Well-foundedness proof for Π^1_1-reflection

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Abstract

In [4] it is shown that an ordinal $\psi_\Omega(\varepsilon_{\mathbb{S}^+ +1})$ is an upper bound for the proof-theoretic ordinal of a set theory $\mathsf{KP}\omega + (M \prec_{\Sigma_1} V)$. In this note we show that $\mathsf{KP}\omega + (M \prec_{\Sigma_1} V)$ proves the well-foundedness up to $\psi_\Omega(\omega_n(\mathbb{S}^+ + 1))$ for each n.

1 Introduction

In [4] the following theorem is shown, where $\mathsf{KP}\omega + (M \prec_{\Sigma_1} V)$ extends $\mathsf{KP}\omega$ with an axiom stating that 'there exists an non-empty and transitive set M such that $M \prec_{\Sigma_1} V$'. $\Omega = \omega_1^{CK}$ and ψ_Ω is a collapsing function such that $\psi_\Omega(\alpha) < \Omega$. $\mathbb{S}$ is an ordinal term denoting a stable ordinal, and $\mathbb{S}^+$ the least admissible ordinal above $\mathbb{S}$ in the theorems.

Theorem 1.1 *Suppose* $\mathsf{KP}\omega + (M \prec_{\Sigma_1} V) \vdash \theta^{L_\Omega}$ *for a* Σ_1*-sentence* θ*. Then we can find an* $n < \omega$ *such that for* $\alpha = \psi_\Omega(\omega_n(\mathbb{S}^+ + 1))$, $L_\alpha \models \theta$.

OT denotes a computable notation system of ordinals in [2] for an ordinal analysis of $\mathsf{KP}\ell^r + (M \prec_{\Sigma_1} V)$, or equivalently of $\Sigma^{1-}_2\text{-CA} + \Pi^1_1\text{-CA}_0$. OT_N is a restriction of OT such that $OT = \bigcup_{0<N<\omega} OT_N$ and $\psi_\Omega(\varepsilon_{\Omega_{\mathbb{S}+N}+1})$ denotes the order type of $OT_N \cap \Omega$. Let $OT(\Pi^1_1) = OT_1$. The aim of this paper is to show the following theorem, thereby the bound in Theorem 1.1 is seen to be tight.

Theorem 1.2 $\mathsf{KP}\omega + (M \prec_{\Sigma_1} V)$ *proves the well-foundedness up to* $\psi_\Omega(\omega_n(\mathbb{S}^+ + 1))$ *for* each n.

The ordinal $\psi_\Omega(\varepsilon_{\mathbb{S}^+ +1})$ is the proof-theoretic ordinal of $\mathsf{KP}\omega + (M \prec_{\Sigma_1} V)$.

Theorem 1.3 $\psi_\Omega(\varepsilon_{\mathbb{S}^+ +1}) = |\mathsf{KP}\omega + (M \prec_{\Sigma_1} V)|_{\Sigma^\Omega_1}$.

To prove the well-foundedness of a computable notation system, we utilize the distinguished class introduced by W. Buchholz [5].

A set theory $\mathsf{KP}\omega + (M \prec_{\Sigma_1} V)$ extends $\mathsf{KP}\omega$ by adding an individual constant M and the axioms for the constant M: M is non-empty $M \neq \emptyset$, transitive $\forall x \in M \forall y \in x(y \in M)$, and stable $M \prec_{\Sigma_1} V$ for the universe V. $M \prec_{\Sigma_1} V$ means that $\varphi(u_1, \ldots, u_n) \land \{u_1, \ldots, u_n\} \subset M \to \varphi^M(u_1, \ldots, u_n)$ for each Σ_1-formula φ in the set-theoretic language.

Since the axiom β does not hold in the theory $\mathsf{KP}\omega + (M \prec_{\Sigma_1} V)$, we need to modify the proof in [2], cf. subsection 3.1. Proofs of propositions and lemmas are omitted when they are found in [1, 2].

2 Ordinals for one stable ordinal

In this section let us recall briefly ordinal notations systems in [2].

For ordinals $\alpha \geq \beta$, $\alpha - \beta$ denotes the ordinal γ such that $\alpha = \beta + \gamma$. Let α and β be ordinals. $\alpha \dot{+} \beta$ denotes the sum $\alpha + \beta$ when $\alpha + \beta$ equals to the commutative (natural) sum $\alpha \# \beta$, i.e., when either $\alpha = 0$ or $\alpha = \alpha_0 + \omega^{\alpha_1}$ with $\omega^{\alpha_1+1} > \beta$.

$\mathbb{S}$ denotes a weakly inaccessible cardinal, and $\Lambda = \mathbb{S}^+$ the next regular cardinal above $\mathbb{S}$.

Definition 2.1 Let $\Lambda = \mathbb{S}^+$. $\varphi_b(\xi)$ denotes the binary Veblen function on Λ^+ with $\varphi_0(\xi) = \omega^\xi$, and $\tilde{\varphi}_b(\xi) := \varphi_b(\Lambda \cdot \xi)$ for the epsilon number Λ.

Let $b, \xi < \Lambda^+$. $\theta_b(\xi)$ $[\tilde{\theta}_b(\xi)]$ denotes a b-th iterate of $\varphi_0(\xi) = \omega^\xi$ [of $\tilde{\varphi}_0(\xi) = \Lambda^\xi$], resp.

Definition 2.2 Let $\xi < \varphi_\Lambda(0)$ be a non-zero ordinal with its normal form:

$$\xi = \sum_{i \leq m} \tilde{\theta}_{b_i}(\xi_i) \cdot a_i =_{NF} \tilde{\theta}_{b_m}(\xi_m) \cdot a_m + \cdots + \tilde{\theta}_{b_0}(\xi_0) \cdot a_0 \tag{1}$$

where $\tilde{\theta}_{b_i}(\xi_i) > \xi_i$, $\tilde{\theta}_{b_m}(\xi_m) > \cdots > \tilde{\theta}_{b_0}(\xi_0)$, $b_i = \omega^{c_i} < \Lambda$, and $0 < a_0, \ldots, a_m < \Lambda$. $SC_\Lambda(\xi) = \bigcup_{i \leq m}(\{a_i\} \cup SC_\Lambda(\xi_i))$.

$\tilde{\theta}_{b_0}(\xi_0)$ is said to be the *tail* of ξ, denoted $\tilde{\theta}_{b_0}(\xi_0) = tl(\xi)$, and $\tilde{\theta}_{b_m}(\xi_m)$ the *head* of ξ, denoted $\tilde{\theta}_{b_m}(\xi_m) = hd(\xi)$.

1. ζ is a *segment* of ξ iff there exists an $n\,(0 \leq n \leq m+1)$ such that $\zeta =_{NF} \sum_{i \geq n} \tilde{\theta}_{b_i}(\xi_i) \cdot a_i = \tilde{\theta}_{b_m}(\xi_m) \cdot a_m + \cdots + \tilde{\theta}_{b_n}(\xi_n) \cdot a_n$ for ξ in (1).

2. Let $\zeta =_{NF} \tilde{\theta}_b(\xi)$ with $\tilde{\theta}_b(\xi) > \xi$ and $b = \omega^{b_0}$, and c be ordinals. An ordinal $\tilde{\theta}_{-c}(\zeta)$ is defined recursively as follows. If $b \geq c$, then $\tilde{\theta}_{-c}(\zeta) = \tilde{\theta}_{b-c}(\xi)$. Let $c > b$. If $\xi > 0$, then $\tilde{\theta}_{-c}(\zeta) = \tilde{\theta}_{-(c-b)}(\tilde{\theta}_{b_m}(\xi_m))$ for the head term $hd(\xi) = \tilde{\theta}_{b_m}(\xi_m)$ of ξ in (1). If $\xi = 0$, then let $\tilde{\theta}_{-c}(\zeta) = 0$.

Definition 2.3 1. A function $f : \Lambda \to \varphi_\Lambda(0)$ with a *finite* support $\mathrm{supp}(f) = \{c < \Lambda : f(c) \neq 0\} \subset \Lambda$ is said to be a *finite function* if $\forall i > 0(a_i = 1)$ and $a_0 = 1$ when $b_0 > 1$ in $f(c) =_{NF} \tilde{\theta}_{b_m}(\xi_m) \cdot a_m + \cdots + \tilde{\theta}_{b_0}(\xi_0) \cdot a_0$ for any $c \in \mathrm{supp}(f)$.

   It is identified with the finite function $f \restriction \mathrm{supp}(f)$. When $c \notin \mathrm{supp}(f)$, let $f(c) := 0$. $SC_\Lambda(f) := \bigcup\{\{c\} \cup SC_\Lambda(f(c))\} : c \in \mathrm{supp}(f)\}$. $f, g, h, \ldots$ range over finite functions.

