The corridor problem with discrete multiple bottlenecks☆

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Abstract

This paper presents a transparent approach to the analysis of dynamic user equilibrium and clarifies the properties of a departure-time choice equilibrium of a corridor problem where discrete multiple bottlenecks exist along a freeway. The basis of our approach is the transformation of the formulation of equilibrium conditions in a conventional "Eulerian coordinate system" into one in a "*Lagrangian-like coordinate system*." This enables us to evaluate dynamic travel times easily, and to achieve a deep understanding of the mathematical structure of the problem, in particular, about the properties of the demand and supply (queuing) sub-models, relations with dynamic system optimal assignment, and differences between the morning and evening rush problems. Building on these foundations, we establish rigorous results on the existence and uniqueness of equilibria.

Keywords: corridor problem, departure time choice equilibrium, complementarity problem, existence and uniqueness of equilibrium

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1. Introduction

Traffic congestion during morning and evening peak periods causes serious economic losses in most large cities. The Vickrey (1969) bottleneck model was proposed to understand and characterize this problem. It describes commuters' departure-time choice equilibrium at a single bottleneck and has been extensively studied by many researchers (e.g., Hendrickson and Kocur, 1981; Smith, 1984; Daganzo, 1985). The model has been applied to many problems because of its simplicity and clarity. For example, it has been used to evaluate traffic policies (e.g., Arnott et al., 1993a; Daganzo and Garcia, 2000; Akamatsu et al., 2006).

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If there is more than one bottleneck, there are two streams of research for analyzing user equilibria. The first uses a straightforward generalization of Vickrey's model, whereas the second considers the dynamic user equilibrium (DUE) on a general network.

In the former stream of research, Kuwahara (1990) and Arnott et al. (1993b) analyzed an user equilibrium in two-tandem bottleneck networks, and Lago and Daganzo (2007) studied a similar problem where there are spillover and merging effects. These generalizations can provide valuable insights into the distribution of congestion among commuters with different origins. However, it is almost impossible to obtain an equilibrium in more general cases where a larger number of bottlenecks exist in a freeway corridor because this stream of studies has only used an analytical approach. Indeed, Arnott and DePalma (2011) studied a variant of the general problem, the so-called *corridor problem*, but they could not provide a complete equilibrium solution.

The latter stream of research applies a more computational approach. Although various mathematical formulations of DUE have been proposed (see, Szeto and Wong, 2011), they are considerably complex ways to analyze properties of an equilibrium because they must handle the complicated nested structure between link and path travel times in a network. Consequently, many issues remain regarding DUE properties, such as their existence, uniqueness, and stability (Iryo, 2013). Moreover, there is no algorithm that ensures convergence to a DUE solution, although many algorithms have been proposed. This is because most algorithms require *monotonicity* of the problem to guarantee convergence. However, this property does not hold for DUE problems, as we shall see later.

This paper presents a transparent approach for analyzing DUE and clarifies the properties of a departure-time choice equilibrium for a corridor problem where discrete multiple bottlenecks exist along a freeway. The basis of our approach is the transformation of the equilibrium conditions from a conventional "Eulerian coordinate system" into a "*Lagrangian-like coordinate system*." This is valuable for the morning rush (many-to-one travel demand) problem and the reverse problem in the evening (one-to-many travel demand problem). We can use this approach to easily evaluate the travel times without the abovementioned complication. We can then achieve deep insights into the mathematical structure of the problem, in terns of the properties of the demand and supply (queuing) sub-models, the relationships with dynamic system optimal assignment, and the differences between the morning and evening rush problems. Building on these foundations, we establish rigorous results on the existence and uniqueness of equilibria.

The remainder of this paper is organized as follows. In Section 2, we describe the dynamic user equilibrium for the corridor problem in an Eulerian coordinate system and discusses why it is difficult to analyze the problem in this form. Section 3 contains our reformulation of the equilibrium conditions in a Lagrangian-like coordinate system. To develop an understanding of the mathematical properties of the problem, we present a preliminary analysis for the deterministic demand case in Section 4. In Section 5, we examine the existence and uniqueness of equilibria. Section 6 provides numerical examples of the equilibrium flow patterns. Finally, Section 7 concludes the paper.

2. The corridor problem

2.1. Assumptions

Consider a freeway corridor consisting of N on-ramps (origin nodes) and a single off-ramp (destination node). These nodes are numbered sequentially from destination node 0 to the most distant on-ramp node N, as shown in Fig.1. We assume that there is a single bottleneck i with a finite capacity μ_i in a segment between each pair $(i-1, i)$ of adjacent ramps. Therefore, there are N bottlenecks in the network numbered sequentially $(1, 2, \ldots, N)$ from downstream to upstream. At each bottleneck, a queue can form when the inflow rate exceeds the capacity. The queue evolution and the associated queuing delay are assumed to be represented by a point queue model (described in 2.2). The distance between each pair $(i-1, i)$ of adjacent bottlenecks measured by the free flow travel time is given by c_i.

From each on-ramp i $(i = 1, \ldots, N)$, Q_i commuters in residential location i $(i = 1, \ldots, N)$ enter the network and reach their destination during the morning rush-hour $\mathcal{T} \in [0, T]$. We assume that the travel demands $\{Q_i\}$ are given constants. Each commuter chooses the departure time of her/his trip from home (on-ramp) to workplace (destination) so as to minimize her/his disutility. This disutility includes travel time, queuing delay at bottlenecks, and a "schedule delay cost" that is associated with deviation from the wished arrival time to the destination. We assume that all commuters are homogeneous, such that they have the same desirable arrival time (work start time) t_w at the workplace, the same value of time, and the same penalty function for the schedule delay. However, for the arrival/departure-time choice principle, we employ a random utility model that describes the heterogeneity of users. The main problem considered in this paper is, under the assumptions mentioned above, to characterize a dynamic user equilibrium distribution of arrivals at the destination where no commuter could reduce her/his disutility by changing arrival times.

2.2. *Arrival/departure time choice equilibrium*

Under the assumptions above, we formulate a model that describes the equilibrium in the corridor network. At equilibrium, the following three conditions should hold.

(*a*) *Queuing conditions at each bottleneck*

We describe the queuing congestion at each bottleneck in the network using a point queue model, where a queue is assumed to form vertically at the entrance of each bottleneck. The model can be represented by the following three conditions. First, the state equation for the number of users queuing at each bottleneck is

$$E_i(t) = A_i(t) - D_i(t), \qquad \forall t \in \mathcal{T},\ \forall i \in \mathcal{N} \tag{2.1a}$$

where $A_i(t)$ $[D_i(t)]$ denotes the cumulative number of users who arrive at [depart from] bottleneck i by time t, $E_i(t)$ is the number of users queuing (existing) in bottleneck i at time t, and $\mathcal{N} \equiv \{1, \ldots, N\}$. Because a queue has no physical length in the point queue model, $E_i(t)$ is the vertical distance between the cumulative arrival curve $\{A_i(t) : t \in \mathcal{T}\}$ and the cumulative departure curve

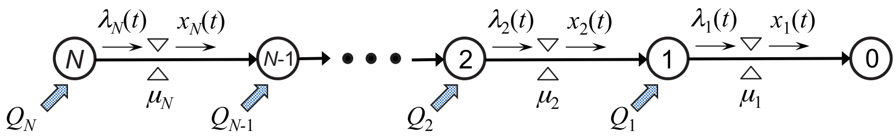

Figure 1: The corridor network with N on-ramps and N bottlenecks

$\{D_i(t) : t \in \mathcal{T}\}$ at time t. Furthermore, the state equation can be represented using the arrival flow rate $\lambda_i(t)$ and the departure (exit) rate $x_i(t)$ at each bottleneck. That is,

$$e_i(t) \equiv \dot{E}_i(t) = \lambda_i(t) - x_i(t), \tag{2.1b}$$

where the "dot" denotes the derivative with respect to time t (e.g., $\dot{A}_i(t) \equiv dA_i(t)/dt$), and $\lambda_i(t)$ and $x_i(t)$ are defined as $\lambda_i(t) \equiv \dot{A}_i(t) \geq 0$ and $x_i(t) \equiv \dot{D}_i(t) \geq 0$, respectively.

The second condition is the exit flow model, which is the most characteristic of the point queue model. The departure flow rate from the bottleneck i at time t is

$$x_i(t) = \begin{cases} \mu_i & \text{if } E_i(t) > 0 \\ \min.[\lambda_i(t), \mu_i] & \text{if } E_i(t) = 0 \end{cases} \qquad \forall t \in \mathcal{T},\ \forall i \in \mathcal{N} \tag{2.2}$$

where μ_i denotes the capacity of bottleneck i, which is a given constant for each bottleneck.

The final condition concerns the relationship between the state variables and the queuing delay (the time taken to pass through the bottleneck). In the point queue model, the queuing delay $d_i(t)$ at bottleneck i for a user arriving at the bottleneck at time t is the horizontal distance between the cumulative arrival curve $\{A_i(t) : t \in \mathcal{T}\}$ and the cumulative departure curve $\{D_i(t) : t \in \mathcal{T}\}$. That is,

$$d_i(t) = E_i(t)/\mu_i \qquad \forall t \in \mathcal{T},\ \forall i \in \mathcal{N}. \tag{2.3a}$$

Because the number of users queuing in the bottleneck $E_i(t)$ satisfies (2.1) and (2.2), the queuing delay (2.3a) can also be written equivalently as

$$\dot{d}_i(t) = \begin{cases} (\lambda_i(t)/\mu_i) - 1 & \text{if } d_i(t) > 0 \\ \max.[0, (\lambda_i(t)/\mu_i) - 1] & \text{if } d_i(t) = 0 \end{cases}. \tag{2.3b}$$

This implies that the queuing delay in the point queue model should satisfy

$$\begin{aligned} &\begin{cases} \dot{d}_i(t) = (\lambda_i(t)/\mu_i) - 1 & \text{if } d_i(t) > 0 \\ \dot{d}_i(t) \geq (\lambda_i(t)/\mu_i) - 1 & \text{if } d_i(t) = 0 \end{cases}, \\ \Leftrightarrow\quad & 0 \leq d_i(t) \perp \dot{d}_i(t) - [(\lambda_i(t)/\mu_i) - 1] \geq 0. \end{aligned} \tag{2.3c}$$

In this paper, we employ the complementarity condition (2.3c)[1] instead of (2.3b) as the queuing delay model. This is because it has an advantage over (2.3b) in terms of the time discretization of the model, and its essential features are consistent with the original point queue model. Note that a natural time-discretization scheme for (2.3c) is given by

$$0 \leq d_i(t + \Delta t) \perp (d_i(t + \Delta t) - d_i(t)) - ((\lambda_i(t + \Delta t)/\mu_i) - 1)\Delta t \geq 0, \tag{2.3d}$$

$$\Leftrightarrow\quad d_i(t + \Delta t) = \max.[0, d_i(t) + ((\lambda_i(t + \Delta t)/\mu_i) - 1)\Delta t], \tag{2.3e}$$

where $\lambda_i(t + \Delta t) \equiv (A_i(t + \Delta t) - A_i(t))/dt$. It is clear that the discretization scheme above has the advantage that it always predicts a non-negative queuing delay, whereas a naïve discretization

[1]The model (2.3c) deviates from the model (2.3b) in the case that $d_i(t) = 0$. However, the model (2.3c) guarantees a non-negativity of the queuing delay and produces a physically meaningful queue dynamics (for the detailed discussions, see Ban et al., 2012).

scheme for (2.3b) could result in a negative queuing delays (for the detailed discussion, see Akamatsu, 2001; Ban et al., 2012).

