary induction on P.

$$P \in M_c(\xi) :\Leftrightarrow \forall f <^c \xi \forall g\, [P \in M_0(g_c) \Rightarrow P \in M_2(M_0(g_c * f^c))] \tag{31}$$

where f, g range over finite functions and

$$M_c(f) := \bigcap\{M_d(f(d)) : d \in \mathrm{supp}(f^c)\} = \bigcap\{M_d(f(d)) : c \leq d \in \mathrm{supp}(f)\}.$$

From Proposition 6.5 we see $\xi < \zeta \Rightarrow M_c(\xi) \supset M_c(\zeta)$.

For classes $\mathcal{X}$ let

$$P \in M_c(\mathcal{X}) :\Leftrightarrow \forall g\, [P \in M_0(g_c) \Rightarrow P \in M_2(M_0(g_c) \cap \mathcal{X})]\,.$$

Then by $M_0(g_c * f^c) = M_0(g_c) \cap M_c(f^c)$, $P \in M_c(\xi) \Leftrightarrow \forall f <^c \xi\, [P \in M_c(M_c(f^c))]$, i.e., $M_c(\xi) = \bigcap_{f <^c \xi} M_c(M_c(f^c))$.

Proposition 6.6 *Suppose $P \in M_c(\xi)$.*

1. *Let $f <^c \xi$. Then $P \in M_c(M_c(f^c))$.*
2. *Let $P \in M_d(\mathcal{X})$ for $d > c$. Then $P \in M_c(M_c(\xi) \cap \mathcal{X})$.*

Proof. 6.6.1. Let g be a function such that $P \in M_0(g_c)$. By the definition (31) of $P \in M_c(\xi)$ we obtain $P \in M_2\,(M_0(g_c) \cap M_c(f^c))$.
6.6.2. Let $P \in M_d(\mathcal{X})$ for $d > c$. Let g be a function such that $P \in M_0(g_c)$. We obtain by $d > c$ with the function $g_c * h$, $P \in M_2\,(M_0(g_c) \cap M_c(\xi) \cap \mathcal{X})$, where $\mathrm{supp}(h) = \{c\}$ and $h(c) = \xi$. □

Lemma 6.7 *Assume $P \in M_d(\xi) \cap M_c(\xi_0)$, $\xi_0 \neq 0$, and $d < c$. Moreover let $\xi_1 \leq \tilde{\theta}_{c-d}(\xi_0)$. Then $P \in M_d(\xi \dot{+} \xi_1) \cap M_d(M_d(\xi \dot{+} \xi_1))$.*

Proof. This is seen as in Lemma 4.11.

We obtain $P \in M_c(\xi_0) \subset M_c(M_c(\emptyset))$ by Proposition 6.6.1. Let $P \in M_d(\xi \dot{+} \xi_1) \cap M_0(g_d)$ for a function g. We show $P \in M_2\left(M_0(g_d) \cap M_d(\xi \dot{+} \xi_1)\right)$. Let $h = g_d \cup \{(d, \xi \dot{+} \xi_1)\}$. Then $P \in M_0(h_c)$ by $d < c$. $P \in M_c(M_c(\emptyset))$ yields $P \in M_2\,(M_0(h_c) \cap M_c(\emptyset))$, and hence $P \in M_2\left(M_0(g_d) \cap M_d(\xi \dot{+} \xi_1)\right)$. Therefore $P \in M_d(M_d(\xi \dot{+} \xi_1))$.

Let f be a finite function such that $f <^d \xi + \xi_1$. We show $P \in M_d(M_d(f^d))$ by main induction on the cardinality of the finite set $\{e \in \mathrm{supp}(f) : e > d\}$ with subsidiary induction on ξ_1.

First let $f <^d \mu$ for a segment μ of ξ. We obtain $P \in M_d(\mu)$ and $P \in M_d(M_d(f^d))$.

In what follows let $f(d) = \xi\dot{+}\zeta$ with $\zeta < \xi_1$. By SIH we obtain $P \in M_d(f(d)) \cap M_d(M_d(f(d)))$. If $\{e \in \mathrm{supp}(f) : e > d\} = \emptyset$, then $M_d(f^d) = M_d(f(d))$, and we are done. Otherwise let $e = \min\{e \in \mathrm{supp}(f) : e > d\}$.

By SIH we can assume $f <^e \tilde{\theta}_{-(e-d)}(tl(\xi_1))$. By $\xi_1 \leq \tilde{\theta}_{c-d}(\xi_0)$, we obtain $f <^e \tilde{\theta}_{-(e-d)}(\tilde{\theta}_{c-d}(\xi_0)) = \tilde{\theta}_{-e}(\tilde{\theta}_c(\xi_0))$. We claim that $P \in M_{c_0}(M_{c_0}(f^{c_0}))$ for $c_0 = \min\{c, e\}$. If $c = e$, then the claim follows from the assumption $P \in M_c(\xi_0)$ and $f <^e \xi_0$. Let $e = c + e_0 > c$. Then $\tilde{\theta}_{-e}(\tilde{\theta}_c(\xi_0)) = \tilde{\theta}_{-e_0}(hd(\xi_0))$, and $f <^c \xi_0$ with $f(c) = 0$ yields the claim. Let $c = e + c_1 > e$. Then $\tilde{\theta}_{-e}(\tilde{\theta}_c(\xi_0)) = \tilde{\theta}_{c_1}(\xi_0)$. MIH yields the claim.

On the other hand we have $M_d(f^d) = M_d(f(d)) \cap M_{c_0}(f^{c_0})$. $P \in M_d(f(d)) \cap M_{c_0}(M_{c_0}(f^{c_0}))$ with $d < c_0$ yields by Proposition 6.6.2, $P \in M_d\left(M_d(f(d)) \cap M_{c_0}(f^{c_0})\right)$, i.e., $P \in M_d(M_d(f^d))$. □

For finite functions f and g,

$$M_0(g) \prec M_0(f) :\Leftrightarrow \forall P \in M_0(f)\, (P \in M_2(M_0(g)))\,.$$

Corollary 6.8 *Let f, g be finite functions and $c \in \mathrm{supp}(f)$. Assume that there exists an ordinal $d < c$ such that $(d, c) \cap \mathrm{supp}(f) = (d, c) \cap \mathrm{supp}(g) = \emptyset$, $g_d = f_d$, $g(d) < f(d)\dot{+}\tilde{\theta}_{c-d}(f(c)) \cdot \omega$, and $g <^c f(c)$. Then $M_0(g) \prec M_0(f)$ holds.*

Proof. By Lemma 6.7. □

Definition 6.9 An *irreducibility* of finite functions f is defined by induction on the cardinality n of the finite set $\mathrm{supp}(f)$. If $n \leq 1$, f is defined to be irreducible. Let $n \geq 2$ and $c < c + d$ be the largest two elements in $\mathrm{supp}(f)$, and let g be a finite function such that $\mathrm{supp}(g) = \mathrm{supp}(f_c) \cup \{c\}$, $g_c = f_c$ and $g(c) = f(c) + \tilde{\theta}_d(f(c + d))$.

Then f is irreducible iff $tl(f(c)) > \tilde{\theta}_d(f(c + d))$ and g is irreducible.

Definition 6.10 Let f, g be irreducible finite functions, and b an ordinal. Let us define a relation $f <^b_{lx} g$ by induction on the cardinality $\#\{e \in \mathrm{supp}(f) \cup \mathrm{supp}(g) : e \geq b\}$ as follows. $f <^b_{lx} g$ holds iff $f^b \neq g^b$ and for the ordinal $c = \min\{c \geq b : f(c) \neq g(c)\}$, one of the following conditions is met:

1. $f(c) < g(c)$ and let μ be the shortest part of $g(c)$ such that $f(c) < \mu$. Then for any $c < c + d \in \mathrm{supp}(f)$, if $tl(\mu) \leq \tilde{\theta}_d(f(c + d))$, then $f <^{c+d}_{lx} g$ holds.

2. $f(c) > g(c)$ and let ν be the shortest part of $f(c)$ such that $\nu > g(c)$. Then there exist a $c < c + d \in \mathrm{supp}(g)$ such that $f <^{c+d}_{lx} g$ and $tl(\nu) \leq \tilde{\theta}_d(g(c + d))$.

Proposition 6.11 *If $f <^0_{lx} g$, then $M_0(f) \prec M_0(g)$.*

Proof. This is seen from Corollary 6.8. □

6.3 Skolem hulls and collapsing functions

Definition 6.12 Let $\mathbb{K} = \omega_{\mathbb{S}+1}$, $a < \varepsilon_{\mathbb{K}+1}$ and $X \subset \Gamma_{\mathbb{K}+1}$.

1. $\mathcal{H}_a(X)$ denotes the Skolem hull of $\{0, \Omega, \mathbb{S}, \mathbb{K}\} \cup X$ under the functions $+, \varphi$, $\beta \mapsto \psi_\Omega(\beta)$ $(\beta < a)$, $\mathbb{S} > \alpha \mapsto \alpha^+$ and $(\pi, b, f) \mapsto \psi_\pi^f(b)$, where $b < a$ and f is a finite function such that $f \in \mathcal{H}_a(X) :\Leftrightarrow SC_{\mathbb{K}}(f) \subset \mathcal{H}_a(X)$.

2. Let $c < \mathbb{K}$, $a < \varepsilon_{\mathbb{K}+1}$ and $\xi < \varphi_{\mathbb{K}}(0)$. $\pi \in Mh_c^a(\xi)$ iff $\{a, c, \xi\} \subset \mathcal{H}_a(\pi)$ and
$$\forall f <^c \xi \forall g\, (SC_{\mathbb{K}}(f) \cup SC_{\mathbb{K}}(g) \subset \mathcal{H}_a(\pi) \,\&\, \pi \in Mh_0^a(g_c) \Rightarrow \pi \in M_2(Mh_0^a(g_c * f^c))) \tag{32}$$
where
$$Mh_c^a(f) := \bigcap\{Mh_d^a(f(d)) : d \in \mathrm{supp}(f^c)\} = \bigcap\{Mh_d^a(f(d)) : c \leq d \in \mathrm{supp}(f)\}.$$

3. $$\psi_\pi^f(a) := \min(\{\pi\} \cup \{\kappa \in Mh_0^a(f) \cap \pi : \mathcal{H}_a(\kappa) \cap \pi \subset \kappa, \{\pi, a\} \cup SC_{\mathbb{K}}(f) \subset \mathcal{H}_a(\kappa)\}) \tag{33}$$

Shrewd cardinals are introduced by [Rathjen05b]. A cardinal κ is *shrewd* iff for any $\eta > 0$, $P \subset V_\kappa$, and formula $\varphi(x, y)$, if $V_{\kappa+\eta} \models \varphi[P, \kappa]$, then there are $0 < \kappa_0, \eta_0 < \kappa$ such that $V_{\kappa_0+\eta_0} \models \varphi[P \cap V_{\kappa_0}, \kappa_0]$. $\tilde{T}$ denotes the extension of ZFC by the axiom stating that $\mathbb{S}$ is a shrewd cardinal.

Lemma 6.13 *$\tilde{T}$ proves that $\mathbb{S} \in Mh_c^a(\xi) \cap M_2(Mh_c^a(\xi))$ for every $a < \varepsilon_{\mathbb{K}+1}$, $c < \mathbb{K}$, $\xi < \varphi_{\mathbb{K}}(0)$ such that $\{a, c, \xi\} \subset \mathcal{H}_a(\mathbb{S})$.*

Proof. We show the lemma by induction on $\xi < \varphi_{\mathbb{K}}(0)$.

Let $\{a, c, \xi\} \cup SC_{\mathbb{K}}(f) \subset \mathcal{H}_a(\mathbb{S})$ and $f <^c \xi$. We show $\mathbb{S} \in Mh_c^a(f^c)$, and $\mathbb{S} \in M_2\,(Mh_0^a(g_c) \cap Mh_c^a(f^c))$ assuming $\mathbb{S} \in Mh_0^a(g_c)$ and $SC_{\mathbb{K}}(g_c) \subset \mathcal{H}_a(\mathbb{S})$.

For each $d \in \mathrm{supp}(f^c)$ we obtain $f(d) < \xi$ by $\tilde{\theta}_{-e}(\zeta) \leq \zeta$. IH yields $\mathbb{S} \in Mh_c^a(f^c)$.

We have to show $\mathbb{S} \in M_2(A \cap B)$ for $A = Mh_0^a(g_c) \cap \mathbb{S}$ and $B = Mh_c^a(f^c) \cap \mathbb{S}$. Let C be a club subset of $\mathbb{S}$.

We have $\mathbb{S} \in Mh_0^a(g_c) \cap Mh_c^a(f^c)$, and $\{a, c\} \cup SC_{\mathbb{K}}(g_c, f^c) \subset \mathcal{H}_a(\mathbb{S})$. Pick a $b < \mathbb{S}$ so that $\{a, c\} \cup SC_{\mathbb{K}}(g_c, f^c) \subset \mathcal{H}_a(b)$, and a bijection $F : \mathbb{S} \to \mathcal{H}_a(\mathbb{S})$. Each $\alpha \in \mathcal{H}_a(\mathbb{S}) \cap \Gamma_{\mathbb{K}+1}$ is identified with its code, denoted by $F^{-1}(\alpha)$. Let P be the class $P = \{(\pi, d, \alpha) \in \mathbb{S}^3 : \pi \in Mh_{F(d)}^a(F(\alpha))\}$, where $F(d) < \mathbb{K}$ and $F(\alpha) < \varphi_{\mathbb{K}}(0)$ with $\{F(d), F(\alpha)\} \subset \mathcal{H}_a(\pi)$. For fixed a, the set $\{(d, \eta) \in \mathbb{K} \times \varphi_{\mathbb{K}}(0) : \mathbb{S} \in Mh_d^a(\eta)\}$ is defined from the class P by recursion on ordinals $d < \mathbb{K}$. Let φ be a formula such that $V_{\mathbb{S}+\mathbb{K}} \models \varphi[P, C, \mathbb{S}, b]$ iff $\mathbb{S} \in Mh_0^a(g_c) \cap Mh_c^a(f^c)$ and C is a club subset of $\mathbb{S}$. Since $\mathbb{S}$ is shrewd, pick $b < \mathbb{S}_0 < \mathbb{K}_0 < \mathbb{S}$ such that $V_{\mathbb{S}_0+\mathbb{K}_0} \models \varphi[P \cap \mathbb{S}_0, C \cap \mathbb{S}_0, \mathbb{S}_0, b]$. We obtain $\mathbb{S}_0 \in A \cap B \cap C$. Therefore $\mathbb{S} \in Mh_c^a(\xi)$ is shown. $\mathbb{S} \in M_2(Mh_c^a(\xi))$ is seen from the shrewdness of $\mathbb{S}$. □

Corollary 6.14 *$\tilde{T}$ proves that $\forall a < \varepsilon_{\mathbb{K}+1} \forall c < \mathbb{K}[\{a, c, \xi\} \subset \mathcal{H}_a(\mathbb{S}) \to \psi^f_{\mathbb{S}}(a) < \mathbb{S})]$ for every $\xi < \varphi_{\mathbb{K}}(0)$ and finite functions f such that $\mathrm{supp}(f) = \{c\}$, $c < \mathbb{K}$ and $f(c) = \xi$.*

Lemma 6.15 *Assume $\mathbb{S} \geq \pi \in Mh^a_d(\xi) \cap Mh^a_c(\xi_0)$, $\xi_0 \neq 0$, and $d < c$. Moreover let $\xi_1 \in \mathcal{H}_a(\pi)$ for $\xi_1 \leq \tilde{\theta}_{c-d}(\xi_0)$. Then $\pi \in Mh^a_d(\xi \dot{+} \xi_1) \cap M^a_d(Mh^a_d(\xi \dot{+} \xi_1))$.*

Proof. As in Lemma 6.7. □

Definition 6.16 For finite functions f and g,

$$Mh^a_0(g) \prec Mh^a_0(f) :\Leftrightarrow \forall \pi \in Mh^a_0(f) \left(SC_{\mathbb{K}}(g) \subset \mathcal{H}_a(\pi) \Rightarrow \pi \in M_2(Mh^a_0(g))\right).$$

Corollary 6.17 *Let f, g be finite functions and $c \in \mathrm{supp}(f)$. Assume that there exists an ordinal $d < c$ such that $(d, c) \cap \mathrm{supp}(f) = (d, c) \cap \mathrm{supp}(g) = \emptyset$, $g_d = f_d$, $g(d) < f(d) \dot{+} \tilde{\theta}_{c-d}(f(c)) \cdot \omega$, and $g <^c f(c)$. Then $Mh^a_0(g) \prec Mh^a_0(f)$ holds. In particular if $\pi \in Mh^a_0(f)$ and $SC_{\mathbb{K}}(g) \subset \mathcal{H}_a(\pi)$, then $\psi^g_\pi(a) < \pi$.*

Proposition 6.18 *Let $f, g : \mathbb{K} \to \varphi_{\mathbb{K}}(0)$. If $f <^0_{lx} g$, then $Mh^a_0(f) \prec Mh^a_0(g)$.*

Proof. This is seen from Corollary 6.17. □

6.4 A Mostowski collapsing

$OT(\Pi^1_1)$ denotes a computable notation system of ordinals with a constant $\mathbb{S}$ for a stable ordinal, collapsing functions $\psi^g_\sigma(a)$ for finite functions g, where $\mathrm{supp}(g) = \{d\}$ for a $d < \mathbb{K} = \mathbb{S}^+$ and $g(d) < \varepsilon_{\mathbb{K}+1}$ if $\sigma = \mathbb{S}$. Let $m(\alpha) = g$ for $\alpha = \psi^g_\sigma(a)$ and $\sigma < \mathbb{S}$. For $g \neq \emptyset$, $\alpha = \psi^g_\sigma(a) \in OT(\Pi^1_1)$ only when g is obtained from $f = m(\sigma)$ as follows, cf. Corollary 6.17. There are c and d such that $d < c \in \mathrm{supp}(f)$, and $(d, c) \cap \mathrm{supp}(f) = \emptyset$. Then $g_d = f_d$, $(d, c) \cap \mathrm{supp}(g) = \emptyset$ $g(d) < f(d) + \tilde{\theta}_{c-d}(f(c)) \cdot \omega$, and $g <^c f(c)$.

In what follows, by ordinals we mean ordinal terms in $OT(\Pi^1_1)$. $\Psi_{\mathbb{S}}$ denotes the set of ordinal terms $\psi^f_\sigma(a)$ for some a, f and $\sigma \in \Psi_{\mathbb{S}} \cup \{\mathbb{S}\}$. Note that in $OT(\Pi^1_1)$, $\psi^f_\sigma(a) \geq \mathbb{S}$ only if $\sigma = \mathbb{K} = \mathbb{S}^+$ and $f = \emptyset$.

We define a Mostowski collapsing $\alpha \mapsto \alpha[\rho/\mathbb{S}]$, which is needed to replace inference rules for stability by ones of reflections. The domain of the collapsing $\alpha \mapsto \alpha[\rho/\mathbb{S}]$ is a subset M_ρ of $E^{\mathbb{S}}_\rho$. For a reason of the restriction, see the beginning of subsection 6.5.

Definition 6.19 For ordinal terms $\psi^f_\sigma(a) \in \Psi_{\mathbb{S}} \subset OT(\Pi^1_1)$, define $m(\psi^f_\sigma(a)) := f$ and $s(\psi^f_\sigma(a)) := \max(\mathrm{supp}(f))$. Also $\mathtt{p}_0(\psi^f_\sigma(a)) = \mathtt{p}_0(\sigma)$ if $\sigma < \mathbb{S}$, and $\mathtt{p}_0(\psi^f_{\mathbb{S}}(a)) = a$.

Definition 6.20 *$M_\rho := \mathcal{H}_b(\rho)$ for $b = \mathtt{p}_0(\rho)$ and $\rho \in \Psi_{\mathbb{S}}$.*

$\alpha = \psi^g_\sigma(a) \in OT(\Pi^1_1)$ only when $\{\sigma, a\} \subset \mathcal{H}_a(\alpha)$ and $SC_{\mathbb{K}}(g) \subset M_\alpha$.

$OT(\Pi^1_1)$ is defined to be closed under $\alpha \mapsto \alpha[\rho/\mathbb{S}]$ for $\alpha \in M_\rho$. Specifically if $\{\alpha, \rho\} \subset OT(\Pi^1_1)$ with $\alpha \in M_\rho$ and $\rho \in \Psi_{\mathbb{S}}$, then $\alpha[\rho/\mathbb{S}] \in OT(\Pi^1_1)$.

Proposition 6.21 *Let $\rho \in \Psi_{\mathbb{S}}$.*

1. *$\mathcal{H}_\gamma(M_\rho) \subset M_\rho$ if $\gamma \leq \mathtt{p}_0(\rho)$.*
2. *$M_\rho \cap \mathbb{S} = \rho$ and $\rho \notin M_\rho$.*
3. *If $\sigma < \rho$ and $\mathtt{p}_0(\sigma) \leq \mathtt{p}_0(\rho)$, then $M_\sigma \subset M_\rho$.*

Definition 6.22 Let $\alpha \in M_\rho$ with $\rho \in \Psi_{\mathbb{S}}$. We define an ordinal $\alpha[\rho/\mathbb{S}]$ recursively as follows. $\alpha[\rho/\mathbb{S}] := \alpha$ when $\alpha < \mathbb{S}$. In what follows assume $\alpha \geq \mathbb{S}$.

$\mathbb{S}[\rho/\mathbb{S}] := \rho$. $\mathbb{K}[\rho/\mathbb{S}] \equiv (\mathbb{S}^+)[\rho/\mathbb{S}] := \rho^+$. $(\psi_{\mathbb{K}}(a))\,[\rho/\mathbb{S}] = (\psi_{\mathbb{S}^+}(a))\,[\rho/\mathbb{S}] = \psi_{\rho^+}(a[\rho/\mathbb{S}])$. The map commutes with $+$ and φ.

Lemma 6.23 *For $\rho \in \Psi_{\mathbb{S}}$, $\{\alpha[\rho/\mathbb{S}] : \alpha \in M_\rho\}$ is a transitive collapse of M_ρ in the sense that $\beta < \alpha \Leftrightarrow \beta[\rho/\mathbb{S}] < \alpha[\rho/\mathbb{S}]$, $\beta \in \mathcal{H}_\alpha(\gamma) \Leftrightarrow \beta[\rho/\mathbb{S}] \in \mathcal{H}_{\alpha[\rho/\mathbb{S}]}(\gamma[\rho/\mathbb{S}]))$ for $\gamma > \mathbb{S}$, and $OT(\Pi^1_1) \cap \alpha[\rho/\mathbb{S}] = \{\beta[\rho/\mathbb{S}] : \beta \in M_\rho \cap \alpha\}$ for $\alpha, \beta, \gamma \in M_\rho$.*

Let $\rho \leq \mathbb{S}$, and ι an RS-term or an RS-formula such that $\mathsf{k}(\iota) \subset M_\rho$, where $M_{\mathbb{S}} = \mathbb{K}$. Then $\iota^{[\rho/\mathbb{S}]}$ denotes the result of replacing each unbounded quantifier Qx by $Qx \in L_{\mathbb{K}[\rho/\mathbb{S}]}$, and each ordinal term $\alpha \in \mathsf{k}(\iota)$ by $\alpha[\rho/\mathbb{S}]$ for the Mostowski collapse in Definition 6.22.

Proposition 6.24 *Let $\rho \in \Psi_{\mathbb{S}} \cup \{\mathbb{S}\}$.*

1. *Let v be an RS-term with $\mathsf{k}(v) \subset M_\rho$, and $\alpha = |v|$. Then $v^{[\rho/\mathbb{S}]}$ is an RS-term of level $\alpha[\rho/\mathbb{S}]$, $\left|v^{[\rho/\mathbb{S}]}\right| = \alpha[\rho/\mathbb{S}]$ and $\mathsf{k}(v^{[\rho/\mathbb{S}]}) = (\mathsf{k}(v))^{[\rho/\mathbb{S}]}$.*
2. *Let $\alpha \leq \mathbb{K}$ be such that $\alpha \in M_\rho$. Then $(Tm(\alpha))^{[\rho/\mathbb{S}]} := \{v^{[\rho/\mathbb{S}]} : v \in Tm(\alpha), \mathsf{k}(v) \subset M_\rho\} = Tm(\alpha[\rho/\mathbb{S}])$.*
3. *Assume $\mathcal{H}_\gamma(\rho) \cap \mathbb{S} \subset \rho$. For an RS-formula A with $\mathsf{k}(A) \subset \mathcal{H}_\gamma(\rho)$, $A^{[\rho/\mathbb{S}]}$ is an RS-formula such that $\mathsf{k}(A^{[\rho/\mathbb{S}]}) \subset \{\alpha[\rho/\mathbb{S}] : \alpha \in \mathsf{k}(A)\} \cup \{\mathbb{K}[\rho/\mathbb{S}]\}$.*

For each sentence A, either a disjunction is assigned as $A \simeq \bigvee(A_\iota)_{\iota\in J}$, or a conjunction is assigned as $A \simeq \bigwedge(A_\iota)_{\iota\in J}$. In the former case A is said to be a $\bigvee$*-formula*, and in the latter A is a $\bigwedge$*-formula*.

Definition 6.25 Let $[\rho]Tm(\alpha) := \{u \in Tm(\alpha) : \mathsf{k}(u) \subset M_\rho\}$.

Proposition 6.26 *Let $\rho \in \Psi_{\mathbb{S}} \cup \{\mathbb{S}\}$. For RS-formulas A, let $A \simeq \bigvee(A_\iota)_{\iota\in J}$ and assume $\mathsf{k}(A) \subset M_\rho$. Then $A^{[\rho/\mathbb{S}]} \simeq \bigvee\left((A_\iota)^{[\rho/\mathbb{S}]}\right)_{\iota\in[\rho]J}$. The case $A \simeq \bigwedge(A_\iota)_{\iota\in J}$ is similar.*

6.5 Operator controlled derivations for Π^1_1-reflection

We define a derivability relation $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_c \Gamma; \Pi^{[\cdot]}$ where $\mathtt{Q}_\Pi$ is a finite set of ordinals in $\Psi_{\mathbb{S}}$, c is a bound of ranks of the inference rules (stbl) and of ranks of cut formulas. The relation depends on an ordinal γ_0, and should be written as $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_{c,\gamma_0} \Gamma; \Pi^{[\cdot]}$. However the ordinal γ_0 will be fixed. So let us omit it.

The rôle of the calculus $\vdash^{*a}_c$ is twofold: first finite proof figures are embedded in the calculus, and second the cut rank c in $\vdash^{*a}_c$ is lowered to $\mathbb{K} = \mathbb{S}^+$. In the next subsection 6.6 the relation $\vdash^{*a}_c$ is embedded in another derivability relation $\vdash^a_{c,e,b_1} A^{(\rho)}$ with caps ρ. In the latter calculus, cut ranks c as well as the ranks of formulas to be reflected are lowered to $\mathbb{S}$, and the inferences for reflections are removed. For this we need to distinguish formulas with smaller ranks$< \mathbb{S}$ from higher ones.

As in Lemma 4.13, in eliminating of inferences for reflections,

$$\frac{\{\mathcal{H}_\gamma[\Theta] \vdash \Gamma^{(\rho)}, \neg\delta^{(\rho)}\}_{\delta\in\Delta} \quad \{\mathcal{H}_\gamma[\Theta\cup\{\sigma\}] \vdash \Gamma^{(\rho)}, \Delta^{(\sigma)}\}_\sigma}{\mathcal{H}_\gamma[\Theta] \vdash^a \Gamma^{(\rho)}}\ (\mathrm{rfl}_\rho)$$

is rewritten to, cf. Recapping 6.47

$$\dfrac{\{\mathcal{H}_\gamma[\Theta] \vdash \neg\theta^{(\kappa)}, \Gamma^{(\kappa)}\}_{\theta\in\Gamma} \quad \dfrac{\begin{matrix}\vdots\ \rho\rightsquigarrow\sigma \\ \{\mathcal{H}_\gamma[\Theta] \vdash \Gamma^{(\sigma)}, \neg\delta^{(\sigma)}\}_{\delta\in\Delta}\end{matrix} \quad \begin{matrix}\vdots\ \rho\rightsquigarrow\kappa \\ \mathcal{H}_\gamma[\Theta\cup\{\sigma\}] \vdash \Gamma^{(\kappa)}, \Delta^{(\sigma)}\end{matrix}}{\{\mathcal{H}_\gamma[\Theta\cup\{\sigma\}] \vdash \Gamma^{(\kappa)}, \Gamma^{(\sigma)}\}_\sigma}\ (cut)}{\mathcal{H}_\gamma[\Theta] \vdash \Gamma^{(\kappa)}}\ (\mathrm{rfl}_\kappa)$$

where $\sigma < \kappa < \rho$. In the rewriting, the inference (rfl_ρ) is replaced by (rfl_κ) for a smaller $\kappa < \rho$. This means that (rfl_ρ) is replaced by (rfl_σ) in the part $\rho \rightsquigarrow \sigma$. κ reflects Γ to some σ, and σ has to reflect Δ, where $\mathrm{rk}(\Delta) > \mathrm{rk}(\Gamma)$ is possible. Therefore the termination of the whole process of removing is seen to be by induction on reflecting ordinals ρ, cf. Lemma 6.48.

The Mahlo degree $g = m(\kappa)$ in $\kappa = \psi^g_\rho(\alpha)$ is obtained by (an iteration of) a stepping-down $(f, d, c) \mapsto g$, where $f = m(\rho)$, $d < c \in \mathrm{supp}(f)$, $(d, c) \cap \mathrm{supp}(f) = \emptyset$, $g_d = f_d$, $(d, c) \cap \mathrm{supp}(g) = \emptyset$, $g(d) < f(d) + \tilde{\theta}_{c-d}(f(c)) \cdot \omega$, and $g <^c f(c)$. g depends on a, ρ and $\mathrm{rk}(\Gamma^{(\rho)}) := \mathrm{rk}(\Gamma)$. In showing

$$SC_{\mathbb{K}}(g) \subset \mathcal{H}_\alpha(\kappa)$$

ρ and $\mathrm{rk}(\Gamma^{(\rho)})$ are harmless since these relates to the given ordinal ρ, while the ordinal a causes trouble, since all of the reflecting ordinals $\rho, \ldots$ share the ordinal depth a of the derivation. We need $a \in \mathcal{H}_{\alpha_0}(\rho)$ if $\rho = \psi^f_\sigma(\alpha_0)$, and $a \in \mathcal{H}_\beta(\tau)$ if $\tau = \psi^h_\lambda(\beta)$, and so forth. This leads us to the set $M_\rho = \mathcal{H}_b(\rho)$ for $b = \mathtt{p}_0(\rho)$, where $\rho = \psi^f_{\ddots_{\psi_{\mathbb{S}}(b)}}(\alpha_0)$, and the condition (35) that a as well as ordinals occurring in the derivation should be in M_ρ for every reflecting ordinal ρ occurring in derivations. Note that $M_\rho = \mathcal{H}_b(\rho) \subset \mathcal{H}_{\alpha_0}(\rho)$ by $b \leq \alpha_0$, but $E^{\mathbb{S}}_\rho \not\subset \mathcal{H}_{\alpha_0}(\rho)$. This is the reason why we restrict the domain of the Mostowski collapsing $\alpha \mapsto \alpha[\rho/\mathbb{S}]$ to $\alpha \in M_\rho \subsetneq E^{\mathbb{S}}_\rho$.

$\mathtt{Q}_\Pi$ in $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_c \Gamma; \Pi^{[\cdot]}$, is the set of ordinals σ which is introduced in a right upper sequent $(\mathcal{H}_\gamma, \Theta \cup \{\sigma\}; \mathtt{Q}_\Pi \cup \{\sigma\}) \vdash^{*a_0}_c \Gamma; \Pi^{[\cdot]}, \neg B(u)^{[\sigma/\mathbb{S}]}$ of an inference (stbl) for stability occurring below $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_c \Gamma; \Pi^{[\cdot]}$, while the set $\Pi^{[\cdot]} = \bigcup\{\Pi^{[\sigma/\mathbb{S}]}_\sigma : \sigma \in \mathtt{Q}_\Pi\}$ is the collection of formulas $\neg B(u)^{[\sigma/\mathbb{S}]}$.

$$\frac{(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a_0}_c \Gamma, B(u); \Pi^{[\cdot]} \quad \{(\mathcal{H}_\gamma, \Theta \cup \{\sigma\}; \mathtt{Q}_\Pi \cup \{\sigma\}) \vdash^{*a_0}_c \Gamma; \Pi^{[\cdot]}, \neg B(u)^{[\sigma/\mathbb{S}]}\}_\sigma}{(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_c \Gamma; \Pi^{[\cdot]}} \text{(stbl)}$$

These motivates the following Definitions 6.27, 6.28 and 6.40.