For an ordinal c, f_c and f^c are restrictions of f to the domains $\mathrm{supp}(f_c) = \{d \in \mathrm{supp}(f) : d < c\}$ and $\mathrm{supp}(f^c) = \{d \in \mathrm{supp}(f) : d \geq c\}$. $g_c * f^c$ denotes the concatenated function such that $\mathrm{supp}(g_c * f^c) = \mathrm{supp}(g_c) \cup \mathrm{supp}(f^c)$, $(g_c * f^c)(a) = g(a)$ for $a < c$, and $(g_c * f^c)(a) = f(a)$ for $a \geq c$.

2. Let f be a finite function and c, ξ ordinals. A relation $f <^c \xi$ is defined by induction on the cardinality of the finite set $\{d \in \mathrm{supp}(f) : d > c\}$ as follows. If $f^c = \emptyset$, then $f <^c \xi$ holds. For $f^c \neq \emptyset$, $f <^c \xi$ iff there exists a segment μ of ξ such that $f(c) < \mu$ and $f <^{c+d} \tilde{\theta}_{-d}(tl(\mu))$ for $d = \min\{c + d \in \mathrm{supp}(f) : d > 0\}$.

Proposition 2.4 $f <^c \xi \leq \zeta \Rightarrow f <^c \zeta$.

In the following Definition 2.5, $\varphi\alpha\beta = \varphi_\alpha(\beta)$ denotes the binary Veblen function on $\Lambda^+ = \mathbb{S}^{++}$, $\tilde{\theta}_b(\xi)$ the function defined in Definition 2.1 for $\Lambda = \mathbb{S}^+$. For $\alpha < \mathbb{S}$, α^+ denotes the next regular cardinal above α.

For $a < \varepsilon_{\Lambda+1}$, $c < \Lambda$, and $\xi < \Gamma_{\Lambda+1}$, define simultaneously classes $\mathcal{H}_a(X) \subset \Gamma_{\Lambda+1}$, $Mh^a_c(\xi) \subset (\mathbb{S} + 1)$, and ordinals $\psi^f_\kappa(a) \leq \kappa$ by recursion on ordinals a as follows.

Definition 2.5 Let $\Lambda = \mathbb{S}^+$. Let $a < \varepsilon_{\Lambda+1}$ and $X \subset \Gamma_{\Lambda+1}$.

1. (Inductive definition of $\mathcal{H}_a(X)$.)

   (a) $\{0, \Omega_1, \mathbb{S}, \mathbb{S}^+\} \cup X \subset \mathcal{H}_a(X)$.

   (b) If $x, y \in \mathcal{H}_a(X)$, then $x + y \in \mathcal{H}_a(X)$, and $\varphi xy \in \mathcal{H}_a(X)$.

   (c) Let $\alpha \in \mathcal{H}_a(X) \cap \mathbb{S}$. Then $\alpha^+ \in \mathcal{H}_a(X)$.

   (d) Let $\alpha = \psi^f_\pi(b)$ with $\{\pi, b\} \subset \mathcal{H}_a(X)$, $b < a$, and a finite function f such that $SC_\Lambda(f) \subset \mathcal{H}_a(X) \cap \mathcal{H}_b(\alpha)$. Then $\alpha \in \mathcal{H}_a(X)$.

2. (Definitions of $Mh^a_c(\xi)$ and $Mh^a_c(f)$)
   The classes $Mh^a_c(\xi)$ are defined for $c < \Lambda$, and ordinals $a < \varepsilon_{\Lambda+1}$, $\xi < \Gamma_{\Lambda+1}$. Let π be a regular ordinal$\leq \mathbb{S}$. Then by main induction on ordinals $\pi \leq \mathbb{S}$ with subsidiary induction on $c < \Lambda$ we define $\pi \in Mh^a_c(\xi)$ iff $\{a, c, \xi\} \subset \mathcal{H}_a(\pi)$ and

   $$\forall f <^c \xi \forall g \, (SC_\Lambda(f, g) \subset \mathcal{H}_a(\pi) \,\&\, \pi \in Mh^a_0(g_c) \Rightarrow \pi \in M(Mh^a_0(g_c * f^c))) \tag{2}$$

   where f, g vary through finite functions, and

   $$\begin{aligned} Mh^a_c(f) &:= \bigcap\{Mh^a_d(f(d)) : d \in \mathrm{supp}(f^c)\} \\ &= \bigcap\{Mh^a_d(f(d)) : c \leq d \in \mathrm{supp}(f)\}. \end{aligned}$$

   In particular $Mh^a_0(g_c) = \bigcap\{Mh^a_d(g(d)) : d \in \mathrm{supp}(g_c)\} = \bigcap\{Mh^a_d(g(d)) : c > d \in \mathrm{supp}(g)\}$. When $f = \emptyset$ or $f^c = \emptyset$, let $Mh^a_c(\emptyset) := \Lambda$.

3. (Definition of $\psi^f_\pi(a)$)
   Let $a < \varepsilon_{\Lambda+1}$ be an ordinal, π a regular ordinal and f a finite function. Then let

$$\psi^f_\pi(a) := \min(\{\pi\}\cup\{\kappa \in Mh^a_0(f)\cap\pi : \mathcal{H}_a(\kappa)\cap\pi \subset \kappa, \{\pi, a\}\cup SC_\Lambda(f) \subset \mathcal{H}_a(\kappa)\}) \tag{3}$$

   For the empty function $\emptyset$, $\psi_\pi(a) := \psi^\emptyset_\pi(a)$.

4. For classes $A \subset (\mathbb{S}+1)$, let $\alpha \in M^a_c(A)$ iff $\alpha \in A$ and

$$\forall g[\alpha \in Mh^a_0(g_c)\,\&\, SC_\Lambda(g_c) \subset \mathcal{H}_a(\alpha) \Rightarrow \alpha \in M\left(Mh^a_0(g_c) \cap A\right)] \tag{4}$$

Assuming an existence of a shrewd cardinal introduced by M. Rathjen [6], we show in [3] that $\psi^f_{\mathbb{S}}(a) < \mathbb{S}$ if $\{a, c, \xi\} \subset \mathcal{H}_a(\mathbb{S})$ with $c < \mathbb{S}^+$, $a, \xi < \varepsilon_{\mathbb{S}^++1}$, and $\mathrm{supp}(f) = \{c\}$ and $f(c) = \xi$. Moreover $\psi^g_\pi(b) < \pi$ provided that $\pi \in Mh^b_0(f)$, $SC_\Lambda(g)\cup\{\pi, b\} \subset \mathcal{H}_b(\pi)$, and g is a finite function defined from a finite function f and ordinals d, c as follows. $d < c \in \mathrm{supp}(f)$ with $(d, c)\cap\mathrm{supp}(f) = (d, c)\cap\mathrm{supp}(g) = \emptyset$, $g_d = f_d$, $g(d) < f(d) + \tilde{\theta}_{c-d}(f(c))\cdot\omega$, and $g <^c f(c)$. Also the following Lemma 2.6 is shown in [3].

Lemma 2.6 *Assume* $\mathbb{S} \geq \pi \in Mh^a_d(\xi)\cap Mh^a_c(\xi_0)$, $\xi_0 \neq 0$, $d < c$, *and* $\{a, c, d\} \subset \mathcal{H}_a(\pi)$. *Moreover let* $\tilde{\theta}_{c-d}(\xi_0) \geq \xi_1 \in \mathcal{H}_a(\pi)$ *and* $tl(\xi) > \xi_1$ *when* $\xi \neq 0$. *Then* $\pi \in Mh^a_d(\xi+\xi_1) \cap M^a_d(Mh^a_d(\xi+\xi_1))$.

2.1 Normal forms in ordinal notations

Definition 2.7 An *irreducibility* of finite functions f is defined by induction on the cardinality n of the finite set $\mathrm{supp}(f)$. If $n \leq 1$, f is defined to be irreducible. Let $n \geq 2$ and $c < c+d$ be the largest two elements in $\mathrm{supp}(f)$, and let g be a finite function such that $\mathrm{supp}(g) = \mathrm{supp}(f_c)\cup\{c\}$, $g_c = f_c$ and $g(c) = f(c) + \tilde{\theta}_d(f(c+d))$. Then f is irreducible iff $tl(f(c)) > \tilde{\theta}_d(f(c+d))$ and g is irreducible.

Definition 2.8 Let f, g be irreducible functions, and b, a ordinals.

1. Let us define a relation $f <^b_{lx} g$ by induction on the cardinality of the finite set $\{e \in \mathrm{supp}(f)\cup\mathrm{supp}(g) : e \geq b\}$ as follows. $f <^b_{lx} g$ holds iff $f^b \neq g^b$ and for the ordinal $c = \min\{c \geq b : f(c) \neq g(c)\}$, one of the following conditions is met:

   (a) $f(c) < g(c)$ and let μ be the shortest segment of $g(c)$ such that $f(c) < \mu$. Then for any $c < c+d \in \mathrm{supp}(f)$, if $tl(\mu) \leq \tilde{\theta}_d(f(c+d))$, then $f <^{c+d}_{lx} g$ holds.