As a final comment on the point queue model, note that the point queue model (i.e., the conditions (2.1), (2.2) and (2.3)) implies the First-In-First-Out (FIFO) and flow propagation conditions at each bottleneck. That is,

$$
\begin{aligned}
& A_i(t) = D_i(t + d_i(t)) \qquad \forall t \in \mathcal{T},\ \forall i \in \mathcal{N} && (2.4a)\\
\Leftrightarrow\quad & \lambda_i(t)/x_i(t + d_i(t)) = 1 + \dot{d}_i(t) \geq 0. && (2.4b)
\end{aligned}
$$

It also follows that the queuing conditions (2.1) and (2.2) combined with the FIFO condition (2.4) imply the queuing delay (2.3).

(b) Flow conservations in the network

The dynamic traffic flows in the network should satisfy the following two conservation conditions. The first one is that the origin-destination (OD) travel demands for each origin must be assigned to each time point in the interval $\mathcal{T} \in [0, T]$. That is, the time-dependent OD demand should satisfy

$$
\hat{Q}_i(T) = Q_i \qquad \forall i \in \mathcal{N} \tag{2.5a}
$$

where Q_i is the number of users who wish to travel from origin i to the destination (i.e., the total OD demand of origin i to the destination), which is a given constant for each OD pair, and $\hat{Q}_i(t)$ denotes the cumulative number of users who depart from origin node i and arrive at the bottleneck i by time t. This can be equivalently written as

$$
\int_0^T \hat{q}_i(t)dt = Q_i \tag{2.5b}
$$

where $\hat{q}_i(t)$ is the arrival rate of the users with origin i at bottleneck i at time t, that is, $\hat{q}_i(t) = d\hat{Q}_i(t)/dt$.

The second condition is that the inflow and outflow at each bottleneck at each time point must be equal. That is,

$$
\begin{aligned}
& D_{i+1}(t - c_{i+1}) + \hat{Q}_i(t) = A_i(t), \quad \hat{Q}_N(t) = A_N(t) \qquad \forall t \in \mathcal{T},\ i = 1, \ldots, N-1 && (2.6a)\\
\Leftrightarrow\quad & x_{i+1}(t - c_{i+1}) + \hat{q}_i(t) = \lambda_i(t), \quad \hat{q}_N(t) = \lambda_N(t), && (2.6b)
\end{aligned}
$$

where c_i is the free-flow travel time from bottleneck i to bottleneck $i-1$, which is a given constant.

(c) Equilibrium conditions for arrival/departure-time choice

In accordance with random utility theory, each user's disutility is expressed as $\hat{v}_i(t) + \hat{\epsilon}_i(t)$, where $\hat{\epsilon}_i(t)$ denotes the disutility representing the users' idiosyncratic choices for arrival/departure times, with the distribution of $\hat{\epsilon}_i(t)$ is assumed to be continuous; and $\hat{v}_i(t)$ denotes the generalized travel cost defined as the sum of the "schedule cost" and the travel time to the destination. The schedule cost for a user is the cost (penalty) caused by the difference between the user's desired arrival time t_w and the actual arrival time t. The desired arrival time t_w is assumed to be the same for all users. The schedule cost (in the unit of travel time) is represented by the function $p(t)$, where t is the destination arrival time. This function is common to all users, and is assumed to

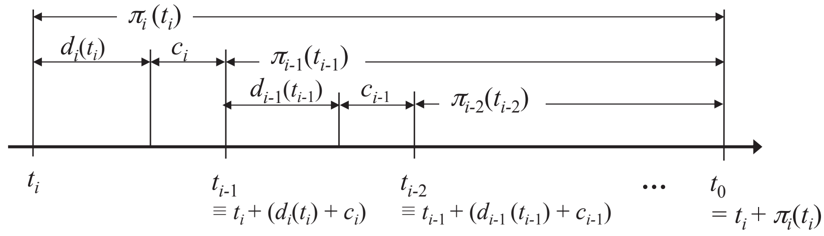

Figure 2: Travel time $\pi_i(t)$ from each bottleneck/origin to the destination

be convex and continuous, following previous studies on the departure time choice equilibrium (e.g., Smith, 1984; Daganzo, 1985; Kuwahara, 1990).

At equilibrium, no one can improve her/his own disutility by changing her/his trip schedule unilaterally. Let $\pi_i(t)$ be the travel time from each bottleneck to the destination, for a user arriving at bottleneck i at time t. It follows from the definition that the user's arrival time at the destination is $t + \pi_i(t)$, and the schedule cost for the user is $p(t + \pi_i(t))$. Therefore, the equilibrium condition can be expressed as

$$\hat{q}_i(t) = Q_i \hat{P}_i(t) \qquad t \in \mathcal{T}, \ \forall i \in \mathcal{N} \tag{2.7a}$$

$$\hat{P}_i(t) \equiv \Pr[\hat{v}_i(t) + \hat{\epsilon}_i(t) \leq \hat{v}_i(t') + \hat{\epsilon}_i(t') \ \forall t' \neq t \in \mathcal{T}] \tag{2.7b}$$

$$\hat{v}_i(t) \equiv p(t + \pi_i(t)) + \pi_i(t) \tag{2.7c}$$

where $\hat{P}_i(t)$ is the fraction of users arriving at bottleneck i at time t.

Note that the equilibrium concept employed in this paper is not the "*reactive* user equilibrium" *but* the "*predictive* user equilibrium." That is, travel time $\pi_i(t)$ is defined as the time that a user arriving at bottleneck i at time t actually experiences during the course of a trip to the destination, which can be defined by the following recursive equations (see Fig.2):

$$\pi_i(t_i) - \pi_{i-1}(t_{i-1}) = d_i(t_i) + c_i, \quad \pi_0(t) = 0 \qquad \forall t \in \mathcal{T}, \ \forall i \in \mathcal{N} \tag{2.8a}$$

$$t_{i-1} - t_i = d_i(t_i) + c_i \qquad \forall i \in \mathcal{N}. \tag{2.8b}$$

2.3. *Difficulties of analyzing the problem in an Eulerian coordinate system*

The recursive equation (2.8a) for the travel time $\pi_i(t)$ has a complicated nested structure with respect to time (i.e., (2.8) is a system of recursive equations for $\pi_i(t)$ with *time delays* that are *state-dependent and time-varying*). This causes extreme difficulties in analyzing the properties of the equilibrium conditions. Indeed, even if we are just evaluating $\pi_i(t)$, we must evaluate every arrival time $\{t_j, \ j = 1, \ldots, i-1\}$ at the downstream bottlenecks for a user arriving at bottleneck i at time t using the recursive equation (2.8b). This, in turn, requires to evaluate all the queuing delays $\{d_j(t_j), \ j = 1, \ldots, i-1\}$, which are *time-varying and dependent on other state variables*. Thus, it is almost impossible to derive theoretical properties of the equilibria defined by travel time function $\{\pi_i(t)\}$ with such troublesome issues[2].

[2]Recently, Han et al. (2013) proved the existence of the simultaneous route-and-departure choice dynamic user equilibrium within an Eulerian coordinate system. However, their approach may not provide more insights on the equilibrium like those presented in Section 4 because of the complicated nested structure between link and path travel times.

One of the reasons for this difficulty is that the problem is formulated in an "Eulerian coordinate system." The Eulerian system is not suitable for dealing with the travel time that a user actually experiences during the course of a trip. To explicitly represent such travel time we must trace the time-space path of each user in the network. To handle such travel time in a simpler and more transparent manner, it is rather natural to employ a "Lagrangian-like coordinate system," in which we can easily trace the time-space paths of users. Thus, in the subsequent section, we reformulate the equilibrium condition into a Lagrangian-like coordinate system.

3. Reformulation in a Lagrangian coordinate system

In this section, we describe the reformulation of the equilibrium conditions for the corridor problem into a Lagrangian-like coordinate system. This formulation is better suited to both the morning-rush problem presented in the previous section and the reverse problem (the evening-rush). After reformulating the morning-rush problem, we briefly describe the evening-rush problem by demonstrating their differences.

3.1. Many-to-one travel demand (morning-rush) problem

The equilibrium concept along with the FIFO discipline of the bottleneck model defined in Section 2.2 implies that users who arrive at the destination at the same time have the same arrival time at any bottleneck on the way to their destination, and that the order of arrival at the destination must be kept at any bottleneck from origins (for the detailed discussions on this property, see Kuwahara, 1990; Kuwahara and Akamatsu, 1993; Akamatsu and Kuwahara, 1999; Akamatsu, 2001). Using this property, we can define the equilibrium arrival time $\tau_i(s)$ at bottleneck i for a user arriving at the destination at time s, and an object period $\mathcal{S} \in [\underline{S}, \overline{S}]$ for the destination arrival time s that is sufficiently long for all demands to be served in $\mathcal{S}$. Letting $w_i(s)$ be the queuing delay at bottleneck i for this user, $\boldsymbol{\tau}(s)$ and $\mathbf{w}(s)$ should satisfy

$$\tau_i(s) = \tau_{i-1}(s) - (w_i(s) + c_i) \qquad \forall s \in \mathcal{S},\ \forall i \in \mathcal{N}. \tag{3.1a}$$

It follows from this and the boundary condition $\tau_0(s) = s\ \forall s \in \mathcal{S}$ (note that $\tau_0(s)$ is the time of arrival at the destination) that $\tau_i(s)$ can be written as an explicit function of s and $\mathbf{w}(s)$, that is,

$$\tau_i(s) = s - \textstyle\sum_{j=1}^{i}(w_j(s) + c_j) \qquad \forall s \in \mathcal{S},\ \forall i \in \mathcal{N}. \tag{3.1b}$$

Similarly, we define $\sigma_i(s)$ as the departure time from bottleneck i for a user arriving at the destination at time s, which obviously satisfies (see Fig. 3)

$$\sigma_i(s) = \tau_i(s) + w_i(s) = \tau_{i-1}(s) - c_i. \tag{3.2}$$

In addition to $\{\boldsymbol{\tau}(s), \mathbf{w}(s) : s \in \mathcal{S}\}$ and $\{\boldsymbol{\sigma}(s) : s \in \mathcal{S}\}$, which represent the time-space paths of users, it is convenient to define the arrival flow rate $y_i(s)$ at bottleneck i for a user arriving at the destination at time s as

$$y_i(s) \equiv dA_i(\tau_i(s))/ds = \lambda_i(\tau_i(s)) \cdot \Delta\tau_i(s), \tag{3.3}$$

where Δ denotes the derivative operation with respect to the *destination-arrival time*, s (e.g., $\Delta\tau_i(s) \equiv d\tau_i(s)/ds$).

Based on these variables labeled by the destination-arrival-time s (instead of the time t in the Eulerian coordinate system), we can reformulate the equilibrium using the following three conditions.