Definition 6.27 Let $\mathtt{Q} \subset \Psi_\mathbb{S}$ be a finite set of ordinals, and $A \simeq \bigvee(A_\iota)_{\iota \in J}$. Define $M_\mathtt{Q} := \bigcap_{\sigma \in \mathtt{Q}} M_\sigma$,

$$[\mathtt{Q}]_A J := [\mathtt{Q}]_{\neg A} J := \{\iota \in J : \mathrm{rk}(A_\iota) \geq \mathbb{S} \Rightarrow \mathsf{k}(\iota) \subset M_\mathtt{Q}\}$$

$$\mathsf{k}^\mathbb{S}(\Gamma) := \bigcup\{\mathsf{k}(A) : A \in \Gamma, \mathrm{rk}(A) \geq \mathbb{S}\}$$

Definition 6.28 Let Θ be a finite set of ordinals, $\gamma \leq \gamma_0$ and a, c ordinals[2], and $\mathtt{Q}_\Pi \subset \Psi_\mathbb{S}$ a finite set of ordinals such that $\mathsf{p}_0(\sigma) \geq \gamma_0$ for each $\sigma \in \mathtt{Q}_\Pi$. Let $\Pi = \bigcup\{\Pi_\sigma : \sigma \in \mathtt{Q}_\Pi\} \subset \Delta_0(\mathbb{K})$ be a set of formulas such that $\mathsf{k}(\Pi_\sigma) \subset M_\sigma$ for each $\sigma \in \mathtt{Q}_\Pi$, $\Pi^{[\cdot]} = \bigcup\{\Pi^{[\sigma/\mathbb{S}]}_\sigma : \sigma \in \mathtt{Q}_\Pi\}$, $\Theta^{(\sigma)} = \Theta \cap M_\sigma$ and $\Theta_{\mathtt{Q}_\Pi} = \Theta \cap M_{\mathtt{Q}_\Pi}$.

$(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_c \Gamma; \Pi^{[\cdot]}$ holds for a set Γ of formulas if

$$\mathsf{k}(\Gamma) \subset \mathcal{H}_\gamma[\Theta] \,\&\, \forall \sigma \in \mathtt{Q}_\Pi \left(\mathsf{k}(\Pi_\sigma) \subset \mathcal{H}_\gamma[\Theta^{(\sigma)}]\right) \tag{34}$$

$$\{\gamma, a, c\} \cup \mathsf{k}^\mathbb{S}(\Gamma) \cup \mathsf{k}^\mathbb{S}(\Pi) \subset \mathcal{H}_\gamma[\Theta_{\mathtt{Q}_\Pi}] \tag{35}$$

and one of the following cases holds:

($\bigvee$) [3] There exist $A \simeq \bigvee(A_\iota)_{\iota \in J}$, an ordinal $a(\iota) < a$ and an $\iota \in J$ such that $A \in \Gamma$, $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a(\iota)}_c \Gamma, A_\iota; \Pi^{[\cdot]}$.

($\bigvee$)$^{[\cdot]}$ There exist $A \equiv B^{[\sigma/\mathbb{S}]} \in \Pi^{[\cdot]}$, $B \simeq \bigvee(B_\iota)_{\iota \in J}$, an ordinal $a(\iota) < a$ and an $\iota \in [\sigma]J$ such that $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a(\iota)}_c \Gamma; \Pi^{[\cdot]}, A_\iota$ with $A_\iota \equiv B^{[\sigma/\mathbb{S}]}_\iota$.

($\bigwedge$) There exist $A \simeq \bigwedge(A_\iota)_{\iota \in J}$, ordinals $a(\iota) < a$ such that $A \in \Gamma$ and $(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\iota); \mathtt{Q}_\Pi) \vdash^{*a(\iota)}_c \Gamma, A_\iota; \Pi^{[\cdot]}$ for each $\iota \in [\mathtt{Q}_\Pi]_A J$.

($\bigwedge$)$^{[\cdot]}$ There exist $A \equiv B^{[\sigma/\mathbb{S}]} \in \Pi^{[\cdot]}$, $B \simeq \bigwedge(B_\iota)_{\iota \in J}$, ordinals $a(\iota) < a$ such that $(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\iota); \mathtt{Q}_\Pi) \vdash^{*a(\iota)}_c \Gamma; A_\iota, \Pi^{[\cdot]}$ for each $\iota \in [\mathtt{Q}_\Pi]_B J \cap [\sigma]J$.

(*cut*) There exist an ordinal $a_0 < a$ and a formula C such that $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a_0}_c \Gamma, \neg C; \Pi^{[\cdot]}$ and $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a_0}_c C, \Gamma; \Pi^{[\cdot]}$ with $\mathrm{rk}(C) < c$.

[2] In this subsection 6.5 we can set $\gamma = \mathbb{S}$.

[3] The condition (4), $|\iota| < a$ is absent in the inference ($\bigvee$), cf. **Case 3** in Lemma 6.44.

(Σ-rfl) There exist ordinals $a_\ell, a_r < a$ and a formula $C \in \Sigma(\pi)$ for a $\pi \in \{\Omega, \mathbb{K} = \mathbb{S}^+\}$ such that $c \geq \pi$, $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a_\ell}_{c} \Gamma, C; \Pi^{[\cdot]}$ and $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a_r}_{c} \neg\exists x < \pi\, C^{(x,\pi)}, \Gamma; \Pi^{[\cdot]}$.

(stbl) There exist an ordinal $a_0 < a$, a $\bigwedge$-formula $B(0) \in \Delta_0(\mathbb{S})$, and a $u \in Tm(\mathbb{K})$ for which the following hold: $\mathbb{S} \leq \mathrm{rk}(B(u)) < c$, $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a_0}_{c} \Gamma, B(u); \Pi^{[\cdot]}$, and $(\mathcal{H}_\gamma, \Theta \cup \{\sigma\}; \mathtt{Q}_\Pi \cup \{\sigma\}) \vdash^{*a_0}_{c} \Gamma; \Pi^{[\cdot]}, \neg B(u)^{[\sigma/\mathbb{S}]}$ holds for every ordinal $\sigma \in \Psi_{\mathbb{S}}$ such that $\Theta \subset M_\sigma$.

$$\frac{(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a_0}_{c} \Gamma, B(u); \Pi^{[\cdot]} \quad \{(\mathcal{H}_\gamma, \Theta \cup \{\sigma\}; \mathtt{Q}_\Pi \cup \{\sigma\}) \vdash^{*a_0}_{c} \Gamma; \Pi^{[\cdot]}, \neg B(u)^{[\sigma/\mathbb{S}]}\}_{\Theta \subset M_\sigma}}{(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_{c} \Gamma; \Pi^{[\cdot]}} \text{(stbl)}$$

Note that $(\Theta \cup \{\sigma\})_{\mathtt{Q}_\Pi \cup \{\sigma\}} = \Theta_{\mathtt{Q}_\Pi}$ if $\Theta_{\mathtt{Q}_\Pi} \subset M_\sigma$.

Proposition 6.29 (Tautology) *Let* $\gamma \in \mathcal{H}_\gamma[\mathsf{k}(A)]$ *and* $d = \mathrm{rk}(A)$.

1. $(\mathcal{H}_\gamma, \mathsf{k}(A); \emptyset) \vdash^{*2d}_{0} \neg A, A; \emptyset$.
2. $(\mathcal{H}_\gamma, \mathsf{k}(A) \cup \{\sigma\}; \{\sigma\}) \vdash^{*2d}_{0} \neg A^{[\sigma/\mathbb{S}]}; A^{[\sigma/\mathbb{S}]}$ *if* $\mathsf{k}(A) \subset M_\sigma$ *and* $\gamma \geq \mathbb{S}$.

Proof. Both are seen by induction on d. Consider Proposition 6.29.2.

We have $(\mathsf{k}(A) \cup \{\sigma\}) \cap M_\sigma = \mathsf{k}(A)$ for (34) and (35), and $\mathsf{k}(A^{[\sigma/\mathbb{S}]}) \subset \mathcal{H}_{\mathbb{S}}((\mathsf{k}(A) \cap \mathbb{S}) \cup \{\sigma\})$ for (34). Note that $\sigma \not\in \mathcal{H}_\gamma[\mathsf{k}(A)]$ since $\sigma \not\in \mathsf{k}(A) \subset M_\sigma$ and $\gamma \leq \gamma_0 \leq \mathsf{p}_0(\sigma)$, and $\mathrm{rk}(A^{[\sigma/\mathbb{S}]}) \not\in \mathcal{H}_\gamma[(\mathsf{k}(A) \cup \{\sigma\}) \cap M_\sigma]$.

Let $A \simeq \bigvee(A_\iota)_{\iota \in J}$. Then $A^{[\sigma/\mathbb{S}]} \simeq \bigvee(A_\iota^{[\sigma/\mathbb{S}]})_{\iota \in [\sigma]J}$ by Proposition 6.26 and $\mathsf{k}(\iota^{[\sigma/\mathbb{S}]}) \subset \mathcal{H}_{\mathbb{S}}[(\mathsf{k}(\iota) \cap \mathbb{S}) \cup \{\sigma\}]$. Let $I = \{\iota^{[\sigma/\mathbb{S}]} : \iota \in [\sigma]J\}$. Then $A^{[\sigma/\mathbb{S}]} \simeq \bigvee(B_\nu)_{\nu \in I}$ with $B_\nu \equiv A_\iota^{[\sigma/\mathbb{S}]}$ for $\nu = \iota^{[\sigma/\mathbb{S}]}$, and $[\{\sigma\}]_{A^{[\sigma/\mathbb{S}]}} I = I$ by $\mathrm{rk}(A^{[\sigma/\mathbb{S}]}) < \mathbb{S}$. For $d_\iota = \mathrm{rk}(A_\iota) \in \mathcal{H}_\gamma[\mathsf{k}(A, \iota)]$ with $\iota \in [\sigma]J = [\{\sigma\}]_{A^{(\sigma)}} J$ we obtain

$$\cfrac{\cfrac{(\mathcal{H}_\gamma, \mathsf{k}(A, \iota) \cup \{\sigma\}; \{\sigma\}) \vdash^{*2d_\iota}_{0} \neg A_\iota^{[\sigma/\mathbb{S}]}; A_\iota^{[\sigma/\mathbb{S}]}}{(\mathcal{H}_\gamma, \mathsf{k}(A, \iota) \cup \{\sigma\}; \{\sigma\}) \vdash^{*2d_\iota+1}_{0} \neg A_\iota^{[\sigma/\mathbb{S}]}; A^{[\sigma/\mathbb{S}]}} (\bigvee)^{[\cdot]}}{(\mathcal{H}_\gamma, \mathsf{k}(A) \cup \{\sigma\}; \{\sigma\}) \vdash^{*2d}_{0} \neg A^{[\sigma/\mathbb{S}]}; A^{[\sigma/\mathbb{S}]}} (\bigwedge)$$

and

$$\cfrac{\cfrac{(\mathcal{H}_\gamma, \mathsf{k}(A) \cup \mathsf{k}(\iota) \cup \{\sigma\}; \{\sigma\}) \vdash^{*2d_\iota}_{0} A_\iota^{[\sigma/\mathbb{S}]}; \neg A_\iota^{[\sigma/\mathbb{S}]}}{(\mathcal{H}_\gamma, \mathsf{k}(A) \cup \mathsf{k}(\iota) \cup \{\sigma\}; \{\sigma\}) \vdash^{*2d_\iota+1}_{0} A^{[\sigma/\mathbb{S}]}; \neg A_\iota^{[\sigma/\mathbb{S}]}} (\bigvee)}{(\mathcal{H}_\gamma, \mathsf{k}(A) \cup \{\sigma\}; \{\sigma\}) \vdash^{*2d}_{0} A^{[\sigma/\mathbb{S}]}; \neg A^{[\sigma/\mathbb{S}]}} (\bigwedge)^{[\cdot]}$$

□

Lemma 6.30 (Embedding of Axioms) *For each axiom* A *in* S_1*, there is an* $m < \omega$ *such that* $(\mathcal{H}_{\mathbb{S}}, \emptyset; \emptyset) \vdash^{*\mathbb{K}\cdot 2}_{\mathbb{K}+m} A;$ *holds for* $\mathbb{K} = \mathbb{S}^+$.

Proof. We show that the axiom $\exists x\, B(x, v) \land v \in L_{\mathbb{S}} \to \exists x \in L_{\mathbb{S}}\, B(x, v)$ $(B \in \Delta_0)$ follows by an inference (stbl). In the proof let us omit the operator $\mathcal{H}_{\mathbb{S}}$. Let $B(0) \in \Delta_0(\mathbb{S})$ be a $\bigwedge$-formula and $u \in Tm(\mathbb{K})$. We may assume that $\mathbb{K} > d = \mathrm{rk}(B(u)) \geq \mathbb{S}$. Let $\mathsf{k}_0 = \mathsf{k}(B(0))$ and $\mathsf{k}_u = \mathsf{k}(u)$. Let $\mathsf{k}_0 \cup \mathsf{k}_u \subset M_\sigma$.

Then for $\exists x \in L_{\mathbb{S}} B(x) \simeq \bigvee(B(v))_{v\in J}$, we obtain $u^{[\sigma/\mathbb{S}]} \in J = Tm(\mathbb{S})$ by $\mathrm{rk}(\exists x \in L_{\mathbb{S}} B(x)) = \mathbb{S}$. We have $B(u^{[\sigma/\mathbb{S}]}) \equiv B(u)^{[\sigma/\mathbb{S}]}$, $\mathsf{k}_u^{[\sigma/\mathbb{S}]} = \mathsf{k}(u^{[\sigma/\mathbb{S}]}) \subset \mathcal{H}_{\mathbb{S}}[\mathsf{k}(u) \cup \{\sigma\}]$, $(\mathsf{k}_0 \cup \mathsf{k}_u)_\emptyset = \mathsf{k}_0 \cup \mathsf{k}_u$ and $(\mathsf{k}_0 \cup \mathsf{k}_u \cup \{\sigma\}) \cap M_\sigma = \mathsf{k}_0 \cup \mathsf{k}_u$.

$$\cfrac{\mathsf{k}_0 \cup \mathsf{k}_u; \vdash_0^{*2d} \neg B(u), B(u); \quad \cfrac{\mathsf{k}_0 \cup \mathsf{k}_u \cup \{\sigma\}; \{\sigma\} \vdash_0^{*2d} B(u^{[\sigma/\mathbb{S}]}); \neg B(u)^{[\sigma/\mathbb{S}]}}{\{\mathsf{k}_0 \cup \mathsf{k}_u \cup \{\sigma\}; \{\sigma\} \vdash_0^{*2d+1} \exists x \in L_{\mathbb{S}} B(x); \neg B(u)^{[\sigma/\mathbb{S}]}\}_{\mathsf{k}_0 \cup \mathsf{k}_u \subset M_\sigma}}\,(\bigvee)}{\cfrac{\mathsf{k}_0 \cup \mathsf{k}_u; \vdash_{\mathbb{K}}^{*\mathbb{K}} \neg B(u), \exists x \in L_{\mathbb{S}} B(x);}{\mathsf{k}_0; \vdash_{\mathbb{K}}^{*\mathbb{K}+1} \neg \exists x\, B(x), \exists x \in L_{\mathbb{S}} B(x);}\,(\bigwedge)}\,(\mathrm{stbl})$$

□

Proposition 6.31 (Inversion) *Let* $A \simeq \bigwedge(A_\iota)_{\iota\in J}$ *with* $A \in \Gamma$, $\iota \in [\mathtt{Q}_\Pi]_A J$ *and* $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash_c^{*a} \Gamma; \Pi^{[\cdot]}$. *Then* $(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\iota); \mathtt{Q}_\Pi) \vdash_c^{*a} \Gamma, A_\iota; \Pi^{[\cdot]}$.

Proposition 6.32 *Let* $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash_c^{*a} \Gamma; \Pi^{[\cdot]}$. *Assume* $\Theta \subset M_\sigma$. *Then* $(\mathcal{H}_\gamma, \Theta \cup \{\sigma\}; \mathtt{Q}_\Pi \cup \{\sigma\}) \vdash_c^{*a} \Gamma; \Pi^{[\cdot]}$.

Proof. By induction on a. We obtain $(\Theta \cup \{\sigma\})_{\mathtt{Q}_\Pi \cup \{\sigma\}} = \Theta_{\mathtt{Q}_\Pi}$ by the assumption. In an inference (stbl), the right upper sequents are restricted to τ such that $\sigma \in M_\tau$. Also we need to prune some branches at $(\bigwedge)$ and $(\bigwedge)^{[\cdot]}$ since $[(\mathtt{Q}_\Pi \cup \{\sigma\})]_A J \subset [\mathtt{Q}_\Pi]_A J$. □

Proposition 6.33 (Reduction) *Let* $C \simeq \bigvee(C_\iota)_{\iota\in J}$ *and* $\mathbb{K} = \mathbb{S}^+ \leq \mathrm{rk}(C) \leq c$. *Assume* $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash_c^{*a} \Gamma, \neg C; \Pi^{[\cdot]}$ *and* $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash_c^{*b} C, \Gamma; \Pi^{[\cdot]}$.

Then $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash_c^{*a+b} \Gamma; \Pi^{[\cdot]}$.

Proof. By induction on b using Inversion 6.31 and Proposition 6.32.

Note that if $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash_c^{*b(\iota)} C_\iota, \Gamma; \Pi^{[\cdot]}$ for an $\iota \in J$ such that $\mathrm{rk}(C_\iota) \geq \mathbb{K}$, we obtain $\mathsf{k}(C_\iota) \subset \mathcal{H}_\gamma[\Theta_{\mathtt{Q}_\Pi(\mathbb{S})}] \subset M_{\mathtt{Q}_\Pi(\mathbb{S})}$ by (35) and Proposition 6.21 with $\gamma \leq \gamma_0 \leq \mathsf{p}_0(\sigma)$ for $\sigma \in \mathtt{Q}_\Pi$. Hence $\iota \in [\mathtt{Q}_\Pi]_C J$ if $\mathsf{k}(\iota) \subset \mathsf{k}(C_\iota)$. □

Proposition 6.34 (Cut-elimination) *Assume* $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash_{c+1}^{*a} \Gamma; \Pi^{[\cdot]}$ *with* $c \geq \mathbb{S}^+ = \mathbb{K}$. *Then* $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash_c^{*\omega^a} \Gamma; \Pi^{[\cdot]}$.

Proof. This is seen by induction on a using Reduction 6.33. □

Lemma 6.35 (Collapsing) *Let* $\Gamma \subset \Sigma$ *be a set of formulas, and* $\Pi \subset \Delta_0(\mathbb{K})$. *Suppose* $\Theta \subset \mathcal{H}_\gamma(\psi_{\mathbb{K}}(\gamma))$ *and* $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash_{\mathbb{K}}^{*a} \Gamma; \Pi^{[\cdot]}$. *Let* $\beta = \psi_{\mathbb{K}}(\hat{a})$ *with* $\hat{a} = \gamma + \omega^a$. *Then* $(\mathcal{H}_{\hat{a}+1}, \Theta; \mathtt{Q}_\Pi) \vdash_\beta^{*\beta} \Gamma^{(\beta,\mathbb{K})}; \Pi^{[\cdot]}$ *holds.*

Proof. By induction on a as in Theorem 1.22. We have $\{\gamma, a\} \subset \mathcal{H}_\gamma[\Theta_{\mathtt{Q}_\Pi}]$ by (35), and $\beta \in \mathcal{H}_{\hat{a}+1}[\Theta_{\mathtt{Q}_\Pi}]$.

When the last inference is a (stbl), let $B(0) \in \Delta_0(\mathbb{S})$ be a $\bigwedge$-formula and a term $u \in Tm(\mathbb{K})$ such that $\mathbb{S} \leq \mathrm{rk}(B(u)) < \mathbb{K}$, $\mathsf{k}(B(u)) \subset \mathcal{H}_\gamma[\Theta]$, and $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash_{\mathbb{K}}^{*a_0} \Gamma, B(u); \Pi^{[\cdot]}$ for an ordinal $a_0 \in \mathcal{H}_\gamma[\Theta_{\mathtt{Q}_\Pi}] \cap a$. Then we obtain $\mathrm{rk}(B(u)) < \beta$.

Consider the case when the last inference is a (Σ-rfl) on $\mathbb{K}$. We have ordinals $a_\ell, a_r < a$ and a formula $C \in \Sigma$ such that $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a_\ell}_{\mathbb{K}} \Gamma, C; \Pi^{[\cdot]}$ and $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a_r}_{\mathbb{K}} \neg\exists x\, C^{(x,\mathbb{K})}, \Gamma; \Pi^{[\cdot]}$.

Let $\beta_\ell = \psi_{\mathbb{K}}(\widehat{a_\ell}) \in \mathcal{H}_{\widehat{a_\ell}+1}[\Theta_{\mathtt{Q}_\Pi}] \cap \beta$ with $\widehat{a_\ell} = \gamma + \omega^{a_\ell}$. IH yields $(\mathcal{H}_{\hat{a}+1}, \Theta; \mathtt{Q}_\Pi) \vdash^{*\beta_\ell}_{\beta} \Gamma^{(\beta,\mathbb{K})}, C^{(\beta_\ell,\mathbb{K})}; \Pi^{[\cdot]}$. On the other, Inversion 6.31 yields $(\mathcal{H}_{\widehat{a_\ell}+1}, \Theta; \mathtt{Q}_\Pi) \vdash^{*a_r}_{\mathbb{K}} \neg C^{(\beta_\ell,\mathbb{K})}, \Gamma; \Pi^{[\cdot]}$. For $\beta_r = \psi_{\mathbb{K}}(\widehat{a_r}) \in \mathcal{H}_{\hat{a}+1}[\Theta_{\mathtt{Q}_\Pi}] \cap \beta$ with $\widehat{a_r} = \widehat{a_\ell} + \omega^{a_r}$, IH yields $(\mathcal{H}_{\hat{a}+1}, \Theta; \mathtt{Q}_\Pi) \vdash^{*\beta_r}_{\beta} \neg C^{(\beta_\ell,\mathbb{K})}, \Gamma^{(\beta,\mathbb{K})}; \Pi^{[\cdot]}$. We obtain $(\mathcal{H}_{\hat{a}+1}, \Theta; \mathtt{Q}_\Pi) \vdash^{*\beta}_{\beta} \Gamma^{(\beta,\mathbb{K})}; \Pi^{[\cdot]}$ by a (cut).

Note that since $\Pi \subset \Delta_0(\mathbb{K})$, inferences $(\bigwedge)^{[\cdot]}$ are harmless for the condition $\Theta \subset \mathcal{H}_\gamma(\psi_{\mathbb{K}}(\gamma))$. □

6.6 Operator controlled derivations with caps

In this subsection we introduce another derivability relation $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a}_{c,e,b_1} \Gamma$, which depends again on an ordinal γ_0, and should be written as $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a}_{c,e,\gamma_0,b_1} \Gamma$. However the ordinal γ_0 will be fixed, and specified in the proof of Theorem 6.51. So let us omit it.

The inference rules (stbl) are replaced by inferences (rfl(ρ, d, f, b_1)) by putting a *cap* ρ on formulas in Lemma 6.44. In $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a}_{c,e,b_1} \Gamma$, c is a bound for cut ranks and e a bound for ordinals ρ in the inferences (rfl(ρ, d, f, b_1)) occurring in the derivation. b_1 is a bound such that $s(\rho) = \max(\mathrm{supp}(m(\rho))) \leq b_1$. Although the capped formula $A^{(\rho)}$ in Definition 6.36, is intended to denote the formula $A^{[\rho/\mathbb{S}]}$, we need to distinguish it from $A^{[\rho/\mathbb{S}]}$. Our main task is to eliminate inferences (rfl(ρ, d, f)) from a resulting derivation $\mathcal{D}_1$. In Recapping 6.47 the cap ρ in inferences (rfl(ρ, d, f, b_1)) are replaced by another cap $\kappa < \rho$. In this process new inferences (rfl(σ, d_1, f_1, b_1)) arise with $\sigma < \kappa$. Iterating this process, we arrive at a derivation $\mathcal{D}_2$ such that $s(\rho) \leq \mathbb{S}$, i.e., $\mathrm{supp}(m(\rho)) \subset \mathbb{S}+1$. Then caps play no rôle, i.e., $A^{(\rho)}$ is 'equivalent' to A for $A \in \Delta_0(\mathbb{S})$. Finally inferences (rfl(ρ, d, f, b_1)) are removed from $\mathcal{D}_2$ by throwing up caps and replacing these by a series of (cut)'s, cf. Lemma 6.48.

The ordinal, i.e., the threshold γ_0 will be specified in the end of this section.

Definition 6.36 By a *capped formula* we mean a pair (A, ρ) of *RS*-sentence A and an ordinal $\rho < \mathbb{S}$ such that $\mathsf{k}(A) \subset M_\rho$. Such a pair is denoted by $A^{(\rho)}$. A *sequent* is a finite set of capped formulas, denoted by $\Gamma_0^{(\rho_0)}, \ldots, \Gamma_n^{(\rho_n)}$, where each formula in the set $\Gamma_i^{(\rho_i)}$ puts on the cap $\rho_i \in \mathbb{S}$. When we write $\Gamma^{(\rho)}$, we tacitly assume that $\mathsf{k}(\Gamma) \subset M_\rho$. A capped formula $A^{(\rho)}$ is said to be a $\Sigma(\pi)$-formula if $A \in \Sigma(\pi)$. Let $\mathsf{k}(A^{(\rho)}) := \mathsf{k}(A)$.

Definition 6.37 Let f be a non-empty (and irreducible) finite function. Then f is said to be *special* if there exists an ordinal α such that $f(c_{\max}) = \alpha + \mathbb{K}$ for $c_{\max} = \max(\mathrm{supp}(f))$. For a special finite function f, f' denotes a finite function such that $\mathrm{supp}(f') = \mathrm{supp}(f)$, $f'(c) = f(c)$ for $c \neq c_{\max}$, and $f'(c_{\max}) = \alpha$ with $f(c_{\max}) = \alpha + \mathbb{K}$.

The ordinal $\mathbb{K}$ in $f(c_{\max}) = \alpha + \mathbb{K}$ is a 'room' to be replaced by a smaller ordinal, cf. Definition 6.45.

Definition 6.38 A finite set $\mathtt{Q} \subset \Psi_{\mathbb{S}}$ is said to be a *finite family* for ordinals γ_0 and b_1 if $\rho \in \mathcal{H}_{\gamma_0+\mathbb{S}} = \mathcal{H}_{\gamma_0+\mathbb{S}}(0)$, $m(\rho) : \mathbb{K} \to \varphi_{\mathbb{K}}(0)$ is special such that $s(\rho) = \max(\mathrm{supp}(m(\rho))) \leq b_1$ and $\mathtt{p}_0(\rho) \geq \gamma_0$ for each $\rho \in \mathtt{Q}$.

The resolvent class $H_\rho(f, b_1, \gamma_0, \Theta)$ in the following Definition 6.39 is the set of ordinals $\sigma < \rho$, which are candidates of substitutes for ρ in the inference $(\mathrm{rfl}(\rho, d, f, b_1))$ for reflection. Note that if $\mathtt{p}_0(\sigma) \leq \mathtt{p}_0(\rho)$ and $\sigma < \rho$, then $M_\sigma \subset M_\rho = \mathcal{H}_{\mathtt{p}_0(\rho)}(\rho)$. Moreover if $\mathtt{p}_0(\sigma) \geq \gamma_0 \geq \gamma$ and $\Theta \subset M_\sigma$, then $\mathcal{H}_\gamma[\Theta] \subset M_\sigma$ by Proposition 6.21.

Definition 6.39 $H_\rho(f, b_1, \gamma_0, \Theta)$ denotes the *resolvent class* for finite functions f, ordinals ρ, b_1, γ_0 and finite sets Θ of ordinals defined by $\sigma \in H_\rho(f, b_1, \gamma_0, \Theta)$ iff $\sigma \in \mathcal{H}_{\gamma_0+\mathbb{S}} \cap \rho$, $SC_{\mathbb{K}}(m(\sigma)) \subset \mathcal{H}_{\gamma_0}[\Theta]$, $\Theta \subset M_\sigma$, $\mathtt{p}_0(\sigma) = \mathtt{p}_0(\rho) \geq \gamma_0$, and $m(\sigma)$ is special such that $s(f) = \max(\mathrm{supp}(f)) \leq s(\sigma) \leq b_1$ and $f' \leq (m(\sigma))'$, where $f \leq g \Leftrightarrow \forall i(f(i) \leq g(i))$.

We define a derivability relation $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_{c,e} \Gamma$, where $\mathbb{S} \leq \gamma \leq \gamma_0$ is an ordinal, Θ a finite set of ordinals, $\mathtt{Q}$ a finite family for γ_0, b_1, and $a, c < \mathbb{K} = \mathbb{S}^+$. c a bound of cut ranks, e a bound of ρ in inference rules $(\mathrm{rfl}(\rho, d, f, b_1))$, and b_1 a bound on $s(\rho)$. The relation $\vdash^a_{c,e}$ depends on fixed ordinals γ_0 and b_1.

For $d = \mathrm{rk}(A) < \mathbb{S}$, it may be $\mathsf{k}(A) \cup \{d\} \not\subset M_{\mathtt{Q}}$. Let us avoid deriving the tautology $\neg A, A$ by a standard derivation to show $\vdash^{2d} \neg A, A$.

Definition 6.40 Let $\Theta^{(\rho)} = \Theta \cap M_\rho$, $[\mathtt{Q}]_{A^{(\rho)}} J = [\mathtt{Q}]_A J \cap [\rho] J$, $\mathbb{S} \leq \gamma \leq \gamma_0$ and $e \in \mathcal{H}_{\gamma_0+\mathbb{S}}(0)$.

$(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_{c,e,\gamma_0,b_1} \Gamma$ holds for a set $\Gamma = \bigcup\{\Gamma^{(\rho)}_\rho : \rho \in \mathtt{Q}\}$ of formulas if

$$\forall \rho \in \mathtt{Q} \left(\mathsf{k}(\Gamma_\rho) \subset \mathcal{H}_\gamma[\Theta^{(\rho)}] \right) \tag{36}$$

$$\{\gamma, a, c, b_1\} \cup \mathsf{k}^{\mathbb{S}}(\Gamma) \subset \mathcal{H}_\gamma[\Theta_{\mathtt{Q}}] \tag{37}$$

and one of the following cases holds:

(Taut) $\{\neg A^{(\rho)}, A^{(\rho)}\} \subset \Gamma$ for a $\rho \in \mathtt{Q}$ and a formula A such that $\mathrm{rk}(A) < \mathbb{S}$.

($\bigvee$) There exist $A \simeq \bigvee(A_\iota)_{\iota \in J}$, a cap $\rho \in \mathtt{Q}$, an ordinal $a_\iota < a$ and an $\iota \in [\rho] J$ such that $A^{(\rho)} \in \Gamma$ and $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a_\iota}_{c,e,\gamma_0,b_1} \Gamma, (A_\iota)^{(\rho)}$.

Note that if $\mathrm{rk}(A_\iota) \geq \mathbb{S}$, then $\mathsf{k}(A_\iota) \subset \mathcal{H}_\gamma[\Theta_{\mathtt{Q}}] \subset M_{\mathtt{Q}}$ by (37). Hence $\iota \in [\mathtt{Q}]_A J = \{\iota \in J : \mathrm{rk}(A_\iota) \geq \mathbb{S} \Rightarrow \mathsf{k}(\iota) \subset M_{\mathtt{Q}}\}$.

($\bigwedge$) There exist $A \simeq \bigwedge(A_\iota)_{\iota \in J}$, a cap $\rho \in \mathtt{Q}$, ordinals $a_\iota < a$ for each $\iota \in [\mathtt{Q}]_{A^{(\rho)}} J$ such that $A^{(\rho)} \in \Gamma$ and $(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\iota), \mathtt{Q}) \vdash^{a_\iota}_{c,e,\gamma_0,b_1} \Gamma, (A_\iota)^{(\rho)}$.

Note that if $\mathrm{rk}(A_\iota) \geq \mathbb{S}$, then $\mathsf{k}(\iota) \subset M_{\mathtt{Q}}$ by $\iota \in [\mathtt{Q}]_{A^{(\rho)}} J$. Hence $\mathsf{k}^{\mathbb{S}}(A_\iota) \subset \mathcal{H}_\gamma[(\Theta \cup \mathsf{k}(\iota))_{\mathtt{Q}}]$ for (37), where $(\Theta \cup \mathsf{k}(\iota))_{\mathtt{Q}} = \Theta_{\mathtt{Q}} \cup \mathsf{k}(\iota)$.