   (b) $f(c) > g(c)$ and let ν be the shortest segment of $f(c)$ such that $\nu > g(c)$. Then there exist a $c < c+d \in \mathrm{supp}(g)$ such that $f <^{c+d}_{lx} g$ and $tl(\nu) \leq \tilde{\theta}_d(g(c+d))$.

2. $Mh^a_b(f) \prec Mh^a_b(g)$ holds iff

$$\forall \pi \in Mh^a_b(g) \forall b_0 \leq b \left(SC_\Lambda(f) \subset \mathcal{H}_a(\pi) \,\&\, \pi \in Mh^a_{b_0}(f_b) \Rightarrow \pi \in M(Mh^a_{b_0}(f)) \right).$$

Lemma 2.9 *Let f, g be irreducible finite functions, and b an ordinal such that $f^b \neq g^b$. If $f <^b_{lx} g$, then $Mh^a_b(f) \prec Mh^a_b(g)$ holds for every ordinal a.*

Proposition 2.10 *Let f, g be irreducible finite functions, and assume that $\psi^f_\pi(b) < \pi$ and $\psi^g_\kappa(a) < \kappa$.*

Then $\psi^f_\pi(b) < \psi^g_\kappa(a)$ iff one of the following cases holds:

1. *$\pi \leq \psi^g_\kappa(a)$.*

2. *$b < a$, $\psi^f_\pi(b) < \kappa$ and $SC_\Lambda(f) \cup \{\pi, b\} \subset \mathcal{H}_a(\psi^g_\kappa(a))$.*

3. *$b > a$ and $SC_\Lambda(g) \cup \{\kappa, a\} \not\subset \mathcal{H}_b(\psi^f_\pi(b))$.*

4. *$b = a$, $\kappa < \pi$ and $\kappa \not\in \mathcal{H}_b(\psi^f_\pi(b))$.*

5. *$b = a$, $\pi = \kappa$, $SC_\Lambda(f) \subset \mathcal{H}_a(\psi^g_\kappa(a))$, and $f <^0_{lx} g$.*

6. *$b = a$, $\pi = \kappa$, $SC_\Lambda(g) \not\subset \mathcal{H}_b(\psi^f_\pi(b))$.*

Definition 2.11 1. $a(\xi)$ denotes an ordinal defined recursively by $a(0) = 0$, and $a(\xi) = \sum_{i \leq m} \tilde{\theta}_{b_i}(\omega \cdot a(\xi_i))$ when $\xi =_{NF} \sum_{i \leq m} \tilde{\theta}_{b_i}(\xi_i) \cdot a_i$ in (1).

2. For irreducible functions f let us associate ordinals $o(f) < \Gamma_{\mathbb{S}^+ + 1}$ as follows. $o(\emptyset) = 0$ for the empty function $f = \emptyset$. Let $\{0\} \cup \mathrm{supp}(f) = \{0 = c_0 < c_1 < \cdots < c_n\}$, $f(c_i) = \xi_i < \Gamma_{\mathbb{S}^+ + 1}$ for $i > 0$, and $\xi_0 = 0$. Define ordinals $\zeta_i = o(f; c_i)$ by $\zeta_n = \omega \cdot a(\xi_n)$, and $\zeta_i = \omega \cdot a(\xi_i) + \tilde{\theta}_{c_{i+1} - c_i}(\zeta_{i+1} + 1)$. Finally let $o(f) = \zeta_0 = o(f; c_0)$.

3. Let $SC_\Lambda(f) < \mu < \Lambda$ be an epsilon number. Then $o_\mu(f)$ is defined from $o(f)$ by replacing the base Λ of $\tilde{\theta}$ in $f(c)$ by μ. This means that Λ is replaced by μ, and $\tilde{\theta}_1(\xi) = \Lambda^\xi$ by μ^ξ.

Lemma 2.12 *Let f be an irreducible finite function defined from an irreducible function g and ordinals c, d as follows. $f_c = g_c$, $c < d \in \mathrm{supp}(g)$ with $(c, d) \cap \mathrm{supp}(g) = (c, d) \cap \mathrm{supp}(f) = \emptyset$, $f(c) < g(c) + \tilde{\theta}_{d-c}(g(d)) \cdot \omega$, and $f <^d g(d)$. Then $o(f) < o(g)$ holds.*

Moreover when $SC_\Lambda(f, g) < \mu < \Lambda$, $o_\mu(f) < o_\mu(g)$ holds.

Lemma 2.13 *For irreducible finite functions f and g, assume $f <^0_{lx} g$. Then $o(f) < o(g)$ holds.*

Moreover when $SC_\Lambda(f, g) < \mu < \Lambda$, $o_\mu(f) < o_\mu(g)$ holds.

By Proposition 2.10 a notation system $OT(\Pi^1_1) = OT_1$ is defined.

Definition 2.14 $OT(\Pi^1_1)$ is closed under $\mathbb{S} > \alpha \mapsto \alpha^+$. There are two cases when an ordinal term $\psi^f_\pi(a)$ is constructed in $OT(\Pi^1_1)$, from $\{\pi, a\} \subset OT(\Pi^1_1)$ and an irreducible function f with $SC_\Lambda(f) \subset OT(\Pi^1_1)$ and $\Lambda = \mathbb{S}^+$. $E_\mathbb{S}(\alpha)$ denotes the set of subterms$< \mathbb{S}$ of α.

1. Let $\xi, a, c \in OT(\Pi^1_1)$, $\xi > 0$, $c < \mathbb{S}^+$ and $\{\xi, a, c\} \subset \mathcal{H}_a(\alpha)$. Then $\alpha = \psi^f_\mathbb{S}(a) \in OT(\Pi^1_1)$ and $\alpha^+ \in OT(\Pi^1_1)$ with $\text{supp}(f) = \{c\}$ and $f(c) = \xi$ if $\max(SC_{\mathbb{S}^+}(f)) \leq \max(SC_{\mathbb{S}^+}(a))$. Let $f = m(\alpha)$.

2. Let $\{a, d, \pi\} \subset OT(\Pi^1_1)$, $f = m(\pi)$, $d < c \in \text{supp}(f)$, and $(d, c) \cap \text{supp}(f) = \emptyset$. Let g be an irreducible function such that $SC_\Lambda(g) = \bigcup\{\{c, g(c)\} : c \in \text{supp}(g)\} \subset OT(\Pi^1_1)$, $g_d = f_d$, $(d, c) \cap \text{supp}(g) = \emptyset$ $g(d) < f(d) + \tilde{\theta}_{c-d}(f(c)) \cdot \omega$, and $g <^c f(c)$. Moreover if $\max(SC_\Lambda(f)) < \mu < \Lambda$ for an epsilon number μ, then $\max(SC_\Lambda(g)) < \mu$.

   Then $\alpha = \psi^g_\pi(a) \in OT(\Pi^1_1)$ and $\alpha^+ \in OT(\Pi^1_1)$ if $\{\pi, a\} \cup SC_\Lambda(f, g) \subset \mathcal{H}_a(\alpha)$, and, cf. Proposition 3.23.

$$SC_\Lambda(g) \subset M_\alpha \tag{5}$$

M_α is defined as follows.

Definition 2.15 For ordinal terms $\psi^f_\sigma(a) \in \Psi_\mathbb{S} \subset OT(\Pi^1_1)$, define $m(\psi^f_\sigma(a)) := f$ and $\mathtt{p}_0(\psi^f_\sigma(a)) = \mathtt{p}_0(\sigma)$ if $\sigma < \mathbb{S}$, and $\mathtt{p}_0(\psi^f_\mathbb{S}(a)) = a$.

Definition 2.16 *$M_\rho := \mathcal{H}_b(\rho)$ for $b = \mathtt{p}_0(\rho)$ and $\rho \in \Psi_\mathbb{S}$.*

Definition 2.17 For $\gamma \prec \mathbb{S}$, an epsilon number $\mathbb{S} < \mu = \Lambda(\gamma) < \mathbb{S}^+$ is defined. Let $\gamma = \psi^f_\sigma(\alpha) \preceq \psi^g_\mathbb{S}(b)$ with $b = \mathtt{p}_0(\gamma)$. Then $\Lambda(\gamma)$ denotes the least epsilon number $\mathbb{S} < \mu < \mathbb{S}^+$ such that $\max(SC_{\mathbb{S}^+}(b)) < \mu$.

From Definition 2.14 we see $\max(SC_{\mathbb{S}^+}(f)) < \Lambda(\gamma)$.

$OT(\Pi^1_1)$ is closed under $\alpha \mapsto \alpha[\rho/\mathbb{S}]$ for $\alpha \in M_\rho$. Specifically if $\{\alpha, \rho\} \subset OT(\Pi^1_1)$ with $\alpha \in M_\rho$ and $\rho \in \Psi_\mathbb{S}$, then $\alpha[\rho/\mathbb{S}] \in OT(\Pi^1_1)$.

Definition 2.18 Let $\alpha \in M_\rho$ with $\rho \in \Psi_\mathbb{S}$. We define an ordinal $\alpha[\rho/\mathbb{S}]$ recursively as follows. $\alpha[\rho/\mathbb{S}] := \alpha$ when $\alpha < \mathbb{S}$. In what follows assume $\alpha \geq \mathbb{S}$.