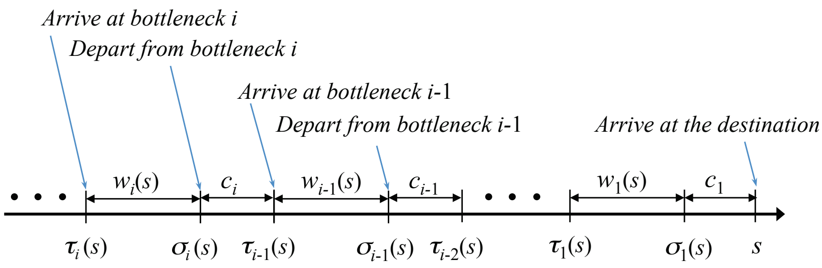

Figure 3: Arrival/departure time at bottlenecks for a user arriving at the destination at time s

(a) *Queuing conditions at each bottleneck*

For users arriving at the destination at time s, the queuing delay $w_i(s)$ at bottleneck i is

$$w_i(s) = d_i(\tau_i(s)), \qquad \forall s \in \mathcal{S}\ \forall i \in \mathcal{N}, \tag{3.4a}$$

which implies

$$\Delta w_i(s) = \dot{d}_i(\tau_i(s)) \cdot \Delta\tau_i(s). \tag{3.4b}$$

Substituting in the queuing delay (complementarity) condition (2.3c) into this, we have

$$\begin{cases} \Delta w_i(s) = (y_i(s)/\mu_i) - \Delta\tau_i(s) & \text{if } \ w_i(s) > 0 \\ \Delta w_i(s) \geq (y_i(s)/\mu_i) - \Delta\tau_i(s) & \text{if } \ w_i(s) = 0 \end{cases}. \tag{3.5}$$

As shown in Section 2.2, the point queue model defined in (2.1) – (2.3) always satisfies the FIFO conditions (2.4) at each bottleneck. Similarly, the queuing delay condition (3.5) combined with the definitional equations (3.1) and (3.2) implies the FIFO condition. We can confirm this as follows. For users arriving at bottleneck i at time $t = \tau_i(s)$, the FIFO condition can be written as

$$A_i(\tau_i(s)) = D_i(\sigma_i(s)), \qquad \forall s \in \mathcal{S},\ \forall i \in \mathcal{N} \tag{3.6a}$$

$$\text{or} \quad y_i(s) = x_i(\sigma_i(s)) \cdot \Delta\sigma_i(s). \tag{3.6b}$$

Recalling the definitions (3.1) and (3.2) of $\boldsymbol{\tau}(s)$ and $\boldsymbol{\sigma}(s)$, we have that $\boldsymbol{\Delta\tau}(s)$ and $\boldsymbol{\Delta\sigma}(s)$ are

$$\begin{aligned} &\Delta\sigma_i(s) = \Delta\tau_i(s) + \Delta w_i(s), \\ &\Delta\tau_i(s) = \Delta\tau_{i-1}(s) - \Delta w_i(s), \quad \Delta\tau_0(s) = 1. \end{aligned} \tag{3.7}$$

Substituting these into (3.6b), we have $y_i(s) = x_i(\tau_i(s) + w_i(s)) \cdot \Delta\tau_{i-1}(s)$, which reduces to

$$\begin{cases} y_i(s) = \mu_i \Delta\tau_{i-1}(s) & \text{if } \ w_i(s) > 0 \\ y_i(s) \leq \mu_i \Delta\tau_{i-1}(s) & \text{if } \ w_i(s) = 0 \end{cases} \tag{3.8a}$$

$$\text{or} \quad \begin{cases} y_i(s) = \mu_i \Delta\sigma_i(s) & \text{if } \ w_i(s) > 0 \\ y_i(s) \leq \mu_i \Delta\sigma_i(s) & \text{if } \ w_i(s) = 0 \end{cases}. \tag{3.8b}$$

It is clear that the queuing delay condition (3.5) combined with (3.7) implies the FIFO condition (3.8), and that the FIFO condition (3.8) combined with (3.7) implies the queuing delay condition (3.5). Thus, (3.5) and (3.8) are mutually interchangeable when formulating the equilibrium condition.

(b) *Flow conservations in the network*

The flow conservation (2.5) for each OD pair in Section 2.2 can be represented as

$$Q_i(\overline{S}) = \hat{Q}_i(\tau_i(\overline{S})) = Q_i \qquad \forall i \in \mathcal{N} \tag{3.9a}$$

where $Q_i(s)$ is the cumulative number of users who depart from origin node i and arrive at the destination by time s, which implies $\hat{Q}_i(\tau_i(s)) = Q_i(s)$. Letting $q_i(s)$ be the arrival flow rate at the destination defined by

$$q_i(s) = \Delta\hat{Q}_i(\tau_i(s)) = \hat{q}_i(\tau_i(s)) \cdot \Delta\tau_i(s),$$

(3.9a) can be equivalently represented as

$$Q_i = \int_{\underline{S}}^{\overline{S}} q_i(s)ds \qquad \forall i \in \mathcal{N}. \tag{3.9b}$$

The flow conservation (2.6) for each bottleneck can also be rewritten in terms of the destination arrival time s. Substituting (3.1), (3.2), and the FIFO condition (3.6a) into the conservation equation (2.6a) at $t = \tau_i(s)$ yields

$$A_{i+1}(\tau_{i+1}(s)) + Q_i(s) = A_i(\tau_i(s)), \quad Q_N(s) = A_N(\tau_N(s)) \qquad \forall s \in \mathcal{S},\ i = 1, \ldots, N-1 \tag{3.10a}$$

which is equivalent to

$$q_i(s) = y_i(s) - y_{i+1}(s), \quad q_N(s) = y_N(s) \qquad \forall s \in \mathcal{S},\ i = 1, \ldots, N-1. \tag{3.10b}$$

(c) *Equilibrium conditions for arrival/departure-time choice*

For a user with origin i who arrives at the destination at time s, the schedule cost is given by $p(s)$, and the travel time is $s - \tau_i(s)$. Therefore, the equilibrium condition can be expressed as

$$q_i(s) = Q_i P_i(s) \qquad s \in \mathcal{S},\ \forall i \in \mathcal{N} \tag{3.11a}$$

$$P_i(s) \equiv \Pr[v_i(s) + \tilde{\varepsilon}_i(s) \le v_i(s') + \tilde{\varepsilon}_i(s')\ \forall s' \ne s \in \mathcal{S}] \tag{3.11b}$$

$$v_i(s) \equiv p(s) + (s - \tau_i(s)) \tag{3.11c}$$

where $\tilde{\varepsilon}_i(s)$ denotes the disutility representing the users' idiosyncratic taste for arrival times, and $P_i(s)$ is the fraction of users with origin i who arrive at the destination at time s. No assumption of a specific functional form of $P_i(s)$ is needed in our proof of uniqueness for the DUE. However, we use the logit choice function to derive a sufficient condition for the existence of the DUE and for the numerical examples in the later sections. That is,

$$P_i(s) \equiv \exp(-\theta v_i(s)) / \int_{\underline{S}}^{\overline{S}} \exp(-\theta v_i(s))ds \tag{3.12}$$

where θ is the inverse of the variance of the idiosyncratic tastes, which implies that the assumption of the $\tilde{\varepsilon}_i(s)$ are i.i.d. Gumbel distributions. As $\theta \to +\infty$ (i.e., $\tilde{\varepsilon}_i(s) \to 0$), the logit type demand condition reduces to that of the homogeneous user (or deterministic demand) case,

$$\begin{cases} p(s) + (s - \tau_i(s)) = \rho_i & \text{if } q_i(s) > 0 \\ p(s) + (s - \tau_i(s)) \ge \rho_i & \text{if } q_i(s) = 0 \end{cases} \qquad \forall s \in \mathcal{S},\ \forall i \in \mathcal{N} \tag{3.13}$$

where ρ_i denotes the equilibrium disutility for a user with origin i.

3.2. One-to-many travel demand (evening-rush) problem

In the evening-rush problem, the corridor network defined in Section 2.1 is used by commuters in reverse. That is, there are N *off*-ramps (*destination* nodes) and a single *on*-ramp (*origin* node). These nodes are numbered sequentially from origin node 0 to the most distant off-ramp node N. Q_i commuters in residential location i *depart from the origin* during evening rush hour, and *exit* from the network at off-ramp i (i.e., one-to-many travel demand). Each commuter chooses the departure time of her trip from the workplace (origin) to home (destination) so as to minimize her disutility. The only difference in the assumptions of two problems is that, in the evening-rush, commuters are assumed to experience a schedule delay cost associated with deviation from *wished departure time from the origin* à la de Palma and Lindsey (2002). Under these assumptions, the one-to-many travel demand (evening-rush) problem describes a DUE distribution of *departures from the origin* such that no commuter could reduce her disutility by changing her *departure time*.

Now, let us formulate the equilibrium conditions for the one-to-many model in a Lagrangian-like coordinate system. In a similar manner to the many-to-one model, we can define the equilibrium *arrival* time $\sigma_i(s)$ at destination i for a user *departing from the origin* at time s. In this network topology, this is also equal to the *departure time from bottleneck* i for the user. The period $\mathcal{S} \equiv [\underline{S}, \overline{S}]$ of the origin departure time s is sufficiently long so that all demands can be served in $\mathcal{S}$. Letting $w_i(s)$ be the queuing delay at bottleneck i for this user, we see that $\boldsymbol{\sigma}(s)$ and $\mathbf{w}(s)$ should satisfy (see Fig.4)

$$\sigma_i(s) = \sigma_{i-1}(s) + (w_i(s) + c_i) \qquad \forall s \in \mathcal{S},\ \forall i \in \mathcal{N}. \tag{3.14a}$$

It follows from this and the boundary condition $\sigma_0(s) \equiv s$ (i.e., $\sigma_0(s)$ is defined as the departure time from the origin) that $\sigma_i(s)$ can be written as an explicit function of s and $\mathbf{w}(s)$:

$$\sigma_i(s) = s + \textstyle\sum_{j=1}^{i}(w_j(s) + c_j) \qquad \forall s \in \mathcal{S},\ \forall i \in \mathcal{N}. \tag{3.14b}$$

The arrival time $\tau_i(s)$ at bottleneck i for a user departing from the origin at time s also satisfies

$$\tau_i(s) = \sigma_i(s) - w_i(s) = \sigma_{i-1}(s) + c_i.$$

We also define $y_i(s)$ as the arrival flow rate at bottleneck i for a user departing from the origin at time s and $q_i(s)$ as the departure flow rate from the origin at time s.

We can now reformulate the equilibrium for the one-to-many model using these variables labeled by the *origin-departure-time* s. This formulation can be obtained by slightly modifying the equilibrium conditions for the many-to-one model, although the definitions (i.e., reference time) of the variables of the two models are different. Specifically, the queuing condition has exactly the same form as (3.8b). Additionally, for the flow conservations in the network, the condition for each OD pair and the condition at each bottleneck have exactly the same forms as (3.9b) and (3.10b), respectively. For the departure-time choice, we can obtain the equilibrium condition for the one-to-many model by replacing the travel time $(s - \tau_i(s))$ in (3.11) and (3.13) by $(\sigma_i(s) - s)$.

4. Preliminary analysis

This section presents a preliminary analysis for the deterministic demand (homogeneous users) case. Although it is a particular case of the general demand condition, the analysis for

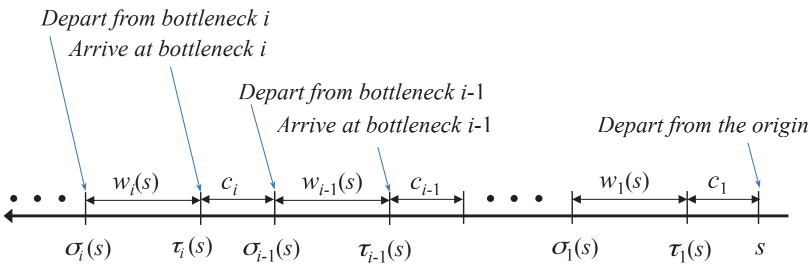

Figure 4: Arrival/departure time at bottlenecks for a user departing from the origin at time s

this case gives several new insights for understanding the mathematical structure of the problem and the properties of the basic components of the underlying problem (i.e., the queuing and the demand models).