(cut) There exist a cap $\rho \in \mathtt{Q}$, an ordinal $a_0 < a$ and a formula C such that $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a_0}_{c,e,\gamma_0,b_1} \Gamma, \neg C^{(\rho)}$ and $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a_0}_{c,e,\gamma_0,b_1} C^{(\rho)}, \Gamma$ with $\mathrm{rk}(C) < c$.

(Σ-rfl(Ω)) There exist a cap $\rho \in \mathtt{Q}$, ordinals $a_\ell, a_r < a$, and an uncapped formula $C \in \Sigma(\Omega)$ such that $c \geq \Omega$, $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a_\ell}_{c,e,\gamma_0,b_1} \Gamma, C^{(\rho)}$ and $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a_r}_{c,e,\gamma_0,b_1} \neg\left(\exists x < \pi\, C^{(x,\Omega)}\right)^{(\rho)}, \Gamma$.

(rfl(ρ, d, f, b_1)) There exist a cap $\rho \in \mathtt{Q}$ such that $\Theta \subset M_\rho$, ordinals $d \in \mathrm{supp}(m(\rho))$, and $a_0 < a$, a special finite function f, and a finite set Δ of uncapped formulas enjoying the following conditions.

- (r0) $\rho < e$ if $s(\rho) > \mathbb{S}$.
- (r1) $\Delta \subset \bigvee(d) := \{\delta : \mathrm{rk}(\delta) < d, \delta \text{ is a } \bigvee\text{-formula}\} \cup \{\delta : \mathrm{rk}(\delta) < \mathbb{S}\}$.
- (r2) For the special finite function $g = m(\rho)$, $s(f) \leq b_1$, $SC_{\mathbb{K}}(f, g) \subset \mathcal{H}_{\gamma_0}[\Theta^{(\rho)}]$ and $f_d = g_d \,\&\, f^d <^d g'(d)$.
- (r3) For each $\delta \in \Delta$, $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a_0}_{c,e,\gamma_0,b_1} \Gamma, \neg\delta^{(\rho)}$.
- (r4) $(\mathcal{H}_\gamma, \Theta \cup \{\sigma\}, \mathtt{Q} \cup \{\sigma\}) \vdash^{a_0}_{c,e,\gamma_0,b_1} \Gamma, \Delta^{(\sigma)}$ holds for every $\sigma \in H_\rho(f, b_1, \gamma_0, \Theta^{(\rho)})$.

$$\frac{\{(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a_0}_{c,e} \Gamma, \neg\delta^{(\rho)}\}_{\delta \in \Delta} \quad \{(\mathcal{H}_\gamma, \Theta \cup \{\sigma\}, \mathtt{Q} \cup \{\sigma\}) \vdash^{a_0}_{c,e} \Gamma, \Delta^{(\sigma)}\}_{\sigma \in H_\rho(f, b_1, \gamma_0, \Theta^{(\rho)})}}{(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a}_{c,e} \Gamma}\ (\mathrm{rfl}(\rho, d, f, b_1))$$

Note that $(\Theta \cup \{\sigma\})_{\mathtt{Q} \cup \{\sigma\}} = \Theta_{\mathtt{Q} \cup \{\sigma\}} = \Theta_{\mathtt{Q}}$ by $\Theta^{(\rho)} \subset M_\sigma$ and $\rho \in \mathtt{Q}$.

$\{e\} \cup \mathtt{Q} \subset \mathcal{H}_\gamma[\Theta]$ need not to hold.

Suppose $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_{c,e} \Gamma$ holds with $A^{(\rho)} \in \Gamma$ and $\rho \in \mathtt{Q}$. By (36) we have $\mathsf{k}(A) \subset \mathcal{H}_\gamma[\Theta^{(\rho)}]$. We obtain $\mathsf{k}(A) \subset M_\rho$ by Proposition 6.21.

In this subsection the ordinals γ_0 and b_1 will be fixed, and we write $\vdash^a_{c,e}$ for $\vdash^a_{c,e,\gamma_0,b_1}$.

Proposition 6.41 (Tautology) *Let* $\{\gamma\} \cup \mathsf{k}^{\mathbb{S}}(A) \subset \mathcal{H}_\gamma[\Theta_{\mathtt{Q}}]$ *and* $\sigma \in \mathtt{Q}$, $\mathsf{k}(A) \subset \mathcal{H}_\gamma[\Theta^{(\sigma)}]$. *Then* $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{2d}_{0,0} \neg A^{(\sigma)}, A^{(\sigma)}$ *holds for* $d = \max\{\mathbb{S}, \mathrm{rk}(A)\}$.

Proof. By induction on d. Let $A \simeq \bigvee(A_\iota)_{\iota \in J}$ with $\mathrm{rk}(A) \geq \mathbb{S}$. For $\iota \in [\mathtt{Q}]_{A^{(\sigma)}} J \subset [\sigma] J$, let $d_\iota = 0$ if $\mathrm{rk}(A_\iota) < \mathbb{S}$. Otherwise $d_\iota = \max\{\mathbb{S}, \mathrm{rk}(A_\iota)\}$. In each case we have $d_\iota < d$. IH yields

$$\cfrac{\cfrac{(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\iota), \mathtt{Q}) \vdash^{2d_\iota}_{0,0} \neg A_\iota^{(\sigma)}, A_\iota^{(\sigma)}}{(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\iota), \mathtt{Q}) \vdash^{2d_\iota+1}_{0,0} \neg A_\iota^{(\sigma)}, A^{(\sigma)}}\ (\bigvee)}{(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{2d}_{0,0} \neg A^{(\sigma)}, A^{(\sigma)}}\ (\bigwedge)$$

□

Proposition 6.42 (Inversion) *Let* $A \simeq \bigwedge(A_\iota)_{\iota \in J}$ *with* $A^{(\rho)} \in \Gamma$ *and* $\mathrm{rk}(A) \geq \mathbb{S}$, $\iota \in [\mathtt{Q}]_{A^{(\rho)}} J$ *with* $\rho \in \mathtt{Q}$ *and* $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_{c,e} \Gamma$. *Then* $(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\iota), \mathtt{Q}) \vdash^a_{c,e} \Gamma, A_\iota$.

Proposition 6.43 (Cut-elimination) *Let* $(\mathcal{H}_\gamma,\Theta,\mathtt{Q})\vdash^a_{c+d,e}\Gamma$ *with* $\mathcal{H}_\gamma[\Theta_{\mathtt{Q}}]\ni c\geq\mathbb{S}$. *Then* $(\mathcal{H}_\gamma,\Theta,\mathtt{Q})\vdash^{\varphi_d(a)}_{c,e}\Gamma$.

Proof. By main induction on d with subsidiary induction on a using an analogue to Reduction 6.33 with (37). Note that $\mathrm{rk}(C)\in\mathcal{H}_\gamma[\Theta_{\mathtt{Q}}]$ when $\mathrm{rk}(C)\geq\mathbb{S}$ and $(\mathcal{H}_\gamma,\Theta,\mathtt{Q})\vdash^a_{c,e}\Gamma,C$. □

Lemma 6.44 (Capping) *Let* $\Gamma\cup\Pi\subset\Delta_0(\mathbb{K})$ *with* $\Pi=\bigcup\{\Pi_\sigma:\sigma\in\mathtt{Q}_\Pi\}$. *Suppose* $(\mathcal{H}_\gamma,\Theta;\mathtt{Q}_\Pi)\vdash^{*a}_{c,\gamma_0}\Gamma;\Pi^{[\cdot]}$ *for* $a,c<\mathbb{K}$ *and* $\Pi^{[\cdot]}=\bigcup\{\Pi_\sigma^{[\sigma/\mathbb{S}]}:\sigma\in\mathtt{Q}_\Pi\}$. *Let* $\rho=\psi^g_{\mathbb{S}}(\gamma_1)$ *be an ordinal such that* $\mathtt{Q}_\Pi\subset\rho$,

$$\Theta\subset M_\rho \tag{38}$$

and $g=m(\rho)$ *a special finite function such that* $\mathrm{supp}(g)=\{c\}$ *and* $g(c)=\alpha_0+\mathbb{K}$, *where* $\mathbb{K}(2a+1)\leq\alpha_0+\mathbb{K}\leq\gamma_0\leq\gamma_1$ *with* $\{\gamma_1,c,\alpha_0\}\subset\mathcal{H}_\gamma[\Theta]\cap\mathcal{H}_{\gamma_0}$, *and* $\mathtt{p}_0(\sigma)\leq\mathtt{p}_0(\rho)=\gamma_1$ *for each* $\sigma\in\mathtt{Q}_\Pi$. *Let* $\widehat{\Gamma}=\bigcup\{A^{(\rho)}:A\in\Gamma\}$, $\widehat{\Pi}=\bigcup\{\Pi_\sigma^{(\sigma)}:\sigma\in\mathtt{Q}_\Pi\}$ *and* $\mathtt{Q}=\mathtt{Q}_\Pi\cup\{\rho\}$.

Then $(\mathcal{H}_\gamma,\Theta_\Pi,\mathtt{Q})\vdash^a_{c,\rho+1,\gamma_0,c}\widehat{\Gamma},\widehat{\Pi}$ *holds holds for* $\Theta_\Pi=\Theta\cup\mathtt{Q}_\Pi$.

Proof. By induction on a. Let us write $\vdash^a_c$ for $\vdash^a_{c,\rho+1,\gamma_0,c}$ in the proof. By assumptions we have $\Theta\subset M_\rho$ and $\mathtt{Q}_\Pi\subset\rho$. Hence $\Theta=\Theta^{(\rho)}$ and $\Theta_{\mathtt{Q}_\Pi}=\Theta_{\mathtt{Q}}$. On the other hand we have $\mathsf{k}(\Gamma)\subset\mathcal{H}_\gamma[\Theta]$ and for $\sigma\in\mathtt{Q}_\Pi$, $\mathsf{k}(\Pi_\sigma)\subset\mathcal{H}_\gamma[\Theta^{(\sigma)}]$ by (34). Therefore (36) is enjoyed. We have $\{\gamma,a,c\}\subset\mathcal{H}_\gamma[\Theta_{\mathtt{Q}_\Pi}]$ by (35). Hence (37) is enjoyed. Moreover we have $SC_{\mathbb{K}}(g)\subset\mathcal{H}_\gamma[\Theta]\subset M_\rho$.

Case 1. First consider the case when the last inference is a (stbl):

$$\frac{(\mathcal{H}_\gamma,\Theta;\mathtt{Q}_\Pi)\vdash^{*a_0}_c\Gamma,B(u);\Pi^{[\cdot]}\quad\{(\mathcal{H}_\gamma,\Theta\cup\{\sigma\};\mathtt{Q}_\Pi\cup\{\sigma\})\vdash^{*a_0}_c\Gamma;\neg B(u)^{[\sigma/\mathbb{S}]},\Pi^{[\cdot]}\}_{\Theta\subset M_\sigma}}{(\mathcal{H}_\gamma,\Theta;\mathtt{Q}_\Pi)\vdash^{*a}_c\Gamma;\Pi^{[\cdot]}}\ (\mathrm{stbl})$$

Note that it may be the formula $B(u)^{[\sigma/\mathbb{S}]}$ is in Γ, cf. Embedding 6.30. σ in $\Theta\cup\{\sigma\}$ ensures us $\mathsf{k}(B(u)^{[\sigma/\mathbb{S}]})\subset\mathcal{H}_\gamma[\Theta\cup\{\sigma\}]$ in (34). This explains the additional set $\mathtt{Q}_\Pi$ in $(\mathcal{H}_\gamma,\Theta_\Pi,\mathtt{Q})\vdash^a_c\widehat{\Gamma},\widehat{\Pi}$, and the addition would be an obstacle to $a\in\Theta_{\mathtt{Q}}$ in (37).

We have an ordinal $a_0<a$, a $\bigwedge$-formula $B(0)\in\Delta_0(\mathbb{S})$, and a term $u\in Tm(\mathbb{K})$ such that $\mathbb{S}\leq\mathrm{rk}(B(u))<c$. We have $(\mathcal{H}_\gamma,\Theta;\mathtt{Q}_\Pi)\vdash^{*a_0}_c\Gamma,B(u);\Pi^{[\cdot]}$. $(\mathcal{H}_\gamma,\Theta_\Pi,\mathtt{Q})\vdash^{a_0}_c\widehat{\Gamma},(B(u))^{(\rho)},\widehat{\Pi}$ follows from IH.

On the other hand we have $(\mathcal{H}_\gamma,\Theta\cup\{\sigma\};\mathtt{Q}_\Pi\cup\{\sigma\})\vdash^{*a_0}_c\Gamma;\neg B(u)^{[\sigma/\mathbb{S}]},\Pi^{[\cdot]}$ for every ordinal σ such that $\Theta\subset M_\sigma$.

Let h be a special finite function such that $\mathrm{supp}(h)=\{c\}$ and $h(c)=\mathbb{K}(2a_0+1)$. Then $h_c=g_c=\emptyset$ and $h^c<^c g'(c)$ by $h(c)=\mathbb{K}(2a_0+1)<\mathbb{K}(2a)\leq\alpha_0=g'(c)$. Let $\sigma\in H_\rho(h,c,\gamma_0,\Theta)$. For example $\sigma=\psi^h_\rho(\gamma_1+\eta)$ with $\eta=\max(\{1\}\cup E_{\mathbb{S}}(\Theta))$, where $E_{\mathbb{S}}(\Theta)=\bigcup_{\alpha\in\Theta}E_{\mathbb{S}}(\alpha)$ with the set $E_{\mathbb{S}}(\alpha)$ of subterms$<\mathbb{S}$ of α. We obtain $\Theta\subset\mathcal{H}_{\gamma_1}(\sigma)=M_\sigma$ by $\Theta\subset M_\rho$, and $\{\gamma_1,c,a_0\}\subset\mathcal{H}_\gamma[\Theta]\subset\mathcal{H}_{\gamma_1}(\sigma)$.

We have $\mathsf{k}^{\mathbb{S}}(B(u))=\mathsf{k}(B(u))\subset\mathcal{H}_\gamma[\Theta_{\mathtt{Q}}]\subset M_\sigma$ for (37), and $(\mathcal{H}_\gamma,\Theta_\Pi\cup\{\sigma\},\mathtt{Q}\cup\{\sigma\})\vdash^{a_0}_c\widehat{\Gamma},\neg B(u)^{(\sigma)},\widehat{\Pi}$ follows from IH with $\sigma\in M_\rho$. Since this holds for every such σ, we obtain $(\mathcal{H}_\gamma, \Theta_\Pi, \mathtt{Q}) \vdash^a_{c,\rho+1} \widehat{\Gamma}, \widehat{\Pi}$ by an inference (rfl(ρ, c, h, c)) with $\mathrm{rk}(B(u)) < c \in \mathrm{supp}(m(\rho))$. In the following figure let us omit the operator $\mathcal{H}_\gamma$.

$$\frac{(\Theta_\Pi, \mathtt{Q}) \vdash^{a_0}_c \widehat{\Gamma}, B(u)^{(\rho)}, \widehat{\Pi} \quad \{(\Theta_\Pi \cup \{\sigma\}, \mathtt{Q} \cup \{\sigma\}) \vdash^{a_0}_c \widehat{\Gamma}, \neg B(u)^{(\sigma)}, \widehat{\Pi}\}_\sigma}{(\Theta_\Pi, \mathtt{Q}) \vdash^a_c \widehat{\Gamma}, \widehat{\Pi}} \ (\mathrm{rfl}(\rho, c, h, c))$$

Case 2. Second the last inference introduces a $\bigvee$-formula A.
Case 2.1. First let $A \in \Gamma$ be introduced by a $(\bigvee)$, and $A \simeq \bigvee (A_\iota)_{\iota \in J}$. There are an $\iota \in J$ an ordinal $a(\iota) < a$ such that $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a(\iota)}_c \Gamma, A_\iota; \Pi^{[\cdot]}$. Let $\mathsf{k}(\iota) \subset \mathsf{k}(A_\iota)$. We obtain $\mathsf{k}(\iota) \subset \mathcal{H}_\gamma[\Theta] \subset M_\rho$ by (34), $\Theta \subset M_\rho$ and $\gamma \leq \gamma_0 \leq \gamma_1$. Hence $\iota \in [\rho]J$. IH yields $(\mathcal{H}_\gamma, \Theta_\Pi, \mathtt{Q}) \vdash^{a(\iota)}_c \widehat{\Pi}, \widehat{\Gamma}, (A_\iota)^{(\rho)}$. $(\mathcal{H}_\gamma, \Theta_\Pi, \mathtt{Q}) \vdash^a_c \widehat{\Pi}, \widehat{\Gamma}$ follows from a $(\bigvee)$.
Case 2.2. Second $A \equiv B^{[\sigma/\mathbb{S}]} \in \Pi^{[\cdot]}$ is introduced by a $(\bigvee)^{[\cdot]}$ with $B^{(\sigma)} \in \widehat{\Pi}$ and $\sigma \in \mathtt{Q}_\Pi$. Let $B \simeq \bigvee (B_\iota)_{\iota \in J}$. Then $A \simeq \bigvee \left(B_\iota^{[\sigma, \mathbb{S}]}\right)_{\iota \in [\sigma]J}$ by Proposition 6.26. There are an $\iota \in [\sigma]J$ and an ordinal $a(\iota) < a$ such that $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{a(\iota)}_c \Gamma; B_\iota^{[\sigma/\mathbb{S}]}, \Pi^{[\cdot]}$ for $A_\iota \equiv B_\iota^{[\sigma/\mathbb{S}]}$. IH yields $(\mathcal{H}_\gamma, \Theta_\Pi, \mathtt{Q}) \vdash^{a(\iota)}_c \widehat{\Pi}, \widehat{\Gamma}, (B_\iota)^{(\sigma)}$. We obtain $(\mathcal{H}_\gamma, \Theta_\Pi, \mathtt{Q}) \vdash^a_c \widehat{\Pi}, \widehat{\Gamma}$ by a $(\bigvee)$.
Case 3. Third the last inference introduces a $\bigwedge$-formula A.
Case 3.1. First let $A \in \Gamma$ be introduced by a $(\bigwedge)$, and $A \simeq \bigwedge (A_\iota)_{\iota \in J}$. For every $\iota \in [\mathtt{Q}_\Pi]_A J$ there exists an $a(\iota) < a$ such that $(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\iota); \mathtt{Q}_\Pi) \vdash^{*a(\iota)}_c \Gamma, A_\iota; \Pi^{[\cdot]}$.

IH yields $(\mathcal{H}_\gamma, \Theta_\Pi \cup \mathsf{k}(\iota), \mathtt{Q}) \vdash^{a(\iota)}_c \widehat{\Pi}, \widehat{\Gamma}, (A_\iota)^{(\rho)}$ for each $\iota \in [\mathtt{Q}]_{A^{(\rho)}} J \subset [\mathtt{Q}_\Pi]_A J$, where $\mathsf{k}(\iota) \subset M_\rho$. We obtain $(\mathcal{H}_\gamma, \Theta_\Pi, \mathtt{Q}) \vdash^a_c \widehat{\Pi}, \widehat{\Gamma}$ by a $(\bigwedge)$.
Case 3.2. Second $A \equiv B^{[\sigma/\mathbb{S}]} \in \Pi^{[\cdot]}$ is introduced by a $(\bigwedge)^{[\cdot]}$ with $B^{(\sigma)} \in \widehat{\Pi}$ and $\sigma \in \mathtt{Q}_\Pi$. Let $B \simeq \bigwedge (B_\iota)_{\iota \in J}$ with $A \simeq \bigwedge \left(B_\iota^{[\sigma/\mathbb{S}]}\right)_{\iota \in [\sigma]J}$. For each $\iota \in [\mathtt{Q}_\Pi]_B J \cap [\sigma]J$ there is an ordinal $a(\iota) < a$ such that $(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\iota); \mathtt{Q}_\Pi) \vdash^{*a(\iota)}_c \Gamma; A_\iota, \Pi^{[\cdot]}$ for $A_\iota \equiv B_\iota^{[\sigma/\mathbb{S}]}$. IH yields $(\mathcal{H}_\gamma, \Theta_\Pi \cup \mathsf{k}(\iota), \mathtt{Q}) \vdash^{a(\iota)}_c \widehat{\Pi}, \widehat{\Gamma}, (B_\iota)^{(\sigma)}$ for each $\iota \in [\mathtt{Q}]_{B^{(\sigma)}} J \subset [\mathtt{Q}_\Pi]_B J \cap [\sigma]J$, where $\mathsf{k}(\iota) \subset M_\sigma \subset M_\rho$. $(\mathcal{H}_\gamma, \Theta_\Pi, \mathtt{Q}) \vdash^a_c \widehat{\Pi}, \widehat{\Gamma}$ follows from a $(\bigwedge)$.

The other cases (cut) or (Σ-rfl) on Ω are seen from IH. □

6.7 Eliminations of inferences (rfl)

In this subsection, (rfl(ρ, c, γ)) are removed from operator controlled derivations of Σ_1-sentences θ^{L_Ω} over Ω.

Definition 6.45 For a special finite function g and ordinals $a < \mathbb{K}$, $b < c_{\max} = \max(\mathrm{supp}(g)) < \mathbb{K}$, let us define a special finite function $h = h^b(g; a)$ as follows. $\max(\mathrm{supp}(h)) = b$, and $h_b = g_b$. To define $h(b)$, let $\{b = b_0 < b_1 < \cdots < b_n = c_{\max}\} = \{b, c_{\max}\} \cup ((b, c_{\max}) \cap \mathrm{supp}(g))$. Define recursively ordinals α_i by $\alpha_n = \alpha + a$ with $g(c_{\max}) = \alpha + \mathbb{K}$. $\alpha_i = g(b_i) + \tilde{\theta}_{c_i}(\alpha_{i+1})$ for $c_i = b_{i+1} - b_i$. Finally put $h(b) = \alpha_0 + \mathbb{K}$.

Proposition 6.46 *Let f and g be special finite functions with $c_{\max} = \max(\text{supp}(g))$.*

1. *Let $b < e < c_{\max}$ and $a_0, a_1 < a$. Then $h^b(h^e(g;a_0);a_1) \leq (h^b(g;a))'$.*
2. *Suppose $f <^d g'(d)$ for a $d \in \text{supp}(g)$. Let $b < d$. Then $f_b = (h^b(g;a))_b$ and $f <^b (h^b(g;a))'(b)$.*

Recall that $s(\rho) = \max(\text{supp}(m(\rho)))$.

Lemma 6.47 (Recapping)
Let $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_{c_1,e,\gamma_0,b_2} \Pi, \widehat{\Gamma}$ for a finite family $\mathtt{Q}$ for γ_0, b_2, $\mathtt{Q}^t \subset \mathtt{Q}$, $\forall \rho \in \mathtt{Q}^t(s(\rho) > \mathbb{S})$ and $\mathtt{Q}^f = \mathtt{Q} \setminus \mathtt{Q}^t$, $\Gamma \cup \Pi \subset \Delta_0(\mathbb{K})$, $\widehat{\Gamma} = \bigcup\{\Gamma^{(\rho)}_\rho : \rho \in \mathtt{Q}^t\}$, where each $\theta \in \Gamma$ is either a $\bigvee$-formula or $\text{rk}(\theta) < \mathbb{S}$, and Π a set of formulas such that $\tau \in \mathtt{Q}^f$ for every $A^{(\tau)} \in \Pi$.

Let $\max\{s(\rho) : \rho \in \mathtt{Q}^t\} \leq b_1$. For each $\rho \in \mathtt{Q}^t$, let $\mathbb{S} \leq b^{(\rho)} \in \mathcal{H}_\gamma[\Theta^{(\rho)}]$ with $\text{rk}(\Gamma_\rho) < b^{(\rho)} < s(\rho)$, and $\kappa(\rho) \in H_\rho(h^{b^{(\rho)}}(m(\rho);\omega(b_1,a)), b_2, \gamma_0, \Theta^{(\rho)})$ with $\omega(b,a) = \omega^{\omega^b}a$. Assume $b_1 \in \mathcal{H}_\gamma[\Theta_{\mathtt{Q}}]$.

Then $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}(\kappa)) \vdash^{\omega(b_1,a)}_{c_{b_1},e^\kappa,\gamma_0,b_2} \Pi, \widehat{\Gamma}_\kappa$ holds, where $\mathtt{Q}(\kappa) = \mathtt{Q}^f \cup \{\kappa(\rho) : \rho \in \mathtt{Q}^t\}$, $c_{b_1} = \max\{c_1, b_1\}$, $e^\kappa = \max(\{\tau \in \mathtt{Q}^f : s(\tau) > \mathbb{S}\} \cup \{\kappa(\rho) : \rho \in \mathtt{Q}^t\}) + 1$, $\widehat{\Gamma}_\kappa = \bigcup\{\Gamma^{(\kappa(\rho))}_\rho : \rho \in \mathtt{Q}^t\}$.

$e^\kappa < e$ holds when $\mathtt{Q}^t = \{\rho \in \mathtt{Q} : s(\rho) > \mathbb{S}\} \neq \emptyset$.

Proof. We show the lemma by main induction on b_1 with subsidiary induction on a. The subscripts γ_0, b_2 are omitted in the proof. We obtain $\{\gamma, b_1, a, c_1\} \cup \mathsf{k}^{\mathbb{S}}(\Pi, \Gamma) \subset \mathcal{H}_\gamma[\Theta_{\mathtt{Q}}]$ by the assumption and (37). Then $\{\gamma, \omega(b_1,a), c_{b_1}\} \cup \mathsf{k}^{\mathbb{S}}(\Pi, \Gamma) \subset \mathcal{H}_\gamma[\Theta_{\mathtt{Q}(\kappa)}]$ since $\Theta^{(\rho)} \subset M_{\kappa(\rho)}$ for each $\rho \in \mathtt{Q}^t$. Hence (37) is enjoyed in $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}(\kappa)) \vdash^{\omega(b_1,a)}_{c_{b_1},e,\gamma_0,b_2} \Pi, \widehat{\Gamma}_\kappa$.

Let $\rho \in \mathtt{Q}^t$. We have $b^{(\rho)} \in \mathcal{H}_\gamma[\Theta^{(\rho)}]$, $SC_{\mathbb{K}}(m(\rho)) \subset \mathcal{H}_{\gamma_0}[\Theta^{(\rho)}]$ and $\Theta^{(\rho)} \subset M_{\kappa(\rho)}$. $SC_{\mathbb{K}}(h^{b^{(\rho)}}(m(\rho);\omega(b_1,a))) \subset \mathcal{H}_{\gamma_0}[\Theta^{(\rho)}]$ follows. Moreover we have $SC_{\mathbb{K}}(m(\kappa(\rho))) \subset \mathcal{H}_{\gamma_0}[\Theta^{(\rho)}] \subset M_{\kappa(\rho)}$.

Consider the case when the last inference is a $(\text{rfl}(\rho, d, f, b_2))$ for a $\rho \in \mathtt{Q}$. The case $\rho \in \mathtt{Q}^f$ is seen from SIH. Assume $\rho \in \mathtt{Q}^t$. Let $b = b^{(\rho)}$, $g = m(\rho)$, $b_1 \geq s(\rho) \geq d \in \text{supp}(g)$, $\kappa = \kappa(\rho)$, $\Gamma = \Gamma_\rho$, $\widehat{\Lambda} = \bigcup_{\rho \neq \tau \in \mathtt{Q}^t}\{\Gamma^{(\tau)}_\tau\}$, and $\widehat{\Lambda}_\kappa = \bigcup_{\rho \neq \tau \in \mathtt{Q}^t}\{\Gamma^{\kappa(\tau)}_\tau\}$. We have a sequent $\Delta \subset \bigvee(d)$ such that $\text{rk}(\Delta) < d \leq s(\rho) \leq b_1$ and $\mathsf{k}^{\mathbb{S}}(\Delta) \subset \mathcal{H}_\gamma[\Theta_{\mathtt{Q}}] \subset M_{\mathtt{Q}}$ by (37) and $\mathsf{k}^{\mathbb{S}}(\Delta) \subset M_{\mathtt{Q}(\kappa)}$ by $\Theta_{\mathtt{Q}} = \Theta_{\mathtt{Q}(\kappa)}$. There is an ordinal $a_0 \in \mathcal{H}_\gamma[\Theta_{\mathtt{Q}}] \cap a$ such that $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a_0}_{c_1,e} \Pi, \widehat{\Lambda}, \Gamma^{(\rho)}, \neg\delta^{(\rho)}$ for each $\delta \in \Delta$. For each $\delta \in \Delta \subset \bigvee(d)$ with $\text{rk}(\delta) \geq \mathbb{S}$, we have $\delta \simeq \bigvee(\delta_\iota)_{\iota \in J}$. Let $b_0 = \max(\{\mathbb{S}\} \cup \{\text{rk}(\delta) : \delta \in \Delta\})$. Then $s(\rho) > b_0 \in \mathcal{H}_\gamma[\Theta_{\mathtt{Q}}]$. Inversion 6.42 yields for $\text{rk}(\delta) \geq \mathbb{S}$

$$(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\iota), \mathtt{Q}) \vdash^{a_0}_{c_1,e} \Pi, \widehat{\Lambda}, \Gamma^{(\rho)}, \neg(\delta_\iota)^{(\rho)} \tag{39}$$

for each $\iota \in [\mathtt{Q}]_{\delta^{(\rho)}} J$, where $J \subset Tm(b_0)$ and $\neg\delta_\iota \in \bigvee(b_0)$ by $\text{rk}(\delta_\iota) < \text{rk}(\delta)$.

On the other side for each $\sigma \in H_\rho(f, b_2, \gamma_0, \Theta^{(\rho)})$

$$(\mathcal{H}_\gamma, \Theta \cup \{\sigma\}, \mathtt{Q} \cup \{\sigma\}) \vdash^{a_0}_{c_1, e} \Pi, \widehat{\Lambda}, \Gamma^{(\rho)}, \Delta^{(\sigma)} \tag{40}$$

f is a special finite function such that $s(f) \leq b_2$, $f_d = g_d$, $f^d <^d g'(d)$ and $SC_{\mathbb{K}}(f) \subset \mathcal{H}_{\gamma_0}[\Theta^{(\rho)}]$. Let $(\mathtt{Q} \cup \{\sigma\})^f = \mathtt{Q}^f \cup \{\sigma\}$.
Case 1. $b_0 < b$: Then $\mathrm{rk}(\Delta) < b$. Let $\mathrm{rk}(\delta) \geq \mathbb{S}$. From (39) we obtain by SIH with $b > b_0 \geq \mathbb{S}$, $(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\iota), \mathtt{Q}(\kappa)) \vdash^{\omega(b_1, a_0)}_{c_{b_1}, e^\kappa} \Pi, \widehat{\Lambda}_\kappa, \Gamma^{(\kappa)}, \neg(\delta_\iota)^{(\kappa)}$ for each $\iota \in [\mathtt{Q}(\kappa)]_{\delta^{(\kappa)}} J \subset [\mathtt{Q}]_{\delta^{(\rho)}} J$. An inference $(\bigwedge)$ yields

$$(\mathcal{H}_\gamma, \Theta, \mathtt{Q}(\kappa)) \vdash^{\omega(b_1, a_0)+1}_{c_{b_1}, e^\kappa} \Pi, \widehat{\Lambda}_\kappa, \Gamma^{(\kappa)}, \neg\delta^{(\kappa)} \tag{41}$$

Moreover SIH yields (41) for $\mathrm{rk}(\delta) < \mathbb{S}$. Let $d_1 = \min\{b, d\}$. Then $\Delta \subset \bigvee(d_1)$ by $b > b_0$.

We claim for the special finite function $h = h^b(g; \omega(b_1, a))$ that

$$f_{d_1} = h_{d_1} \,\&\, f^{d_1} <^{d_1} h'(d_1) \tag{42}$$

If $d_1 = d \leq b$, then $h_d = g_d$ and $g'(d) = g(d) \leq h'(d)$. Proposition 6.5 yields the claim. If $d_1 = b < d$, then Proposition 6.46.2 yields the claim.