$\mathbb{S}[\rho/\mathbb{S}] := \rho$. $\mathbb{K}[\rho/\mathbb{S}] \equiv (\mathbb{S}^+)[\rho/\mathbb{S}] := \rho^+$. $(\psi_\mathbb{K}(a))\,[\rho/\mathbb{S}] = (\psi_{\mathbb{S}^+}(a))\,[\rho/\mathbb{S}] = \psi_{\rho^+}(a[\rho/\mathbb{S}])$. The map commutes with $+$ and φ.

Lemma 2.19 *For $\rho \in \Psi_\mathbb{S}$, $\{\alpha[\rho/\mathbb{S}] : \alpha \in M_\rho\}$ is a transitive collapse of M_ρ in the sense that $\beta < \alpha \Leftrightarrow \beta[\rho/\mathbb{S}] < \alpha[\rho/\mathbb{S}]$, $\beta \in \mathcal{H}_\alpha(\gamma) \Leftrightarrow \beta[\rho/\mathbb{S}] \in \mathcal{H}_{\alpha[\rho/\mathbb{S}]}(\gamma[\rho/\mathbb{S}])$ for $\gamma > \mathbb{S}$, and $OT(\Pi^1_1) \cap \alpha[\rho/\mathbb{S}] = \{\beta[\rho/\mathbb{S}] : \beta \in M_\rho \cap \alpha\}$ for $\alpha, \beta, \gamma \in M_\rho$.*

Proposition 2.20 *Let $\rho \in \Psi_\mathbb{S}$.*

1. *$\mathcal{H}_\gamma(M_\rho) \subset M_\rho$ if $\gamma \leq \mathtt{p}_0(\rho)$.*

2. *$M_\rho \cap \mathbb{S} = \rho$ and $\rho \notin M_\rho$.*

3. *If $\sigma < \rho$ and $\mathtt{p}_0(\sigma) \leq \mathtt{p}_0(\rho)$, then $M_\sigma \subset M_\rho$.*

3 Well-foundedness proof with the maximal distinguished set

In this section working in the set theory $\mathsf{KP}\omega + (M \prec_{\Sigma_1} V)$, we show the well-foundedness of the notation system $OT(\Pi^1_1)$ up to each $\psi_\Omega(\omega_n(\mathbb{S}^+ + 1))$. Let us write $L_{\mathbb{S}}$ for M, i.e., $L_{\mathbb{S}} \prec_{\Sigma_1} L$. The proof is based on distinguished classes, which was first introduced by Buchholz [5].

3.1 Distinguished sets

$X, Y, \ldots$ range over subsets of $OT(\Pi^1_1)$. We define sets $\mathcal{C}^\alpha(X) \subset OT(\Pi^1_1)$ for $\alpha \in OT(\Pi^1_1)$ and $X \subset OT(\Pi^1_1)$ as follows.

Definition 3.1 Let $\alpha, \beta \in OT(\Pi^1_1)$ and $X \subset OT(\Pi^1_1)$.

$\mathcal{C}^\alpha(X)$ denotes the closure of $\{0, \Omega, \mathbb{S}, \mathbb{S}^+\} \cup (X \cap \alpha)$ under $+$, $\sigma \mapsto \sigma^+$, $(\beta, \gamma) \mapsto \varphi\beta\gamma$, and $(\sigma, \beta, f) \mapsto \psi^f_\sigma(\beta)$ for $\sigma > \alpha$ in $OT(\Pi^1_1)$.

The last clause says that, $\psi^f_\sigma(\beta) \in \mathcal{C}^\alpha(X)$ if $\{\sigma, \beta\} \cup SC_\Lambda(f) \subset \mathcal{C}^\alpha(X)$ and $\sigma > \alpha$.

Proposition 3.2 *Assume* $\forall\gamma \in X[\gamma \in \mathcal{C}^\gamma(X)]$ *for a set* $X \subset OT(\Pi^1_1)$.

1. $\alpha \leq \beta \Rightarrow \mathcal{C}^\beta(X) \subset \mathcal{C}^\alpha(X)$.
2. $\alpha < \beta < \alpha^+ \Rightarrow \mathcal{C}^\beta(X) = \mathcal{C}^\alpha(X)$.

Definition 3.3

1. $Prg[X, Y] :\Leftrightarrow \forall\alpha \in X(X \cap \alpha \subset Y \to \alpha \in Y)$.
2. For a definable class $\mathcal{X}$, $TI[\mathcal{X}]$ denotes the schema: $TI[\mathcal{X}] :\Leftrightarrow Prg[\mathcal{X}, \mathcal{Y}] \to \mathcal{X} \subset \mathcal{Y}$ holds for any definable classes $\mathcal{Y}$.
3. For $X \subset OT(\Pi^1_1)$, $W(X)$ denotes the *well-founded part* of X.
4. $Wo[X] :\Leftrightarrow X \subset W(X)$.
5. $\alpha \in W_\Sigma(X)$ denotes a Σ_1-formula saying that $\alpha \in X$ and 'there exists an embedding $f : X \cap (\alpha+1) \to ON$', i.e., $\exists f \in {}^\omega ON \forall\beta, \gamma \in X \cap (\alpha+1)(\beta < \gamma \to f(\beta) < f(\gamma))$, where ON is the class of all ordinals, $\beta < \gamma$ in $OT(\Pi^1_1)$ and $f(\beta) < f(\gamma)$ in ON.
6. $Wo_\Sigma[X]$ denotes a Σ_1-formula saying that 'there exists an embedding $f : X \to ON$', i.e., $\exists f \in {}^\omega ON \forall\beta, \gamma \in X(\beta < \gamma \to f(\beta) < f(\gamma))$.

Note that for $\alpha \in OT(\Pi^1_1)$, $W(X) \cap \alpha = W(X \cap \alpha)$. Also $\mathsf{KP}\omega \vdash \alpha \in W_\Sigma(X) \Rightarrow \alpha \in W(X)$, and $\mathsf{KP}\ell \vdash \alpha \in W(X) \Rightarrow \alpha \in W_\Sigma(X)$.

Definition 3.4 For $X \subset OT(\Pi^1_1)$ and $\alpha \in OT(\Pi^1_1)$,

1. $D[X] :\Leftrightarrow \forall\alpha(\alpha \leq X \to W(\mathcal{C}^\alpha(X)) \cap \alpha^+ = X \cap \alpha^+)$.

   A set X is said to be a *distinguished set* if $D[X]$.

2. $D_\Sigma[X]$ is a Σ-formula defined by

$$D_\Sigma[X] :\Leftrightarrow \forall\alpha(\alpha \leq X \to W(\mathcal{C}^\alpha(X)) \cap \alpha^+ \subset X \cap \alpha^+ \subset W_\Sigma(\mathcal{C}^\alpha(X)) \cap \alpha^+) \tag{6}$$

3. $\mathcal{W} := \bigcup\{X : D_\Sigma[X]\}$.

From $\mathsf{KP}\omega \vdash \alpha \in W_\Sigma(X) \Rightarrow \alpha \in W(X)$ we see $D_\Sigma[X] \Rightarrow D[X]$ for any X.

Let $\alpha \in X$ for a Σ-distinguished set X. Then $W(\mathcal{C}^\alpha(X)) \cap \alpha^+ = X \cap \alpha^+$. Hence X is a well order. Although $\bigcup\{X : D[X]\}$ might be a proper class, it turns out that $\mathcal{W}$ is a set.

Proposition 3.5 *Let $X \in L_\mathbb{S}$.*

1. *$\alpha \in W(X) \Leftrightarrow L_\mathbb{S} \models \alpha \in W(X)$.*
2. *$\alpha \in W_\Sigma(X) \Leftrightarrow L_\mathbb{S} \models \alpha \in W_\Sigma(X)$.*
3. *$D_\Sigma[X] \Leftrightarrow L_\mathbb{S} \models D_\Sigma[X]$.*
4. *$D_\Sigma[X] \leftrightarrow D[X]$.*
5. *$\mathcal{W} = \bigcup\{X \in L_\mathbb{S} : D[X]\}$ and $\exists X(X = \mathcal{W})$.*