4.1. Equivalent linear complementarity problems

In contrast to the previous section, we begin with analyzing the One-to-Many model (DUE-E), and show that the model can be cast into a standard linear complementarity problem. We then apply a similar analysis to the Many-to-One model (DUE-M), which reveals that DUE-M has a slightly more complicated structure than DUE-E.

The equilibrium problem [DUE-E] for the *One-to-Many* model with homogeneous users consists of five conditions (i.e., (3.14), (3.8b), (3.9b), (3.10b), and (3.13)) and associated unknowns $\{\mathbf{w}(s) = [w_1(s), \ldots, w_N(s)]^T, \boldsymbol{\tau}(s) = [\tau_1(s), \ldots, \tau_N(s)]^T, \mathbf{q}(s) = [q_1(s), \ldots, q_N(s)]^T, \mathbf{y}(s) = [y_1(s), \ldots, y_N(s)]^T :$ $s \in \mathcal{S}\}$ and $\boldsymbol{\rho} = [\rho_1, \ldots, \rho_N]^T$. However, they are somewhat redundant. Note that the flow conservation holds as an equality between $\mathbf{y}(s)$ and $\mathbf{q}(s)$, i.e.,

$$\mathbf{N}\mathbf{y}(s) = -\mathbf{q}(s).$$

Here, $\mathbf{N}$ denotes the reduced link-node incidence matrix for the corridor network, in which a row corresponding to the destination node is deleted, and is invertible. That is,

$$\mathbf{N} = \begin{bmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ & & & -1 \end{bmatrix} \Leftrightarrow \quad \mathbf{N}^{-1} = -\begin{bmatrix} 1 & \cdots & \cdots & 1 \\ & 1 & \cdots & 1 \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} = -\mathbf{L}^T,$$

where $\mathbf{L}$ is a lower triangular matrix with all non-zero entries equal to 1. This implies that $\mathbf{N}$ is an M-matrix, and hence, $\mathbf{y}(s)$ can be uniquely determined from $\mathbf{q}(s)$:

$$\mathbf{y}(s) = -\mathbf{N}^{-1}\mathbf{q}(s) = \mathbf{L}^T\mathbf{q}(s). \tag{4.1}$$

Note that the non-negativity condition $\mathbf{y}(s) \geq \mathbf{0}$ is "automatically" satisfied if $\mathbf{q}(s) \geq \mathbf{0}$. Hence, by substituting (4.1) into (3.8b), we can eliminate $\mathbf{y}(s)$ from the problem [DUE-E]. Similarly, the travel time $\hat{\boldsymbol{\sigma}}(s) \equiv \boldsymbol{\sigma}(s) - \mathbf{s}$ for each OD pair and the queuing delay $\mathbf{w}(s)$ at each bottleneck always satisfy

$$-\mathbf{N}^T\hat{\boldsymbol{\sigma}}(s) = \mathbf{w}(s) + \mathbf{c} \tag{4.2}$$

where $\boldsymbol{\sigma}(s) = [\sigma_1(s), \ldots, \sigma_N(s)]^T$, $\mathbf{1} = [1, \ldots, 1]^T$, $\mathbf{s} = s\mathbf{1}$, and $\mathbf{c} = [c_1, \ldots, c_N]^T$. Because $-\mathbf{N}^T$ is an M-matrix, $\hat{\boldsymbol{\sigma}}(s)$ can be uniquely determined from $\mathbf{w}(s)$:

$$\hat{\boldsymbol{\sigma}}(s) = -(\mathbf{N}^T)^{-1}(\mathbf{w}(s) + \mathbf{c}) = \mathbf{L}(\mathbf{w}(s) + \mathbf{c}), \tag{4.3}$$

where the non-negativity condition $\hat{\boldsymbol{\sigma}} \geq \mathbf{0}$ is satisfied if $\mathbf{w}(s) + \mathbf{c} \geq \mathbf{0}$. It follows that changes in the departure time from each bottleneck $\boldsymbol{\Delta\sigma}(s)$ can be represented by $\boldsymbol{\Delta}\mathbf{w}(s)$. That is,

$$\boldsymbol{\Delta\sigma}(s) = \mathbf{1} - (\mathbf{N}^T)^{-1}\boldsymbol{\Delta}\mathbf{w}(s) = \mathbf{1} + \mathbf{L}\boldsymbol{\Delta}\mathbf{w}(s) \tag{4.4}$$

where $\boldsymbol{\Delta\sigma}(s) = [\Delta\sigma_1(s), \ldots, \Delta\sigma_N(s)]^T$ and $\boldsymbol{\Delta}\mathbf{w}(s) = [\Delta w_1(s), \ldots, \Delta w_N(s)]^T$.

Thus, we can obtain a more concise expression for the problem [DUE-E] by eliminating these redundant variables $\{\mathbf{y}(s), \boldsymbol{\sigma}(s)\}$. This, together with the observation that [DUE-E] only consists of linear equations and linear complementarity conditions for the homogeneous users case, leads to the following proposition:

Proposition 4.1. *The arrival/departure-time choice equilibrium problem [DUE-E] for the corridor network is equivalent to the following linear complementarity problem (LCP):* Find $\{\mathbf{q}(s), \mathbf{w}(s) : s \in \mathcal{S}\}$ and $\boldsymbol{\rho}$ such that

$$\mathbf{0} \leq \mathbf{q}(s) \perp p(s)\mathbf{1} + \mathbf{L}(\mathbf{c} + \mathbf{w}(s)) - \boldsymbol{\rho} \geq \mathbf{0}, \qquad \forall s \in \mathcal{S} \tag{4.5a}$$

$$\mathbf{0} \leq \mathbf{w}(s) \perp \mathbf{C}\boldsymbol{\Delta\sigma}(s) - \mathbf{L}^T\mathbf{q}(s) \geq \mathbf{0}, \qquad \forall s \in \mathcal{S} \tag{4.5b}$$

$$\mathbf{0} \leq \boldsymbol{\rho} \perp \int_{\underline{S}}^{\overline{S}} \mathbf{q}(s)ds - \overline{\mathbf{Q}} \geq \mathbf{0} \tag{4.5c}$$

where $\boldsymbol{\Delta\sigma}(s) \equiv \mathbf{1} + \mathbf{L}\boldsymbol{\Delta}\mathbf{w}(s)$, $\boldsymbol{\sigma}(s) \equiv \mathbf{L}(\mathbf{w}(s) + \mathbf{c}) + \mathbf{s}$, $\mathbf{C} \in \mathbb{R}^{N\times N}$ is a diagonal matrix whose i-th diagonal element designates the capacity μ_i of bottleneck i, and $\overline{\mathbf{Q}} = [Q_1, \ldots, Q_N]^T$.

Proof. It is obvious that [DUE-E] implies the LCP above. Hence, it is sufficient for us to prove that the LCP implies [DUE-E]. The proof is by contradiction. Suppose, on the contrary, that there is a flow pattern $\{q_j(s)\}$ that satisfies the complementarity condition (4.5c) but does not satisfy the equality (3.9b) for some j. That is,

$$\rho_j \cdot (\int_{\underline{S}}^{\overline{S}} q_j(s)ds - Q_j) = 0, \ \rho_j \geq 0, \text{ and } \int_{\underline{S}}^{\overline{S}} q_j(s)ds - Q_j > 0.$$

It then follows that $\rho_j = 0$ and $q_j(s) > 0$ for this j and some s because $Q_j > 0$. For these particular s and j with positive flow $q_j(s) > 0$, the equilibrium condition (4.5a) reduces to $p(s) + [\mathbf{L}(\mathbf{w}(s) + \mathbf{c})]_j = 0$. However, $[\mathbf{L}(\mathbf{w}(s) + \mathbf{c})]_j = \sum_{i=1}^{j}(w_i(s) + c_i) > 0$ because $\mathbf{c} > \mathbf{0}$ and $\mathbf{w}(s) \geq \mathbf{0}$. This contradicts the assumption that $p(s) \geq 0$. **QED**.

In what follows, we restrict ourselves to the cases in which the arrival time at the destination s is discretized. That is, the underlying time period s is divided into a finite number of intervals, K, labeled $s \in \mathcal{S} \equiv \{1, 2, \ldots, K\}$, which have a finite duration δs. We use this restriction because rigorous discussion of existence and uniqueness issues for continuous time models would lead us into unnecessary mathematical complications, and such digression would obscure the essential structure of the model.

In accordance with the discretization of s, the time-dependent variables $\{\mathbf{q}(s) : s \in \mathcal{S}\}$, $\{\mathbf{w}(s) : s \in \mathcal{S}\}$, and $\{\Delta\mathbf{w}(s) : s \in \mathcal{S}\}$, and the schedule delay function $\{p(s) : s \in \mathcal{S}\}$ are represented as finite dimensional column vectors:

$$\mathbf{q} \equiv [\mathbf{q}(s)]_{s=1}^{K} \in \mathbb{R}^{NK},\ \mathbf{w} \equiv [\mathbf{w}(s)]_{s=1}^{K} \in \mathbb{R}^{NK},\ \Delta\mathbf{w} \equiv [\Delta\mathbf{w}(s)]_{s=1}^{K} \in \mathbb{R}^{NK},\ \text{and } \mathbf{p} \equiv [p(s)\mathbf{1}]_{s=1}^{K} \in \mathbb{R}^{NK}.$$

Here, $\Delta\mathbf{w}(k) \in \mathbb{R}^{N}$ *should be defined as the backward-difference*:

$$\Delta\mathbf{w}(s) \equiv \mathbf{w}(s) - \mathbf{w}(s-1) \ \text{ for } \ s = 1, \ldots, K, \text{ and } \Delta\mathbf{w}(0) \equiv \mathbf{0},$$

because the discretized formulation of the queuing condition (4.5b) should correspond to the discretization scheme (2.3d) of the point queue model in Section 2. Note that the variable $\Delta\mathbf{w} \in \mathbb{R}^{NK}$ defined above can be represented by $\Delta\mathbf{w} \equiv [\Delta_K \otimes \mathbf{I}]\mathbf{w}$, where $\otimes$ denotes the Kronecker product, $\mathbf{I}$ is the N by N identity matrix, and Δ_K is the K by K matrix representing the first order *backward difference* operator:

$$\Delta_K = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}.$$

In this discrete time setting, the Proposition 4.1 immediately implies the following result.

Proposition 4.2A. (One-to-Many model) *The DUE problem for the One-to-Many model in the discretized time period $\mathcal{S}$ can be represented as a standard finite dimensional linear complementarity problem:*

[DUE-E-LCP] Find $\mathbf{X} \equiv [\mathbf{q}, \mathbf{w}, \boldsymbol{\rho}]^T$ such that $\mathbf{0} \leq \mathbf{X} \perp \mathbf{F}(\mathbf{X}) \equiv \mathbf{M}\mathbf{X} + \mathbf{b} \geq \mathbf{0}$,

where $\mathbf{M} \in \mathbb{R}^{(2K+1)N \times (2K+1)N}$ and $\mathbf{b} \in \mathbb{R}^{(2K+1)N}$ are defined as

$$\mathbf{M} \equiv \left[\begin{array}{c:c:c} & \mathbf{I}_K \otimes \mathbf{L} & -\mathbf{1}_K \otimes \mathbf{I} \\ \hdashline -\mathbf{I}_K \otimes \mathbf{L}^T & \Delta_K \otimes \mathbf{CL} & \\ \hdashline \mathbf{1}_K^T \otimes \mathbf{I} & & \end{array}\right], \quad \mathbf{b} \equiv \left[\begin{array}{c} \mathbf{p} + \mathbf{1}_K \otimes \mathbf{Lc} \\ \hdashline \mathbf{1}_K \otimes \mathbf{C1} \\ \hdashline -\mathbf{Q} \end{array}\right], \tag{4.6}$$

and $\mathbf{I}_K \equiv \mathrm{diag}[1, \ldots, 1] \in \mathbb{R}^{K \times K}$, $\mathbf{1}_K \equiv [1, \ldots, 1]^T \in \mathbb{R}^{K}$, and $\mathbf{Q} \equiv [Q_1/\delta s, \ldots, Q_N/\delta s]^T$.