On the other hand, for each $\sigma \in H_\kappa(f, b_2, \gamma_0, \Theta^{(\rho)}) \subset H_\rho(f, b_2, \gamma_0, \Theta^{(\rho)})$ we have by (40) and SIH,

$$(\mathcal{H}_\gamma, \Theta \cup \{\sigma\}, \mathtt{Q}(\kappa) \cup \{\sigma\}) \vdash^{\omega(b_1, a_0)}_{c_{b_1}, e^\kappa} \Pi, \widehat{\Lambda}_\kappa, \Gamma^{(\kappa)}, \Delta^{(\sigma)} \tag{43}$$

We have $\kappa = \kappa(\rho) < \kappa(\rho) + 1 \leq e^\kappa$ for (r0). An inference $(\mathrm{rfl}(\kappa, d_1, f, b_2))$ with (42), (41) and (43) yields $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}(\kappa)) \vdash^{\omega(b_1, a)}_{c_{b_1}, e^\kappa} \Pi, \widehat{\Lambda}_\kappa, \Gamma^{(\kappa)}$, where $d_1 \in \mathrm{supp}(m(\kappa))$ and $\mathsf{k}^{\mathbb{S}}(\Delta) \subset \mathcal{H}_\gamma[\Theta_{\mathtt{Q}(\kappa)}]$.
Case 2. $b \leq b_0$: When $b = b_0$, let $\tau = \kappa$. When $b < b_0$, let $\tau \in H_\rho(h, b_2, \gamma_0, \Theta^{(\rho)})$ be such that $\kappa < \tau$ and $m(\tau) = h = h^{b_0}(g; a_1)$ with $a_1 = \omega(b_1, a_0) + 1$.

Let $\sigma \in H_\tau(f, b_2, \gamma_0, \Theta^{(\rho)})$. SIH with (40) and $b_0 < s(\rho)$ yields

$$(\mathcal{H}_\gamma, \Theta \cup \{\sigma\}, \mathtt{Q}_\tau \cup \{\sigma\}) \vdash^{\omega(b_1, a_0)}_{c_{b_1}, e^\tau} \Delta^{(\sigma)}, \Pi, \widehat{\Lambda}_\kappa, \Gamma^{(\tau)} \tag{44}$$

where $\mathtt{Q}_\tau = \mathtt{Q}^f \cup \{\kappa(\lambda) : \rho \neq \lambda \in \mathtt{Q}^t\} \cup \{\tau\}$, and $e^\tau = \max(\{\lambda \in \mathtt{Q}^f : s(\lambda) > \mathbb{S}\} \cup \{\kappa(\lambda) : \rho \neq \lambda \in \mathtt{Q}^t\} \cup \{\tau\}) + 1$. Let $\sigma \in R := \{\sigma \in H_\tau(f, b_2, \gamma_0, \Theta^{(\rho)}) : (m(\sigma))' \geq (h^{b_0}(g; \omega(b_1, a_0)))'\}$. We see $\sigma \in H_\rho(h^{b_0}(g; \omega(b_1, a_0)), b_2, \gamma_0, \Theta^{(\rho)})$. Moreover $\mathrm{rk}(\neg\delta_\iota) < b_0$ if $\mathrm{rk}(\delta) \geq \mathbb{S}$, and $\mathrm{rk}(\neg\delta) < b_0$ if $\mathrm{rk}(\delta) < \mathbb{S} \leq b_0$.

For each $\iota \in [\mathtt{Q}]_{\delta^{(\rho)}} J$ and $\mathrm{rk}(\delta) \geq \mathbb{S}$, we obtain $(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\iota), \mathtt{Q}_\sigma) \vdash^{\omega(b_1, a_0)}_{c_{b_1}, e^\sigma} \Pi, \widehat{\Lambda}_\kappa, \Gamma^{(\sigma)}, \neg(\delta_\iota)^{(\sigma)}$ by $\mathrm{rk}(\neg\delta_\iota) < b_0$, SIH and (39), where $\mathtt{Q}_\sigma \cup \{\tau\} = \mathtt{Q}_\tau \cup \{\sigma\}$. A $(\bigwedge)$ yields $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}_\sigma) \vdash^{\omega(b_1, a_0)+1}_{c_{b_1}, e^\sigma} \Pi, \widehat{\Lambda}_\kappa, \Gamma^{(\sigma)}, \neg\delta^{(\sigma)}$. When $\mathrm{rk}(\delta) < \mathbb{S}$, this follows from SIH. Also $M_{\mathtt{Q}_\sigma} = M_{\mathtt{Q}_\sigma \cup \{\tau\}}$ and $e^\sigma \leq e^\tau$ by $\tau > \sigma$. Therefore

$$(\mathcal{H}_\gamma, \Theta, \mathtt{Q}_\tau \cup \{\sigma\}) \vdash^{\omega(b_1, a_0)+1}_{c_{b_1}, e^\tau} \Pi, \widehat{\Lambda}_\kappa, \Gamma^{(\sigma)}, \neg\delta^{(\sigma)} \tag{45}$$

From (44) and (45) by several (cut)'s of δ with $\mathrm{rk}(\delta) < d \leq b_1 \leq c_{b_1}$ we obtain for a $p < \omega$,

$$\forall \sigma \in R\left[(\mathcal{H}_\gamma, \Theta \cup \{\sigma\}, \mathtt{Q}_\tau \cup \{\sigma\}) \vdash^{\omega(b_1,a_0)+p}_{c_{b_1},e^\tau} \Pi, \widehat{\Lambda}_\kappa, \Gamma^{(\sigma)}, \Gamma^{(\tau)}\right] \tag{46}$$

On the other hand we have $r = \max\{\mathbb{S}, \mathrm{rk}(\Gamma)\} \leq b < b_1$ and $\mathsf{k}^{\mathbb{S}}(\Gamma) \subset \mathcal{H}_\gamma[\Theta_{\mathtt{Q}}] = \mathcal{H}_\gamma[\Theta_{\mathtt{Q}_\tau}] \subset M_{\mathtt{Q}_\tau}$ by (37), where $\Theta_{\mathtt{Q}} = \Theta_{\mathtt{Q}_\tau}$ by $\Theta^{(\rho)} \subset M_\tau$. Tautology 6.41 yields for each $\theta \in \Gamma$

$$(\mathcal{H}_\gamma, \Theta, \mathtt{Q}_\tau) \vdash^{2r}_{0,0} \Gamma^{(\tau)}, \neg\theta^{(\tau)} \tag{47}$$

Let us define a finite function h by $\mathrm{supp}(h) = \mathrm{supp}(g_{b_0}) \cup \mathrm{supp}(f^{b_0+1}) \cup \{b_0\}$, $h_{b_0} = g_{b_0}$ and $h^{b_0+1} = f^{b_0+1}$. Let $(h^{b_0}(g; \omega(b_1, a_0)))(b_0) = \alpha + \mathbb{K}$. Then $h(b_0) = \alpha$ if $f^{b_0+1} \neq \emptyset$. Otherwise $h(b_0) = \alpha + \mathbb{K}$. We see that $R = H_\tau(h, \gamma_0, \Theta^{(\rho)})$, and $h^{b_0} <^{b_0} (m(\tau))'(b_0)$.

By an inference $(\mathrm{rfl}(\tau, b_0, h, b_2))$ with its resolvent class $R = H_\tau(h, b_2, \gamma_0, \Theta^{(\rho)})$ and $\Gamma \subset \bigvee(b_0)$ we conclude from (47) and (46) for $\mathrm{rk}(\Gamma) < b \leq b_0 \leq s(\tau)$

$$(\mathcal{H}_\gamma, \Theta, \mathtt{Q}_\tau) \vdash^{a_2}_{c_{b_1},e^\tau} \Pi, \widehat{\Lambda}_\kappa, \Gamma^{(\tau)} \tag{48}$$

where $a_2 = \max\{2r, \omega(b_1, a_0) + p\} + 1 < \omega(b_1, a) = \omega^{\omega^{b_1}} a$. If $b_0 = b$, we are done. In what follows assume $b < b_0$. We have $a_1 < \omega(b_1, a)$ and $\omega(b_0, a_2) = \omega^{\omega^{b_0}} a_2 < \omega(b_1, a)$ by $b_0 < b_1$. Moreover Proposition 6.46.1 for $m(\tau) = h^{b_0}(g; a_1)$ yields $(h^b(m(\tau); \omega(b_0, a_2)))' = (h^b(h^{b_0}(g; a_1); \omega(b_0, a_2)))' \leq (h^b(g; \omega(b_1, a)))'$.

Let $(\mathtt{Q}_\tau)^t = \{\tau\}$ and $\kappa(\tau) = \kappa(\rho) = \kappa$. Then $(e^\tau)^\kappa = \max(\{\lambda \in (\mathtt{Q}_\tau)^f : s(\lambda) > \mathbb{S}\} \cup \{\kappa\}) + 1 = e^\kappa$. We have $\mathsf{k}^{\mathbb{S}}(\Gamma) \cup \{b_0\} \subset \mathcal{H}_\gamma[\Theta_{\mathtt{Q}_\tau}]$, $\mathrm{rk}(\Gamma_\rho) < b^{(\rho)} = b < b_0 = s(\tau) < b_1$ for $\Gamma = \Gamma_\rho$ and $b \in \mathcal{H}_\gamma[\Theta^{(\tau)}]$, $\omega(b_0, a_2) < \omega(b_1, a)$ and $\max\{c_{b_1}, b_0\} = c_{b_1}$. Also $\kappa \in H_\rho(h^b(g; \omega(b_1, a)), b_2, \gamma_0, \Theta^{(\rho)}) \cap \tau \subset H_\tau(h^b(m(\tau); \omega(b_1, a_2)), b_2, \gamma_0, \Theta^{(\rho)})$. MIH with (48) yields $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}(\kappa)) \vdash^{\omega(b_1,a)}_{c_{b_1},e^\kappa} \Pi, \Gamma^{(\kappa)}$.

Second consider the case when the last inference $(\bigvee)$ introduces a $\bigvee$-formula B: If $B \in \Pi$, SIH yields the lemma. Assume that $B \equiv A^{(\rho)} \in \Gamma^{(\rho)}_\rho$ with $A \simeq \bigvee(A_\iota)_{\iota \in J}$ and $\rho \in \mathtt{Q}$. We may assume $\rho \in \mathtt{Q}^t$. We have $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a_0}_{c_1,e} \Pi, \widehat{\Gamma}, (A_\iota)^{(\rho)}$, where $a_0 < a$, $\iota \in [\rho]J$. We claim that $\iota \in [\kappa(\rho)]J$. We may assume $\mathsf{k}(\iota) \subset \mathsf{k}(A_\iota)$. We have $\mathsf{k}(A_\iota) \subset \mathcal{H}_\gamma[\Theta^{(\rho)}]$ by (36). $\Theta^{(\rho)} \subset M_{\kappa(\rho)}$ yields $\mathsf{k}(A_\iota) \subset M_{\kappa(\rho)}$.

Let $A_\iota \simeq \bigwedge(B_\nu)_{\nu \in I}$ for $\bigvee$-formulas B_ν, and assume $\mathrm{rk}(A_\iota) \geq \mathbb{S}$. Inversion 7.25 yields for each $\nu \in [\mathtt{Q}]_{A_\iota^{(\rho)}} I$, $(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\nu), \mathtt{Q}) \vdash^{a_0}_{c_1,e} \Pi, \widehat{\Gamma}, (B_\nu)^{(\rho)}$.

SIH yields for each $\nu \in [\mathtt{Q}(\kappa)]_{A_\iota^{(\rho)}} I \subset [\mathtt{Q}]_{A_\iota^{(\rho)}} I$ that $(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\nu), \mathtt{Q}(\kappa)) \vdash^{\omega(b_1,a_0)}_{c_{b_1},e^\kappa} \Pi, \widehat{\Gamma}_\kappa, (B_\nu)^{(\kappa)}$. $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}(\kappa)) \vdash^{\omega(b_1,a_0)+1}_{c_{b_1},e^\kappa} \Pi, \widehat{\Gamma}_\kappa, (A_\iota)^{(\kappa)}$ follows from a $(\bigwedge)$. An inference $(\bigvee)$ yields $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}(\kappa)) \vdash^{\omega(b_1,a)}_{c_{b_1},e^\kappa} \Pi, \widehat{\Gamma}_\kappa$.

Other cases are seen from SIH. □

For $c \leq \mathbb{S}$, $(\mathcal{H}_\gamma, \Theta) \vdash^{*a}_c \Gamma$ denotes $(\mathcal{H}_\gamma, \Theta; \emptyset) \vdash^{*a}_c \Gamma; \emptyset$. Since $\Theta_\emptyset = \Theta$, (34) and (35) amount to (3) $\{\gamma, a, c\} \cup \mathsf{k}(\Gamma) \subset \mathcal{H}_\gamma[\Theta]$, and there occurs no inferences

$(\bigvee)^{[\cdot]}$, $(\bigwedge)^{[\cdot]}$ nor (stbl). The inference (Σ-rfl) is only on Ω. This means that $(\mathcal{H}_\gamma, \Theta) \vdash^{*a}_c \Gamma$ is equivalent to $\mathcal{H}_\gamma[\Theta] \vdash^a_c \Gamma$ in Definition 1.16.

Lemma 6.48 (Elimination of inferences (rfl))
Let $\mathtt{Q}$ *be a finite family for* γ_0 *and* $b_1 \geq \mathbb{S}$. *Let* $\max(\mathrm{rk}(\Gamma)) < \mathbb{S}$, $\widehat{\Gamma} = \bigcup\{\Gamma^{(\rho)}_\rho : \rho \in \mathtt{Q}\}$ *and* $\Gamma = \bigcup\{\Gamma_\rho : \rho \in \mathtt{Q}\}$, *where* $\mathsf{k}(\Gamma_\rho) \subset M_\rho$. *Suppose* $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_{\mathbb{S},e,\gamma_0,b_1} \widehat{\Gamma}$.

Then $(\mathcal{H}_{\gamma_1}, \Theta) \vdash^{*\tilde{a}}_{\mathbb{S}} \Gamma$ *holds for* $\gamma_1 = \gamma_0 + \mathbb{S}$, $\tilde{a} = \varphi_e(b_1 + a)$.

Proof. By main induction on e with subsidiary induction on a. We have $\{e\} \cup \mathtt{Q} \subset \mathcal{H}_{\gamma_1}$ by Definitions 6.40 and 6.38, $b_1 \in \mathcal{H}_\gamma[\Theta_{\mathtt{Q}}]$ by (37), and $\emptyset = \mathsf{k}^{\mathbb{S}}(\Gamma) \subset \mathcal{H}_\gamma[\Theta_{\mathtt{Q}}]$.
Case 1. First let $\{\neg A^{(\sigma)}, A^{(\sigma)}\} \subset \widehat{\Gamma}$ with $\mathrm{rk}(A) < \mathbb{S}$ by (Taut). Then $(\mathcal{H}_0, \mathsf{k}(A)) \vdash^{*\mathbb{S}}_0 \neg A, A$ by Tautology 6.29.1 and $(\mathcal{H}_{\gamma_1}, \Theta) \vdash^{*\tilde{a}}_{\mathbb{S}} \Gamma$ by $\tilde{a} > \mathbb{S}$.
Case 2. Second consider the case when the last inference is a $(\mathrm{rfl}(\rho, d, f, b_1))$ for a $\rho \in \mathtt{Q}$. Let $\mathtt{Q}^t = \{\tau \in \mathtt{Q} : s(\tau) > \mathbb{S}\}$, $\mathtt{Q}^f = \mathtt{Q} \setminus \mathtt{Q}^t$, and $\kappa(\tau) \in H_\tau(h^{\mathbb{S}}(m(\tau); \omega(b, a)), b_1, \gamma_0, \Theta^{(\tau)})$ for each $\tau \in \mathtt{Q}^t$. Let $g = m(\rho)$, $s(\rho) \geq d \in \mathrm{supp}(g)$, $\kappa = \kappa(\rho)$ when $\rho \in \mathtt{Q}^t$, $\widehat{\Pi} = \bigcup_{\rho \neq \tau \in \mathtt{Q}^f} \Gamma^{(\tau)}_\tau$, $\widehat{\Lambda} = \bigcup_{\rho \neq \tau \in \mathtt{Q}^t} \Gamma^{(\tau)}_\tau$, and $\widehat{\Lambda}_\kappa = \bigcup_{\rho \neq \tau \in \mathtt{Q}^t} \Gamma^{\kappa(\tau)}_\tau$. We have a sequent $\Delta \subset \bigvee(d)$ and an ordinal $a_0 < a$ such that $\mathrm{rk}(\Delta) < d \leq s(\rho)$ and $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a_0}_{\mathbb{S},e,\gamma_0,b_1} \widehat{\Pi}, \widehat{\Lambda}, \Gamma^{(\rho)}_\rho, \neg\delta^{(\rho)}$ for each $\delta \in \Delta$. On the other hand we have $(\mathcal{H}_\gamma, \Theta \cup \{\sigma\}, \mathtt{Q} \cup \{\sigma\}) \vdash^{a_0}_{\mathbb{S},e,\gamma_0,b_1} \widehat{\Pi}, \widehat{\Lambda}, \Gamma^{(\rho)}_\rho, \Delta^{(\sigma)}$, where $\sigma \in H_\rho(f, b_1, \gamma_0, \Theta^{(\rho)})$, f is a special finite function such that $s(f) \leq b_1$, $f_d = g_d$, $f^d <^d g'(d)$ and $SC_{\mathbb{K}}(f) \subset \mathcal{H}_{\gamma_0}[\Theta^{(\rho)}]$.
Case 2.1 $s(\rho) \leq \mathbb{S}$: We have $\mathrm{rk}(\Delta) < d \leq s(\rho) \leq \mathbb{S}$. Let $\tilde{a}_0 = \varphi_e(b_1 + a_0)$. By SIH we obtain $(\mathcal{H}_{\gamma_1}, \Theta) \vdash^{*\tilde{a}_0}_{\mathbb{S}} \Pi, \Lambda, \Gamma_\rho, \neg\delta$ for each $\delta \in \Delta$, and $(\mathcal{H}_{\gamma_1}, \Theta \cup \{\sigma\}) \vdash^{*\tilde{a}_0}_{\mathbb{S}} \Pi, \Lambda, \Gamma_\rho, \Delta$, where $\sigma \in \mathcal{H}_{\gamma_0+\mathbb{S}} \subset \mathcal{H}_{\gamma_1}[\Theta]$. Several (cut)'s of $\mathrm{rk}(\delta) < \mathbb{S}$ yields $(\mathcal{H}_{\gamma_1}, \Theta) \vdash^{*\tilde{a}}_{\mathbb{S}} \Pi, \Lambda, \Gamma_\rho$ for $\Gamma = \Pi \cup \Lambda \cup \Gamma_\rho$.
Case 2.2. $s(\rho) > \mathbb{S}$: Then $\rho \in \mathtt{Q}^t \neq \emptyset$. $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}(\kappa)) \vdash^{\omega(b_1,a)}_{b_1,e^\kappa,\gamma_0,b_1} \widehat{\Pi}, \widehat{\Lambda}_\kappa, \Gamma^{(\kappa)}_\rho$ follows by Recapping 6.47, where $b_1 \geq \mathbb{S}$ and $e^\kappa < e$. Cut-elimination 6.43 yields for $a_1 = \varphi_{b_1}(\omega(b_1, a))$, $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}(\kappa)) \vdash^{a_1}_{\mathbb{S},e^\kappa,\gamma_0,b_1} \widehat{\Pi}, \widehat{\Lambda}_\kappa, \Gamma^{(\kappa)}_\rho$. MIH then yields $(\mathcal{H}_{\gamma_1}, \Theta) \vdash^{*\tilde{a}_1}_{\mathbb{S}} \Gamma$, where $\Gamma = \Pi \cup \Lambda \cup \Gamma_\rho$ and $\tilde{a}_1 = \varphi_{e^\kappa}(b_1 + a_1) < \varphi_e(b_1 + a) = \tilde{a}$ by $e^\kappa < e$ and $a, b_1 < \tilde{a}$.
Case 3. The last inference is a $(\bigwedge)$: We have $a(\iota) < a$, $A^{(\rho)} \in \widehat{\Gamma}$ with $A \simeq \bigwedge(A_\iota)_{\iota \in J}$, and $(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\iota), \mathtt{Q}) \vdash^{a(\iota)}_{\mathbb{S},e,\gamma_0,b_1} \widehat{\Gamma}, (A_\iota)^{(\rho)}$ for each $\iota \in [\mathtt{Q}]_{A^{(\rho)}} J$. Since $A \in \Delta_0(\mathbb{S})$, we obtain $[\mathtt{Q}]_{A^{(\rho)}} J = [\rho] J = J$. SIH yields $(\mathcal{H}_{\gamma_1}, \Theta) \vdash^{*\tilde{a}(\iota)}_{\mathbb{S}} \Gamma, A_\iota$ for each $\iota \in J$, where $\tilde{a}(\iota) = \varphi_e(b_1 + a(\iota)) < \tilde{a}$. A $(\bigwedge)$ yields $(\mathcal{H}_{\gamma_1}, \Theta) \vdash^{*\tilde{a}}_{\mathbb{S}} \Gamma$.

Other cases are seen from SIH. □

Proposition 6.49 (Collapsing) *Suppose* $\Theta \subset \mathcal{H}_\gamma(\psi_\Omega(\gamma))$, $(\mathcal{H}_\gamma, \Theta) \vdash^{*a}_\Omega \Gamma$ *and* $\Gamma \subset \Sigma(\Omega)$. *Then for* $\hat{a} = \gamma + \omega^a$ *and* $\beta = \psi_\Omega(\hat{a})$, $(\mathcal{H}_{\hat{a}+1}, \Theta) \vdash^{*\beta}_\beta \Gamma^{(\beta,\Omega)}$ *holds.*

Proposition 6.50 (Cut-elimination) *Suppose* $(\mathcal{H}_\gamma, \Theta) \vdash^{*a}_{c+d} \Gamma$ *with* $c + d \leq \mathbb{S}$ *and* $\neg(c < \Omega \leq c + d)$. *Then* $(\mathcal{H}_\gamma, \Theta) \vdash^{*\theta_d(a)}_c \Gamma$.

Theorem 6.51 *Assume* $S_1 \vdash \theta^{L_\Omega}$ *for* $\theta \in \Sigma$. *Then there exists an* $n < \omega$ *such that* $L_\alpha \models \theta$ *for* $\alpha = \psi_\Omega(\omega_n(\mathbb{K} + 1))$ *in* $OT(\Pi^1_1)$.

Proof. Let $S_1 \vdash \theta^{L_\Omega}$ for a Σ-sentence θ. By Embedding 6.30 pick an m so that $(\mathcal{H}_{\mathbb{S}}, \emptyset; \emptyset) \vdash^{*\mathbb{K}\cdot 2+m}_{\mathbb{K}+m} \theta^{L_\Omega}; \emptyset$. Cut-elimination 6.34 yields $(\mathcal{H}_{\mathbb{S}}, \emptyset; \emptyset) \vdash^{*a}_{\mathbb{K}} \theta^{L_\Omega}$ for $a = \omega_m(\mathbb{K}\cdot 2 + m) < \omega_{m+1}(\mathbb{K}+1)$. Now let $\gamma_0 = \omega_{m+2}(\mathbb{K}+1)$. Let $\beta = \psi_{\mathbb{K}}(\omega^a) > \mathbb{S}$, where $\omega^a < \gamma_0 = \omega_{m+2}(\mathbb{K}+1)$. Collapsing 7.18 yields $(\mathcal{H}_{\omega^a+1}, \emptyset; \emptyset) \vdash^{*\beta}_{\beta} \theta^{L_\Omega}; \emptyset$.

Let $\rho = \psi^{g}_{\mathbb{S}}(\gamma_0)$ with $g = \{(\beta, \beta + \mathbb{K})\}$, where $\mathbb{K}(\beta+1) = \beta + \mathbb{K}$. We obtain $(\mathcal{H}_{\omega^a+1}, \emptyset, \{\rho\}) \vdash^{\beta}_{\beta,\rho+1,\gamma_0,\beta} (\theta^{L_\Omega})^{(\rho)}$ by Capping 6.44. Cut-elimination 6.43 yields $(\mathcal{H}_{\omega^a+1}, \emptyset, \{\rho\}) \vdash^{a_1}_{\mathbb{S},\rho+1,\gamma_0,\beta} (\theta^{L_\Omega})^{(\rho)}$ for $a_1 = \varphi_\beta(\beta)$.

We obtain $(\mathcal{H}_{\gamma_1}, \emptyset) \vdash^{*a_2}_{\mathbb{S}} \theta^{L_\Omega}$ by Lemma 6.48, where $a_2 = \varphi_{\rho+1}(\beta + a_1)$ and $\gamma_1 = \gamma_0 + \mathbb{S}$. Cut-elimination 6.50 yields $(\mathcal{H}_{\gamma_1}, \emptyset) \vdash^{*a_3}_{\Omega} \theta^{L_\Omega}$ for $a_3 = \theta_{\mathbb{S}}(a_2)$. Collapsing 6.49 yields $(\mathcal{H}_{\gamma_1+a_3+1}, \emptyset) \vdash^{*\eta}_{\eta} \theta^{L_\eta}$ for $\eta = \psi_\Omega(\gamma_1 + a_3) < \psi_\Omega(\omega_{m+3}(\mathbb{K}+1))$. Cut-elimination 6.50 yields $(\mathcal{H}_{\gamma_1+a_3+1}, \emptyset) \vdash^{*\theta_\eta(\eta)}_{0} \theta^{L_\eta}$. We then see $L_\eta \models \theta$ by induction up to $\theta_\eta(\eta)$. $\Box$

Actually the bound is shown to be tight.

Theorem 6.52 [A∞d]
$\mathsf{KP}\omega + (M \prec_{\Sigma_1} V)$ *proves the well-foundedness up to* $\psi_\Omega(\omega_n(\mathbb{S}^+ + 1))$ *for* each n.

$\mathsf{KP}\omega + (M \prec_{\Sigma_1} V)$ proves an axiom of Σ_1-Separation with parameters from M. $\exists b\, [b = \{x \in a : \varphi(x,c)\} = \{x \in a : M \models \varphi(x,c)\}]$, where $c \in M$, $a \in M \cup \{M\}$ and $\varphi \in \Sigma_1$. However it is open for us whether the parameter-free Σ^1_2-Comprehension Axiom holds in $\mathsf{KP}\omega + (M \prec_{\Sigma_1} V)$.

7 Π_1-Collection

The axioms of the set theory $\mathsf{KP}\omega + \Pi_1$-Collection $+ (V = L)$ consist of those of $\mathsf{KP}\omega + (V = L)$ plus the axiom schema Π_1-Collection: for each Π_1-formula $A(x,y)$ in the language of set theory, $\forall x \in a \exists y A(x,y) \to \exists b \forall x \in a \exists y \in b A(x,y)$. It is easy to see that the second order arithmetic $\Sigma^1_3{-}\mathrm{DC} + \mathrm{BI}$ is interpreted to $\mathsf{KP}\omega + \Pi_1$-Collection $+ (V = L)$ canonically.

Next we show that $\mathsf{KP}\omega + \Pi_1$-Collection $+ (V = L)$ is contained in a set theory $S_{\mathbb{I}}$. The language of the theory $S_{\mathbb{I}}$ is $\{\in, St, \Omega\}$ with a unary predicate constant St and an individual constant Ω. $\Delta_0(St)$ denotes the set of bounded formulas in the language $\{\in, St, \Omega\}$, in which atomic formulas $St(t)$ may occur. Similarly $\Sigma_1(St)$ the set of Σ_1-formulas in the expanded language. $St(\alpha)$ is intended to denote the fact that α is a stable ordinal, $L_\alpha \prec_{\Sigma_1} L$, and $\Omega = \omega_1^{CK}$. The axioms of $S_{\mathbb{I}}$ are obtained from those [4] of $\mathsf{KP}\omega$ by adding the following axioms. Let ON denote the class of all ordinals. For ordinals α, $\alpha^\dagger$ denotes the least stable ordinal above α. A *successor stable ordinal* is an ordinal $\alpha^\dagger$ for an α. Note that the least stable ordinal $0^\dagger$ is a successor stable ordinal.

[4] In the axiom schemata Δ_0-Separation and Δ_0-Collection, Δ_0-formulas remain to mean a Δ_0-formula in which St does not occur, while the axiom of foundation may be applied to a formula in which St may occur.

1. $V = L$, and the axioms for recursively regularity of Ω.
2. $\Delta_0(St)$-collection:
$$\forall x \in a \exists y\, \theta(x,y) \to \exists b \forall x \in a \exists y \in b\, \theta(x,y)$$
for each $\Delta_0(St)$-formula θ in which the predicate St may occurs.
3. $L = \bigcup\{L_\sigma : St(\sigma)\}$, i.e.,
$$\forall \alpha \in ON \exists \sigma\, (\alpha < \sigma \land St(\sigma)) \tag{49}$$
.
4. For a successor stable ordinal $\sigma < \mathbb{I}$, $L_\sigma \prec_{\Sigma_1} L = L_{\mathbb{I}}$:
$$SSt(\sigma) \land \varphi(u) \land u \in L_\sigma \to \varphi^{L_\sigma}(u) \tag{50}$$
for each Σ_1-formula φ in the language of set theory, i.e., the constant St does not occur in φ.

Lemma 7.1 *$S_{\mathbb{I}}$ is an extension of* $\mathsf{KP}\omega + \Pi_1$-Collection $+ (V = L)$*. Namely $S_{\mathbb{I}}$ proves* Π_1-Collection*.*

Proof. Argue in $S_{\mathbb{I}}$. Let $A(x,y)$ be a Π_1-formula in the language of set theory. We obtain by the axioms (49) and (50)
$$A(x,y) \leftrightarrow \exists \beta (St(\beta^\dagger) \land x, y \in L_{\beta^\dagger} \land A^{L_{\beta^\dagger}}(x,y)) \tag{51}$$
Assume $\forall x \in a \exists y A(x,y)$. Then we obtain $\forall x \in a \exists y \exists \beta (St(\beta^\dagger) \land x, y \in L_{\beta^\dagger} \land A^{L_{\beta^\dagger}}(x,y))$ by (51). Since $St(\beta^\dagger) \land x, y \in L_{\beta^\dagger} \land A^{L_{\beta^\dagger}}(x,y)$ is a $\Sigma_1(St)$-formula, pick a set c such that $\forall x \in a \exists y \in c \exists \beta \in c (St(\beta^\dagger) \land x, y \in L_{\beta^\dagger} \land A^{L_{\beta^\dagger}}(x,y))$ by $\Delta_0(St)$-Collection. Again by (51) we obtain $\forall x \in a \exists y \in c A(x,y)$. □

Conversely in $\mathsf{KP}\omega + \Pi_1$-Collection $+ (V = L)$, the predicate $St(\alpha)$ is defined by a Π_1-formula $st(\alpha)$ so that (50) is provable, and $\Delta_0(St)$-collection follows from Π_1-Collection.

Lemma 7.2 $\mathsf{KP}\omega + \Pi_1$-Collection *proves each of* Σ_1-Separation*,* Δ_2-Separation *and* Σ_2-Replacement*.*

Proof. We show that $\{x \in a : \varphi(x)\}$ exists as a set for a Σ_1-formula $\varphi \equiv \exists y \theta(x,y)$ with a Δ_0 matrix θ. We have by logic $\forall x \in a \exists y (\exists z \theta(x,z) \to \theta(x,y))$. By Π_1-Collection pick a set b so that $\forall x \in a \exists y \in b(\varphi(x) \to \theta(x,y))$. In other words, $\{x \in a : \varphi(x)\} = \{x \in a : \exists y \in b\, \theta(x,y)\}$. □

Let $\mathrm{Hull}_{\Sigma_1}(\alpha)$ denote the Σ_1-Skolem hull $\mathrm{Hull}_{\Sigma_1}(\alpha)$ of an ordinal α. $\mathrm{Hull}_{\Sigma_1}(\alpha)$ is the collection of Σ_1-definable elements from parameters$< \alpha$ in the universe.

Specifically let $\{\varphi_i : i \in \omega\}$ denote an enumeration of Σ_1-formulas. Each is of the form $\varphi_i \equiv \exists y \theta_i(x, y; u)$ $(\theta \in \Delta_0)$ with fixed variables x, y, u. Set for $b \in \alpha$

$$\begin{array}{rcl} r(i,b) & \simeq & \text{the } <_L \text{-least } c \in L \text{ such that } L \models \theta_i((c)_0, (c)_1; b) \\ h(i,b) & \simeq & (r(i,b))_0 \\ \mathrm{Hull}_{\Sigma_1}(\alpha) & = & \{h(i,b) \in L : i \in \omega, b \in \alpha\} \end{array}$$

The domain of the partial Δ_1-map r is a Σ_1-subset of $\omega \times \alpha$, and from Lemma 7.2 (Σ_1-Separation) we see that the domain exists as a set, and so does $\mathrm{Hull}_{\Sigma_1}(\alpha)$. Therefore its Mostowski collapse[5] ordinal $\beta \geq \alpha$. This shows (49).