Proof. 3.5.1. Since $\alpha \in W(X)$ is a Π_1-formula, it suffices to show $\alpha \in W(X)$ assuming $L_\mathbb{S} \models \alpha \in W(X)$. We obtain $L_\mathbb{S} \models (\alpha \in W(X) \leftrightarrow \alpha \in W_\Sigma(X))$ by $L_\mathbb{S} \models \mathsf{KP}\ell$. Hence $\alpha \in W_\Sigma(X)$ and $\alpha \in W(X)$.
3.5.2. Assume $\alpha \in W_\Sigma(X)$. Since $\alpha \in W_\Sigma(X)$ is a Σ_1-formula, we obtain $L_\mathbb{S} \models \alpha \in W_\Sigma(X)$ by $L_\mathbb{S} \prec_{\Sigma_1} L$. The other direction follows from the persistency of Σ_1-formulas.
3.5.3 follows from Propositions 3.5.1 and 3.5.2.
3.5.4. From $W_\Sigma(\mathcal{C}^\alpha(X)) \subset W(\mathcal{C}^\alpha(X))$ we see $D_\Sigma[X] \to D[X]$. Assume $D[X]$, $\alpha \leq X$ and $\beta \in X \cap \alpha^+$. Then $\beta \in W(\mathcal{C}^\alpha(X)) \cap \alpha^+$ by $D[X]$. We obtain $\beta \in W_\Sigma(\mathcal{C}^\alpha(X)) \cap \alpha^+$ by $L_\mathbb{S} \models \beta \in W(\mathcal{C}^\alpha(X)) \to \beta \in W_\Sigma(\mathcal{C}^\alpha(X))$ and Propositions 3.5.1 and 3.5.2.
3.5.5. By Proposition 3.5.4 we obtain $\bigcup\{X \in L_\mathbb{S} : D[X]\} \subset \mathcal{W}$. Let $\alpha \in \mathcal{W}$. This means a Σ_1-formula $\exists X(\alpha \in X \wedge D_\Sigma[X])$ holds. We obtain $L_\mathbb{S} \models \exists X(\alpha \in X \wedge D_\Sigma[X])$ by $L_\mathbb{S} \prec_{\Sigma_1} L$. By Propositions 3.5.3 and 3.5.4 we obtain $\alpha \in \bigcup\{X \in L_\mathbb{S} : L_\mathbb{S} \models D_\Sigma[X]\} = \bigcup\{X \in L_\mathbb{S} : D_\Sigma[X]\} = \bigcup\{X \in L_\mathbb{S} : D[X]\}$. Δ_0-separation yields $\exists X(X = \mathcal{W})$. □

Proposition 3.6 *Let $X \in L_\mathbb{S}$ be a distinguished set. Then $\alpha \in X \Rightarrow \forall\beta[\alpha \in \mathcal{C}^\beta(X)]$.*

Proposition 3.7 *For any distinguished sets X and Y in $L_\mathbb{S}$, $X \cap \alpha = Y \cap \alpha \Rightarrow \forall\beta < \alpha^+\{\mathcal{C}^\beta(X) \cap \beta^+ = \mathcal{C}^\beta(Y) \cap \beta^+\}$ holds*

Proposition 3.8 *For distinguished sets X and Y in $L_\mathbb{S}$, $\alpha \leq X \,\&\, \alpha \leq Y \Rightarrow X \cap \alpha^+ = Y \cap \alpha^+$.*

Proposition 3.9 *$\mathcal{W}$ is the maximal distinguished* set*, i.e., $D[\mathcal{W}]$ and $\exists X(X = \mathcal{W})$.*

Proof. First we show $\forall\gamma \in \mathcal{W}(\gamma \in \mathcal{C}^\gamma(\mathcal{W}))$. Let $\gamma \in \mathcal{W}$, and pick a distinguished set $X \in L_{\mathbb{S}}$ such that $\gamma \in X$ by Proposition 3.5.5. Then $\gamma \in \mathcal{C}^\gamma(X) \subset \mathcal{C}^\gamma(\mathcal{W})$ by $X \subset \mathcal{W}$.

Let $\alpha \leq \mathcal{W}$. Pick a distinguished set $X \in L_{\mathbb{S}}$ such that $\alpha \leq X$. We claim that $\mathcal{W}\cap\alpha^+ = X\cap\alpha^+$. Let $Y \in L_{\mathbb{S}}$ be a distinguished set and $\beta \in Y\cap\alpha^+$. Then $\beta \in Y \cap \beta^+ = X \cap \beta^+$ by Proposition 3.8. The claim yields $W(\mathcal{C}^\alpha(\mathcal{W})) \cap \alpha^+ = W(\mathcal{C}^\alpha(X)) \cap \alpha^+ = X \cap \alpha^+ = \mathcal{W} \cap \alpha^+$. Hence $D[\mathcal{W}]$. $\square$

From $\mathcal{W} \cap \alpha^+ = X \cap \alpha^+$ for a Σ-distinguished set X, We see $\mathcal{W} \cap \alpha^+ = X \cap \alpha^+ \subset W_\Sigma(\mathcal{C}^\alpha(X)) \cap \alpha^+ = W_\Sigma(\mathcal{C}^\alpha(\mathcal{W})) \cap \alpha^+$. Hence $D_\Sigma[\mathcal{W}]$.

3.2 Sets $\mathcal{C}^\alpha(\mathcal{W}_{\mathbb{S}})$ and $\mathcal{G}$

Definition 3.10 $\mathcal{G}(Y) := \{\alpha \in OT(\Pi^1_1) : \alpha \in \mathcal{C}^\alpha(Y) \,\&\, \mathcal{C}^\alpha(Y) \cap \alpha \subset Y\}$.

Lemma 3.11 *For $D[X]$, $X \subset \mathcal{G}(X)$.*

Lemma 3.12 *Suppose $D[Y]$ and $\alpha \in \mathcal{G}(Y)$ for $Y \in L_{\mathbb{S}}$. Let $X = W(\mathcal{C}^\alpha(Y)) \cap \alpha^+ \in L_{\mathbb{S}}$. Assume that the following condition (7) is fulfilled. Then $\alpha \in X$ and $D[X]$.*

$$\forall\beta < \mathbb{S}\left(Y \cap \alpha^+ < \beta \,\&\, \beta^+ < \alpha^+ \to W(\mathcal{C}^\beta(Y)) \cap \beta^+ \subset Y\right) \tag{7}$$

Proposition 3.13 *Let $D[X]$.*

1. *Let $\{\alpha, \beta\} \subset X$ with $\alpha + \beta = \alpha\#\beta$ and $\alpha > 0$. Then $\gamma = \alpha + \beta \in X$.*
2. *If $\{\alpha, \beta\} \subset X$, then $\varphi_\alpha(\beta) \in X$.*

Proposition 3.14
1. *$0 \in \mathcal{W}$.*
2. *Let either $\sigma = 0$ or $\sigma = \psi^f_{\mathbb{S}}(a)$ or $\sigma = \psi^f_\pi(a)$. Assume $\sigma \in \mathcal{W}$. Then $\sigma^+ \in \mathcal{W}$.*

Proof. Each is seen from Lemma 3.12 as follows.
3.14.1. We see $0 \in Y = W(\mathcal{C}^0(\emptyset)) \cap \Omega \in L_{\mathbb{S}}$ with $\Omega = 0^+$ and $D[Y]$.
3.14.2. Let $\sigma \in Y \in L_{\mathbb{S}}$ with $D[Y]$. We see $\sigma^+ \in X = W(\mathcal{C}^{\sigma^+}(Y)) \cap \sigma^{++} \in L_{\mathbb{S}}$ and $D[X]$. $\square$

Lemma 3.15 *Suppose $D[Y]$ with $\{0, \Omega\} \subset Y \in L_{\mathbb{S}}$, and for $\eta \in OT(\Pi^1_1)\cap(\mathbb{S}+1)$*

$$\eta \in \mathcal{G}(Y) \tag{8}$$

and

$$\forall\gamma \prec \eta(\gamma \in \mathcal{G}(Y) \Rightarrow \gamma \in Y) \tag{9}$$

Let $X = W(\mathcal{C}^\eta(Y)) \cap \eta^+$. Then $\eta \in X \in L_{\mathbb{S}}$ and $D[X]$.

3.3 Mahlo universes

In this subsection we consider the maximal distinguished class $\mathcal{W}^P$ *inside* a set $P \in L_{\mathbb{S}}$ as in [1]. Let ad denote a Π_3^--sentence such that a transitive set z is admissible iff $(z; \in) \models ad$. Let $lmtad :\Leftrightarrow \forall x \exists y (x \in y \land ad^y)$. Observe that $lmtad$ is a Π_2^--sentence.

Definition 3.16 1. By a *universe* we mean either $L_{\mathbb{S}}$ or a transitive set $Q \in L_{\mathbb{S}}$ with $\omega \in Q$. Universes are denoted by $P, Q, \ldots$

2. For a universe P and a set-theoretic sentence φ, $P \models \varphi :\Leftrightarrow (P; \in) \models \varphi$.

3. A universe P is said to be a *limit universe* if $lmtad^P$ holds, i.e., P is a limit of admissible sets. The class of limit universes is denoted by $Lmtad$.

Lemma 3.17 *$W(\mathcal{C}^\alpha(X))$ as well as $D[X]$ are absolute for limit universes P.*

Definition 3.18 For a universe P, let $\mathcal{W}^P := \bigcup\{X \in P : D[X]\}$.

We see $\mathcal{W} = \mathcal{W}^{L_{\mathbb{S}}}$ from Proposition 3.5.5.