We then proceed to the analysis of the *Many-to-One* model, in which the reduced incidence matrix $\mathbf{N}$ and its inverse are given by

$$\mathbf{N} = \begin{bmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{N}^{-1} = \begin{bmatrix} 1 & \cdots & \cdots & 1 \\ & 1 & \cdots & 1 \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} = \mathbf{L}^T.$$

In exactly the same manner as the One-to-Many model, we can eliminate $\mathbf{y}(s)$ from the problem [DUE-M] using

$$\mathbf{y}(s) = \mathbf{N}^{-1}\mathbf{q}(s) = \mathbf{L}^T\mathbf{q}(s).$$

Similarly, the travel time $\hat{\boldsymbol{\tau}}(s) \equiv \mathbf{s} - \boldsymbol{\tau}(s)$ for each OD pair and the queuing delay $\mathbf{w}(s)$ at each bottleneck always satisfy

$$\mathbf{N}^T \hat{\boldsymbol{\tau}}(s) = \mathbf{w}(s) + \mathbf{c},$$

where $\boldsymbol{\tau}(s) = [\tau_1(s), \dots, \tau_N(s)]^T$. Because $\mathbf{N}^T$ is an M-matrix, $\hat{\boldsymbol{\tau}}(s)$ can be uniquely determined from $\mathbf{w}(s)$:

$$\hat{\boldsymbol{\tau}}(s) = (\mathbf{N}^T)^{-1}(\mathbf{w}(s) + \mathbf{c}) = \mathbf{L}(\mathbf{w}(s) + \mathbf{c}),$$

where the non-negativity condition $\hat{\boldsymbol{\tau}}(s) \geq \mathbf{0}$ is satisfied if $\mathbf{w}(s) + \mathbf{c} \geq \mathbf{0}$. It follows that changes to the departure time from each bottleneck $\boldsymbol{\Delta\sigma}(s)$ can also be represented by $\boldsymbol{\Delta}\mathbf{w}(s)$. That is,

$$\boldsymbol{\Delta\sigma}(s) \equiv \boldsymbol{\Delta\tau}(s) + \boldsymbol{\Delta}\mathbf{w}(s) = \mathbf{1} - \boldsymbol{\Delta}\hat{\boldsymbol{\tau}}(s) + \boldsymbol{\Delta}\mathbf{w}(s) = \mathbf{1} + [\mathbf{I} - \mathbf{L}]\boldsymbol{\Delta}\mathbf{w}(s)$$

where $\boldsymbol{\Delta\tau}(s) = [\Delta\tau_1(s), \dots, \Delta\tau_N(s)]^T$. Thus, by eliminating these redundant variables $\{\mathbf{y}(s), \boldsymbol{\tau}(s)\}$, we can represent the equilibrium conditions for the Many-to-One model as a linear complementarity problem that has exactly the same form as in Proposition 4.1. Similarly, in the discrete time setting, we obtain the linear complementarity problem:

[DUE-M(B)-LCP] Find $\mathbf{X} \equiv [\mathbf{q}, \mathbf{w}, \boldsymbol{\rho}]^T$ such that $\mathbf{0} \leq \mathbf{X} \perp \mathbf{F}(\mathbf{X}) \equiv \mathbf{M}\mathbf{X} + \mathbf{b} \geq \mathbf{0}$, where $\mathbf{M} \in \mathbb{R}^{(2K+1)N \times (2K+1)N}$ and $\mathbf{b} \in \mathbb{R}^{(2K+1)N}$ are defined as

$$\mathbf{M} \equiv \left[\begin{array}{c:c:c} & \mathbf{I}_K \otimes \mathbf{L} & -\mathbf{1}_K \otimes \mathbf{I} \\ \hdashline -\mathbf{I}_K \otimes \mathbf{L}^T & \boldsymbol{\Delta}_K \otimes \mathbf{C}[\mathbf{I} - \mathbf{L}] & \\ \hdashline \mathbf{1}_K^T \otimes \mathbf{I} & & \end{array}\right], \quad \mathbf{b} \equiv \left[\begin{array}{c} \mathbf{p} + \mathbf{1}_K \otimes \mathbf{L}\mathbf{c} \\ \hdashline \mathbf{1}_K \otimes \mathbf{C}\mathbf{1} \\ \hdashline -\mathbf{Q} \end{array}\right]. \tag{4.7}$$

Although this LCP is a natural and concise expression of the DUE for the Many-to-One model, some caution is needed. For the queuing model to be self-consistent, the cumulative arrival curve at each bottleneck should not be backward-bending. In our formulation, this condition is expressed as $\Delta\tau_i(s) \geq 0 \ \forall i \in \mathcal{N}, \forall s \in \mathcal{S}$, which trivially holds for the [DUE-E-LCP] of the One-to-Many model. Indeed, this is true because the queuing condition in the LCP ensures that $\boldsymbol{\Delta\sigma}(s) \geq \mathbf{C}^{-1}\mathbf{L}^T\mathbf{q}(s) \geq \mathbf{0}$ always hold, and $\boldsymbol{\Delta\tau}$ is given from the origin to destinations as

$$\Delta\sigma_0(s) \equiv \Delta\tau_1(s) \equiv 1, \quad \Delta\sigma_i(s) \equiv \Delta\tau_{i+1}(s) \equiv 1 + (\Delta w_1(s) + \cdots + \Delta w_i(s)) \quad \text{for } i = 1, \dots, N.$$

The queuing condition of [DUE-M(B)-LCP], however, does not ensure $\Delta\tau_N(s) \geq 0$ for the most upstream bottleneck. This is because $\boldsymbol{\Delta\tau}$ is defined from the destination to origins in a "*backward*" manner (i.e, the direction opposite to the direction of travel). That is,

$$\Delta\sigma_1(s) \equiv \Delta\tau_0(s) = 1, \quad \Delta\sigma_i(s) \equiv \Delta\tau_{i-1}(s) \equiv 1 - (\Delta w_1(s) + \cdots + \Delta w_i(s)) \quad \text{for } i = 2, \dots, N.$$

Since the complementarity queuing condition of [DUE-M(B)-LCP] ensures $\boldsymbol{\Delta\sigma}(s) \geq \mathbf{0}$, the condition $\Delta\tau_i(s) \geq 0$ holds for $i = 0, 1, \dots, N-1$, but *the condition for* $\Delta\tau_N(s)$ *is missing*. Thus, for the complete description of the problem, we should add a "consistency condition" to [DUE-M(B)-LCP]:

$$\Delta\tau_N(s) = 1 - (\Delta w_1(s) + \cdots + \Delta w_N(s)), \text{ and } \Delta\tau_N(s) \geq 0. \tag{4.8}$$

As an alternative to the "*backward type*" formulation in which information on $\Delta\tau_N(s)$ is missing from the complementarity queuing condition, we can also consider another ("*forward type*") type of formulation. The new formulation first represents the arrival time $\tau_i(s)$ at each bottleneck from the most upstream bottleneck to the destination in a forward direction,

$$\tau_i(s) = \tau_N(s) + (w_N(s) + \cdots + w_{i+1}(s)) \qquad \text{for } i = 0, 1, \dots, N-1.$$

This implies

$$\Delta\sigma_i(s) \equiv \Delta\tau_{i-1}(s) = \Delta\tau_N(s) + (\Delta w_N(s) + \cdots + \Delta w_i(s)) \qquad \forall i \in \mathcal{N}.$$

Substituting this into the queuing condition (4.5b), we can derive a forward type representation of the queuing model that explicitly includes $\Delta\tau_N(s)$.

$$\mathbf{0} \leq \mathbf{w}(s) \perp \mathbf{C}[\Delta\tau_N(s)\mathbf{1} + \mathbf{L}^T\mathbf{\Delta w}(s)] - \mathbf{L}^T\mathbf{q}(s) \geq \mathbf{0}.$$

Finally, we must have $\tau_0(s) = s$ and $\tau_N(s) \geq 0$ for this formulation to be consistent with the definition of the destination arrival time based variables. This reduces to

$$\Delta\tau_N(s) + (\Delta w_N(s) + \cdots + \Delta w_1(s)) = 1, \text{ and } \Delta\tau_N(s) \geq 0. \tag{4.9}$$

The following proposition summarizes the above discussion:

Proposition 4.2B. (Many-to-One model) *The DUE problem for the Many-to-One model in the discretized time period $\mathcal{S}$ can be represented as the following two types of complementarity problems:*

a) Backward type: [DUE-M(B)-LCP] and the consistency condition (4.8)*.*

b) Forward type: [DUE-M(F)-MCP] and the consistency condition $\mathbf{\Delta\tau}_N \geq \mathbf{0}$

[DUE-M(F)-MCP] Find $\mathbf{X} \equiv [\mathbf{q}, \mathbf{w}, \boldsymbol{\rho}]^T$ and $\mathbf{\Delta\tau}_N$ such that

$\mathbf{0} \leq \mathbf{X} \perp \mathbf{F}(\mathbf{X}) \equiv \mathbf{MX} + \mathbf{b} \geq \mathbf{0},$ and $\mathbf{\Delta\tau}_N + (\mathbf{\Delta w}_N + \cdots + \mathbf{\Delta w}_1) = \mathbf{1}$

where $\mathbf{M} \in \mathbb{R}^{(2K+1)N\times(2K+1)N}$ and $\mathbf{b} \in \mathbb{R}^{(2K+1)N}$ are defined as

$$\mathbf{M} \equiv \left[\begin{array}{c:c:c} & \mathbf{I}_K \otimes \mathbf{L} & -\mathbf{1}_K \otimes \mathbf{I} \\ \hdashline -\mathbf{I}_K \otimes \mathbf{L}^T & \Delta_K \otimes \mathbf{CL}^T & \\ \hdashline \mathbf{1}_K^T \otimes \mathbf{I} & & \end{array}\right], \quad \mathbf{b} \equiv \left[\begin{array}{c} \mathbf{p} + \mathbf{1}_K \otimes \mathbf{Lc} \\ \hdashline \mathbf{\Delta\hat{\tau}}_N \\ \hdashline -\mathbf{Q} \end{array}\right], \tag{4.10}$$

and $\mathbf{\Delta\hat{\tau}}_N \equiv [\mathbf{C\Delta\hat{\tau}}_N(1)\mathbf{1}, \dots, \mathbf{C\Delta\hat{\tau}}_N(K)\mathbf{1}]^T \in \mathbb{R}^{NK}$.

4.2. Connections with dynamic system optimal assignment

In the equivalent LCPs in Propositions 4.2A and 4.2B, the matrix $\mathbf{M}$ has a skew-symmetric structure. Based on this property, we can convert the DUE problem into a variational inequality problem with only cost variables $(\mathbf{w}(s), \boldsymbol{\rho})$.