Note that a limit of admissible ordinals need not to be admissible since there exists a Π_3^--formula ad such that for any transitive set x, x is admissible iff ad^x holds. On the other side every limit κ of stable ordinals is stable: for $c \in L_\kappa$, pick a stable ordinal $\sigma < \kappa$ such that $c \in L_\sigma$. Then for Σ_1-formula A, $L \models A(c) \Rightarrow L_\sigma \models A(c) \Rightarrow L_\kappa \models A(c)$.

7.1 Ordinals for Π_1-Collection

In this subsection up to subsection 7.2 we work in a set theory $\mathsf{ZFC}(St)$, where St is a unary predicate symbol. We assume that St is an unbounded class of ordinals below $\mathbb{I}$ such that the least element $\mathbb{S}_0$ of St is larger than Ω. $\alpha^\dagger$ denotes the least ordinal$> \alpha$ in the class St when $\alpha < \mathbb{I}$. $\alpha^\dagger := \mathbb{I}$ if $\alpha \geq \mathbb{I}$. Then $\mathbb{S}_0 = \Omega^\dagger$. Let $SSt := \{\alpha^\dagger : \alpha \in ON\}$ and $LS = St \setminus SSt$. For natural numbers k, $\alpha^{\dagger k}$ is defined recursively by $\alpha^{\dagger 0} = \alpha$ and $\alpha^{\dagger(k+1)} = (\alpha^{\dagger k})^\dagger$.

$\varphi_b(\xi)$ denotes the binary Veblen function on $\mathbb{I}^+ = \omega_{\mathbb{I}+1}$ with $\varphi_0(\xi) = \omega^\xi$ Let $\Lambda \leq \mathbb{I}$ be a strongly critical number. As in Definition 6.2, $\tilde{\varphi}_b(\xi) := \varphi_b(\mathbb{I} \cdot \xi)$. Let $b, \xi < \mathbb{I}^+$. $\theta_b(\xi)$ $[\tilde{\theta}_b(\xi)]$ denotes a b-th iterate of $\varphi_0(\xi) = \omega^\xi$ [of $\tilde{\varphi}_0(\xi) = \mathbb{I}^\xi$], resp.

Definition 7.3 A finite function $f : \mathbb{I} \to \varphi_{\mathbb{I}}(0)$ is said to be a *finite function* if $\forall i > 0(a_i = 1)$ and $a_0 = 1$ when $b_0 > 1$ in $f(c) =_{NF} \tilde{\theta}_{b_m}(\xi_m) \cdot a_m + \cdots + \tilde{\theta}_{b_0}(\xi_0) \cdot a_0$ for any $c \in \mathrm{supp}(f)$. Let $SC_{\mathbb{I}}(f) := \bigcup \{\{c\} \cup SC_{\mathbb{I}}(f(c)) : c \in \mathrm{supp}(f)\}$.

For a finite function f, $c < \mathbb{I}$ and $\xi < \varphi_{\mathbb{I}}(0)$. A relation $f <^c_{\mathbb{I}} \xi$ is defined by induction on the cardinality of the finite set $\{d \in \mathrm{supp}(f) : d > c\}$ as in Definition 6.4.2.

Definition 7.4 Let $A \subset \mathbb{I}$ be a set, and $\alpha \leq \mathbb{I}$ a limit ordinal.

$$\alpha \in M(A) :\Leftrightarrow A \cap \alpha \text{ is stationary in } \alpha \Leftrightarrow \text{ every club subset of } \alpha \text{ meets } A.$$

Classes $\mathcal{H}_a(X) \subset \Gamma_{\mathbb{I}+1}$, $Mh^a_c(\xi) \subset (\mathbb{I}+1)$, and ordinals $\psi^f_\kappa(a) \leq \kappa$ are defined simultaneously as follows.

$\mathcal{H}_a(X)$ denotes the closure of $\{0, \Omega, \mathbb{I}\} \cup X$ under $+, \varphi$, $a \mapsto \psi_\Omega(a)$, $a \mapsto \psi_{\mathbb{I}}(a) \in LS$, $\alpha \mapsto \alpha^\dagger \in SSt$, and $(\pi, b, f) \mapsto \psi^f_\pi(b)$.

[5] The collapse coincides with L_β for the least ordinal β not in $\mathrm{Hull}_{\Sigma_1}(\alpha)$.

$\pi \in Mh^a_c(\xi)$ iff $\{a, c, \xi\} \subset \mathcal{H}_a(\pi)$ and the following condition is met for any finite functions $f, g : \mathbb{I} \to \varphi_{\mathbb{I}}(0)$ such that $f <^c_{\mathbb{I}} \xi$

$$SC_{\mathbb{I}}(f, g) \subset \mathcal{H}_a(\pi) \,\&\, \pi \in Mh^a_0(g_c) \Rightarrow \pi \in M(Mh^a_0(g_c * f^c))$$

where

$$\begin{array}{rcl} Mh^a_c(f) & := & \bigcap\{Mh^a_d(f(d)) : d \in \mathrm{supp}(f^c)\} \\ & = & \bigcap\{Mh^a_d(f(d)) : c \leq d \in \mathrm{supp}(f)\} \end{array}$$

Let a, π ordinals and $f : \mathbb{I} \to \varphi_{\mathbb{I}}(0)$ a finite function. Then $\psi^f_\pi(a)$ denotes the least ordinal $\kappa < \pi$ such that

$$\kappa \in Mh^a_0(f) \,\&\, \mathcal{H}_a(\kappa) \cap \pi \subset \kappa \,\&\, \{\pi, a\} \cup SC_{\mathbb{I}}(f) \subset \mathcal{H}_a(\kappa) \tag{52}$$

if such a κ exists. Otherwise set $\psi^f_\pi(a) = \pi$.

$$\psi_{\mathbb{I}}(a) := \min(\{\mathbb{I}\} \cup \{\kappa \in LS : \mathcal{H}_a(\kappa) \cap \mathbb{I} \subset \kappa\}) \tag{53}$$

For classes $A \subset \mathbb{I}$, let $\alpha \in M^a_c(A)$ iff $\alpha \in A$ and for any finite functions $g : \mathbb{I} \to \varphi_{\mathbb{I}}(0)$

$$\alpha \in Mh^a_0(g_c) \,\&\, SC_{\mathbb{I}}(g_c) \subset \mathcal{H}_a(\alpha) \Rightarrow \alpha \in M\left(Mh^a_0(g_c) \cap A\right) \tag{54}$$

Proposition 7.5 *Each of $x \in \mathcal{H}_a(y)$, $x \in Mh^a_c(f)$ and $x = \psi^f_\kappa(a)$ is a $\Delta_1(St)$-predicate in* $\mathsf{ZFC}(St)$.

7.2 A small large cardinal hypothesis

It is convenient for us to assume the existence of a small large cardinal in justification of the above definition.

Subtle cardinals are introduced by R. Jensen and K. Kunen. It is shown in Lemma 2.7 of [Rathjen05b] that the set of shrewd cardinals in V_π is stationary in a subtle cardinal π. From this fact we see that the set of shrewd limits of shrewd cardinals in V_π is also stationary in a subtle cardinal π, where for a shrewd cardinal κ in V_π, κ is a shrewd limit iff κ is a limit of shrewd cardinals in V_π.

Let C be a closed subset of π, and $C_0 \subset C$ be a subset defined by $\kappa \in C_0$ iff $\kappa \in C$ and κ is a limit of shrewd cardinals. Since the set of shrewd cardinals is stationary in V_π, C_0 is a club subset of π. Hence the exists a shrewd cardinal in C_0.

In this subsection we work in an extension T of ZFC by adding the axiom stating that there exists a regular cardinal $\mathbb{I}$ such that the set St of shrewd cardinals in $V_{\mathbb{I}}$ is stationary in $\mathbb{I}$. In this subsection Ω denotes the least uncountable ordinal ω_1, and LS denotes the set of shrewd limits in $V_{\mathbb{I}}$. The class LS is stationary in $\mathbb{I}$. A *successor shrewd cardinal* is a shrewd cardinal in $V_{\mathbb{I}}$, not in LS.

Lemma 7.6 $\forall a[\psi_{\mathbb{I}}(a) < \mathbb{I}]$.

Proof. The set $C = \{\kappa < \mathbb{I} : \mathcal{H}_a(\kappa) \cap \mathbb{I} \subset \kappa\}$ is a club subset of the regular cardinal $\mathbb{I}$. This shows the existence of a $\kappa \in LS \cap C$, and hence $\psi_{\mathbb{I}}(a) < \mathbb{I}$ by the definition (53). □

Lemma 7.7 *Let $\mathbb{S}$ be a shrewd cardinal, $a < \varepsilon(\mathbb{I})$, $h : \mathbb{I} \to \varphi_{\mathbb{I}}(0)$ a finite function with $\{a\} \cup SC_{\mathbb{I}}(h) \subset \mathcal{H}_a(\mathbb{S})$. Then $\mathbb{S} \in Mh_0^a(h) \cap M(Mh_0^a(h))$.*

Proof. By induction on $\xi < \varphi_{\mathbb{I}}(0)$ we show $\mathbb{S} \in Mh_c^a(\xi)$ for $\{a, c, \xi\} \subset \mathcal{H}_a(\mathbb{S})$ as in Lemma 6.13. □

Lemma 7.8 *Let $\mathbb{S}$ be a shrewd cardinal, a an ordinal, and $f : \mathbb{I} \to \varphi_{\mathbb{I}}(0)$ a finite function such that $\{a\} \cup SC_{\mathbb{I}}(f) \subset \mathcal{H}_a(\mathbb{S})$. Then $\psi_{\mathbb{S}}^f(a) < \mathbb{S}$ holds.*

Corollary 7.9 *Let $f, g : \mathbb{I} \to \varphi_{\mathbb{I}}(0)$ be finite functions and $c \in \mathrm{supp}(f)$. Assume that there exists an ordinal $d < c$ such that $(d, c) \cap \mathrm{supp}(f) = (d, c) \cap \mathrm{supp}(g) = \emptyset$, $g_d = f_d$, $g(d) < f(d) + \tilde{\theta}_{c-d}(f(c); \mathbb{I}) \cdot \omega$, and $g <_{\mathbb{I}}^c f(c)$.*

Then $Mh_0^a(g) \prec Mh_0^a(f)$ holds. In particular if $\pi \in Mh_0^a(f)$ and $SC_{\mathbb{I}}(g) \subset \mathcal{H}_a(\pi)$, then $\psi_\pi^g(a) < \pi$.

Proof. This is seen as in Corollary 6.17. □

An *irreducibility* of finite functions $f : \mathbb{I} \to \varphi_{\mathbb{I}}(0)$ is defined as in Definition 6.9, and a lexicographic order $f <_{lx}^b g$ on finite functions f, g as in Definition 6.10. Then $f <_{lx}^0 g \Rightarrow Mh_0^a(f) \prec Mh_0^a(g)$ is seen as in Proposition 6.18.

A computable notation system $OT(\mathbb{I})$ for Π_1-collection is defined so as to be closed under Mostowski collapsings. A new constructor $\mathbb{I}[\cdot]$ is used to generate terms in $OT(\mathbb{I})$. Note that there is no clause for constructing $\kappa = \psi_{\mathbb{S}}(a)$ from a for $\mathbb{S} \in LS$.

Definition 7.10

1. $\{(\rho, \sigma) : \rho \prec \sigma\}$ denotes the transitive closure of the relation $\{(\rho, \sigma) : \exists f, a(\rho = \psi_\sigma^f(a))\}$. Let $\rho \preceq \sigma :\Leftrightarrow \rho \prec \sigma \lor \rho = \sigma$.
2. Let $\alpha \prec \mathbb{S}$ for an $\mathbb{S} \in SSt$ and $b = \mathtt{p}_0(\alpha)$. Then let
$$M_\alpha := \mathcal{H}_b(\alpha).$$
3. For $\alpha \in \Psi$ an ordinal $\mathtt{p}_0(\alpha)$ is defined.
   (a) Let $\alpha \preceq \psi_{\mathbb{S}}^g(b)$ for an $\mathbb{S} \in SSt$. Then $\mathtt{p}_0(\alpha) = b$.
   (b) There exists an $\mathbb{S} = \mathbb{T}^\dagger \in SSt$ and a $\mathbb{T} < \tau < \mathbb{S}$ such that $\alpha \prec \tau^{\dagger k}$ for a $k > 0$. Let $\rho \prec \mathbb{S}$ be such that $\alpha = \beta[\rho/\mathbb{S}]$ for a $\beta \in M_\rho$. Let $\mathtt{p}_0(\alpha) = \mathtt{p}_0(\beta)$.
   (c) $\mathtt{p}_0(\alpha) = 0$ otherwise.

$\alpha = \psi_{\mathbb{S}}^{f}(a) \in OT(\mathbb{I})$ only if

$$SC_{\mathbb{I}}(f) \subset \mathcal{H}_a(SC_{\mathbb{I}}(a)) \tag{55}$$

where $a = \mathtt{p}_0(\alpha)$.

Let $\{\pi, a, d\} \subset OT(\mathbb{I})$ with $\pi \prec \mathbb{S} \in SSt$, $m(\pi) = f$, $d < c \in \mathrm{supp}(f)$, and $(d, c) \cap \mathrm{supp}(f) = \emptyset$.

When $g \neq \emptyset$, let g be an irreducible finite function such that $SC_{\mathbb{I}}(g) \subset OT(\mathbb{I})$, $g_d = f_d$, $(d, c) \cap \mathrm{supp}(g) = \emptyset$, $g(d) < f(d) + \tilde{\theta}_{c-d}(f(c); \mathbb{I}) \cdot \omega$, and $g <_{\mathbb{I}}^{c} f(c)$.

Then $\alpha = \psi_{\pi}^{g}(a) \in OT(\mathbb{I})$ only if

$$SC_{\mathbb{I}}(g) \subset M_\alpha \tag{56}$$

The Mostowski collapsing $\alpha \mapsto \alpha[\rho/\mathbb{S}]$ $(\alpha \in M_\rho)$ is defined as follows. $(\mathbb{S})[\rho/\mathbb{S}] := \rho$, $(\mathbb{S}^\dagger)[\rho/\mathbb{S}] := \rho^\dagger$, and $(\mathbb{I})[\rho/\mathbb{S}] := \mathbb{I}[\rho]$. $(\tau^\dagger)[\rho/\mathbb{S}] = (\tau[\rho/\mathbb{S}])^\dagger$, where $\mathbb{S} < \tau^\dagger$. $(\mathbb{I}[\tau])[\rho/\mathbb{S}] = \mathbb{I}[\tau[\rho/\mathbb{S}]]$, where $\mathbb{I}[\tau] \neq \mathbb{I}$.

A relation $\alpha < \beta$ for $\alpha, \beta \in OT(\mathbb{I})$ is defined so that $\psi_{\kappa}^{f}(a) < \kappa$ and $\rho < \psi_{\rho^\dagger}^{g}(b) < \rho^\dagger < \tau = \psi_{\mathbb{I}[\rho]}(c) < \psi_{\tau^\dagger}^{h}(d) < \tau^\dagger < \mathbb{I}[\rho]$ for every κ, ρ, a, b, c, d and f, g, h.

Proposition 7.11 *There is no $\psi_{\sigma}^{f}(a) \in OT(\mathbb{I})$ such that $\rho < \psi_{\sigma}^{f}(a) \leq \rho^\dagger < \sigma$.*

Lemma 7.12 *For $\rho \prec \mathbb{S}$ and $\mathbb{S} \in SSt$, $\{\alpha[\rho/\mathbb{S}] : \alpha \in M_\rho\}$ is a transitive collapse of M_ρ as in Lemma 6.23.*

7.3 Operator controlled derivations for Π_1-Collection

We consider *RS*-formulas in a language with a unary predicate $St(a)$, where $a = L_\kappa$ for a stable ordinal κ. Specifically $St(a) :\simeq \bigvee((\forall x \in \iota(x \in a)) \wedge (\forall x \in a(x \in \iota)))_{\iota \in J}$ with $J = \{L_\kappa : \kappa \in St \cap (|a| + 1)\}$ for $St \subset OT(\mathbb{I})$.

Definition 7.13 A *finite family* is a finite function $\mathtt{Q} \subset \coprod_{\mathbb{S}} \Psi_{\mathbb{S}}$ such that its domain $dom(\mathtt{Q})$ is a finite set of successor stable ordinals, and $\mathtt{Q}(\mathbb{S})$ is a finite set of ordinals in $\Psi_{\mathbb{S}}$ for each $\mathbb{S} \in dom(\mathtt{Q})$. Let $\mathtt{Q}(\mathbb{T}) = \emptyset$ for $\mathbb{T} \notin dom(\mathtt{Q})$ and $\bigcup \mathtt{Q} = \bigcup_{\mathbb{S} \in dom(\mathtt{Q})} \mathtt{Q}(\mathbb{S})$. Define $M_{\mathtt{Q}(\mathbb{S})} = \bigcap_{\sigma \in \mathtt{Q}(\mathbb{S})} M_\sigma$.

For $A \simeq \bigvee(A_\iota)_{\iota \in J}$ and $\iota \in J$

$$\iota \in [\mathtt{Q}]_A J = [\mathtt{Q}]_{\neg A} J :\Leftrightarrow \forall \mathbb{U} \in dom(\mathtt{Q}) \left(\mathrm{rk}(A_\iota) \geq \mathbb{U} \Rightarrow \mathsf{k}(\iota) \subset M_{\mathtt{Q}(\mathbb{U})}\right)$$

We define a derivability relation $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash_c^{*a} \Gamma; \Pi^{\cdot]}$ where c is a bound of ranks of the inference rules (stbl) and of ranks of cut formulas. The relation depends on an ordinal γ_0, and should be written as $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash_{c,\gamma_0}^{*a} \Gamma; \Pi^{\cdot]}$. However the ordinal γ_0 will be fixed. So let us omit it.

Definition 7.14 Let Θ a finite set of ordinals, a, c ordinals, and $\mathtt{Q}_\Pi$ a finite family such that $\gamma_0 \leq \mathtt{p}_0(\sigma)$ for each $(\mathbb{S}, \sigma) \in \mathtt{Q}_\Pi$. Let $\Pi = \bigcup_{\sigma \in \bigcup \mathtt{Q}_\Pi} \Pi_\sigma \subset \Delta_0(\mathbb{I})$ be a set of formulas such that $\mathsf{k}(\Pi_\sigma) \subset M_\sigma$ for each $(\mathbb{S}, \sigma) \in \mathtt{Q}_\Pi$. Let $\Pi^{[\cdot]} = \bigcup_{\sigma \in \bigcup \mathtt{Q}_\Pi} \Pi_\sigma^{[\sigma/\mathbb{S}]}$ and $\Theta_{\mathtt{Q}_\Pi(\mathbb{S})} = \Theta \cap M_{\mathtt{Q}_\Pi(\mathbb{S})}$.

$(\mathcal{H}_\gamma, \Theta; \mathsf{Q}_\Pi) \vdash^{*a}_c \Gamma; \Pi^{[\cdot]}$ holds for a set Γ of formulas if $\gamma \leq \gamma_0$

$$\mathsf{k}(\Gamma) \subset \mathcal{H}_\gamma[\Theta] \,\&\, \forall \sigma \in \bigcup \mathsf{Q}_\Pi \left(\mathsf{k}(\Pi_\sigma) \subset \mathcal{H}_\gamma[\Theta^{(\sigma)}] \right) \tag{57}$$

$$\forall \mathbb{S} \in dom(\mathsf{Q}_\Pi) \left(\{\gamma, a, c, \gamma_0\} \cup \mathsf{k}^{\mathbb{S}}(\Gamma, \Pi) \subset \mathcal{H}_\gamma[\Theta_{\mathsf{Q}_\Pi(\mathbb{S})}] \right) {}^{6} \tag{58}$$

$$\forall \{\mathbb{U} \leq \mathbb{S}\} \subset dom(\mathsf{Q}_\Pi) \left(\mathbb{S} \in \mathcal{H}_\gamma[\Theta_{\mathsf{Q}_\Pi(\mathbb{U})}] \right) \tag{59}$$

and one of the following cases holds:

($\bigvee$) [7] There exist $A \simeq \bigvee(A_\iota)_{\iota \in J}$, an ordinal $a(\iota) < a$ and an $\iota \in J$ such that $A \in \Gamma$ and $(\mathcal{H}_\gamma, \Theta; \mathsf{Q}_\Pi) \vdash^{*a(\iota)}_c \Gamma, A_\iota; \Pi^{[\cdot]}$.

$(\bigvee)^{[\cdot]}$ There exist $\sigma \in \bigcup \mathsf{Q}_\Pi$, $A \simeq \bigvee(A_\iota)_{\iota \in J}$, an ordinal $a(\iota) < a$ and an $\iota \in [\sigma]J$ such that $A^{[\sigma/\mathbb{S}]} \in \Pi^{[\cdot]}$, $(\mathcal{H}_\gamma, \Theta; \mathsf{Q}_\Pi) \vdash^{*a(\iota)}_c \Gamma; (A_\iota)^{[\sigma/\mathbb{S}]}, \Pi^{[\cdot]}$.

($\bigwedge$) There exist $A \simeq \bigwedge(A_\iota)_{\iota \in J}$, ordinals $a(\iota) < a$ such that $A \in \Gamma$ and for each $\iota \in [\mathsf{Q}_\Pi]_A J$, $(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\iota); \mathsf{Q}_\Pi) \vdash^{*a(\iota)}_c \Gamma, A_\iota; \Pi^{[\cdot]}$.

$(\bigwedge)^{[\cdot]}$ There exist $\sigma \in \bigcup \mathsf{Q}_\Pi$, $A \simeq \bigwedge(A_\iota)_{\iota \in J}$, ordinals $a(\iota) < a$ such that $A^{[\sigma/\mathbb{S}]} \in \Pi^{[\cdot]}$, and $(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\iota); \mathsf{Q}_\Pi) \vdash^{*a(\iota)}_c \Gamma; \Pi^{[\cdot]}, (A_\iota)^{[\sigma/\mathbb{S}]}$ for each $\iota \in [\mathsf{Q}_\Pi]_A J \cap [\sigma]J$.

(cut) There exist an ordinal $a_0 < a$ and a formula C such that $(\mathcal{H}_\gamma, \Theta; \mathsf{Q}_\Pi) \vdash^{*a_0}_c \Gamma, \neg C; \Pi^{[\cdot]}$ and $(\mathcal{H}_\gamma, \Theta; \mathsf{Q}_\Pi) \vdash^{*a_0}_c C, \Gamma; \Pi^{[\cdot]}$ with $\mathrm{rk}(C) < c$.

($\Sigma(St)$-rfl) There exist ordinals $a_\ell, a_r < a$ and a formula $C \in \Sigma(St)$ such that $c \geq \mathbb{I}$, $(\mathcal{H}_\gamma, \Theta; \mathsf{Q}_\Pi) \vdash^{*a_\ell}_c \Gamma, C; \Pi^{[\cdot]}$ and $(\mathcal{H}_\gamma, \Theta; \mathsf{Q}_\Pi) \vdash^{*a_r}_c \neg \exists x\, C^{(x,\mathbb{I})}, \Gamma; \Pi^{[\cdot]}$.

($\Sigma(\Omega)$-rfl) There exist ordinals $a_\ell, a_r < a$ and a formula $C \in \Sigma(\Omega)$ such that $c \geq \Omega$, $(\mathcal{H}_\gamma, \Theta; \mathsf{Q}_\Pi) \vdash^{*a_\ell}_c \Gamma, C; \Pi^{[\cdot]}$ and $(\mathcal{H}_\gamma, \Theta; \mathsf{Q}_\Pi) \vdash^{*a_r}_c \neg \exists x < \Omega\, C^{(x,\Omega)}, \Gamma; \Pi^{[\cdot]}$.

(stbl($\mathbb{S}$)) There exist an ordinal $a_0 < a$, a successor stable ordinal $\mathbb{S}$, a $\bigwedge$-formula $B(0) \in \Delta_0(\mathbb{S})$ and a $u \in Tm(\mathbb{I})$ for which the following hold:

$$\mathbb{S} \in \mathcal{H}_\gamma[\Theta_{\mathsf{Q}_\Pi(\mathbb{S})}] \,\&\, \forall \mathbb{U} \in dom(\mathsf{Q}_\Pi) \cap \mathbb{S} \left(\mathbb{S} \in \mathcal{H}_\gamma[\Theta_{\mathsf{Q}_\Pi(\mathbb{U})}] \right) \tag{60}$$

$\mathbb{S} \leq \mathrm{rk}(B(u)) < c$, $(\mathcal{H}_\gamma, \Theta; \mathsf{Q}_\Pi) \vdash^{*a_0}_c \Gamma, B(u); \Pi^{[\cdot]}$, and $(\mathcal{H}_\gamma, \Theta \cup \{\sigma\}; \mathsf{Q}_\Pi \cup \{(\mathbb{S}, \sigma)\}) \vdash^{*a_0}_c \Gamma; \neg B(u)^{[\sigma/\mathbb{S}]}, \Pi^{[\cdot]}$ holds for every ordinal $\sigma \in \Psi_{\mathbb{S}}$ such that $\mathsf{p}_0(\sigma) \geq \gamma_0$ and

$$\Theta \cup \{\mathbb{S}\} \subset M_\sigma \tag{61}$$

where $dom(\mathsf{Q}_\Pi \cup \{(\mathbb{S}, \sigma)\}) = dom(\mathsf{Q}_\Pi) \cup \{\mathbb{S}\}$, and $(\mathsf{Q}_\Pi \cup \{(\mathbb{S}, \sigma)\})(\mathbb{S}) = \mathsf{Q}_\Pi(\mathbb{S}) \cup \{\sigma\}$.

$$\frac{(\mathcal{H}_\gamma, \Theta; \mathsf{Q}_\Pi) \vdash^{*a_0}_c \Gamma, B(u); \Pi^{[\cdot]} \quad \{(\mathcal{H}_\gamma, \Theta \cup \{\sigma\}; \mathsf{Q}_\Pi \cup \{(\mathbb{S}, \sigma)\}) \vdash^{*a_0}_c \Gamma; \neg B(u)^{[\sigma/\mathbb{S}]}, \Pi^{[\cdot]}\}_\sigma}{(\mathcal{H}_\gamma, \Theta; \mathsf{Q}_\Pi) \vdash^{*a}_c \Gamma; \Pi^{[\cdot]}}$$

Assume (60) and (61). Then $(\Theta \cup \{\sigma\})_{(\mathsf{Q}_\Pi \cup \{(\mathbb{S},\sigma)\})(\mathbb{S})} = \Theta_{\mathsf{Q}_\Pi(\mathbb{S})}$, and $(\Theta \cup \{\sigma\})_{(\mathsf{Q}_\Pi \cup \{(\mathbb{S},\sigma)\})(\mathbb{U})} = (\Theta \cup \{\sigma\})_{\mathsf{Q}_\Pi(\mathbb{U})} \supset \Theta_{\mathsf{Q}_\Pi(\mathbb{U})}$ for $\mathbb{U} \in dom(\mathsf{Q}_\Pi) \cap \mathbb{S}$.

[6] (58) means $\{\gamma, a, c, \gamma_0\} \subset \mathcal{H}_\gamma[\Theta]$ when $dom(\mathsf{Q}_\Pi) = \emptyset$.

[7] The condition $|\iota| < a$ is absent in the inference ($\bigvee$).

Lemma 7.15 (Tautology) *Let* $\gamma \in \mathcal{H}_\gamma[\mathsf{k}(A)]$ *and* $d = \mathrm{rk}(A)$.

1. $(\mathcal{H}_\gamma, \mathsf{k}(A); \emptyset) \vdash^{*2d}_0 \neg A, A; \emptyset$.
2. $(\mathcal{H}_\gamma, \mathsf{k}(A) \cup \{\mathbb{S}, \sigma\}; \{(\mathbb{S}, \sigma)\}) \vdash^{*2d}_0 \neg A^{[\sigma/\mathbb{S}]}; A^{[\sigma/\mathbb{S}]}$ *if* $\mathsf{k}(A) \cup \{\mathbb{S}\} \subset M_\sigma$ *and* $\gamma \geq \mathbb{S}$.

Proof. Each is seen by induction on $d = \mathrm{rk}(A)$. For example consider the lemma 7.15.2. We have $\mathrm{rk}(A^{[\sigma/\mathbb{S}]}) < \mathbb{S}$ and $(\mathsf{k}(A) \cup \{\mathbb{S}, \sigma\}) \cap M_\sigma = \mathsf{k}(A) \cup \{\mathbb{S}\}$ for (58) and (59), and $\mathsf{k}(A^{[\sigma/\mathbb{S}]}) \subset \mathcal{H}_\mathbb{S}((\mathsf{k}(A) \cap \mathbb{S}) \cup \{\sigma\})$ for (57). □

Lemma 7.16 (Embedding of Axioms) *For each axiom* A *in* $S_\mathbb{I}$ *there is an* $m < \omega$ *such that* $(\mathcal{H}_\mathbb{I}, \emptyset; \emptyset) \vdash^{*\mathbb{I}\cdot 2}_{\mathbb{I}+m} A; \emptyset$ *holds.*

Proof. Let us suppress the operator $\mathcal{H}_\mathbb{I}$. We show first that the axiom (50), $SSt(\sigma) \wedge \varphi(u) \wedge u \in L_\sigma \to \varphi^{L_\sigma}(u)$ by an inference (stbl($\mathbb{S}$)) for successor stable ordinals $\mathbb{S} < \mathbb{I}$. Let $B(0) \in \Delta_0(\mathbb{S})$ be a $\bigwedge$-formula, and $u \in Tm(\mathbb{I})$.

We may assume that $\mathbb{I} > d = \mathrm{rk}(B(u)) \geq \mathbb{S}$. Let $\mathsf{k}_0 = \mathsf{k}(B(0))$ and $\mathsf{k}_u = \mathsf{k}(u)$. Then $\mathsf{k}(B(0)) \subset \mathcal{H}_0(\mathsf{k}_0)$. Let $\sigma \in \Psi_\mathbb{S}$ be an ordinal such that $\mathsf{k}_0 \cup \mathsf{k}_u \cup \{\mathbb{S}\} \subset M_\sigma$ and $\gamma_0 \leq \mathsf{p}_0(\sigma)$.

$$\dfrac{\dfrac{\mathsf{k}_0 \cup \mathsf{k}_u; \vdash^{*2d}_0 \neg B(u), B(u); \quad \dfrac{\mathsf{k}_0 \cup \mathsf{k}_u \cup \{\mathbb{S}, \sigma\}; \{(\mathbb{S}, \sigma)\} \vdash^{*2d}_0 B(u^{[\sigma/\mathbb{S}]}); \neg B(u)^{[\sigma/\mathbb{S}]}}{\{\mathsf{k}_0 \cup \mathsf{k}_u \cup \{\mathbb{S}, \sigma\}; \{(\mathbb{S}, \sigma)\} \vdash^{*2d+1}_0 \exists x \in L_\mathbb{S} B(x); \neg B(u)^{[\sigma/\mathbb{S}]}\}_\sigma}\,(\bigvee)}{\mathsf{k}_0 \cup \mathsf{k}_u \cup \{\mathbb{S}\}; \vdash^{*\mathbb{I}}_\mathbb{I} \neg B(u), \exists x \in L_\mathbb{S} B(x);}\,(\mathrm{stbl}(\mathbb{S}))}{\mathsf{k}_0 \cup \{\mathbb{S}\}; \vdash^{*\mathbb{I}+1}_\mathbb{I} \neg \exists x\, B(x), \exists x \in L_\mathbb{S} B(x);}\,(\bigwedge)$$

Therefore $(\mathcal{H}_\mathbb{I}, \emptyset; \emptyset) \vdash^{*\mathbb{I}+\omega}_\mathbb{I} \forall \mathbb{S}, v \left[SSt(\mathbb{S}) \wedge A(v) \wedge v \in L_\mathbb{S} \to A^{(\mathbb{S},\mathbb{I})}(v)\right]; \emptyset$, where $SSt(\alpha) :\Leftrightarrow (St(\alpha) \wedge \exists \beta < \alpha \forall \gamma < \alpha (St(\gamma) \to \gamma \leq \beta)])$.