Lemma 3.19 *Let P be a universe closed under finite unions, and $\alpha \in OT(\Pi^1_1)$.*

1. *There is a finite set $K(\alpha) \subset OT(\Pi^1_1)$ such that $\forall Y \in P \forall \gamma [K(\alpha) \cap Y = K(\alpha) \cap \mathcal{W}^P \Rightarrow \big(\alpha \in \mathcal{C}^\gamma(\mathcal{W}^P) \Leftrightarrow \alpha \in \mathcal{C}^\gamma(Y)\big)]$.*

2. *There exists a distinguished set $X \in P$ such that $\forall Y \in P \forall \gamma [X \subset Y \,\&\, D[Y] \Rightarrow (\alpha \in \mathcal{C}^\gamma(\mathcal{W}^P) \Leftrightarrow \alpha \in \mathcal{C}^\gamma(Y))]$.*

Proposition 3.20 *For each limit universe P, $D[\mathcal{W}^P]$ holds, and $\exists X \in L_{\mathbb{S}}(X = \mathcal{W}^P)$ if $P \in L_{\mathbb{S}}$.*

For a universal Π_n-formula $\Pi_n(a)$ $(n > 0)$ uniformly on admissibles, let

$$P \in M_2(\mathcal{C}) :\Leftrightarrow P \in Lmtad \,\&\, \forall b \in P[P \models \Pi_2(b) \to \exists Q \in \mathcal{C} \cap P(Q \models \Pi_2(b))].$$

Definition 3.21 Let $\gamma = \psi^f_\sigma(\alpha) \prec \mathbb{S}$ and $\mu = \Lambda(\gamma) < \mathbb{S}^+$ be the ordinal in Definition 2.17. Let $O(\gamma) = o_\mu(f) < \mathbb{S}^+$, where $o_\mu(f)$ is the ordinal defined in Definition 2.11 from the epsilon number μ.

Let $O(\Omega) = 1$, $O(\mathbb{S}) = \mathbb{S}^+$ and $O(\gamma) = 0$ else.

Lemma 3.22 *Let $\mathcal{C}$ be a Π^1_0-class such that $\mathcal{C} \subset Lmtad$. Suppose $P \in M_2(\mathcal{C})$, $\alpha \in \mathcal{G}(\mathcal{W}^P)$ and $O(\alpha) \in W(\mathcal{C}^{\mathbb{S}}(\mathcal{W}^P))$ Then there exists a universe $Q \in \mathcal{C}$ such that $\alpha \in \mathcal{G}(\mathcal{W}^Q)$ and $O(\alpha) \in W(\mathcal{C}^{\mathbb{S}}(\mathcal{W}^Q))$.*

Proof. Suppose $P \in M_2(\mathcal{C})$, $\alpha \in \mathcal{G}(\mathcal{W}^P)$ and $O(\alpha) \in W(\mathcal{C}^{\mathbb{S}}(\mathcal{W}^P))$. First by $\alpha \in \mathcal{C}^{\alpha}(\mathcal{W}^P)$, $O(\alpha) \in \mathcal{C}^{\mathbb{S}}(\mathcal{W}^P)$ and Lemma 3.19 pick a distinguished set $X_0 \in P$ such that $\alpha \in \mathcal{C}^{\alpha}(X_0)$, $O(\alpha) \in \mathcal{C}^{\mathbb{S}}(X_0)$ and $K(\alpha) \cap \mathcal{W}^P \subset X_0$. Then for any universe $X_0 \in Q \in P$, we obtain $O(\alpha) \in W(\mathcal{C}^{\mathbb{S}}(\mathcal{W}^Q))$ by $\mathcal{W}^Q \subset \mathcal{W}^P$.

Next writing $\mathcal{C}^{\alpha}(\mathcal{W}^P) \cap \alpha \subset \mathcal{W}^P$ analytically we have

$$\forall \beta < \alpha[\beta \in \mathcal{C}^{\alpha}(\mathcal{W}^P) \Rightarrow \exists Y \in P(D[Y] \,\&\, \beta \in Y)]$$

By Lemma 3.19 we obtain $\beta \in \mathcal{C}^{\alpha}(\mathcal{W}^P) \Leftrightarrow \exists X \in P\{D[X] \,\&\, K(\beta) \cap \mathcal{W}^P \subset X \,\&\, \beta \in \mathcal{C}^{\alpha}(X)\}$. Hence for any $\beta < \alpha$ and any distinguished set $X \in P$, there are $\gamma \in K(\beta)$, $Z \in P$ and a distinguished set $Y \in P$ such that if $\gamma \in Z \,\&\, D[Z] \to \gamma \in X$ and $\beta \in \mathcal{C}^{\alpha}(X)$, then $\beta \in Y$. By Lemma 3.17 $D[X]$ is absolute for limit universes. Hence the following Π_2-predicate holds in the universe $P \in M_2(\mathcal{C})$:

$$\forall \beta < \alpha \forall X \exists \gamma \in K(\beta) \exists Z \exists Y[\{D[X] \,\&\, (\gamma \in Z \,\&\, D[Z] \to \gamma \in X) \,\&\, \beta \in \mathcal{C}^{\alpha}(X)\} \Rightarrow (D[Y] \,\&\, \beta \in Y)] \tag{10}$$

Now pick a universe $Q \in \mathcal{C} \cap P$ with $X_0 \in Q$ and $Q \models (10)$. Tracing the above argument backwards in the limit universe Q we obtain $\mathcal{C}^{\alpha}(\mathcal{W}^Q) \cap \alpha \subset \mathcal{W}^Q$ and $X_0 \subset \mathcal{W}^Q = \bigcup\{X \in Q : Q \models D[X]\} \in P$. Thus Lemma 3.19 yields $\alpha \in \mathcal{C}^{\alpha}(\mathcal{W}^Q)$. We obtain $\alpha \in \mathcal{G}(\mathcal{W}^Q)$. □

Proposition 3.23 *Let $\gamma = \psi^f_{\sigma}(\alpha) \in \mathcal{G}(Y)$ and $\gamma \preceq \gamma_0 = \psi^g_{\mathbb{S}}(b)$ with $b = \mathtt{p}_0(\gamma)$. Then $O(\gamma) \in \mathcal{C}^{\mathbb{S}}(Y)$.*

Proof. We have $\gamma \in \mathcal{C}^{\gamma}(Y)$ and $\mathcal{C}^{\gamma}(Y) \cap \gamma \subset Y$. We obtain $\{\sigma, b\} \cup SC_{\mathbb{S}^+}(f) \subset \mathcal{C}^{\gamma}(Y)$. We obtain $SC_{\mathbb{S}^+}(f, b) < \mu = \Lambda(\gamma)$ by Definition 2.14. $E_{\mathbb{S}}(SC_{\mathbb{S}^+}(f, b)) \subset \mathcal{C}^{\gamma}(Y)$ follows from $\gamma < \mathbb{S}$. On the other hand we have $SC_{\mathbb{S}^+}(f, b) \subset \mathcal{H}_b(\gamma)$ for $b = \mathtt{p}_0(\gamma)$ by (5). This yields $E_{\mathbb{S}}(SC_{\mathbb{S}^+}(f, b)) \subset \mathcal{H}_b(\gamma) \cap \mathbb{S} \subset \gamma$. We obtain $E_{\mathbb{S}}(SC_{\mathbb{S}^+}(f, b)) \subset \mathcal{C}^{\gamma}(Y) \cap \gamma \subset Y$. Hence $SC_{\mathbb{S}^+}(f, b) \subset \mathcal{C}^{\mathbb{S}}(Y)$. From $SC_{\mathbb{S}^+}(f, b) \subset \mathcal{C}^{\mathbb{S}}(Y)$ we see $O(\gamma) = o_{\mu}(f) \in \mathcal{C}^{\mathbb{S}}(Y)$ for $\gamma = \psi^f_{\sigma}(\alpha) \in \mathcal{G}(Y)$. □

Definition 3.24 We define the class $M_2(\alpha)$ of α-recursively Mahlo universes for $\mathbb{S} \geq \alpha \in OT(\Pi^1_1)$ as follows:

$$P \in M_2(\alpha) \Leftrightarrow P \in Lmtad \,\&\, \forall \beta \prec \alpha[O(\beta) \in W(\mathcal{C}^{\mathbb{S}}(\mathcal{W}^P)) \Rightarrow P \in M_2(M_2(\beta))] \tag{11}$$

$M_2(\alpha)$ is a Π_3-class.