Proposition 4.3. *The problem [DUE-M(B)-LCP] is equivalent to the following variational inequality problem (VIP):*

[DUE-VIP] Find $(\mathbf{w}^*(s), \boldsymbol{\rho}^*) \in \Omega$ such that

$$\textstyle\sum_{s=1}^K (\mathbf{w}(s) - \mathbf{w}^*(s)) \cdot \mathbf{\Delta\sigma}^*(s) - (\boldsymbol{\rho} - \boldsymbol{\rho}^*) \cdot \mathbf{Q} \geq 0 \qquad \forall (\mathbf{w}(s), \boldsymbol{\rho}) \in \Omega$$

where $\mathbf{\Delta\sigma}^*(s) \equiv \mathbf{1} + [\mathbf{I} - \mathbf{L}]\mathbf{\Delta w}^*(s),$

$$\Omega \equiv \{(\mathbf{w}(s), \boldsymbol{\rho}) \geq \mathbf{0} : p(s)\mathbf{1} + \mathbf{L}(\mathbf{w}(s) + \mathbf{c}) - \boldsymbol{\rho} \geq \mathbf{0}\ \forall s \in \mathcal{S}\}.$$

It is clear from the Proposition 4.3 that the above variational inequality problem can be associated with linear programming (LP) problems:

Corollary 4.3. *If the value of* $\{\Delta\sigma^*(s) : s \in \mathcal{S}\}$ *at equilibrium is known in advance, the problem [DUE-VIP] reduces to the following parametric linear programming problem [DUE-PLP-D]:*

$$\min_{(\mathbf{w},\rho)\geq\mathbf{0}} .F_{PLP-D} \equiv \textstyle\sum_{s=1}^{K} \mathbf{w}(s)\cdot \mathbf{C}\Delta\sigma^*(s) - \rho\cdot\mathbf{Q} \quad \text{subject to} \quad p(s)\mathbf{1} + \mathbf{L}(\mathbf{w}(s)+\mathbf{c}) - \rho \geq \mathbf{0} \qquad \forall s \in \mathcal{S}$$

and its dual problem [DUE-PLP-P] is given by

$$\min_{\mathbf{q}\geq\mathbf{0}} .F_{PLP-P} \equiv \textstyle\sum_{s=1}^{K} (p(s)\mathbf{1} + \mathbf{Lc})\cdot\mathbf{q}(s)$$

$$\text{subject to} \qquad \mathbf{C}\Delta\sigma^*(s) - \mathbf{L}^T\mathbf{q}(s) \geq \mathbf{0} \quad \forall s \in \mathcal{S}, \text{ and } \textstyle\sum_{s=1}^{K}\mathbf{q}(s) - \mathbf{Q} = \mathbf{0}.$$

A few remarks are in order. First, for the special case of $N = 1$ (i.e., a network with a single bottleneck) for the Many-to-One model, the equivalent variational inequality problem in Proposition 4.3 always reduces to the LP in Corollary 4.3 without any assumptions on $\Delta\sigma^*$. Specifically, the vector $\Delta\sigma^*$ for this case reduces to $\Delta\sigma_1(s) \equiv d\tau_0(s)/ds$, and $\Delta\sigma_1(s) \equiv 1$ always holds because $\sigma_1(s) \equiv \tau_0(s) \equiv s$, by definition. It follows that the equilibrium solution of the problem [DUE] for $N = 1$ is obtained by simply solving the LP with unknowns $\{w_1(s)\}$ and ρ_1. This result is consistent with the finding of Iryo and Yoshii (2007). Second, note that [DUE-PLP-P] can be interpreted as a particular type of dynamic system optimal assignment. The OD flow pattern $\{\mathbf{q}(s)\}$ is controlled so as to minimize the total disutility of the commuters in the network, which consists of the total schedule delay $p(s)\mathbf{1}\cdot\mathbf{q}(s)$ and the total free flow travel time $\mathbf{Lc}\cdot\mathbf{q}(s)$. The constraints are the physical conditions that dynamic flows in the corridor network should satisfy. The second constraint is the conservation of the OD demand. The first constraint concerns the traffic capacity, and prevents the queues build up at each bottleneck. We can interpret this as the inflow rate $\mathbf{L}^T\mathbf{q}(s) = \mathbf{y}(s)$ is always less than $\mathbf{C}\Delta\sigma^*(s)$, *so the capacity is given by* $\mathbf{C}\Delta\sigma^*(s)$ *rather than the actual capacity* $\mathbf{C1}$. Finally, the solution $\{\mathbf{q}(s)\}$ of the problem [DUE-PLP-P] is not necessarily unique and can be a convex set, which implies that the equilibrium OD flow pattern for the DUE problem with the deterministic demand model (i.e., the solutions of the LCPs in Propositions 4.2A and 4.2B) is not necessarily unique, even if the equilibrium cost pattern $\{\mathbf{w}(s)\}$ is unique.

4.3. Properties of the queuing models

For the One-to-Many model, the queuing sub-model is given by

$$\mathbf{0} \leq \mathbf{w}(s) \perp \mathbf{C}\Delta\sigma(s) - \mathbf{L}^T\mathbf{q}(s) \geq \mathbf{0} \qquad\qquad \forall s \in \mathcal{S}$$

where $\Delta\sigma(s) \equiv \mathbf{1} + \mathbf{L}\Delta\mathbf{w}(s)$. This can be interpreted as an implicit functional relationship $\mathbf{w}(\mathbf{q})$ from $\mathbf{q}$ to $\mathbf{w}$. More specifically, suppose that an OD flow pattern $\mathbf{q} \in \mathbb{R}^{NK}$ is given. Then, the corresponding queuing delay pattern $\mathbf{w} \in \mathbb{R}^{NK}$ is given as the solution of the following LCP:

$$\textbf{[PQ-E]} \qquad \mathbf{0} \leq \mathbf{w} \perp [\Delta_K \otimes \mathbf{CL}]\mathbf{w} + \mathbf{b} \geq \mathbf{0}, \text{ where } \mathbf{b} \equiv \mathbf{1}_K \otimes \mathbf{C1} - [\mathbf{I}_K \otimes \mathbf{L}^T]\mathbf{q}.$$

By inspecting this LCP, we see that the function $\mathbf{w}(\mathbf{q})$ has the following useful property.

Proposition 4.4A. (One-to-Many queuing model) *For a given OD flow vector* $\mathbf{q} \in \mathbb{R}^{NK}$*, the solution* $\mathbf{w} \in \mathbb{R}^{NK}$ *(i.e., the queuing delays) of the problems [PQ-E] is uniquely determined. Furthermore, the solution* $\mathbf{w}(\mathbf{q})$ *is Lipschitz continuous with respect to changes in* $\mathbf{q}$.

Proof. It is well known in the theory of linear complementarity problems that if $\mathbf{M}$ is a P-matrix then the LCP $\mathbf{0} \leq \mathbf{X} \perp \mathbf{MX}+\mathbf{b} \geq \mathbf{0}$ has a unique solution, and the solution is Lipschitz continuous with respect to changes in $\mathbf{b}$ (Mangasarian and Shiau, 1987). Therefore, it suffices for us to prove that $\mathbf{M} \equiv [\mathbf{\Delta}_K \otimes \mathbf{CL}]$ is a P-matrix. Note that $\mathbf{M}$ is a lower triangular matrix, in which all diagonal entries $\{\mu_i\}$ are positive. This implies that all the eigenvalues of $\mathbf{M}$ and its principal submatrices are positive, and hence, $\mathbf{M}$ is a P-matrix. **QED**.

Similarly, we can analyze the properties of the queuing condition for the *Many-to-One* model, which has two types of formulation (i.e., backward and forward). For each formulation, $\mathbf{\Delta}\boldsymbol{\sigma}(s)$ is defined as

$$\begin{aligned} &\text{Backward type:} \quad \mathbf{\Delta}\boldsymbol{\sigma}(s) \equiv \mathbf{1} + [\mathbf{I} - \mathbf{L}]\mathbf{\Delta w}(s) \\ &\text{Forward type:} \quad \mathbf{\Delta}\boldsymbol{\sigma}(s) \equiv \Delta\tau_N(s)\mathbf{1} + \mathbf{L}^T\mathbf{\Delta w}(s). \end{aligned}$$

Thus, in the backward type formulation, the queuing delay pattern $\mathbf{w}$ for a given OD flow pattern $\mathbf{q}$ is the solution to the following LCP:

$$\textbf{[PQ-M(B)]} \qquad \mathbf{0} \leq \mathbf{w} \perp [\mathbf{\Delta}_K \otimes \mathbf{C}[\mathbf{I} - \mathbf{L}]]\mathbf{w} + \mathbf{b} \geq \mathbf{0}, \text{ where } \mathbf{b} \equiv \mathbf{1}_K \otimes \mathbf{C1} - [\mathbf{I}_K \otimes \mathbf{L}^T]\mathbf{q}.$$

For the forward type formulation, suppose that an OD flow pattern $\mathbf{q}$ and a parameter vector $\mathbf{\Delta}\boldsymbol{\tau}_N$ for the consistency condition are given. Then, the corresponding queuing delay pattern $\mathbf{w}$ is the solution to the following LCP:

$$\textbf{[PQ-M(F)]} \qquad \mathbf{0} \leq \mathbf{w} \perp [\mathbf{\Delta}_K \otimes \mathbf{CL}^T]\mathbf{w} + \mathbf{b} \geq \mathbf{0}, \text{ where } \mathbf{b} \equiv \mathbf{\Delta}\hat{\boldsymbol{\tau}}_N - [\mathbf{I}_K \otimes \mathbf{L}^T]\mathbf{q}.$$

From these LCP representations, we have the following proposition concerning the properties of the Many-to-One queuing model.

Proposition 4.4B. (Many-to-One queuing model) *For a given OD flow vector $\mathbf{q} \in \mathbb{R}^{NK}$ and a parameter vector $\mathbf{\Delta}\boldsymbol{\tau}_N \in \mathbb{R}^K$ for the consistency condition of the forward-type formulation, the solution $\mathbf{w} \in \mathbb{R}^{NK}$ (i.e., the queuing delays) of the problem [PQ-M(F)] is uniquely determined, whereas the solution to the problem [PQ-M(B)] is not necessarily unique. Furthermore, the solution $\mathbf{w} \in \mathbb{R}^{NK}$ to the problem [PQ-M(F)] is Lipschitz continuous with respect to changes in $\mathbf{q}$ and $\mathbf{\Delta}\boldsymbol{\tau}_N$.*

Proof. As in the proof of Proposition 4.4A, it suffices to show that the LCP mapping $\mathbf{M}$ for [PQ-M(F)] is a P-matrix, and that it is not for [PQ-M(B)]. For the problem [PQ-M(F)], $\mathbf{M} \equiv [\mathbf{\Delta}_K \otimes \mathbf{CL}^T]$ is a block lower triangular matrix in which each diagonal block $\mathbf{CL}^T$ is an upper triangular matrix with positive diagonal entries $\{\mu_i\}$. This implies that all the eigenvalues of $\mathbf{M}$ and its principal submatrices are positive, and hence, that $\mathbf{M}$ is a P-matrix. For the problem [PQ-M(B)], $\mathbf{M} \equiv [\mathbf{\Delta}_K \otimes \mathbf{C}[\mathbf{I} - \mathbf{L}]]$ is a lower triangular matrix with diagonal entries equal to zero, which implies that all the eigenvalues of $\mathbf{M}$ are zero, and hence, that $\mathbf{M}$ is a P_0-matrix but is not a P-matrix. **QED**.