Next we show the axiom (49). Let α be an ordinal such that $\alpha < \mathbb{I}$. We obtain for $\alpha < \alpha^\dagger < \mathbb{I}$ with $d_0 = \mathrm{rk}(\alpha < \alpha^\dagger)$ and $\alpha^\dagger \leq d_1 = \mathrm{rk}(St(\alpha^\dagger)) < d_2 = \omega(\alpha^\dagger + 1)$ with $\alpha^\dagger \in \mathcal{H}_0[\{\alpha\}]$

$$\dfrac{\dfrac{\dfrac{\{\alpha\}; \emptyset \vdash^{*d_0}_0 \alpha < \alpha^\dagger; \emptyset \quad \{\alpha\}; \emptyset \vdash^{*2d_1}_0 St(\alpha^\dagger); \emptyset}{\{\alpha\}; \emptyset \vdash^{*d_2}_0 \alpha < \alpha^\dagger \wedge St(\alpha^\dagger); \emptyset}\,(\bigwedge)}{\{\alpha\}; \emptyset \vdash^{*d_2+1}_0 \exists \sigma\, (\alpha < \sigma \wedge St(\sigma)); \emptyset}\,(\bigvee)}{\emptyset; \emptyset \vdash^{*\mathbb{I}}_0 \forall \alpha \in ON \exists \sigma\, (\alpha < \sigma \wedge St(\sigma)); \emptyset}\,(\bigwedge)$$

□

Lemma 7.17 (Cut-elimination) *Assume* $(\mathcal{H}_\gamma, \Theta; \mathsf{Q}_\Pi) \vdash^{*a}_{c+1} \Gamma; \Pi^{[\cdot]}$ *with* $c \geq \mathbb{I}$. *Then* $(\mathcal{H}_\gamma, \Theta; \mathsf{Q}_\Pi) \vdash^{*\omega^a}_c \Gamma; \Pi^{[\cdot]}$.

Proof. Use the fact: if $(\mathcal{H}_\gamma, \Theta; \mathsf{Q}_\Pi) \vdash^{*a}_c \Gamma; \Pi^{[\cdot]}$ and $\Theta \cup \{\mathbb{S}\} \subset M_\sigma$, then $(\mathcal{H}_\gamma, \Theta \cup \{\sigma\}; \mathsf{Q}_\Pi \cup \{(\mathbb{S}, \sigma)\}) \vdash^{*a}_c \Gamma; \Pi^{[\cdot]}$. □

Lemma 7.18 (Collapsing) *Let* $\Gamma \subset \Sigma(St)$ *be a set of formulas. Suppose* $\Theta \subset \mathcal{H}_\gamma(\psi_\mathbb{I}(\gamma))$ *and* $(\mathcal{H}_\gamma, \Theta; \mathsf{Q}_\Pi) \vdash^{*a}_\mathbb{I} \Gamma; \Pi^{[\cdot]}$. *Let* $\beta = \psi_\mathbb{I}(\hat{a})$ *with* $\hat{a} = \gamma + \omega^a$. *Then* $(\mathcal{H}_{\hat{a}+1}, \Theta; \mathsf{Q}_\Pi) \vdash^{*\beta}_\beta \Gamma^{(\beta,\mathbb{I})}; \Pi^{[\cdot]}$ *holds.*

Proof. By induction on a. We have $\{\gamma, a\} \subset \mathcal{H}_\gamma[\Theta_{\mathtt{Q}_\Pi(\mathbb{S})}]$ by (58), and $\beta \in \mathcal{H}_{\hat{a}+1}[\Theta_{\mathtt{Q}_\Pi(\mathbb{S})}]$ for $\mathbb{S} \in dom(\mathtt{Q}_\Pi)$

When the last inference is a (stbl($\mathbb{S}$)), let $B(0) \in \Delta_0(\mathbb{S})$ be a $\bigwedge$-formula and a term $u \in Tm(\mathbb{I})$ such that $\mathbb{S} \leq \mathrm{rk}(B(u)) < \mathbb{I}$, $\mathsf{k}(B(u)) \subset \mathcal{H}_\gamma[\Theta]$, and $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a_0}_{\mathbb{I}} \Gamma, B(u); \Pi^{[\cdot]}$ for an ordinal $a_0 \in \mathcal{H}_\gamma[\Theta_{\mathtt{Q}_\Pi}] \cap a$. Then we obtain $\mathbb{S} \leq \mathrm{rk}(B(u)) < \beta$. □

7.4 Operator controlled derivations with caps

Let $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_{\mathbb{K}} \Gamma; \Pi^{[\cdot]}$ in the calculus for Π^1_1-reflection in subsection 6.5. In Capping 6.44, each formula $A \in \Gamma$ puts on a cap ρ such that $\mathtt{Q}_\Pi \subset \rho$ and (38), $\Theta \subset M_\rho$. (38) is needed in **Case 3.1** of the proof. Namely when $\Gamma \ni A \simeq \bigvee (A_\iota)_{\iota \in J}$ is introduced by a ($\bigvee$) such that $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a(\iota)}_{\mathbb{K}} \Gamma, A_\iota; \Pi^{[\cdot]}$, we need $\iota \in [\rho]J$, i.e., $\mathsf{k}(\iota) \subset M_\rho$, which follows from $\mathsf{k}(A_\iota) \subset \mathcal{H}_\gamma[\Theta] \subset M_\rho$ by (34) and $\Theta \subset M_\rho$.

We are concerned here with several stable ordinals $\mathbb{S}, \mathbb{T}, \ldots$. It is convenient for us to regard *uncapped formulas* A as capped formulas $A^{(\mathtt{u})}$ with its cap $\mathtt{u}$. Let $M_{\mathtt{u}} = OT(\mathbb{I})$.

In Capping 7.29 Γ is classified into $\Gamma = \Gamma_{\mathtt{u}} \cup \bigcup_{\mathbb{S} \in dom(\mathtt{Q}_\Pi)} \Gamma_{\mathbb{S}}$. $\Gamma_{\mathbb{S}}$ is the set of formulas $B(u)$ in inferences for the stability of a successor stable ordinal $\mathbb{S}$.

$$\frac{(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi \cup \{\mathbb{S}\}) \vdash^{*a_0}_{c} \Gamma, B(u); \Pi^{[\cdot]} \quad \{(\mathcal{H}_\gamma, \Theta \cup \{\sigma\}; \mathtt{Q}_\Pi \cup \{(\mathbb{S}, \sigma)\}) \vdash^{*a_0}_{c} \Gamma; \neg B(u)^{[\sigma/\mathbb{S}]}, \Pi^{[\cdot]}\}_\sigma}{(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_{c} \Gamma; \Pi^{[\cdot]}}$$

Each formula $A \in \Gamma_{\mathbb{S}}$ puts on a cap $\rho_{\mathbb{S}}$ for the stable ordinal $\mathbb{S}$. Then (38) runs $\Theta \subset M_{\rho_{\mathbb{S}}}$ for every $\mathbb{S} \in dom(\mathtt{Q}_\Pi)$. This means $\Theta \subset M_{\partial\mathtt{Q}} := \bigcap_{\kappa \in \partial\mathtt{Q}} M_\kappa$, where

$$\partial\mathtt{Q} = \{\max(\mathtt{Q}(\mathbb{S})) : \mathbb{S} \in dom(\mathtt{Q}), \mathtt{Q}(\mathbb{S}) \neq \emptyset\}.$$

Ordinals occurring in derivations are restricted to the set $M_{\partial\mathtt{Q}}$.

In section 6 for Π^1_1-reflection, an ordinal γ_0 is a threshold, which means that every ordinal occurring in derivations is in $\mathcal{H}_{\gamma_0}(0)$ and the subscript $\gamma \leq \gamma_0$ in $\mathcal{H}_\gamma$, while each $\rho \in \mathtt{Q}$ exceeds γ_0 in such a way that $\mathtt{p}_0(\rho) \geq \gamma_0$. This ensures us that $\mathcal{H}_\gamma(M_\rho) \subset M_\rho$. In the end, inferences (rfl(ρ, d, f)) are removed in Lemma 6.48 by moving outside $\mathcal{H}_{\gamma_0}(0)$. Specifically $\mathtt{Q} \subset \mathcal{H}_{\gamma_0+\mathbb{S}}(0)$.

Now we have several (successor) stable ordinals $\mathbb{S}, \mathbb{T}, \ldots \in dom(\mathtt{Q})$. Inferences (stbl($\mathbb{S}$)) and their children (rfl$_{\mathbb{S}}(\rho, d, f)$) are eliminated first for bigger $\mathbb{S} > \mathbb{T}$, and then smaller ones (stbl($\mathbb{T}$)). Therefore we need assignment $dom(\mathtt{Q}) \ni \mathbb{S} \mapsto \gamma^{\mathtt{Q}}_{\mathbb{S}}$ for thresholds so that $\gamma^{\mathtt{Q}}_{\mathbb{S}} < \gamma^{\mathtt{Q}}_{\mathbb{T}}$ if $\mathbb{S} > \mathbb{T}$. This is done by gapping, i.e., a gap $\mathbb{I} \cdot 2^a$ between $\gamma^{\mathtt{Q}}_{\mathbb{S}}$ and $\gamma^{\mathtt{Q}}_{\mathbb{T}}$ in advance, when $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_{c} \Gamma; \Pi^{[\cdot]}$ is embedded to $(\mathcal{H}_\gamma, \Theta_\Pi, \mathtt{Q}) \vdash^{a}_{c,c,\gamma_0} \widehat{\Gamma}, \widehat{\Pi}$, cf. Capping 7.29.

Definition 7.19 A triple $(\mathtt{Q}, \gamma^{\mathtt{Q}}, e^{\mathtt{Q}})$ is said to be a *finite family for ordinals* γ_0 *and* b_1 if $\mathtt{Q}$ is a finite family in the sense of Definition 7.13 and the following conditions are met:

1. $\gamma^{\mathtt{Q}}$ is a map $dom(\mathtt{Q}) \ni \mathbb{S} \mapsto \gamma^{\mathtt{Q}}_{\mathbb{S}}$ such that $\gamma_0 + \mathbb{I}^2 > \gamma^{\mathtt{Q}}_{\mathbb{S}} \geq \gamma_0$, $\gamma^{\mathtt{Q}}_{\mathbb{S}} \geq \gamma^{\mathtt{Q}}_{\mathbb{T}} + \mathbb{I}$ for $\{\mathbb{S} < \mathbb{T}\} \subset dom(\mathtt{Q})$ and $\mathbb{S} \in \mathcal{H}_{\gamma^{\mathtt{Q}}_{\mathbb{S}}+\mathbb{I}}$ for $\mathbb{S} \in dom(\mathtt{Q})$.

   $\mathtt{Q}$ is said to have *gaps* η if $\gamma^{\mathtt{Q}}_{\mathbb{S}} \geq \gamma^{\mathtt{Q}}_{\mathbb{T}} + \mathbb{I} \cdot \eta$ holds for $\{\mathbb{S} < \mathbb{T}\} \subset dom(\mathtt{Q})$, and $\gamma^{\mathtt{Q}}_{\mathbb{S}} \geq \gamma_0 + \mathbb{I} \cdot \eta$ for $\mathbb{S} \in dom(\mathtt{Q})$.

2. For each $\rho \in \mathtt{Q}(\mathbb{S})$, $m(\rho) : \mathbb{I} \to \varphi_{\mathbb{I}}(0)$ is special, $s(\rho) \leq b_1$, $\rho \in \mathcal{H}_{\gamma^{\mathtt{Q}}_{\mathbb{S}}+\mathbb{I}}(0)$, and $\gamma^{\mathtt{Q}}_{\mathbb{S}} \leq \mathtt{p}_0(\rho)$.

3. $e^{\mathtt{Q}}$ assigns an ordinal $e^{\mathtt{Q}}_{\mathbb{S}} \in \mathcal{H}_{\gamma^{\mathtt{Q}}_{\mathbb{S}}+\mathbb{I}} \cap (\mathbb{S}+1)$ to each $\mathbb{S} \in dom(\mathtt{Q})$ such that

$$\max(\{0\} \cup \{\rho \in \mathtt{Q}(\mathbb{S}) : s(\rho) > \mathbb{S}\}) < e^{\mathtt{Q}}_{\mathbb{S}} \tag{62}$$

   Let $e^{\mathtt{Q}}_{\mathbb{S}} = \mathbb{S}$ when $\mathbb{S} \not\in dom(\mathtt{Q})$.

Definition 7.20 For a finite family $\mathtt{Q}$, and for $A \simeq \bigvee(A_\iota)_{\iota \in J}$

$$[\mathtt{Q}]_{A^{(\rho)}} J = [\mathtt{Q}]_{\neg A^{(\rho)}} J = [\mathtt{Q}]_A J \cap [\partial\mathtt{Q}] J \cap [\rho] J$$

where $[\mathtt{u}]J = J$ and

$$[\partial\mathtt{Q}]J = \bigcap_{\kappa \in \partial\mathtt{Q}} [\kappa] J.$$

Definition 7.21 1. For a finite family $\mathtt{Q}$, let $\partial\mathtt{Q} = \{\max(\mathtt{Q}(\mathbb{S})) : \mathbb{S} \in dom(\mathtt{Q}), \mathtt{Q}(\mathbb{S}) \neq \emptyset\}$ and $M_{\partial\mathtt{Q}} = \bigcap_{\kappa \in \partial\mathtt{Q}} M_\kappa$.

2.

$$[\mathtt{Q}]_{A^{(\rho)}} J = [\mathtt{Q}]_{\neg A^{(\rho)}} J = [\mathtt{Q}]_A J \cap [\partial\mathtt{Q}] J \cap [\rho] J$$

   where $[\mathtt{u}]J = J$ and $[\partial\mathtt{Q}]J = \bigcap_{\kappa \in \partial\mathtt{Q}} [\kappa] J$.

Definition 7.22 $H^{\mathtt{Q}}_\rho(f, b_1, \gamma, \Theta)$ denotes the *resolvent class* for $\mathtt{Q}$, ρ, special functions f, ordinals b_1, γ, and finite sets Θ of ordinals defined as follows: $\sigma \in H^{\mathtt{Q}}_\rho(f, \gamma, \Theta)$ iff $\sigma \in \mathcal{H}_{\gamma+\mathbb{I}}(0) \cap \rho \cap M_{\partial\mathtt{Q}}$, $SC_{\mathbb{I}}(m(\sigma)) \subset \mathcal{H}_\gamma[\Theta]$, $\Theta \subset M_\sigma$, $\gamma \leq \mathtt{p}_0(\sigma) \leq \mathtt{p}_0(\rho)$ and $m(\sigma)$ is special such that $s(f) \leq s(m(\sigma)) \leq b_1$, $f' \leq (m(\sigma))'$, where $\sigma, \rho \prec \mathbb{S}$ and $f \leq g \Leftrightarrow \forall i (f(i) \leq g(i))$.

We define another derivability relation $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_{c,\xi,\gamma_0,b_1} \Gamma$, where c is a bound of ranks of cut formulas, and ξ a bound of ordinals $\mathbb{S}$ in the inference rules $(\mathrm{rfl}_{\mathbb{S}}(\rho, d, f, b_1))$.

Definition 7.23 Let $\Theta^{(\rho)} = \Theta \cap M_\rho$ and $\Theta_{\partial\mathtt{Q}} = \Theta \cap M_{\partial\mathtt{Q}}$. Let $a, b, c, \xi < \mathbb{I}$, a finite set $\Theta \subset \mathbb{I}$, and $\mathtt{Q}$ be a finite family for γ_0, b_1 such that $dom(\mathtt{Q}) \subset (\xi + 1)$.

$(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_{c,\xi,\gamma_0,b_1} \Gamma$ holds for a sequent $\Gamma = \bigcup\{\Gamma^{(\rho)}_\rho : \rho \in \{\mathtt{u}\} \cup \bigcup \mathtt{Q}\}$ if $\gamma \leq \gamma_0$

$$\forall \rho \in \{\mathtt{u}\} \cup \bigcup \mathtt{Q} \left(\mathsf{k}(\Gamma_\rho) \subset \mathcal{H}_\gamma[\Theta^{(\rho)}] \cap \mathcal{H}_\gamma[\Theta_{\partial\mathtt{Q}}] \right) \tag{63}$$

$$\forall \mathbb{S}\in dom(\mathtt{Q})\left(\{\gamma,a,c,\xi,\gamma_0,b_1\}\subset\mathcal{H}_\gamma[\Theta_{\mathtt{Q}(\mathbb{S})}]\cap\mathcal{H}_\gamma[\Theta_{\partial\mathtt{Q}}]\right)^{8} \tag{64}$$

$$\forall\{\mathbb{U}\leq\mathbb{S}\}\subset dom(\mathtt{Q})\left(\{\mathbb{S},\gamma^{\mathtt{Q}}_{\mathbb{S}}\}\subset\mathcal{H}_\gamma[\Theta_{\mathtt{Q}(\mathbb{U})}]\cap\mathcal{H}_\gamma[\Theta_{\partial\mathtt{Q}}]\right) \tag{65}$$

$$\forall\rho\in\{\mathtt{u}\}\cup\bigcup\mathtt{Q}\forall\mathbb{S}\in dom(\mathtt{Q})\left(\mathsf{k}^{\mathbb{S}}(\Gamma_\rho)\subset\mathcal{H}_\gamma\left[\Theta_{\mathtt{Q}(\mathbb{S})}\right]\cap\mathcal{H}_\gamma[\Theta_{\partial\mathtt{Q}}]\right) \tag{66}$$

$$\forall(\mathbb{S},\rho)\in\mathtt{Q}\left(SC_{\mathbb{I}}(m(\rho))\subset\mathcal{H}_{\gamma^{\mathtt{Q}}_{\mathbb{S}}}\left[\Theta^{(\rho)}\cup\{\mathbb{S}\}\cup\Theta_{\partial\mathtt{Q}}\right]\right) \tag{67}$$

and one of the following cases holds:

(Taut) $\{\neg A^{(\rho)},A^{(\rho)}\}\subset\Gamma$ for a $\rho\in\{\mathtt{u}\}\cup\bigcup\mathtt{Q}$ and a formula A such that $\mathrm{rk}(A)<\mathbb{S}\leq\xi$ for some successor stable ordinal $\mathbb{S}$.

If $\mathrm{rk}(A)<\mathbb{S}$, then $(\mathcal{H}_\gamma,\Theta,\mathtt{Q})\vdash^{0}_{0,\mathbb{S},\gamma_0,b_1}\neg A^{(\sigma)},A^{(\sigma)}$ by (Taut) provided that (64) and (66) are met.

($\bigvee$) There exist $A\simeq\bigvee(A_\iota)_{\iota\in J}$, a cap $\rho\in\{\mathtt{u}\}\cup\bigcup\mathtt{Q}$, an ordinal $a(\iota)<a$ and an $\iota\in[\rho]J\cap[\partial\mathtt{Q}]J$ such that $A^{(\rho)}\in\Gamma$ and $(\mathcal{H}_\gamma,\Theta,\mathtt{Q})\vdash^{a(\iota)}_{c,\xi,\gamma_0,b_1}\Gamma,(A_\iota)^{(\rho)}$.

($\bigwedge$) There exist $A\simeq\bigwedge(A_\iota)_{\iota\in J}$, a cap $\rho\in\{\mathtt{u}\}\cup\bigcup\mathtt{Q}$, ordinals $a(\iota)<a$ for each $\iota\in[\mathtt{Q}]_{A^{(\rho)}}J$ such that $A^{(\rho)}\in\Gamma$ and $(\mathcal{H}_\gamma,\Theta\cup\mathsf{k}(\iota),\mathtt{Q})\vdash^{a(\iota)}_{c,\xi,\gamma_0,b_1}\Gamma,(A_\iota)^{(\rho)}$.

(cut) There exist a cap $\rho\in\{\mathtt{u}\}\cup\bigcup\mathtt{Q}$, ordinals $a_0<a$ and a formula C such that $\mathrm{rk}(C)<c$, $(\mathcal{H}_\gamma,\Theta,\mathtt{Q})\vdash^{a_0}_{c,\xi,\gamma_0,b_1}\Gamma,\neg C^{(\rho)}$ and $(\mathcal{H}_\gamma,\Theta,\mathtt{Q})\vdash^{a_0}_{c,\xi,\gamma_0,b_1}C^{(\rho)},\Gamma$.

($\Sigma(\Omega)$-rfl) There exist ordinals $a_\ell,a_r<a$ and an uncapped formula $C\in\Sigma(\Omega)$ such that $c\geq\Omega$, $(\mathcal{H}_\gamma,\Theta,\mathtt{Q})\vdash^{a_\ell}_{c,\xi,\gamma_0,b_1}\Gamma,C$ and $(\mathcal{H}_\gamma,\Theta,\mathtt{Q})\vdash^{a_r}_{c,\xi,\gamma_0,b_1}\neg\exists x<\pi\,C^{(x,\Omega)},\Gamma$.

($\mathrm{rfl}_{\mathbb{S}}(\rho,d,f,b_1)$) There exist a successor stable ordinal $\mathbb{S}\leq\xi$ and an ordinal $\rho\prec\mathbb{S}$ such that

$$\Theta_{\mathtt{Q}(\mathbb{S})}\cup\{\mathbb{S}\}\cup\Theta_{\partial\mathtt{Q}}\subset M_\rho \tag{68}$$

and $\rho\in\mathtt{Q}(\mathbb{S})$ if $\mathbb{S}\in dom(\mathtt{Q})$. Let $\mathtt{R}=\mathtt{Q}$ if $\mathbb{S}\in dom(\mathtt{Q})$. Otherwise $\mathtt{R}=\mathtt{Q}\cup\{(\mathbb{S},\rho)\}$, where $\mathtt{Q}\cup\{(\mathbb{S},\rho)\}$ is a finite family for γ_0 extending $\mathtt{Q}$ such that $dom(\mathtt{R})=dom(\mathtt{Q})\cup\{\mathbb{S}\}$, $\mathtt{R}(\mathbb{S})=\mathtt{Q}(\mathbb{S})\cup\{\rho\}$, $e^{\mathtt{R}}_{\mathbb{T}}=e^{\mathtt{Q}}_{\mathbb{T}}$ for $\mathbb{S}\neq\mathbb{T}\in dom(\mathtt{Q})$, $\gamma^{\mathtt{Q}}_{\mathbb{T}}\geq\gamma^{\mathtt{R}}_{\mathbb{S}}+\mathbb{I}$ for every $\mathbb{S}>\mathbb{T}\in dom(\mathtt{Q})$ and $\gamma^{\mathtt{R}}_{\mathbb{S}}\geq\gamma_0+\mathbb{I}$.

Also there exist an ordinal $d\in\mathrm{supp}(m(\rho))$, a special function f, an ordinal $a_0<a$, and a finite set Δ of uncapped formulas enjoying the following conditions.

(r0) $\rho<e^{\mathtt{R}}_{\mathbb{S}}$ if $s(\rho)=\max(\mathrm{supp}(m(\rho)))>\mathbb{S}$.

(r1) $\Delta\subset\bigvee_{\mathbb{S}}(d):=\{\delta:\mathrm{rk}(\delta)<d,\delta\text{ is a }\bigvee\text{-formula}\}\cup\{\delta:\mathrm{rk}(\delta)<\mathbb{S}\}$.

[8] (64) means $\{\gamma,a,c,\xi,\gamma_0\}\subset\mathcal{H}_\gamma[\Theta]$ when $dom(\mathtt{Q})=\emptyset$.

(r2) For $g = m(\rho)$, $s(f) \leq b_1$, $SC_{\mathbb{I}}(f) \cup SC_{\mathbb{I}}(g) \subset \mathcal{H}_{\gamma^{\mathtt{R}}_{\mathbb{S}}}[\Theta^{(\rho)}]$ and $f_d = g_d \,\&\, f^d <^d_{\mathbb{I}} g'(d)$.

(r3) For each $\delta \in \Delta$, $(\mathcal{H}_\gamma, \Theta, \mathtt{R}) \vdash^{a_0}_{c,\xi,\gamma_0} \Gamma, \neg\delta^{(\rho)}$.

(r4) Let $\gamma^{\mathtt{R}\cup\{(\mathbb{S},\sigma)\}} = \gamma^{\mathtt{R}}$, $e^{\mathtt{R}\cup\{(\mathbb{S},\sigma)\}} = e^{\mathtt{R}}$ and $\sigma \in H^{\mathtt{R}}_\rho(f, b_1, \gamma^{\mathtt{R}}_{\mathbb{S}}, \Theta^{(\rho)} \cup \Theta_{\partial\mathtt{Q}})$. Then $(\mathcal{H}_\gamma, \Theta\cup\{\sigma\}, \mathtt{R}\cup\{(\mathbb{S},\sigma)\}) \vdash^{a_0}_{c,\xi,\gamma_0} \Gamma, \Delta^{(\sigma)}$ holds.
In particular $\sigma < e^{\mathtt{R}}_{\mathbb{S}}$ if $s(\sigma) > \mathbb{S}$ by (62).

Note that $\bigcup \mathtt{Q} \subset \mathcal{H}_\gamma[\Theta]$ need not to hold. Moreover $(\Theta\cup\{\sigma\})_{(\mathtt{R}(\mathbb{S})\cup\{\sigma\})} = \Theta_{\mathtt{R}(\mathbb{S})} = \Theta_{\mathtt{Q}(\mathbb{S})}$ and $\Theta_{\partial\mathtt{R}} = \Theta_{\partial\mathtt{Q}}$ by $\Theta^{(\rho)} \subset M_\sigma$ and (68).

In this subsection the ordinals γ_0 and b_1 will be fixed, and we write $\vdash^a_{c,\xi}$ for $\vdash^a_{c,\xi,\gamma_0,b_1}$.

Lemma 7.24 (Tautology) *Let* $\{\gamma,\gamma_0,\mathbb{S}\} \cup \mathsf{k}^{\mathbb{T}}(A) \subset \mathcal{H}_\gamma[\Theta_{\mathtt{Q}(\mathbb{T})}] \cap \mathcal{H}_\gamma[\Theta_{\partial\mathtt{Q}}]$ *for every* $\mathbb{T} \in dom(\mathtt{Q}) \subset (\mathbb{S}{+}1)$, $\sigma \in \{\mathtt{u}\}\cup\bigcup\mathtt{Q}$ *and* $\mathsf{k}(A) \subset M_\sigma$. *Then* $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{2d}_{0,\mathbb{S}} \neg A^{(\sigma)}, A^{(\sigma)}$ *holds for* $d = \max\{\mathbb{S}, \mathrm{rk}(A)\}$.

Lemma 7.25 (Inversion) *Let* $A \simeq \bigwedge(A_\iota)_{\iota\in J}$ *and* $(\mathcal{H}, \Theta, \mathtt{Q}) \vdash^a_{c,\xi} \Gamma$ *with* $A^{(\rho)} \in \Gamma$ *and there is no* $\mathbb{S} \in SSt$ *such that* $\mathrm{rk}(A) < \mathbb{S} \leq \xi$. *Then for any* $\iota \in [\mathtt{Q}]_{A^{(\rho)}} J$, $(\mathcal{H}, \Theta\cup\mathsf{k}(\iota), \mathtt{Q}) \vdash^a_{c,\xi} \Gamma, (A_\iota)^{(\rho)}$.

Proof. We need to assume that there is no $\mathbb{S} \in SSt$ such that $\mathrm{rk}(A) < \mathbb{S} \leq \xi$ due to (Taut). □

Lemma 7.26 (Reduction) *Let* $C \simeq \bigvee(C_\iota)_{\iota\in J}$ *and* $\Omega \leq \mathrm{rk}(C) \leq c$. *Assume* $(\mathcal{H}_\gamma, \Theta, \mathtt{Q} \vdash^a_{c,\xi} \Gamma, \neg C^{(\tau)}$ *and* $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^b_{c,\xi} C^{(\tau)}, \Gamma$ *with* $SSt \cap (c, \xi] = \emptyset$.

Then $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a+b}_{c,\xi} \Gamma$.

Lemma 7.27 (Cut-elimination) *If* $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_{c+c_1,\xi} \Gamma$ *with* $\Omega \leq c < \mathbb{I}$, $\forall\mathbb{S} \in dom(\mathtt{Q})(c \in \mathcal{H}_\gamma[\Theta_{\mathtt{Q}(\mathbb{S})}] \cap \mathcal{H}_\gamma[\Theta_{\partial\mathtt{Q}}])$ *and* $SSt\cap(c,\xi] = \emptyset$, *then* $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{\varphi_{c_1}(a)}_{c,\xi} \Gamma$.

Lemma 7.28 (Collapsing) *Let* $\Gamma \subset \Sigma(\Omega)$ *be a sets of uncapped formulas. Suppose* $\Theta \subset \mathcal{H}_\gamma(\psi_\Omega(\gamma))$ *and* $(\mathcal{H}_\gamma, \Theta, \emptyset) \vdash^a_{\Omega,0} \Gamma$. *Let* $\beta = \psi_\Omega(\hat{a})$ *with* $\hat{a} = \gamma + \omega^a < \gamma_0$. *Then* $(\mathcal{H}_{\hat{a}+1}, \Theta, \emptyset) \vdash^\beta_{\beta,0} \Gamma^{(\beta,\Omega)}$ *holds.*

7.5 Eliminations of stable ordinals

Lemma 7.29 (Capping) *Let* $\Gamma \cup \Pi \subset \Delta_0(\mathbb{I})$ *be a set of uncapped formulas. Suppose* $(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_{c,\gamma_0} \Gamma; \Pi^{[\cdot]}$, *where* $a, c < \mathbb{I}$, $dom(\mathtt{Q}_\Pi) \subset c$, $\Gamma = \Gamma_{\mathtt{u}} \cup \bigcup_{\mathbb{S}\in dom(\mathtt{Q}_\Pi)} \Gamma_{\mathbb{S}}$, $\Pi^{[\cdot]} = \bigcup_{(\mathbb{S},\sigma)\in\mathtt{Q}_\Pi} \Pi^{[\sigma/\mathbb{S}]}_\sigma$.

For each $\mathbb{S} \in dom(\mathtt{Q}_\Pi)$, *let* $\rho_{\mathbb{S}} = \psi^{g_{\mathbb{S}}}_{\mathbb{S}}(\delta_{\mathbb{S}})$ *be an ordinal with an ordinal* $\delta_{\mathbb{S}} \in \mathcal{H}_\gamma[\Theta]$ *and a special finite function* $g_{\mathbb{S}} = m(\rho_{\mathbb{S}}) : \mathbb{I} \to \varphi_{\mathbb{I}}(0)$ *such that* $\mathrm{supp}(g_{\mathbb{S}}) = \{c\}$ *with* $g_{\mathbb{S}}(c) = \alpha_{\mathbb{S}} + \mathbb{I}$, $\mathbb{I}(2a+1) \leq \alpha_{\mathbb{S}} + \mathbb{I}$, $SC_{\mathbb{I}}(g_{\mathbb{S}}) = SC_{\mathbb{I}}(c, \alpha_{\mathbb{S}}) \subset \mathcal{H}_0(SC_{\mathbb{I}}(\delta_{\mathbb{S}})) \cap \mathcal{H}_\gamma[\Theta]$, *cf. (55) and (67). Also let* $\widehat{\Pi} = \bigcup_{(\mathbb{S},\sigma)\in\mathtt{Q}_\Pi} \Pi^{(\sigma)}_\sigma$, $\widehat{\Gamma} = \Gamma^{(\mathtt{u})}_{\mathtt{u}} \cup \bigcup_{\mathbb{S}\in dom(\mathtt{Q}_\Pi)} \Gamma^{(\rho_{\mathbb{S}})}_{\mathbb{S}}$.