Lemma 3.25 *If $\mathbb{S} \geq \eta \in \mathcal{G}(\mathcal{W}^P)$, $O(\eta) \in W(\mathcal{C}^{\mathbb{S}}(\mathcal{W}^P))$ and $P \in M_2(M_2(\eta))$ with $P \in L_{\mathbb{S}}$, then $\eta \in \mathcal{W}^P$.*

Proof. We show this by induction on $\in$. Suppose, as IH, the lemma holds for any $Q \in P$. By Lemma 3.22 pick a $Q \in P$ such that $Q \in M_2(\eta)$, and for $Y = \mathcal{W}^Q \in P$, $\{0, \Omega\} \subset Y$, $O(\eta) \in W(\mathcal{C}^{\mathbb{S}}(Y))$ and

$$\eta \in \mathcal{G}(Y) \tag{8}$$

On the other the definition (11) yields $\forall \gamma \prec \eta[O(\gamma) \in W(\mathcal{C}^{\mathbb{S}}(\mathcal{W}^Q)) \Rightarrow Q \in M_2(M_2(\gamma))]$. IH yields with $Y = \mathcal{W}^Q$

$$\forall \gamma \prec \eta(\gamma \in \mathcal{G}(Y) \,\&\, O(\gamma) \in W(\mathcal{C}^{\mathbb{S}}(Y)) \Rightarrow \gamma \in Y)$$

On the other $O(\eta) \in W(\mathcal{C}^{\mathbb{S}}(Y))$ yields $O(\gamma) \in W(\mathcal{C}^{\mathbb{S}}(Y))$ for $\mathcal{G}(Y) \ni \gamma \prec \eta$ by Proposition 3.23. Therefore

$$\forall \gamma \prec \eta(\gamma \in \mathcal{G}(Y) \Rightarrow \gamma \in Y) \tag{9}$$

Therefore by Lemma 3.15 we conclude $\eta \in X$ and $D[X]$ for $X = W(\mathcal{C}^{\eta}(Y)) \cap \eta^+$.
$X \in P$ follows from $Y \in P \in Lmtad$. Consequently $\eta \in \mathcal{W}^P$. □

Lemma 3.26 $\forall \eta \leq \mathbb{S}[O(\eta) \in W(\mathcal{C}^{\mathbb{S}}(\mathcal{W})) \Rightarrow L_{\mathbb{S}} \in M_2(M_2(\eta))]$.

Proof. We show the lemma by induction on $O(\eta) \in W(\mathcal{C}^{\mathbb{S}}(\mathcal{W}))$. Suppose $O(\eta) \in W(\mathcal{C}^{\mathbb{S}}(\mathcal{W}))$ and $L_{\mathbb{S}} \models \Pi_2(b)$ for a $b \in L_{\mathbb{S}}$. We have to find a universe $Q \in L_{\mathbb{S}}$ such that $b \in Q$, $Q \in M_2(\eta)$ and $Q \models \Pi_2(b)$.

By the definition (11) $L_{\mathbb{S}} \in M_2(\eta)$ is equivalent to $\forall \gamma \prec \eta[O(\gamma) \in W(\mathcal{C}^{\mathbb{S}}(\mathcal{W})) \Rightarrow L_{\mathbb{S}} \in M_2(M_2(\gamma))]$, where $\mathcal{W} = \mathcal{W}^{L_{\mathbb{S}}}$ by Proposition 3.5.5. We obtain $\gamma \prec \eta \Rightarrow O(\gamma) < O(\eta)$. Thus IH yields $L_{\mathbb{S}} \in M_2(\eta)$. Let g be a primitive recursive function in the sense of set theory such that $L \in M_2(\eta) \Leftrightarrow P \models \Pi_3(g(\eta))$. Then $L_{\mathbb{S}} \models \Pi_2(b) \wedge \Pi_3(g(\eta))$. Since this is a Π_3-formula which holds in a Π_3-reflecting universe $L_{\mathbb{S}}$, we conclude for some $Q \in L_{\mathbb{S}}$, $Q \models \Pi_2(b) \wedge \Pi_3(g(\eta))$ and hence $Q \in M_2(\eta)$. We are done. □

Lemma 3.27 $\forall \eta \leq \mathbb{S}\left[\eta \in \mathcal{G}(\mathcal{W}) \,\&\, O(\eta) \in W(\mathcal{C}^{\mathbb{S}}(\mathcal{W})) \Rightarrow \eta \in \mathcal{W}\right]$.

Proof. Assume $\mathbb{S} \geq \eta \in \mathcal{G}(\mathcal{W})$ and $O(\eta) \in W(\mathcal{C}^{\mathbb{S}}(\mathcal{W}))$. Lemma 3.26 yields $L_{\mathbb{S}} \in M_2(M_2(\eta))$. From this we see $L_{\mathbb{S}} \in M_2(\mathcal{C})$ with $\mathcal{C} = M_2(M_2(\eta))$ as in the proof of Lemma 3.26 using Π_3-reflection of $L_{\mathbb{S}}$ once again. Then by Lemma 3.22 pick a set $P \in L_{\mathbb{S}}$ such that $\eta \in \mathcal{G}(\mathcal{W}^P)$, $O(\eta) \in W(\mathcal{C}^{\mathbb{S}}(\mathcal{W}^P))$ and $P \in \mathcal{C} = M_2(M_2(\eta))$. Lemma 3.25 yields $\eta \in \mathcal{W}^P \subset \mathcal{W}$. □

Definition 3.28 Let $\mathcal{W}_1 := W(\mathcal{C}^{\mathbb{S}}(\mathcal{W}))$.

Proposition 3.29 *1.* $\mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1) \cap \mathbb{S}^+ = \mathcal{W}_1 \cap \mathbb{S}^+$ *and* $\mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1) \cap \mathbb{S} = \mathcal{C}^{\mathbb{S}}(\mathcal{W}) \cap \mathbb{S} = \mathcal{W}_1 \cap \mathbb{S} = \mathcal{W} \cap \mathbb{S}$.

2. $\mathbb{S} \in \mathcal{W}_1$.

3. $TI[\mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1) \cap \omega_n(\mathbb{S}^+ + 1)]$ *for* each $n < \omega$.

Proof. 3.29.1 and 3.29.2. Since there is no regular ordinal$> \mathbb{S}^+$, $\mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1)\cap\mathbb{S}^+ = \mathcal{W}_1\cap\mathbb{S}^+$. We see $\mathcal{C}^{\mathbb{S}}(\mathcal{W})\cap\mathbb{S} = \mathcal{W}\cap\mathbb{S} = \mathcal{W}_1\cap\mathbb{S}$ from $\psi_{\mathbb{S}^+}(a) > \mathbb{S}$ and $W(\mathcal{W}) = \mathcal{W}$. Hence $\mathbb{S} \in \mathcal{W}_1$.
3.29.3. $TI[\mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1)\cap\mathbb{S}^+]$ follows from $\mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1)\cap\mathbb{S}^+ = \mathcal{W}_1\cap\mathbb{S}^+$. By meta-induction on $n < \omega$, we see $TI[\mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1)\cap\omega_n(\mathbb{S}^+ + 1)]$ using the Gentzen's jump set. □

Lemma 3.30 $a \in \mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1)\cap\omega_n(\mathbb{S}^+ + 1)\,\&\,\psi_{\mathbb{S}^+}(a) \in OT(\Pi^1_1) \Rightarrow \psi_{\mathbb{S}^+}(a) \in \mathcal{W}_1$ *for* each $n < \omega$.

Proof. By Proposition 3.29.3 it suffices to show that $Prg[\mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1), B]$ for $B(a) :\Leftrightarrow \left[\psi_{\mathbb{S}^+}(a) \in OT(\Pi^1_1) \Rightarrow \psi_{\mathbb{S}^+}(a) \in \mathcal{W}_1\right]$. Assume $a \in \mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1)$. We obtain $\psi_{\mathbb{S}^+}(a) \in \mathcal{C}^{\mathbb{S}}(\mathcal{W})$ by Propositions 3.2.1 and 3.29.1.

Next we show $\beta \in \mathcal{C}^{\mathbb{S}}(\mathcal{W})\cap\mathcal{H}_a(\psi_{\mathbb{S}^+}(a)) \Rightarrow \beta \in \mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1)$ by induction on $\ell\beta$. By Proposition 3.29 we may assume $\beta = \psi_{\mathbb{S}^+}(b)$. Then $b \in \mathcal{C}^{\mathbb{S}}(\mathcal{W})\cap\mathcal{H}_a(\psi_{\mathbb{S}^+}(a))$, and $b \in \mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1)$ by IH on lengths. Moreover $b < a$. Hence IH yields $\beta \in \mathcal{W}_1\cap\mathbb{S}^+ \subset \mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1)$.