A few remarks are in order here. First, Propositions 4.4A and 4.4B state some positive aspects (uniqueness and Lipschitz continuity) of the function $\mathbf{w}(\mathbf{q})$, but they do not imply stronger properties such as monotonicity (that has been the abused assumption in the dynamic traffic assignment literature). Indeed, the function $\mathbf{w}(\mathbf{q})$ derived from the queuing model is not generally monotone. Second, these propositions show that the Many-to-One model is not exactly the reverse of the One-to-Many model. Indeed, the One-to-Many model always yields a unique cost pattern $\mathbf{w}(\mathbf{q})$ for a given flow pattern $\mathbf{q}$, but the Many-to-One model cannot give such a tight relationship between $\mathbf{q}$ and $\mathbf{w}$. Although the forward type formulation [PQ-M(F)] admits a unique solution, it requires information on not only the flow pattern $\mathbf{q}$ but also $\boldsymbol{\Delta\tau}_N$ for the most upstream bottleneck.

4.4. Non-monotonicity of the LCP mapping

Propositions 4.4A and 4.4B reveal that the matrices $\boldsymbol{\Delta}_K \otimes \mathbf{CL}$ and $\boldsymbol{\Delta}_K \otimes \mathbf{CL}^T$ for the queuing sub-model are P-matrices, so it is reasonable to expect that the matrices $\mathbf{M}$ for the overall equilibrium conditions [DUE-E-LCP] or [DUE-M-LCP] also have such a useful property. The following proposition, however, reveals that this conjecture is not true.

Proposition 4.5. *Each mapping* $\mathbf{F}(\mathbf{X}) \equiv \mathbf{MX} + \mathbf{b}$ *of the LCPs, [DUE-E-LCP] and [DUE-M-LCP], in Propositions 4.2A and 4.2B is not monotone. Furthermore,*
a) the One-to-Many model: $\mathbf{F}(\mathbf{X})$ *is not a P-function in general,*
b) the Many-to-One model: $\mathbf{F}(\mathbf{X})$ *is not a* P_0*-function.*

Proof. If $\mathbf{F}(\mathbf{X})$ is monotone, then it implies that $\mathbf{F}(\mathbf{X})$ is a P-function (but the converse is not true). Hence, it is sufficient to show a counter example for the matrix $\mathbf{M}$ being P-matrix (or P_0-matrix). For this, we show analytically that some of the principal minors of $\mathbf{M}$ can be negative even in a small sized problem of $N = K = 3$.
a) the One-to-Many model: for $\overline{\alpha} = \{5, 7, 12, 14, 16\}$, $\det(M(\alpha)) = (2\mu_1 - \mu_2)\mu_2 < 0$ if $\mu_2 > 2\mu_1$
b) the Many-to-One model:

$$\text{(Backward type)} \quad \text{for } \overline{\alpha} = \{3, 4, 8, 12, 14, 18, 21\}, \det(M(\alpha)) = -\mu_2^2 < 0$$
$$\text{(Forward type)} \quad \text{for } \overline{\alpha} = \{2, 4, 9, 12, 14, 18, 21\}, \det(M(\alpha)) = -\mu_2^2 < 0$$

where $\alpha \subseteq \mathcal{S} \equiv \{1, \ldots, K\}$ denotes an index set of the matrix $\mathbf{M}$, $\overline{\alpha}$ is the complement of α, and $M(\alpha) = [M_{ij}]$ denotes a principal submatrix of $\mathbf{M}$ in which $i \in \alpha$ and $j \in \alpha$. **QED**.

The above proposition implies that it is very difficult to prove the uniqueness of the equilibria of [DUE-E] or [DUE-M] for the homogeneous user case. One reason for this difficulty seems to lie in the demand condition, in which $\mathbf{q}$ is not a smooth function of $\mathbf{w}$ but is just an upper hemi-continuous correspondence (recall that [DUE-PLP-P] in Corollary 4.3 does not necessarily have a unique solution). To obtain sharper results, we will examine the uniqueness of the equilibria for a generalized case of heterogeneous users, which we describe in the next section.

5. Existence and uniqueness

5.1. Existence

To establish the existence of equilibria for the problem [DUE], we characterize the equilibrium as a fixed point problem consisting of the demand-side and supply-side conditions discussed in the previous sections. The demand side condition for the problem [DUE] can be generally represented for both cases of the deterministic demand (or homogeneous users) and the stochastic demand (or heterogeneous users) as follows.

$$\mathbf{q} = D(\mathbf{w}) \tag{5.1}$$

where $\boldsymbol{D}(\mathbf{w})$ is the set of time-dependent OD flow patterns that can be generated from the departure time choice model when the queuing delay pattern is $\mathbf{w}$. Specifically, the demand function $\boldsymbol{D} : Y \to X$ is defined in (3.11) for the heterogeneous users case, and is defined in the complementarity condition (3.13) for the homogeneous users case. X denotes the set of feasible OD flow patterns generated from a total OD demand $\mathbf{Q}$, which is defined as $X \equiv \{\mathbf{q} : \sum_{s=1}^{K} \mathbf{q}(s) = \mathbf{Q}, \text{ and } \mathbf{q} \geq \mathbf{0}\}$. The set X is non-empty, compact and convex for finite $\mathbf{Q}$. Y is the set of queuing delay patterns $\mathbf{w}$ determined from the supply side (queuing) condition mentioned below.

The supply side condition and the equilibrium of the problem [DUE] should be described separately for the One-to-Many model and the Many-to-One model because the two models have subtle differences (as discussed in the previous section).

(*A*) *One-to-Many model*

For the One-to-Many model, the supply side condition can be generally expressed as

$$\mathbf{w} = \boldsymbol{W}(\mathbf{q}). \tag{5.2}$$

As shown in Proposition 4.4A, the function $\boldsymbol{W} : X \to Y$ is defined as the solution of the linear complementarity problems [PQ-E] for a given OD flow pattern $\mathbf{q}$ is given. It is a many-to-one mapping and is Lipschitz continuous with respect to changes in $\mathbf{q}$. This, combined with the boundedness and convexity of the set X implies that Y is a non-empty, compact set. Combining (5.1) and (5.2), the equilibrium flow pattern $\mathbf{q}^*$ of [DUE-E] for the One-to-Many model can be defined as the solution of the following fixed point problem:

$$\mathbf{q}^* = \boldsymbol{D}(\boldsymbol{W}(\mathbf{q}^*)) \in X \tag{5.3}$$

and the equilibrium queuing delay patterns $\mathbf{w}^*$ are given by $\boldsymbol{W}(\mathbf{q}^*)$.

By applying Kakutani's fixed point theorem (Kakutani, 1941), we have the following existence result:

Theorem 5.1A. *For the One-to-Many DUE model, the equilibrium* $(\mathbf{q}^*, \mathbf{w}^*)$ *defined in* (5.3) *exists for both cases of heterogeneous users and homogeneous users.*

Proof. The mapping $\boldsymbol{D}(\mathbf{w})$ is (*a*) upper hemi-continuous (for the proof, see for example, Daganzo, 1983; Mas-Colell et al., 1995) for the homogeneous users case, and (*b*) continuous for the heterogeneous users case. It sends every $\mathbf{w}$ into a closed and convex set in X. Because the mapping $\boldsymbol{W}(\mathbf{q})$ is Lipschitz continuous, the composition $\boldsymbol{D}(\boldsymbol{W}(\mathbf{q}))$ of $\boldsymbol{D}(\mathbf{w})$ and $\boldsymbol{W}(\mathbf{q})$ shares the properties $\boldsymbol{D}(\mathbf{w})$ for all $\mathbf{q}$ for which $\boldsymbol{W}(\mathbf{q})$ is Lipschitz continuous. Thus, (5.3) defines an upper hemi-continuous mappings within X that is non-empty, compact and convex set. Then, Kakutani's fixed point theorem guarantees the existence of $\mathbf{q}^*$. **QED**.

(*B*) *Many-to-One model*

For the Many-to-One model, we can similarly define the supply side condition as

$$\mathbf{w} = \boldsymbol{W}(\mathbf{q}, \boldsymbol{\Delta\tau}_N) \tag{5.4}$$

where we employ the forward type formulation (or the LCP [PQ-M(F)]) discussed in 4.3, and $\boldsymbol{\Delta\tau}_N$ should satisfy (4.9), which can be written as

$$\boldsymbol{\Delta\tau}_N = \boldsymbol{L}(\mathbf{w}) \equiv \mathbf{1} - \textstyle\sum_{i=1}^{N} \boldsymbol{\Delta}\mathbf{w}_i. \tag{5.5}$$

Thus, the equilibrium for the Many-to-One model can be defined as the following fixed point problem with respect to $\hat{\mathbf{q}} = [\mathbf{q}, \boldsymbol{\Delta\tau}_N]$:

$$\hat{\mathbf{q}}^* = [\boldsymbol{D}(\boldsymbol{W}(\hat{\mathbf{q}}^*)), \boldsymbol{L}(\boldsymbol{W}(\hat{\mathbf{q}}^*))] \in \hat{X} \tag{5.6}$$

and the equilibrium queuing delay pattern $\mathbf{w}^*$ is given by $\boldsymbol{W}(\hat{\mathbf{q}}^*)$, where $\hat{X} = X \times T$, and T denotes the feasible set of $\boldsymbol{\Delta\tau}_N$, which is defined as $T = \{\boldsymbol{\Delta\tau}_N : \mathbf{0} \leq \boldsymbol{\Delta\tau}_N \leq \mathbf{1}\beta\}$ for some finite constant β.

The assumption that the set T is bounded from above (i.e., $\boldsymbol{\Delta\tau}_N \leq \mathbf{1}\beta$) is always satisfied for the queuing model (and finite $\mathbf{Q}$) in this paper.

Proposition 5.1. *For a finite OD demand pattern $\mathbf{Q}$ and a positive capacity pattern $\boldsymbol{\mu} \neq \mathbf{0}$, the upper bound of $\boldsymbol{\Delta\tau}_N$ is at most $1 + \sum_{i=1}^{N}\sum_{j=N}^{i} Q_j/\mu_i$.*

Proof. From the definition of the queuing model, we have

$$\Delta\tau_i(s) = \Delta\tau_{i-1}(s) - \Delta w_i(s) \text{ and } -\Delta w_i(s) = w_i(s-1) - w_i(s) \leq w_i(s-1) \leq \textstyle\sum_{j=N}^{i} Q_j/\mu_i$$

This implies $\Delta\tau_i(s) \leq \Delta\tau_{i-1}(s) + \sum_{j=N}^{i} Q_j/\mu_i$ for $i = N, \ldots, 1$. "Solving" the recursive inequalities in backward direction from $i = N$ to 1, we obtain

$$\begin{aligned}\Delta\tau_N(s) &\leq \Delta\tau_{N-1}(s) + Q_N/\mu_N \leq \Delta\tau_{N-2}(s) + Q_N/\mu_N + \textstyle\sum_{j=N}^{N-1} Q_j/\mu_{N-1} \\ &\leq \cdots \leq \Delta\tau_0(s) + \textstyle\sum_{i=1}^{N}\sum_{j=N}^{i} Q_j/\mu_i = 1 + \sum_{i=1}^{N}\sum_{j=N}^{i} Q_j/\mu_i \leq \beta \qquad \forall s \in \mathcal{S}.\end{aligned}$$ **QED.**

For the lower bound of the set T (or the consistency condition $\boldsymbol{\Delta\tau}_N \geq \mathbf{0}$), we require the following assumption regarding the schedule delay function.