Let $\mathtt{Q}$ *be a finite family for* $\gamma_0 \geq \gamma$ *such that* $\mathtt{Q}(\mathbb{S}) = \mathtt{Q}_\Pi(\mathbb{S}) \cup \{\rho_\mathbb{S}\}$ *for* $\mathbb{S} \in dom(\mathtt{Q}_\Pi) = dom(\mathtt{Q})$*,* $\rho_\mathbb{S} \in \mathcal{H}_{\gamma^{\mathtt{Q}}_\mathbb{S}+\mathbb{I}}(0)$ *for* $\mathbb{S} \in dom(\mathtt{Q})$*, and* $\alpha_\mathbb{S} + \mathbb{I} \leq \gamma^{\mathtt{Q}}_\mathbb{S} \leq \delta_\mathbb{S} < \gamma^{\mathtt{Q}}_\mathbb{S} + \mathbb{I}$*. Also* $e^{\mathtt{Q}}_\mathbb{S} = \rho_\mathbb{S} + 1$*.*

Assume $\forall \mathbb{S} \in dom(\mathtt{Q}_\Pi)(\gamma^{\mathtt{Q}}_\mathbb{S} \in \mathcal{H}_\gamma[\Theta])$*,* $\mathtt{Q}_\Pi(\mathbb{S}) \subset \rho_\mathbb{S}$*,* $\Theta \cup \{\mathbb{S}\} \subset M_{\rho_\mathbb{S}}$*,* $\mathtt{p}_0(\sigma) \leq \mathtt{p}_0(\rho_\mathbb{S}) = \delta_\mathbb{S}$ *and* $SC_\mathbb{I}(m(\sigma)) \subset \mathcal{H}_{\gamma^{\mathtt{Q}}_\mathbb{S}}[\Theta \cup \{\mathbb{S}\}]$ *for each* $(\mathbb{S}, \sigma) \in \mathtt{Q}_\Pi$*,* $\forall \{\mathbb{U} < \mathbb{S}\} \subset dom(\mathtt{Q}_\Pi)(\rho_\mathbb{S} \in M_{\rho_\mathbb{U}})$*, and* $\mathtt{Q}$ *has gaps* 2^a*.*

Then $(\mathcal{H}_\gamma, \Theta_\Pi, \mathtt{Q}) \vdash^a_{c,c,\gamma_0,c} \widehat{\Gamma}, \widehat{\Pi}$ *holds for* $\Theta_\Pi = \Theta \cup \bigcup \mathtt{Q}_\Pi$*.*

Remark 7.30 When $\alpha_\mathbb{S} = \mathbb{I}(2a)$ and $\Theta = \emptyset$, $\delta_\mathbb{S} < \gamma^{\mathtt{Q}}_\mathbb{S} + \mathbb{I}$ denotes the natural sum $\gamma^{\mathtt{Q}}_\mathbb{S} \# a \# c$. Then $\Theta \cup \{\mathbb{S}\} \subset M_{\rho_\mathbb{S}}$ and $\{a, c\} \subset \mathcal{H}_0(SC_\mathbb{I}(\delta_\mathbb{S}))$. Hence (55) is enjoyed for $\rho_\mathbb{S}$. Namely $SC_\mathbb{I}(g_\mathbb{S}) = \{c, \alpha_\mathbb{S} + \mathbb{I}\} \subset \mathcal{H}_0(SC_\mathbb{I}(\delta_\mathbb{S})) \subset \mathcal{H}_{\delta_\mathbb{S}}(SC_\mathbb{I}(\delta_\mathbb{S}))$ holds.

Let $\mathbb{U} \in dom(\mathtt{Q}_\Pi) \cap \mathbb{S}$. We have $\{\gamma_0, \mathbb{S}, a, c\} \subset \mathcal{H}_\gamma[\Theta_{\mathtt{Q}_\Pi(\mathbb{U})}]$ by (58). We intend to be $\gamma^{\mathtt{Q}}_\mathbb{S} = \gamma_0 + \mathbb{I} \cdot 2^a \cdot n$ for $n = \#\{\mathbb{T} \in dom(\mathtt{Q}) : \mathbb{T} \geq \mathbb{S}\}$. Then $\{\mathbb{S}, a, c, \gamma^{\mathtt{Q}}_\mathbb{S}\} \subset \mathcal{H}_\gamma[\Theta_{\mathtt{Q}_\Pi(\mathbb{U})}] \cap \mathcal{H}_\gamma[\Theta_{\partial\mathtt{Q}}]$ for (64) and (65).

On the other hand we have $\mathtt{Q}_\Pi(\mathbb{S}) \subset \rho_\mathbb{S}$, and $\rho_\mathbb{S} = \max(\mathtt{Q}(\mathbb{S}))$, i.e., $\partial\mathtt{Q} = \{\rho_\mathbb{S} : \mathbb{S} \in dom(\mathtt{Q}_\Pi)\}$. Also $\{\mathbb{S}, \delta_\mathbb{S}\} \cup SC_\mathbb{I}(g_\mathbb{S}) \subset \mathcal{H}_0(\{\mathbb{S}, a, c, \gamma^{\mathtt{Q}}_\mathbb{S}\} \cup \Theta) \subset M_{\rho_\mathbb{U}} = \mathcal{H}_{\delta_\mathbb{U}}(\rho_\mathbb{U})$ for $\mathbb{U} \leq \mathbb{S}$. Therefore $\rho_\mathbb{S} \in M_{\rho_\mathbb{U}}$ for $\mathbb{U} < \mathbb{S}$ by $\delta_\mathbb{S}, \gamma^{\mathtt{Q}}_\mathbb{S} + \mathbb{I} \leq \gamma^{\mathtt{Q}}_\mathbb{U}$. Moreover $\rho_\mathbb{S} \in M_{\rho_\mathbb{U}}$ for $\mathbb{U} > \mathbb{S}$ since $\rho_\mathbb{S} < \mathbb{S} < \rho_\mathbb{U}$.

Proof of Lemma 7.29. This is seen by induction on a as in Capping 6.44. Let us write $\vdash^a_c$ for $\vdash^a_{c,c,\gamma_0,c}$ in the proof. By assumptions we have $\mathtt{Q}_\Pi(\mathbb{S}) \subset \rho_\mathbb{S}$ and $\Theta \subset M_{\rho_\mathbb{S}}$. Hence $\Theta = \Theta^{(\rho_\mathbb{S})} = \Theta_{\partial\mathtt{Q}}$ and $\Theta_{\mathtt{Q}_\Pi(\mathbb{S})} = \Theta_{\mathtt{Q}(\mathbb{S})}$. On the other hand we have $\mathsf{k}(\Gamma) \subset \mathcal{H}_\gamma[\Theta]$ and for $\sigma \in \bigcup \mathtt{Q}_\Pi$, $\mathsf{k}(\Pi_\sigma) \subset \mathcal{H}_\gamma[\Theta^{(\sigma)}]$ by (57). Therefore (63) and (66) are enjoyed. We have $\{\gamma, a, c, \gamma_0, \gamma^{\mathtt{Q}}_\mathbb{S}, \mathbb{S}\} \subset \mathcal{H}_\gamma[\Theta_{\mathtt{Q}_\Pi(\mathbb{U})}] \cap \mathcal{H}_\gamma[\Theta_{\partial\mathtt{Q}}]$ for every $\{\mathbb{U} \leq \mathbb{S}\} \subset dom(\mathtt{Q}) = dom(\mathtt{Q}_\Pi)$ by the assumption, (58) and (59). Hence (64) and (65) are enjoyed. Moreover for (67) we have $SC_\mathbb{I}(m(\rho_\mathbb{S})) \subset \mathcal{H}_\gamma[\Theta]$ and $\gamma \leq \gamma^{\mathtt{Q}}_\mathbb{S}$.

Case 1. First consider the case when the last inference is a (stbl($\mathbb{S}$)): We have a successor stable ordinal $\mathbb{S}$, an ordinal $a_0 < a$, a $\bigwedge$-formula $B(0) \in \Delta_0(\mathbb{S})$, and a term $u \in Tm(\mathbb{I})$ with $\mathbb{S} \leq \mathrm{rk}(B(u)) < c$.

For every ordinal σ such that $\Theta \cup \{\mathbb{S}\} \subset M_\sigma$ and $\mathtt{p}_0(\sigma) \geq \gamma_0$

$$\frac{(\mathcal{H}_\gamma, \Theta, \mathtt{Q}_\Pi) \vdash^{*a_0}_c \Gamma, B(u); \Pi^{[\cdot]} \quad (\mathcal{H}_\gamma, \Theta \cup \{\mathbb{S}, \sigma\}; \mathtt{Q}_\Pi \cup \{(\mathbb{S}, \sigma)\}) \vdash^{*a_0}_c \Gamma; \neg B(u)^{[\sigma/\mathbb{S}]}, \Pi^{[\cdot]}}{(\mathcal{H}_\gamma, \Theta; \mathtt{Q}_\Pi) \vdash^{*a}_c \Gamma; \Pi^{[\cdot]}}$$

Let h be a special finite function such that $\mathrm{supp}(h) = \{c\}$ and $h(c) = \mathbb{I}(2a_0+1)$. Then $h_c = (g_\mathbb{S})_c = \emptyset$ and $h^c <^c_\mathbb{I} (g_\mathbb{S})'(c)$ by $h(c) = \mathbb{I}(2a_0+1) < \mathbb{I}(2a) \leq \alpha_0 = (g_\mathbb{S})'(c)$. Let $\mathtt{R} = \mathtt{Q} \cup \{(\mathbb{S}, \rho_\mathbb{S})\}$ and $\sigma \in H^{\mathtt{R}}_{\rho_\mathbb{S}}(h, c, \gamma^{\mathtt{R}}_\mathbb{S}, \Theta^{(\rho_\mathbb{S})} \cup \{\mathbb{S}\} \cup \Theta_{\partial\mathtt{Q}})$, where $\Theta^{(\rho_\mathbb{S})} \cup \Theta_{\partial\mathtt{Q}} = \Theta$.

For example let $\sigma = \psi^h_{\rho_\mathbb{S}}(\delta_\mathbb{S} + \eta)$ with $\eta = \max(\{1\} \cup E_\mathbb{S}(\Theta))$. We obtain $\Theta \cup \{\mathbb{S}\} \subset \mathcal{H}_{\delta_\mathbb{S}}(\sigma) = M_\sigma$ by $\Theta \cup \{\mathbb{S}\} \subset M_\rho$, and $\{\delta_\mathbb{S}, a_0, c\} \subset \mathcal{H}_\gamma[\Theta]$. Let $\rho_\mathbb{U} \in \partial\mathtt{R}$. We claim that $\sigma \in M_{\rho_\mathbb{U}}$. If $\mathbb{U} \geq \mathbb{S}$, then $\sigma < \rho_\mathbb{U}$. Let $\mathbb{U} < \mathbb{S}$. Then we have $\rho_\mathbb{S} \in M_{\rho_\mathbb{U}}$ by the assumption, and $\sigma \in M_{\rho_\mathbb{U}}$ follows from $\{c, a_0, \delta_\mathbb{S}\} \cup \Theta \subset \mathcal{H}_\gamma[\Theta] \subset \mathcal{H}_{\delta_\mathbb{U}}(\rho_\mathbb{U})$ and $\delta_\mathbb{S} + \eta < \gamma^{\mathtt{Q}}_\mathbb{S} + \mathbb{I} \leq \gamma^{\mathtt{Q}}_\mathbb{U} \leq \delta_\mathbb{U}$. Therefore $\sigma \in H^{\mathtt{R}}_{\rho_\mathbb{S}}(h, c, \gamma^{\mathtt{R}}_\mathbb{S}, \Theta^{(\rho_\mathbb{S})} \cup \{\mathbb{S}\} \cup \Theta_{\partial\mathtt{Q}})$.

Since $\mathtt{Q}$ is assumed to have gaps 2^a, we may assume that $\mathtt{R}\cup\{(\mathbb{S},\sigma)\}$ as well as $\mathtt{R}$ has gaps 2^{a_0}.

IH yields $(\mathcal{H}_\gamma,\Theta_\Pi,\mathtt{R})\vdash^{a_0}_c \widehat{\Gamma}, B(u)^{(\rho_\mathbb{S})},\widehat{\Pi}$, and for $u^{[\sigma/\mathbb{S}]}\in Tm(\mathbb{S})$ and $B(u^{[\sigma,\mathbb{S}]})\equiv B(u)^{[\sigma,\mathbb{S}]}$, $(\mathcal{H}_\gamma,\Theta_\Pi\cup\{\mathbb{S},\sigma\},\mathtt{R}\cup\{(\mathbb{S},\sigma)\})\vdash^{a_0}_c \widehat{\Gamma},\neg B(u)^{(\sigma)},\widehat{\Pi}$ follows, where $\rho_\mathbb{S}>\sigma\in M_{\rho_\mathbb{S}}$ and we have by (59), $\mathsf{k}(B(u))\subset\mathcal{H}_\gamma[\Theta_{\mathtt{Q}_\Pi(\mathbb{T})}]\cap\mathcal{H}_\gamma[\Theta_{\partial\mathtt{Q}}]$ if $\mathrm{rk}(B(u))\geq\mathbb{T}$. Hence $\mathsf{k}(B(u))\subset\mathcal{H}_\gamma[\Theta_{\mathtt{R}(\mathbb{T})}]\cap\mathcal{H}_\gamma[\Theta_{\partial\mathtt{Q}}]$ by $\Theta_{\mathtt{R}(\mathbb{T})}=\Theta_{\mathtt{Q}_\Pi(\mathbb{T})}$ for (59). Moreover we have $\mathbb{S}\in\mathcal{H}_\gamma[\Theta_{\mathtt{Q}(\mathbb{T})}]$ for every $\mathbb{T}<c$, $\Theta_{\mathtt{Q}_\Pi(\mathbb{S})}\cup\Theta_{\partial\mathtt{Q}}\subset M_{\rho_\mathbb{S}}$ for (68), $\rho_\mathbb{S}<e^{\mathtt{Q}}_\mathbb{S}$ for (r0), $\mathrm{rk}(B(u))<c$ and $s(\rho_\mathbb{S})\leq c$ for (r1).

We obtain by an inference $(\mathrm{rfl}_\mathbb{S}(\rho_\mathbb{S},c,h,c))$

$$\frac{(\mathcal{H}_\gamma,\Theta_\Pi,\mathtt{R})\vdash^{a_0}_c \widehat{\Gamma},B(u)^{(\rho_\mathbb{S})},\widehat{\Pi}\quad (\mathcal{H}_\gamma,\Theta_\Pi\cup\{\mathbb{S},\sigma\},\mathtt{R}\cup\{(\mathbb{S},\sigma)\})\vdash^{a_0}_c \widehat{\Gamma},\neg B(u)^{(\sigma)},\widehat{\Pi}}{(\mathcal{H}_\gamma,\Theta_\Pi,\mathtt{Q})\vdash^a_c \widehat{\Gamma},\widehat{\Pi}}$$

in the right upper sequents σ ranges over the resolvent class $\sigma\in H^{\mathtt{R}}_{\rho_\mathbb{S}}(h,c,\gamma^{\mathtt{R}}_\mathbb{S},\Theta^{(\rho_\mathbb{S})}\cup\{\mathbb{S}\}\cup\Theta_{\partial\mathtt{Q}})$.

Case 2. When the last inference is a (*cut*): There exist $a_0<a$ and C such that $\mathrm{rk}(C)<c$, $(\mathcal{H}_\gamma,\Theta;\mathtt{Q}_\Pi)\vdash^{*a_0}_c\Gamma,\neg C;\Pi^{[\cdot]}$ and $(\mathcal{H}_\gamma,\Theta;\mathtt{Q}_\Pi)\vdash^{*a_0}_c\Gamma,C;\Pi^{[\cdot]}$. IH followed by a (*cut*) with an uncapped cut formula $C^{(\mathtt{u})}$ yields the lemma.

Case 3. Third the last inference introduces a $\bigvee$-formula A in Γ. Let $A\simeq\bigvee(A_\iota)_{\iota\in J}$. Then $A^{(\rho_\mathbb{S})}\in\Gamma^{(\rho_\mathbb{S})}_\mathbb{S}$. There are an $\iota\in J$, an ordinal $a(\iota)<a$ such that $(\mathcal{H}_\gamma,\Theta;\mathtt{Q}_\Pi)\vdash^{*a(\iota)}_c\Gamma,A_\iota;\Pi^{[\cdot]}$. We can assume $\mathsf{k}(\iota)\subset\mathsf{k}(A_\iota)$, and claim that $\iota\in[\partial\mathtt{Q}]J$ with $\rho_\mathbb{S}\in\partial\mathtt{Q}$. We obtain $\mathsf{k}(\iota)\subset\mathcal{H}_\gamma[\Theta_{\partial\mathtt{Q}}]\subset M_{\partial\mathtt{Q}}$ by (57) for $\Theta_{\partial\mathtt{Q}}=\Theta$ and $\gamma\leq\gamma_0\leq\gamma^{\mathtt{Q}}_\mathbb{S}\leq\delta_\mathbb{S}\leq\mathtt{p}_0(\rho_\mathbb{S})$.

IH yields $(\mathcal{H}_\gamma,\Theta,\mathtt{Q})\vdash^{a(\iota)}_c\widehat{\Gamma},(A_\iota)^{(\rho_\mathbb{S})},\widehat{\Pi}$. $(\mathcal{H}_\gamma,\Theta,\mathtt{Q})\vdash^a_c\widehat{\Gamma},\widehat{\Pi}$ follows from a $(\bigvee)$.

Other cases are seen from IH as in Capping 6.44. □

Lemma 7.31 (Recapping)
Let $\mathbb{S}$ be a successor stable ordinal, $(\mathcal{H}_\gamma,\Theta,\mathtt{Q})\vdash^a_{c_1,\mathbb{S},\gamma_0,b_2}\Pi,\widehat{\Gamma}$ with a finite family $\mathtt{Q}$ for γ_0,b_2, $\Gamma\cup\Pi\subset\Delta_0(\mathbb{I})$, and $\widehat{\Gamma}=\bigcup\{\Gamma^{(\rho)}_\rho:\rho\in\mathtt{Q}^t(\mathbb{S})\}$, where each $\theta\in\widehat{\Gamma}$ is either a $\bigvee$-formula or $\mathrm{rk}(\theta)<\mathbb{S}$, $\mathtt{Q}^t\subset\mathtt{Q}$ such that $\mathtt{Q}^t(\mathbb{S})\subset\mathtt{Q}(\mathbb{S})$ with $dom(\mathtt{Q}^t)\subset\{\mathbb{S}\}$ and $\forall\rho\in\mathtt{Q}^t(\mathbb{S})(s(\rho)>\mathbb{S})$, and $\mathtt{Q}^f$ is a family such that $\mathtt{Q}^f(\mathbb{S})=\mathtt{Q}(\mathbb{S})\setminus\mathtt{Q}^t(\mathbb{S})$ and $\mathtt{Q}^f(\mathbb{T})=\mathtt{Q}(\mathbb{T})$ for $\mathbb{T}\neq\mathbb{S}$. Π is a set of formulas such that $\tau\in\{\mathtt{u}\}\cup\bigcup\mathtt{Q}^f$ for every $A^{(\tau)}\in\Pi$.

Let $\max\{s(\rho):\rho\in\mathtt{Q}^t(\mathbb{S})\}\leq b_1$ and $\omega(b,a)=\omega^{\omega^b}a$. For each $\rho\in\mathtt{Q}^t(\mathbb{S})$, let $\mathbb{S}\leq b^{(\rho)}\in\mathcal{H}_\gamma[\Theta^{(\rho)}]\cap\mathcal{H}_\gamma[\Theta_{\partial\mathtt{Q}}]$ with $\mathrm{rk}(\Gamma_\rho)<b^{(\rho)}<s(\rho)$, and $\kappa(\rho)$ be ordinals such that $\kappa(\rho)\in H^{\mathtt{Q}}_\rho(h^{b^{(\rho)}}(m(\rho);\omega(b_1,a)),b_2,\gamma^{\mathtt{Q}}_\mathbb{S},\Theta^{(\rho)}\cup\{\mathbb{S}\}\cup\Theta_{\partial\mathtt{Q}})$. Assume $\forall\mathbb{T}\leq\mathbb{S}(b_1\in\mathcal{H}_\gamma[\Theta_{\mathtt{Q}(\mathbb{T})}]\cap\mathcal{H}_\gamma[\Theta_{\partial\mathtt{Q}}])$.

Then $(\mathcal{H}_\gamma,\Theta,\mathtt{Q}^\kappa)\vdash^{\omega(b_1,a)}_{c_{b_1},\mathbb{S},\gamma_0,b_2}\Pi,\widehat{\Gamma}_\kappa$ holds, where $\widehat{\Gamma}_\kappa=\bigcup\{\Gamma^{(\kappa(\rho))}_\rho:\rho\in\mathtt{Q}^t(\mathbb{S})\}$, $c_{b_1}=\max\{c_1,b_1\}$, $\mathtt{Q}^\kappa=\mathtt{Q}^f\cup\{(\mathbb{S},\kappa(\rho)):\rho\in\mathtt{Q}^t(\mathbb{S})\}$, $\gamma^{\mathtt{Q}^\kappa}_\mathbb{T}=\gamma^{\mathtt{Q}}_\mathbb{T}$, $e^{\mathtt{Q}^\kappa}_\mathbb{T}=e^{\mathtt{Q}}_\mathbb{T}$ for $\mathbb{T}\neq\mathbb{S}$ and $e^{\mathtt{Q}^\kappa}_\mathbb{S}=\max(\{\tau\in\mathtt{Q}^f(\mathbb{S}):s(\tau)>\mathbb{S}\}\cup\{\kappa(\rho):\rho\in\mathtt{Q}^t(\mathbb{S})\})+1$.

$e^{\mathtt{Q}^\kappa}_\mathbb{S}<e^{\mathtt{Q}}_\mathbb{S}$ holds when $\mathtt{Q}^t=\{(\mathbb{S},\rho)\in\mathtt{Q}:s(\rho)>\mathbb{S}\}\neq\emptyset$.

Proof. This is shown by main induction on b_1 with subsidiary induction on a as in Recapping 6.47. □

Lemma 7.32 (Elimination of one stable ordinal)
Let $\mathbb{S} = \mathbb{T}^\dagger$ *be a successor stable ordinal and* $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^a_{\mathbb{S},\mathbb{S},\gamma_0,b_1} \Pi, \widehat{\Gamma}$ *with a finite family* $\mathtt{Q}$ *for* γ_0 *and* $b_1 \geq \mathbb{S}$, $\Pi \subset \Delta_0(\mathbb{I})$, $\Gamma \subset \Delta_0(\mathbb{S})$, $\widehat{\Gamma} = \bigcup\{\Gamma^{(\rho)}_\rho : \rho \in \mathtt{Q}(\mathbb{S})\}$*, and* $\mathtt{Q}^t = \{(\mathbb{S}, \tau) \in \mathtt{Q} : s(\tau) > \mathbb{S}\}$, $\mathtt{Q}^f = \mathtt{Q} \setminus \mathtt{Q}^t$. Π *is a set of formulas such that for each* $A^{(\tau)} \in \Pi$, $\tau \in \{\mathtt{u}\} \cup \bigcup_{\mathbb{U}<\mathbb{S}} \mathtt{Q}(\mathbb{U})$.

Let $\tilde{a} = \varphi_{b_1+e^{\mathtt{Q}}_{\mathbb{S}}}(a)$, $\mathtt{Q}_1 = \mathtt{Q} \restriction \mathbb{S} = \{(\mathbb{T}, \rho) \in \mathtt{Q} : \mathbb{T} < \mathbb{S}\}$ *and* $\gamma_1 = \gamma^{\mathtt{Q}}_{\mathbb{S}} + \mathbb{I} < \gamma_0 + \mathbb{I}^2$.

Then $\mathtt{Q}_1$ *is a finite family for* γ_1, b_1 *and* $(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{Q}_1) \vdash^{\tilde{a}}_{\mathbb{T},\mathbb{T},\gamma_1,b_1} \Pi, \Gamma^{(\mathtt{u})}$ *holds for* $\Gamma^{(\mathtt{u})} = \bigcup\{\Gamma^{(\mathtt{u})}_\rho : \rho \in \mathtt{Q}(\mathbb{S})\}$.

Proof. This is seen by main induction on $e^{\mathtt{Q}}_{\mathbb{S}}$ with subsidiary induction on a as in Lemma 6.48. When $\mathbb{S} \in dom(\mathtt{Q})$, we have $\mathtt{Q}(\mathbb{S}) \subset \mathcal{H}_{\gamma_1}$ and $e^{\mathtt{Q}}_{\mathbb{S}} \in \mathcal{H}_{\gamma_1}$ for $\gamma_1 = \gamma^{\mathtt{Q}}_{\mathbb{S}} + \mathbb{I}$ by Definition 7.19. $\mathtt{Q}_1$ is a finite family for γ_1, b_1. Then $\gamma_1 \in \mathcal{H}_\gamma[\Theta_{\mathtt{Q}_1(\mathbb{T})}] \cap \mathcal{H}_\gamma[\Theta_{\partial\mathtt{Q}}]$ for every $\mathbb{T} \in dom(\mathtt{Q}_1)$ by (64).

First assume $\mathtt{Q}^t(\mathbb{S}) \neq \emptyset$. For each $\rho \in \mathtt{Q}^t(\mathbb{S})$, let $\kappa(\rho)$ be an ordinal such that $\kappa(\rho) \in H^{\mathtt{Q}}_\rho(h^{\mathbb{S}}(m(\rho); \omega(b_1, a)), b_1, \gamma^{\mathtt{Q}}_{\mathbb{S}}, \Theta^{(\rho)} \cup \{\mathbb{S}\} \cup \Theta_{\partial\mathtt{Q}})$ with $\omega(b, a) = \omega^{\omega^b} a$. We obtain $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}^\kappa) \vdash^{\omega(b_1,a)}_{b_1,\mathbb{S},\gamma_0,b_1} \Pi, \widehat{\Gamma}_\kappa$ by Recapping 7.31. Cut-elimination 7.27 with $SSt \cap (\mathbb{S}, \mathbb{S}] = \emptyset$ yields for $a_1 = \varphi_{b_1}(\omega(b_1, a))$, $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}^\kappa) \vdash^{a_1}_{\mathbb{S},\mathbb{S},\gamma_0,b_1} \Pi, \widehat{\Gamma}_\kappa$, where $e^{\mathtt{Q}^\kappa}_{\mathbb{S}} = \max\{\kappa(\rho) : \rho \in \mathtt{Q}^t(\mathbb{S})\} + 1 < e^{\mathtt{Q}}_{\mathbb{S}}$. MIH yields $(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{Q}_1) \vdash^{\tilde{a}_1}_{\mathbb{T},\mathbb{T},\gamma_1,b_1} \Pi, \Gamma^{(\mathtt{u})}$, where $\tilde{a}_1 = \varphi_{b_1+e^{\mathtt{Q}^\kappa}_{\mathbb{S}}}(a_1) < \varphi_{b_1+e^{\mathtt{Q}}_{\mathbb{S}}}(a)$ and $\gamma_1 = \gamma^{\mathtt{Q}}_{\mathbb{S}} + \mathbb{I}$.

In what follows assume $\mathtt{Q}^t(\mathbb{S}) = \emptyset$.

Case 1. First let $\{\neg A^{(\sigma)}, A^{(\sigma)}\} \subset \Pi \cup \widehat{\Gamma}$ with $\sigma \in \{\mathtt{u}\} \cup \bigcup \mathtt{Q}$ and $d = \mathrm{rk}(A) < \mathbb{S}$ by (Taut). If $d < \mathbb{T}$, then $(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{Q}_1) \vdash^{\tilde{a}}_{\mathbb{T},\mathbb{T},\gamma_1,b_1} \Pi, \Gamma^{(\mathtt{u})}$ by (Taut).

Let $\mathbb{T} \leq d < \mathbb{S}$. Then $(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{Q}_1) \vdash^{2d}_{0,\mathbb{T},\gamma_1,b_1} \Pi, \Gamma^{(\mathtt{u})}$ by Tautology 7.24 and $(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{Q}_1) \vdash^{\tilde{a}}_{0,\mathbb{T},\gamma_1,b_1} \Pi, \Gamma^{(\mathtt{u})}$ by $\tilde{a} > \mathbb{S} > d$.

Case 2. Second consider the case when the last inference is a $(\mathrm{rfl}_{\mathbb{U}}(\rho, d, f, b_1))$. If $\mathbb{U} \leq \mathbb{T}$, then SIH followed by a $(\mathrm{rfl}_{\mathbb{U}}(\rho, d, f, b_1))$ yields the lemma. Let $\mathbb{U} = \mathbb{S}$.

Let $g = m(\rho)$ and $s(\rho) \geq d \in \mathrm{supp}(g)$. Let $\mathtt{R} = \mathtt{Q} \cup \{(\mathbb{S}, \rho)\}$ and $\gamma_1 = \gamma^{\mathtt{R}}_{\mathbb{S}} + \mathbb{I}$. We have a sequent $\Delta \subset \bigvee_{\mathbb{S}}(d)$ and an ordinal $a_0 < a$ such that $\mathrm{rk}(\Delta) < d \leq s(\rho)$ and $(\mathcal{H}_\gamma, \Theta, \mathtt{R}) \vdash^{a_0}_{\mathbb{S},\mathbb{S},\gamma_0,b_1} \Pi, \widehat{\Gamma}, \neg\delta^{(\rho)}$ for each $\delta \in \Delta$. On the other hand we have $(\mathcal{H}_\gamma, \Theta \cup \{\sigma\}, \mathtt{R} \cup \{(\mathbb{S}, \sigma)\}) \vdash^{a_0}_{\mathbb{S},\mathbb{S},\gamma_0,b_1} \Pi, \widehat{\Gamma}, \Delta^{(\sigma)}$, where $\sigma \in H^{\mathtt{Q}}_\rho(f, b_1, \gamma^{\mathtt{R}}_{\mathbb{S}}, \Theta^{(\rho)} \cup \{\mathbb{S}\} \cup \Theta_{\partial\mathtt{Q}})$, f is a special finite function such that $s(f) \leq b_1$, $f_d = g_d$, $f^d <^d g'(d)$ and $SC_{\mathbb{I}}(f) \subset \mathcal{H}_{\gamma^{\mathtt{R}}_{\mathbb{S}}}[\Theta^{(\rho)}]$.

Case 2.1. $s(\rho) \leq \mathbb{S}$: Then $\Delta \subset \Delta_0(\mathbb{S})$. Let $\tilde{a}_0 = \varphi_{b_1+e^{\mathtt{R}}_{\mathbb{S}}}(a_0)$. SIH yields $(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{Q}_1) \vdash^{\tilde{a}_0}_{\mathbb{T},\mathbb{T},\gamma_1,b_1} \Pi, \Gamma^{(\mathtt{u})}, \neg\delta^{(\mathtt{u})}$ for each $\delta \in \Delta$, and $(\mathcal{H}_{\gamma_1}, \Theta \cup \{\sigma\}, \mathtt{Q}_1) \vdash^{\tilde{a}_0}_{\mathbb{T},\mathbb{T},\gamma_1,b_1} \Pi, \Gamma^{(\mathtt{u})}, \Delta^{(\mathtt{u})}$ for $\sigma \in \mathcal{H}_{\gamma^{\mathtt{R}}_{\mathbb{S}}+\mathbb{I}} = \mathcal{H}_{\gamma_1}$. We obtain $(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{Q}_1) \vdash^{\tilde{a}_0+p}_{\mathbb{S},\mathbb{T},\gamma_1,b_1} \Pi, \Gamma^{(\mathtt{u})}$ by several (cut)'s for a $p < \omega$. Cut-elimination 7.27 with $SSt \cap (\mathbb{T}, \mathbb{T}] = \emptyset$ yields $(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{Q}_1) \vdash^{\varphi_{\mathbb{S}}(\tilde{a}_0+p)}_{\mathbb{T},\mathbb{T},\gamma_1,b_1} \Pi, \Gamma^{(\mathtt{u})}$, where $\varphi_{\mathbb{S}}(\tilde{a}_0 + p) < \tilde{a} = \varphi_{b_1+e^{\mathtt{Q}}_{\mathbb{S}}}(a)$ by $b_1 + e^{\mathtt{Q}}_{\mathbb{S}} > \mathbb{S}$.

Case 2.2. $s(\rho) > \mathbb{S}$: Then $\mathbb{S} \notin dom(\mathtt{Q})$ and $\Gamma = \emptyset$. We have $(\mathcal{H}_\gamma, \Theta, \mathtt{R}) \vdash^a_{\mathbb{S},\mathbb{S},\gamma_0,b_1} \Pi$. Let $\mathtt{R}^t = \{(\mathbb{S}, \rho)\}$. Recapping 7.31 yields $(\mathcal{H}_\gamma, \Theta, \mathtt{R}^\kappa) \vdash^{\omega(b_1,a)}_{\mathbb{S},\mathbb{S},\gamma_0,b_1} \Pi$ and $e^{\mathtt{R}^\kappa}_{\mathbb{S}} = \kappa + 1 < \rho < e^{\mathtt{R}}_{\mathbb{S}}$. MIH yields $(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{Q}_1) \vdash^{a_1}_{\mathbb{T},\mathbb{T},\gamma_1,b_1} \Pi$ with $a_1 = \varphi_{b_1+e^{\mathtt{R}^\kappa}_{\mathbb{S}}}(\omega(b_1, a)) < \varphi_{b_1+e^{\mathtt{Q}}_{\mathbb{S}}}(a) = \tilde{a}$ by $e^{\mathtt{R}^\kappa}_{\mathbb{S}} < \mathbb{S} = e^{\mathtt{Q}}_{\mathbb{S}}$.