In particular we obtain $\mathcal{C}^{\mathbb{S}}(\mathcal{W})\cap\psi_{\mathbb{S}^+}(a) \subset \mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1)\cap\mathbb{S}^+ \subset \mathcal{W}_1$. Therefore $\psi_{\mathbb{S}^+}(a) \in W(\mathcal{C}^{\mathbb{S}}(\mathcal{W})) = \mathcal{W}_1$. □

3.4 Well-foundedness proof concluded

Definition 3.31 For irreducible functions f let

$$f \in J :\Leftrightarrow SC_{\mathbb{S}^+}(f) \subset \mathcal{W}_1.$$

For $a \in OT(\Pi^1_1)$ and irreducible functions f, define:

$$\begin{aligned} A(a,f) &:\Leftrightarrow \forall\sigma \in \mathcal{W}_1\cap\mathbb{S}^+[\psi^f_\sigma(a) \in OT(\Pi^1_1)\,\&\,O(\psi^f_\sigma(a)) \in \mathcal{W}_1 \Rightarrow \psi^f_\sigma(a) \in \mathcal{W}]. \\ \mathrm{MIH}(a) &:\Leftrightarrow \forall b \in \mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1)\cap a\forall f \in J\, A(b,f). \\ \mathrm{SIH}(a,f) &:\Leftrightarrow \forall g \in J[g <^0_{lx} f \Rightarrow A(a,g)]. \end{aligned}$$

Lemma 3.32 *Assume* $a \in \mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1)\cap\omega_n(\mathbb{S}^+ + 1)$, $f \in J$, $\mathrm{MIH}(a)$, *and* $\mathrm{SIH}(a,f)$ *in Definition 3.31. Then*

$$\forall\kappa \in \mathcal{W}_1\cap\mathbb{S}^+[\psi^f_\kappa(a) \in OT(\Pi^1_1)\,\&\,O(\psi^f_\kappa(a)) \in \mathcal{W}_1 \Rightarrow \psi^f_\kappa(a) \in \mathcal{W}].$$

Proof. This is seen as in [1, 2] from Lemma 3.27. Let $\alpha_1 = \psi^f_\kappa(a) \in OT(\Pi^1_1)$ be such that $O(\alpha_1) \in \mathcal{W}_1$, $a \in \mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1)$, $\mathbb{S} \geq \kappa \in \mathcal{W}_1$ and $f \in J$. By Lemma 3.27 it suffices to show $\alpha_1 \in \mathcal{G}(\mathcal{W})$.

By Proposition 3.2.1 we have $\{\kappa, a\}\cup SC_{\mathbb{S}^+}(f) \subset \mathcal{C}^{\alpha_1}(\mathcal{W})$, and hence $\alpha_1 \in \mathcal{C}^{\alpha_1}(\mathcal{W})$. It suffices to show the following claim.

$$\forall\beta_1 \in \mathcal{C}^{\alpha_1}(\mathcal{W})\cap\alpha_1[\beta_1 \in \mathcal{W}]. \tag{12}$$

Proof of (12) by induction on $\ell\beta_1$. Assume $\beta_1 \in \mathcal{C}^{\alpha_1}(\mathcal{W})\cap\alpha_1$ and let

$$\mathrm{LIH} :\Leftrightarrow \forall\gamma \in \mathcal{C}^{\alpha_1}(\mathcal{W})\cap\alpha_1[\ell\gamma < \ell\beta_1 \Rightarrow \gamma \in \mathcal{W}].$$

We show $\beta_1 \in \mathcal{W}$. We may assume that $\beta_1 = \psi^g_\pi(b)$ for some π, b, g such that $\{\pi, b\} \cup SC_{\mathbb{S}^+}(g) \subset \mathcal{C}^{\alpha_1}(\mathcal{W})$ and $\alpha_1 < \pi \leq \mathbb{S}$.

Case 1. $b < a$, $\beta_1 < \kappa$ and $\{\pi, b\} \cup SC_{\mathbb{S}^+}(g) \subset \mathcal{H}_a(\alpha_1)$: Let B denote a set of subterms of β_1 defined recursively as follows. First $\{\pi, b\} \cup SC_{\mathbb{S}^+}(g) \subset B$. Let $\alpha_1 \leq \beta \in B$. If $\beta =_{NF} \gamma_m + \cdots + \gamma_0$, then $\{\gamma_i : i \leq m\} \subset B$. If $\beta =_{NF} \varphi\gamma\delta$, then $\{\gamma, \delta\} \subset B$. If $\beta =_{NF} \gamma^+$, then $\gamma \in B$. If $\beta = \psi^h_\sigma(c)$ with $\sigma > \alpha_1$, then $\{\sigma, c\} \cup SC_{\mathbb{S}^+}(h) \subset B$.

Then from $\{\pi, b\} \cup SC_{\mathbb{S}^+}(g) \subset \mathcal{C}^{\alpha_1}(\mathcal{W})$ we see inductively that $B \subset \mathcal{C}^{\alpha_1}(\mathcal{W})$. Hence by LIH we obtain $B \cap \alpha_1 \subset \mathcal{W}$. Moreover if $\alpha_1 \leq \psi^h_\sigma(c) \in B$, then $c < a$. We claim that

$$\forall \beta \in B(\beta \in \mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1)) \tag{13}$$

Proof of (13) by induction on $\ell\beta$. Let $\beta \in B$. We can assume that $\alpha_1 \leq \beta = \psi^h_\sigma(c)$ by LIH. Then by induction hypothesis we have $\{\sigma, c\} \cup SC_{\mathbb{S}^+}(h) \subset \mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1)$. On the other hand we have $c < a < \omega_n(\mathbb{S}^+ + 1)$. If $\sigma = \mathbb{S}^+$, then Lemma 3.30 yields $\beta = \psi_{\mathbb{S}^+}(c) \in \mathcal{W}_1$. Let $\sigma \leq \mathbb{S}$. Then $\{\sigma\} \cup SC_{\mathbb{S}^+}(h) \subset \mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1) \cap \mathbb{S}^+ = \mathcal{W}_1 \cap \mathbb{S}^+$. Let $\alpha_1 \leq \beta = \psi^h_\sigma(c) \preceq \psi^{h_0}_{\mathbb{S}}(c_0)$. Then $SC_{\mathbb{S}^+}(c_0) \subset B$ and $SC_{\mathbb{S}^+}(c_0) \subset \mathcal{W}_1$ by induction hypothesis on lengths. Hence $\Lambda(\beta) \in \mathcal{W}_1$ for the least epsilon number $\Lambda(\beta) > \max(SC_{\mathbb{S}^+}(c_0))$. We obtain $O(\beta) \in \mathcal{W}_1$ by $SC_{\mathbb{S}^+}(h) \cup \{\Lambda(\beta)\} \subset \mathcal{W}_1$. MIH($a$) yields $\beta \in \mathcal{W}$. Thus (13) is shown. □

In particular we obtain $\{\pi, b, \Lambda(\beta_1)\} \cup SC_{\mathbb{S}^+}(g) \subset \mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1)$. Moreover we have $b < a$. Therefore once again MIH(a) yields $\beta_1 \in \mathcal{W}$.

Case 2. $b = a$, $\pi = \kappa$, $SC_{\mathbb{S}^+}(g) \subset \mathcal{H}_a(\alpha_1)$ and $g <^0_{lx} f$: As in (13) we see that $SC_{\mathbb{S}^+}(g) \subset \mathcal{W}_1$ from Lemma 3.30 and MIH(a). SIH(a, f) yields $\beta_1 \in \mathcal{W}$.

Case 3. $a \leq b$ and $SC_{\mathbb{S}^+}(f) \cup \{\kappa, a\} \not\subset \mathcal{H}_b(\beta_1)$: As in as in [1,2] we see that there exists a γ such that $\beta_1 \leq \gamma \in \mathcal{W} \cap \alpha_1$. Then $\beta_1 \in \mathcal{W}$ follows from $\beta_1 \in \mathcal{C}^{\alpha_1}(\mathcal{W})$.

This completes a proof of (12) and of the lemma. □

Lemma 3.33 *For $\psi^f_\kappa(a) \in OT(\Pi^1_1)$, if $a \in \mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1) \cap \omega_n(\mathbb{S}^+ + 1)$, $\{\kappa\} \cup SC_{\mathbb{S}^+}(f) \subset \mathcal{W}_1 \cap \mathbb{S}^+$ and $O(\psi^f_\kappa(a)) \in \mathcal{W}_1$, then $\psi^f_\kappa(a) \in \mathcal{W}$.*

Proof. This is seen from Lemma 3.32 and Proposition 3.29.3. Note that if $\beta = \psi^g_\kappa(a) < \psi^f_\kappa(a) = \alpha$ by $g <^0_{lx} f$, then $\mathtt{p}_0(\beta) = \mathtt{p}_0(\alpha)$, $\Lambda(\beta) = \Lambda(\alpha)$ and $O(\beta) < O(\alpha)$ by Lemma 2.13. □

Lemma 3.34 *For* each $\alpha \in OT(\Pi^1_1)$, $\alpha \in \mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1)$.

Proof. This is seen by meta-induction on $\ell\alpha$ using Propositions 3.13 and 3.14, and Lemmas 3.30 and 3.33. □

Proof of Theorem 1.2. For each $\alpha \in OT(\Pi^1_1)$ we obtain $\alpha \in \mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1)$ by Lemma 3.34. Therefore by Proposition 3.29.1 we obtain for each $n < \omega$, $\psi_\Omega(\omega_n(\mathbb{S}^+ + 1)) \in \mathcal{C}^{\mathbb{S}^+}(\mathcal{W}_1) \cap \Omega = \mathcal{W} \cap \Omega = W(\mathcal{C}^0(\emptyset)) \cap \Omega$, where $W(\mathcal{C}^0(\emptyset)) = W(OT(\Pi^1_1))$.

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