Case 3. The last inference is a $(\bigwedge)$: We have $a(\iota) < a$, $A^{(\rho)} \in \widehat{\Gamma}$ and for each $\iota \in [\mathtt{Q}]_{A^{(\rho)}} J$ with $A \simeq \bigwedge(A_\iota)_{\iota \in J}$, we have $(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\iota), \mathtt{Q}) \vdash^{a(\iota)}_{\mathbb{S},\mathbb{S},\gamma_0,b_1} \Pi, \widehat{\Gamma}, (A_\iota)^{(\rho)}$. Since $A \in \Delta_0(\mathbb{S})$, we obtain $\mathsf{k}(A) \subset \mathcal{H}_\gamma[\Theta^{(\rho)}] \cap \mathbb{S} \subset M_\rho \cap \mathbb{S} = \rho$ for $\rho \in \mathtt{Q}(\mathbb{S})$. This means $A \in \Delta_0(\rho)$, and $[\rho]J = J$. Hence $[\mathtt{Q}]_{A^{(\rho)}} J = [\mathtt{Q}_1]_{A^{(\mathtt{u})}} J$. SIH yields $(\mathcal{H}_{\gamma_1}, \Theta \cup \mathsf{k}(\iota), \mathtt{Q}_1) \vdash^{\tilde{a}(\iota)}_{\mathbb{T},\mathbb{T},\gamma_1,b_1} \Pi, \Gamma^{(\mathtt{u})}, (A_\iota)^{(\mathtt{u})}$ for each $\iota \in [\mathtt{Q}]_A J$, where $\tilde{a}(\iota) = \varphi_{b_1+e^{\mathtt{Q}}_{\mathbb{S}}}(b + a(\iota)) < \tilde{a}$. A $(\bigwedge)$ yields $(\mathcal{H}_{\gamma_1}, \Theta, \mathtt{Q}_1) \vdash^{\tilde{a}}_{\mathbb{T},\mathbb{T},\gamma_1,b_1} \Pi, \Gamma^{(\mathtt{u})}$.

Other cases are seen from SIH. □

Definition 7.33 We define the *S-rank* $\mathrm{srk}(A^{(\rho)})$ of a capped formula $A^{(\rho)}$ as follows. Let $\mathrm{srk}(A^{(\mathtt{u})}) = 0$, and $\mathrm{srk}(A^{(\rho)}) = \mathbb{S}$ for $\rho \prec \mathbb{S} \in SSt$.

$\mathrm{srk}(\Gamma) = \max\{\mathrm{srk}(A^{(\rho)}) : A^{(\rho)} \in \Gamma\}$.

Lemma 7.34 (Elimination of stable ordinals)
Suppose $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a}_{\xi,\xi,\gamma_0,b_1} \Gamma$ *and* $\mathrm{srk}(\Gamma) \leq \mathbb{S} < \xi \leq b_1 < \mathbb{I}$, *where* $\mathbb{S}$ *is either a stable ordinal or* $\mathbb{S} = \Omega$ *such that* $\forall \mathbb{U} \in dom(\mathtt{Q}_{\mathbb{S}})(\mathbb{S} \in \mathcal{H}_\gamma[\Theta_{\mathtt{Q}(\mathbb{U})}] \cap \mathcal{H}_\gamma[\Theta_{\partial\mathtt{Q}}])$ *for* $\mathtt{Q}_{\mathbb{S}} = \mathtt{Q} \restriction \mathbb{S}$.

Then there exists an ordinal $\gamma_0 \leq \gamma_{\mathbb{S}} < \gamma_0 + \mathbb{I}^2$ *such that* $\mathtt{Q}_{\mathbb{S}}$ *is a finite family for* $\gamma_{\mathbb{S}}, b_1$ *and* $(\mathcal{H}_{\gamma_{\mathbb{S}}}, \Theta, \mathtt{Q}_{\mathbb{S}}) \vdash^{f(\xi,a)}_{\mathbb{S},\mathbb{S},\gamma_{\mathbb{S}},b_1} \Gamma$ *holds for* $f(\xi, a) = \varphi_{b_1+\xi+1}(a)$.

Proof. By main induction on ξ with subsidiary induction on a. (64) in $(\mathcal{H}_{\gamma_{\mathbb{S}}}, \Theta, \mathtt{Q}_{\mathbb{S}}) \vdash^{f(\xi,a)}_{\mathbb{S},\mathbb{S},\gamma_{\mathbb{S}},b_1} \Gamma$ follows from (64) and (65) in $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a}_{\xi,\xi,\gamma_0,b_1} \Gamma$.
Case 1. Consider the case when the last inference is a $(\mathrm{rfl}_{\mathbb{T}}(\rho, d, f, b_1))$ for a $\mathbb{T} = \mathbb{U}^\dagger \leq \xi$. If $\mathbb{T} \leq \mathbb{S}$, then SIH yields the lemma. Let $\mathbb{S} < \mathbb{T} \in dom(\mathtt{R})$ for $\mathtt{R} = \mathtt{Q} \cup \{(\mathbb{T}, \rho)\}$. We have $\forall \mathbb{U} \in dom(\mathtt{Q}_{\mathbb{T}})(\mathbb{T} \in \mathcal{H}_\gamma[\Theta_{\mathtt{Q}(\mathbb{U})}] \cap \mathcal{H}_\gamma[\Theta_{\partial\mathtt{Q}}])$ by (64). Let Δ be a finite set of sentences such that $(\mathcal{H}_\gamma, \Theta, \mathtt{R}) \vdash^{a_0}_{\xi,\xi,\gamma_0,b_1} \Gamma, \neg\delta^{(\rho)}$ for each $\delta \in \Delta$, and $(\mathcal{H}_\gamma, \Theta, \mathtt{R} \cup \{(\mathbb{T}, \sigma)\}) \vdash^{a_0}_{\xi,\xi,\gamma_0,b_1} \Gamma, \Delta^{(\sigma)}$ for each $\sigma \in H^{\mathtt{Q}}_\rho(f, b_1\gamma^{\mathtt{R}}_{\mathbb{T}}, \Theta^{(\rho)} \cup \{\mathbb{T}\} \cup \Theta_{\partial\mathtt{Q}})$, and $a_0 < a$. We have $\mathrm{srk}(\delta^{(\rho)}) = \mathrm{srk}(\Delta^{(\sigma)}) = \mathbb{T}$. By SIH there exists a $\gamma_{\mathbb{T}} < \gamma_0 + \mathbb{I}^2$ such that for $a_1 = f(\xi, a_0) = \varphi_{b_1+\xi+1}(a_0)$, $(\mathcal{H}_{\gamma_{\mathbb{T}}}, \Theta, \mathtt{Q}_{\mathbb{T}}) \vdash^{a_1}_{\mathbb{T},\mathbb{T},\gamma_{\mathbb{T}},b_1} \Gamma, \neg\delta^{(\rho)}$ for each $\delta \in \Delta$, and $(\mathcal{H}_{\gamma_{\mathbb{T}}}, \Theta, \mathtt{Q}_{\mathbb{T}} \cup \{(\mathbb{T}, \sigma)\}) \vdash^{a_1}_{\mathbb{T},\mathbb{T},\gamma_{\mathbb{T}},b_1} \Gamma, \Delta^{(\sigma)}$. $(\mathrm{rfl}_{\mathbb{T}}(\rho, d, f, b_1))$ yields $(\mathcal{H}_{\gamma_{\mathbb{T}}}, \Theta, \mathtt{Q}_{\mathbb{T}}) \vdash^{a_2}_{\mathbb{T},\mathbb{T},\gamma_{\mathbb{T}},b_1} \Gamma$ for $a_2 = a_1 + 1$.

On the other hand we have $\mathrm{srk}(\Gamma) \leq \mathbb{S} < \mathbb{T} = \mathbb{U}^\dagger \leq \xi$. By Lemma 7.32 pick a $\gamma_{\mathbb{U}} < \gamma_{\mathbb{T}} + \mathbb{I}^2 = \gamma_0 + \mathbb{I}^2$ such that $(\mathcal{H}_{\gamma_{\mathbb{U}}}, \Theta, \mathtt{Q}_{\mathbb{U}}) \vdash^{a_3}_{\mathbb{U},\mathbb{U},\gamma_{\mathbb{U}},b_1} \Gamma$, where $a_3 = \varphi_{b_1+e^{\mathtt{Q}_1}_{\mathbb{T}}}(a_2) = \varphi_{b_1+e^{\mathtt{Q}_1}_{\mathbb{T}}}(f(\xi, a_0) + 1) < \varphi_{b_1+\xi+1}(a) = f(\xi, a)$ by $e^{\mathtt{Q}_1}_{\mathbb{T}} \leq \mathbb{T} \leq \xi$. If $\mathbb{S} = \mathbb{U}$, then we are done. Let $\mathbb{S} < \mathbb{U}$ with $\mathbb{U} < \xi$. Then by MIH pick a $\gamma_{\mathbb{S}}$ such that $(\mathcal{H}_{\gamma_{\mathbb{S}}}, \Theta, \mathtt{Q}_{\mathbb{S}}) \vdash^{a_4}_{\mathbb{S},\mathbb{S},\gamma_{\mathbb{S}},b_1} \Gamma$ for $a_4 = f(\mathbb{U}, a_3) = \varphi_{b_1+\mathbb{U}+1}(a_3) < \varphi_{b_1+\xi+1}(a) = f(\xi, a)$ by $\mathbb{U} < \xi$.
Case 2. Next consider the case when the last inference is a (cut) of a cut formula $C^{(\sigma)}$ wth $\mathrm{rk}(C) < \xi$ and $\mathbb{T} = \mathrm{srk}(C^{(\sigma)}) \leq \xi$. We have an ordinal $a_0 < a$ such that $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a_0}_{\xi,\xi,\gamma_0,b_1} \Gamma, \neg C^{(\sigma)}$ and $(\mathcal{H}_\gamma, \Theta, \mathtt{Q}) \vdash^{a_0}_{\xi,\xi,\gamma_0,b_1} C^{(\sigma)}, \Gamma$.

Let $\mathbb{U} = \max\{\mathbb{S}, \mathbb{T}\}$. First assume $\mathbb{U} < \xi$. By SIH pick a $\gamma_{\mathbb{U}}$ such that $(\mathcal{H}_{\gamma_{\mathbb{U}}}, \Theta, \mathtt{Q}_{\mathbb{U}}) \vdash^{a_1}_{\mathbb{U},\mathbb{U},\gamma_{\mathbb{U}},b_1} \Gamma, \neg C^{(\sigma)}$ and $(\mathcal{H}_{\gamma_{\mathbb{U}}}, \Theta, \mathtt{Q}_{\mathbb{U}}) \vdash^{a_1}_{\mathbb{U},\mathbb{U},\gamma_{\mathbb{U}},b_1} C^{(\sigma)}, \Gamma$, where $a_1 = f(\xi, a_0) = \varphi_{b_1+\xi+1}(a_0)$. A (cut) yields $(\mathcal{H}_{\gamma_{\mathbb{U}}}, \Theta, \mathtt{Q}_{\mathbb{U}}) \vdash^{a_1+1}_{\xi,\mathbb{U},\gamma_{\mathbb{U}},b_1} \Gamma$. Cut-elimination 7.27 with $SSt \cap (\mathbb{U}, \mathbb{U}] = \emptyset$ yields $(\mathcal{H}_{\gamma_{\mathbb{U}}}, \Theta, \mathtt{Q}_{\mathbb{U}}) \vdash^{a_2}_{\mathbb{U},\mathbb{U},\gamma_{\mathbb{U}},b_1} \Gamma$, where $a_2 = \varphi_\xi(a_1 + 1) < \varphi_{b_1+\xi+1}(a) = f(\xi, a)$ by $\xi < b_1 + \xi + 1$. If $\mathbb{U} = \mathbb{S}$, then we are done. Let

$\mathbb{U} = \mathbb{T} > \mathbb{S}$. By MIH with $\mathbb{U} < \xi$ we obtain $(\mathcal{H}_{\gamma_{\mathbb{S}}}, \Theta, \mathtt{Q}_{\mathbb{S}}) \vdash^{a_3}_{\mathbb{S},\mathbb{S},\gamma_{\mathbb{S}},b_1} \Gamma$ for a $\gamma_{\mathbb{S}}$, where $a_3 = f(\mathbb{U}, a_2) = \varphi_{b_1+\mathbb{U}+1}(a_2) < \varphi_{b_1+\xi+1}(a) = f(\xi, a)$ by $\mathbb{U} < \xi$.

Second let $\mathbb{T} = \mathbb{U} = \xi = \mathbb{W}^{\dagger} > \mathbb{S}$. Then $C \in \Delta_0(\mathbb{T})$. By Lemma 7.32 pick a $\gamma_{\mathbb{W}}$ such that $(\mathcal{H}_{\gamma_{\mathbb{W}}}, \Theta, \mathtt{Q}_{\mathbb{W}}) \vdash^{\tilde{a}_0}_{\mathbb{W},\mathbb{W},\gamma_{\mathbb{W}},b_1} \Gamma, \neg C^{(\mathtt{u})}$ and $(\mathcal{H}_{\gamma_{\mathbb{W}}}, \Theta, \mathtt{Q}_{\mathbb{W}}) \vdash^{\tilde{a}_0}_{\mathbb{W},\mathbb{W},\gamma_{\mathbb{W}},b_1} C^{(\mathtt{u})}, \Gamma$, where $\tilde{a}_0 = \varphi_{b_1+e^{\mathtt{Q}}_{\mathbb{T}}}(a_0)$. A (cut) yields $(\mathcal{H}_{\gamma_{\mathbb{W}}}, \Theta, \mathtt{Q}_{\mathbb{W}}) \vdash^{\tilde{a}_0+1}_{\mathbb{T},\mathbb{W},\gamma_{\mathbb{W}},b_1} \Gamma$, and we obtain $(\mathcal{H}_{\gamma_{\mathbb{W}}}, \Theta, \mathtt{Q}_{\mathbb{W}}) \vdash^{a_4}_{\mathbb{W},\mathbb{W},\gamma_{\mathbb{W}},b_1} \Gamma$ by Cut-elimination 7.27, where $a_4 = \varphi_{\mathbb{T}}(\tilde{a}_0+1)$ and $SSt \cap (\mathbb{W}, \mathbb{W}] = \emptyset$. By MIH pick a $\gamma_{\mathbb{S}}$ such that $(\mathcal{H}_{\gamma_{\mathbb{S}}}, \Theta, \mathtt{Q}_{\mathbb{S}}) \vdash^{a_5}_{\mathbb{S},\mathbb{S},\gamma_{\mathbb{S}},b_1} \Gamma$ for $\mathbb{W} < \xi$ and $a_5 = f(\mathbb{W}, a_4) = \varphi_{b_1+\mathbb{W}+1}(a_4) < \varphi_{b_1+\xi+1}(a)$ by $\mathbb{W} < \xi$, $\mathbb{T} = \xi < b_1 + \xi + 1$, $e^{\mathtt{Q}}_{\mathbb{T}} \leq \mathbb{T} = \xi < \xi + 1$ and $a_0 < a$.

Case 3. There exists an A such that $\{\neg A^{(\rho)}, A^{(\rho)}\} \subset \Gamma$ with $\mathrm{srk}(A^{(\rho)}) \leq \mathbb{S}$ and $d = \mathrm{rk}(A) < \mathbb{T} \leq \xi$ for a $\mathbb{T} \in SSt$ by (Taut). We may assume $d \geq \mathbb{S}$. Then $(\mathcal{H}_{\gamma}, \Theta, \mathtt{Q}_{\mathbb{S}}) \vdash^{2d}_{0,\mathbb{S},\gamma_0,b_1} \Gamma$ by Tautology 7.24 and the lemma follows from $d < \xi < f(\xi, a)$.

Other cases are seen from SIH. □

Theorem 7.35 *Suppose* $\mathsf{KP}\omega + \Pi_1$-Collection $+ (V = L) \vdash \theta^{L_\Omega}$ *for a* Σ_1-*sentence* θ. *Then* $L_{\psi_\Omega(\varepsilon_{\mathbb{I}+1})} \models \theta$ *holds.*

Proof. Let $S_{\mathbb{I}} \vdash \theta^{L_\Omega}$ for a Σ-sentence θ. By Embedding 7.16 pick an $m > 0$ so that $(\mathcal{H}_{\mathbb{I}}, \emptyset; \emptyset) \vdash^{*\mathbb{I}\cdot 2+m}_{\mathbb{I}+m} \theta^{L_\Omega}$. Cut-elimination 7.17 yields $(\mathcal{H}_{\mathbb{I}}, \emptyset; \emptyset) \vdash^{*a}_{\mathbb{I}} \theta^{L_\Omega}$ for $a = \omega_m(\mathbb{I}\cdot 2+m) < \omega_{m+1}(\mathbb{I}+1)$. Then Collapsing 7.18 yields $(\mathcal{H}_{\hat{a}+1}, \emptyset; \emptyset) \vdash^{*\beta}_{\beta} \theta^{L_\Omega}$ for $\beta = \psi_{\mathbb{I}}(\hat{a}) \in LS$ with $\hat{a} = \omega^{\mathbb{I}+a} = \omega_{m+1}(\mathbb{I} \cdot 2 + m) > \beta$. Capping 7.29 then yields $(\mathcal{H}_{\hat{a}+1}, \emptyset, \emptyset) \vdash^{\beta}_{\beta,\beta,\gamma_0,\beta} \theta^{L_\Omega}$ where $\gamma_0 = \hat{a} + 1$ and $\theta^{L_\Omega} \equiv (\theta^{L_\Omega})^{(\mathtt{u})}$.

Let $\alpha = \varphi_{\beta\cdot 2+1}(\beta)$. By Lemma 7.34 we obtain $(\mathcal{H}_{\gamma_\Omega}, \emptyset, \emptyset) \vdash^{\alpha}_{\Omega,\Omega,\gamma_\Omega,\beta} \theta^{L_\Omega}$ for a $\gamma_\Omega < \gamma_0 + \mathbb{I}^2$. This means $(\mathcal{H}_{\gamma_\Omega}, \emptyset, \emptyset) \vdash^{\alpha}_{\Omega,0,\gamma_\Omega,\beta} \theta^{L_\Omega}$. $(\mathcal{H}_{\gamma_\Omega+\alpha+1}, \emptyset, \emptyset) \vdash^{\delta}_{\delta,0,\gamma_\Omega,\beta} \theta^{L_\delta}$ follows from Collapsing 7.28 for $\delta = \psi_\Omega(\gamma_\Omega + \alpha)$ with $\omega^\alpha = \alpha$. Cut-elimination 7.27 yields $(\mathcal{H}_{\gamma_\Omega+\alpha+1}, \emptyset, \emptyset) \vdash^{\varphi_\delta(\delta)}_{0,0,\gamma_\Omega,\beta} \theta^{L_\delta}$. We see that θ^{L_δ} is true by induction up to $\varphi_\delta(\delta)$, where $\delta < \psi_\Omega(\omega_{m+2}(\mathbb{I} + 1)) < \psi_\Omega(\varepsilon_{\mathbb{I}+1})$.

7.6 Well-foundedness proof in Σ^1_3-DC+BI

Theorem 7.36 [A∞c]
Σ^1_3-DC+BI $\vdash Wo[\alpha]$ *for* each $\alpha < \psi_\Omega(\varepsilon_{\mathbb{I}+1})$.

To prove Theorem 7.36, let us introduce 1-distinguished sets $D_1[X]$, which is obtained from Definition 3.5.1 of distinguished sets $D[X]$, first by replacing the next regular α^+ by the next stable $\alpha^\dagger$, and second by changing the well-founded part $W(\mathcal{C}^\alpha(X))$ to the maximal distinguished set $\mathcal{W}^\alpha_1(X) = \bigcup\{P : D^\alpha_0[P; X]\}$ relative to α and X, where $P \cap \alpha = X \cap \alpha$ if $D^\alpha_0[P; X]$ and α is stable. We see that $\mathcal{W} = \bigcup\{X : D_1[X]\}$ is the maximal 1-distinguished and Σ^1_3-class.

In this subsection let us sketch a part of a well-foundeness proof in Σ^1_3-DC+BI by pinpointing the lemma for which we need Σ^1_3-DC.

An ordinal term σ in $OT(\mathbb{I})$ is said to be *regular* if $\psi^f_\sigma(a)$ is in $OT(\mathbb{I})$ for some f and a. *Reg* denotes the set of regular terms. In this section we need the next regular ordinal above an ordinal α in defining distinguished sets. Although it is customarily denoted by α^+, it is hard to discriminate α^+ from the next stable ordinal $\alpha^\dagger$. Therefore let us write for $\alpha < \mathbb{I}$, $\alpha^{+^1} = \min\{\sigma \in SSt : \sigma > \alpha\}$ for the next stable ordinal $\alpha^\dagger$, and $\alpha^{+^0} = \min\{\sigma \in Reg : \sigma > \alpha\}$ for the next regular ordinal α^+. Let $\alpha^{+^1} := \alpha^{+^0} := \infty$ if $\alpha \geq \mathbb{I}$. Let $\alpha^{-^1} := \max\{\sigma \in St_{\mathbb{I}} \cup \{0\} : \sigma \leq \alpha\}$ when $\alpha < \mathbb{I}$, and $\alpha^{-^1} := \mathbb{I}$ if $\alpha \geq \mathbb{I}$. Since $SSt \subset Reg$, we obtain $\alpha^{+^0} \leq \alpha^{+^1}$ and $\beta^{+^0} < \sigma$ if $\beta < \sigma \in St$ since each $\sigma \in St$ is a limit of regular ordinals.

Definition 7.37 $\mathcal{C}^\alpha(X)$ is the closure of $\{0, \Omega, \mathbb{I}\} \cup (X \cap \alpha)$ under $+, \varphi$, $\{\sigma, \beta\} \cup SC_{\mathbb{I}}(f) \mapsto \psi_\sigma^f(\beta)$ for $\sigma > \alpha$, and $\rho \mapsto \mathbb{I}[\rho], \rho^\dagger$ for $\mathbb{I}[\rho], \rho^\dagger \geq \alpha$ in $OT(\mathbb{I})$.

Definition 7.38 For $P, X \subset OT(\mathbb{I})$ and $\gamma \in OT(\mathbb{I}) \cap \mathbb{I}$, let

$$\begin{array}{rcl} W_0^\alpha(P) & := & W(\mathcal{C}^\alpha(P)) \\ D_0^\gamma[P; X] & :\Leftrightarrow & P \cap \gamma^{-^1} = X \cap \gamma^{-^1} \,\&\, Wo[X \cap \gamma^{-^1}] \,\& \\ & & \forall\alpha \left(\gamma^{-^1} \leq \alpha \leq P \to W_0^\alpha(P) \cap \alpha^{+^0} = P \cap \alpha^{+^0}\right) \end{array} \quad (69)$$

$$\begin{array}{rcl} \mathcal{W}_1^\gamma(X) & := & \bigcup\{P \subset OT(\mathbb{I}) : D_0^\gamma[P; X]\} \\ D_1[X] & :\Leftrightarrow & Wo[X] \,\&\, \forall\gamma \left(\gamma \leq X \to \mathcal{W}_1^\gamma(X) \cap \gamma^{+^1} = X \cap \gamma^{+^1}\right) \quad (70) \\ \mathcal{W}_2 & := & \bigcup\{X \subset OT(\mathbb{I}) : D_1[X]\} \end{array}$$

A set P is said to be a 0-*distinguished set* for γ and X if $D_0^\gamma[P; X]$, and a set X is a 1-*distinguished set* if $D_1[X]$.

Observe that in Σ_2^1-AC, $W_0^\alpha(P)$ is Π_1^1, $D_0^\gamma[P; X]$ is Δ_2^1, $\mathcal{W}_1^\gamma(X)$ is Σ_2^1, and $D_1[X]$ is Δ_3^1. Hence $\mathcal{W}_2$ is a Σ_3^1-class.

Let $\alpha \in P$ for a 0-distinguished set P for $\gamma < \mathbb{I}$ and X. If $\alpha < \gamma^{-^1}$, then $\alpha \in X$ with $Wo[X]$. Otherwise $W(\mathcal{C}^\alpha(P)) \cap \alpha^{+^0} = W_0^\alpha(P) \cap \alpha^{+^0} = P \cap \alpha^{+^0}$ with $\alpha < \alpha^{+^0}$. Hence P is a well order.

Lemma 7.39 (Σ_2^1-CA)
Suppose $Wo[X \cap \gamma^{-^1}]$. Then $\mathcal{W}_1^\gamma(X)$ is the maximal 0-*distinguished* set *for γ and X, i.e., $D_0^\gamma[\mathcal{W}_1^\gamma(X); X]$ and $\exists Y(Y = \mathcal{W}_1^\gamma(X))$.*

Proof. This is seen as in Proposition 3.9. □

Lemma 7.40 *1. Let X and Y be* 1-*distinguished sets.*
Then $\gamma \leq X \,\&\, \gamma \leq Y \Rightarrow X \cap \gamma^{+^1} = Y \cap \gamma^{+^1}$.

2. $\mathcal{W}_2$ is the 1-*maximal distinguished class, i.e., $D_1[\mathcal{W}_2]$.*

3. For a family $\{Y_j\}_{j \in J}$ of 1-*distinguished sets, the union $Y = \bigcup_{j \in J} Y_j$ is also a* 1-*distinguished set.*

Lemma 7.41 1. $\mathcal{C}^{\mathbb{I}}(\mathcal{W}_2)\cap\mathbb{I}=\mathcal{W}_2\cap\mathbb{I}=W(\mathcal{C}^{\mathbb{I}}(\mathcal{W}_2))\cap\mathbb{I}$.

2. (BI) *For* each $n<\omega$, $TI[\mathcal{C}^{\mathbb{I}}(\mathcal{W}_2)\cap\omega_n(\mathbb{I}+1)]$, *i.e., for each class* $\mathcal{X}$, $Prg[\mathcal{C}^{\mathbb{I}}(\mathcal{W}_2),\mathcal{X}]\to\mathcal{C}^{\mathbb{I}}(\mathcal{W}_2)\cap\omega_n(\mathbb{I}+1)\subset\mathcal{X}$.

3. *For* each $n<\omega$, $\mathcal{C}^{\mathbb{I}}(\mathcal{W}_2)\cap\omega_n(\mathbb{I}+1)\subset W(\mathcal{C}^{\mathbb{I}}(\mathcal{W}_2))$. *In particular* $\{\mathbb{I},\omega_n(\mathbb{I}+1)\}\subset W(\mathcal{C}^{\mathbb{I}}(\mathcal{W}_2))$.

As in Definition 3.10, $\mathcal{G}^X:=\{\alpha\in OT(\mathbb{I}):\alpha\in\mathcal{C}^\alpha(X)\,\&\,\mathcal{C}^\alpha(X)\cap\alpha\subset X\}$.

Lemma 7.42 (Σ^1_2-CA)
Suppose $D_1[Y]$ *and* $\alpha\in\mathcal{G}^Y$. *Let* $X=\mathcal{W}^\alpha_1(Y)\cap\alpha^{+^1}$. *Assume that one of the following conditions (71) and (72) is fulfilled. Then* $\alpha\in X$ *and* $D_1[X]$. *In particular* $\alpha\in\mathcal{W}_2$ *holds. Moreover if* $\alpha^{-^1}\leq Y$, *then* $\alpha\in Y$ *holds.*

$$\forall\beta\left(Y\cap\alpha^{+^1}<\beta\,\&\,\beta^{+^0}<\alpha^{+^0}\to W^\beta_0(Y)\cap\beta^{+^0}\subset Y\right)\tag{71}$$

$$\begin{aligned}&\forall\beta\geq\alpha^{-^1}\left(Y\cap\alpha^{+^1}<\beta\,\&\,\beta^{+^0}<\alpha^{+^0}\to W^\beta_0(Y)\cap\beta^{+^0}\subset Y\right)\\ &\&\,\forall\beta<\alpha^{-^1}\exists\gamma(\beta<\gamma^{+^1}\,\&\,\gamma^{-^1}\leq Y)\end{aligned}\tag{72}$$

Proof. This is seen as in Lemma 3.15 by showing that $D^\alpha_0[P;Y]$, $\alpha\in X$ and $D_1[X]$ for $P=W^\alpha_0(Y)\cap\alpha^{+^0}=W(\mathcal{C}^\alpha(Y))\cap\alpha^{+^0}$. □

Lemma 7.43 *Assume* $D_1[Y]$, $\mathbb{I}>\mathbb{S}\in Y\cap(St\cup\{0\})$ *and* $\{0,\Omega\}\subset Y$. *Then* $\mathbb{S}^{+^1}=\mathbb{S}^\dagger\in\mathcal{W}_2$.

Proof. Since the condition (72) in Lemma 3.15 is fulfilled with $(\mathbb{S}^{+^1})^{-^0}=(\mathbb{S}^{+^1})^{-^1}=\mathbb{S}^{+^1}$ and $\mathbb{S}^{-^1}=\mathbb{S}$, it suffices to show that $\mathbb{S}^{+^1}\in\mathcal{G}^Y$. Let $\alpha=\mathbb{S}^{+^1}$. $\alpha\in\mathcal{C}^\alpha(Y)$ follows from $\mathbb{S}\in Y\cap\alpha$. Moreover $\gamma\in\mathcal{C}^\alpha(Y)\cap\alpha\Rightarrow\gamma\in Y$ is seen by induction on $\ell\gamma$ using the assumption $\{0,\Omega\}\subset Y$. Therefore $\alpha\in\mathcal{G}^Y$. □

Lemma 7.44 (Σ^1_3-DC)
If $\alpha\in\mathcal{G}^{\mathcal{W}_2}$, *then there exists a* 1*-distinguished* set Z *such that* $\{0,\Omega\}\subset Z$, $\alpha\in\mathcal{G}^Z$ *and* $\forall\mathbb{S}\in Z\cap(St\cup\{\Omega\})[\mathbb{S}^\dagger\in Z]$.

Proof. Let $\alpha\in\mathcal{G}^{\mathcal{W}_2}$. We have $\alpha\in\mathcal{C}^\alpha(\mathcal{W}_2)$. Pick a 1-distinguished set X_0 such that $\alpha\in\mathcal{C}^\alpha(X_0)$. We can assume $\{0,\Omega\}\subset X_0$. On the other hand we have $\mathcal{C}^\alpha(\mathcal{W}_2)\cap\alpha\subset\mathcal{W}_2$ and $\forall\mathbb{S}\in\mathcal{W}_2\cap(St_{\mathbb{I}}\cup\{\Omega\})[\mathbb{S}^\dagger\in\mathcal{W}_2]$ by Lemma 7.43. We obtain

$$\begin{aligned}&\forall n\forall X\exists Y\{D_1[X]\to D_1[Y]\\ \wedge\ &\forall\beta\in OT(\mathbb{I})\,(\ell\beta\leq n\wedge\beta\in\mathcal{C}^\alpha(X)\cap\alpha\to\beta\in Y)\\ \wedge\ &\forall\mathbb{S}\in(St\cup\{\Omega\})\left(\ell\mathbb{S}\leq n\wedge\mathbb{S}\in X\to\mathbb{S}^\dagger\in Y\right)\}\end{aligned}$$

Since $D_1[X]$ is Δ^1_3, Σ^1_3-DC yields a set Z such that $Z_0 = X_0$ and

$$
\begin{array}{ll}
& \forall n\{D_1[Z_n] \to D_1[Z_{n+1}] \\
\land & \forall \beta \in OT(\mathbb{I})\, (\ell\beta \leq n \land \beta \in \mathcal{C}^\alpha(Z_n) \cap \alpha \to \beta \in Z_{n+1}) \\
\land & \forall \mathbb{S} \in (St \cup \{\Omega\})\, \big(\ell\mathbb{S} \leq n \land \mathbb{S} \in Z_n \to \mathbb{S}^\dagger \in Z_{n+1}\big)\}
\end{array}
$$

Let $Z = \bigcup_n Z_n$. We see by induction on n that $D_1[Z_n]$ for every n. Lemma 7.40.3 yields $D_1[Z]$. Let $\beta \in \mathcal{C}^\alpha(Z) \cap \alpha$. Pick an n such that $\beta \in \mathcal{C}^\alpha(Z_n)$ and $\ell\beta \leq n$. We obtain $\beta \in Z_{n+1} \subset Z$. Therefore $\alpha \in \mathcal{G}^Z$. Furthermore let $\mathbb{S} \in Z \cap (St \cup \{\Omega\})$. Pick an n such that $\mathbb{S} \in Z_n$ and $\ell\mathbb{S} \leq n$. We obtain $\mathbb{S}^\dagger \in Z_{n+1} \subset Z$. □

Remark 7.45 Lemma 7.44 is a Σ^1_4-statement, which is proved in Σ^1_3-DC. Alternatively we could prove the lemma in Σ^1_3-AC if we assign fundamental sequences to limit ordinals as in [Jäger83].

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