Assumption 5.1B. *There exists an appropriately defined schedule delay function $p(s)$ for which the equilibrium of (5.6) always satisfies $\Delta\tau_N(s) \geq 0 \; \forall s \in \mathcal{S}$.*

Although it is difficult to give a necessary and sufficient condition for this assumption to be true in the problem with general demand functions, we can provide a sufficient condition for some typical demand functions.

Proposition 5.2. *Suppose that the schedule delay function $p(s)$ satisfies the following condition*

$$\text{for (a) deterministic demand function case: } \Delta p(s) \geq -1 \qquad \forall s \in \mathcal{S}, \tag{5.7a}$$

$$\text{for (b) logit demand function case: } \Delta p(s) \geq -1 + (1/\theta) \qquad \forall s \in \mathcal{S}. \tag{5.7b}$$

Then, Assumption 5.1B always holds at equilibrium.

Proof. The logit demand model always yields an interior solution with respect to $\mathbf{q}$, so the equilibrium condition for users with origin N can be written as

$$(s - \tau_N(s)) + p(s) + (1/\theta)\ln q_N(s) - \rho_N = 0. \qquad \forall s \in \mathcal{S}$$

This implies that

$$\Delta\tau_N(s) = 1 + \Delta p(s) + \epsilon(s) \qquad \forall s \in \mathcal{S} \tag{5.8}$$

$$\text{where}\quad \epsilon(s) \equiv \frac{1}{\theta}\ln\left(\frac{q_N(s)}{q_N(s-1)}\right) = \frac{1}{\theta}\ln\left(1+\frac{\Delta q_N(s)}{q_N(s-1)}\right) \approx \frac{1}{\theta}\frac{\Delta q_N(s)}{q_N(s-1)} = \frac{1}{\theta}\left(\frac{q_N(s)}{q_N(s-1)}-1\right) > -\frac{1}{\theta}.$$

Taking the limit as $\theta \to +\infty$ of (5.8), we can also obtain $\Delta\tau_N(s) = 1+\Delta p(s)$ for the deterministic demand case. Thus, the condition (5.7) implies $\Delta\tau_N(s) \geq 0$. **QED**.

Based on the fixed point problem (5.6), we can obtain the following existence result for the Many-to-One model:

Theorem 5.1B. *For the Many-to-One DUE model, if Assumption 5.1B holds, the equilibrium* $(\hat{\mathbf{q}}^*, \mathbf{w}^*)$ *defined in* (5.6) *exists for both cases of heterogeneous users and homogeneous users.*

Proof. The proof is almost the same as that of the One-to-Many model. The difference is that $\boldsymbol{W}(\mathbf{q})$ is replaced with $\boldsymbol{W}(\hat{\mathbf{q}})$, and the mapping $\boldsymbol{L}(\boldsymbol{W}(\hat{\mathbf{q}}))$ is introduced. The mapping $\boldsymbol{W}(\hat{\mathbf{q}})$ is Lipschitz continuous with respect to $\hat{\mathbf{q}}$ because $\boldsymbol{\Delta\tau}_N$ and $\mathbf{q}$ are linear terms constituting the constant vector $\mathbf{b}$ of the LCP [PQ-M(F)] (see Propositon4.4B). Using this, and because $\boldsymbol{L}(\mathbf{w})$ is a linear mapping of $\mathbf{w}$, we find that the composition mapping $\boldsymbol{L}(\boldsymbol{W}(\hat{\mathbf{q}}))$ is Lipschitz continuous with respect to $\hat{\mathbf{q}}$. It also follows that $(\boldsymbol{D}(\boldsymbol{W}(\hat{\mathbf{q}})), \boldsymbol{L}(\boldsymbol{W}(\hat{\mathbf{q}})))$ is a mapping from $\hat{X}$ to $\hat{X}$ itself, because Proposition 5.1 and Assumption 5.1B imply that $\boldsymbol{L}(\boldsymbol{W}(\hat{\mathbf{q}}))$ sends every $\hat{\mathbf{q}}$ into the closed and convex set T. Thus, (5.6) defines an upper hemi-continuous mapping within $\hat{X}$ that is non-empty, compact and convex set. Then, Kakutani's fixed point theorem guarantees the existence of $\hat{\mathbf{q}}^*$. **QED**.

The following remarks conclude this subsection. First, condition (5.7a) in the Many-to-One model is consistent with the sufficient condition for the existence of equilibria in Vickrey's bottleneck model (Smith, 1984). This condition limits the decreasing rate of schedule cost function and prevents the cumulative arrival curve at the most upstream bottleneck from being backward-bending. Second, for the One-to-Many model, we do not need a similar condition on the schedule cost to guarantee the existence of equilibria. The reason for this difference in the two models can be clearly understood from structures of queuing models discussed in Section 4.3. It is also consistent with the finding of de Palma and Lindsey (2002) for a single bottleneck.

5.2. Uniqueness

As shown in the previous section, it is difficult to prove the uniqueness of the solution of [DUE] for the homogeneous users case because the equivalent LCPs do not have monotonicity or P-property. In this subsection, we examine the uniqueness of the equilibria for the heterogeneous users case. Specifically, we assume that the demand condition is given by (3.11), which is based on the random utility theory. Note that we do not impose any particular type of functional form (such as logit or probit) for the choice function in the analysis below.

For the convenience of exposition, we slightly change the notation in this subsection: $\mathbf{q}_i \in \mathbb{R}^K$ and $\mathbf{w}_i \in \mathbb{R}^K$ $(i = 1, \ldots, N)$ are column vectors with elements $q_i(s)$ and $w_i(s)$ $(s = 1, \ldots, K)$, respectively; $\mathbf{q} \in \mathbb{R}^{NK}$ and $\mathbf{w} \in \mathbb{R}^{NK}$ denote column vectors with block elements $\mathbf{q}_i$ and $\mathbf{w}_i$ $(i = 1, \ldots, N)$, respectively.

(A) One-to-Many model

The demand-side condition for the heterogeneous users defined in (3.11) is represented as $\mathbf{q}_i = \mathbf{Q}_i\mathbf{P}_i(\mathbf{v}_i(\mathbf{w}))$, where $\mathbf{v}_i(\mathbf{w})$ is a vector (for i th origin) with elements $v_i(s) \equiv \sum_{j=1}^{i}(w_j(s) + c_j) + p(s)$ $(s = 1, \ldots, K)$, $\mathbf{P}_i : \mathbb{R}^{NK} \to \mathbb{R}^K$ is a column vector $[P_i(1), \ldots, P_i(K)]^T$ of the choice probability function $P_i(s) : \mathbb{R}^{NK} \to \mathbb{R}$, which is defined in (3.11b). This can be written as the following complementarity condition:

$$\mathbf{0} \leq \mathbf{q} \perp \mathbf{q} - \mathbf{Q}\mathbf{P}(\mathbf{w}) \geq \mathbf{0} \tag{5.9}$$

where $\mathbf{P} : \mathbb{R}^{NK} \to \mathbb{R}^{NK}$ is a column vector $[\mathbf{P}_1, \ldots, \mathbf{P}_N]^T$, and $\mathbf{Q} \equiv \mathrm{diag}[Q_1\mathbf{I}_K, \ldots, Q_N\mathbf{I}_K] \in \mathbb{R}^{NK \times NK}$. The supply-side (queuing) condition for the One-to-Many model can be written as

$$\mathbf{0} \leq \mathbf{w} \perp [\mathbf{CL} \otimes \boldsymbol{\Delta}_K]\mathbf{w} + \mathbf{C1} \otimes \mathbf{1}_K - [\mathbf{L}^T \otimes \mathbf{I}_K]\mathbf{q} \geq \mathbf{0}. \tag{5.10}$$

Thus, we can put the demand-supply equilibrium condition in a single nonlinear complementarity problem (NCP).

Proposition 5.3A. (One-to-Many model) *The problem [DUE-E] for the One-to-Many model with heterogeneous users is equivalent to the following NCP, and has a solution.*

[DUE-E-NCP]*:* Find $\mathbf{X} \equiv [\mathbf{q}, \mathbf{w}]^T \in \mathbb{R}^{2NK}$ such that

$$\mathbf{0} \leq \mathbf{X} \perp \mathbf{G}(\mathbf{X}) \equiv \left[\begin{array}{c:c} \mathbf{I} \otimes \mathbf{I}_K & \mathbf{0} \\ \hdashline -\mathbf{L}^T \otimes \mathbf{I}_K & \mathbf{CL} \otimes \boldsymbol{\Delta}_K \end{array}\right] \begin{bmatrix} \mathbf{q} \\ \mathbf{w} \end{bmatrix} + \begin{bmatrix} -\mathbf{QP}(\mathbf{w}) \\ \mathbf{C1} \otimes \mathbf{1}_K \end{bmatrix} \geq \mathbf{0}. \tag{5.11}$$

We prove the uniqueness of the solution of [DUE-E-NCP] by using the following lemma, which is based on the Poincare-Hopf's index theorem and the theory of complementarity problems.

Lemma 5.1. (Mas-Colell, 1979; Kolstad and Mathiesen, 1987; Simsek et al., 2007): *For a nonlinear complementarity problem:*

$$\textit{Find } \mathbf{x} \textit{ such that } \mathbf{0} \leq \mathbf{x} \perp \mathbf{F}(\mathbf{x}) \geq \mathbf{0} \tag{5.12}$$

where $\mathbf{F} : \mathbb{R}^N_+ \to \mathbb{R}^N$ *is continuously differentiable, define* $\overline{N}$ *by* $\overline{N} = \{1, \ldots, N\}$, $B(\mathbf{x}) = \{i \in \overline{N} \,|x_i > 0\}$, $\mathbf{J}_B(\mathbf{F}, \mathbf{x})$ *to the principal sub matrix of the Jacobian matrix of* $\mathbf{F}$ *corresponding to the indices of* $B(\mathbf{x})$*, and* $\det(\mathbf{J}_B(\mathbf{F}, \mathbf{x}))$ *its determinant. If* $B(\mathbf{x}) = \emptyset$*, then define* $\det(\mathbf{J}_B(\mathbf{F}, \mathbf{x})) = 1$. *Suppose a solution to the problem* (5.12) *exists, and at each solution* x_i^* *to the NCP that strict complementarity condition holds, (i.e.,* $x_i^* = 0$ *implies* $F_i(\mathbf{x}^*) > 0$*). Then (1) if we have* $\det(\mathbf{J}_B(\mathbf{F}, \mathbf{x})) > 0$ *for all solutions to the NCP then there is precisely one solution; and conversely, (2) If there is only one solution to the NCP then* $\det(\mathbf{J}_B(\mathbf{F}, \mathbf{x})) \geq 0$.

In order to apply this lemma to [DUE-E-NCP], we first need the Jacobian matrix of $\mathbf{G}(\mathbf{X})$. A straightforward calculation leads to the following lemma:

Lemma 5.2A. *The Jacobian matrix* $\nabla\mathbf{G}(\mathbf{X})$ *of* $\mathbf{G}(\mathbf{X})$ *in [DUE-E-NCP] is given by*

$$\nabla\mathbf{G}(\mathbf{X}) = \left[\begin{array}{c:c} \mathbf{I} \otimes \mathbf{I}_K & -\mathbf{H}(\mathbf{L} \otimes \mathbf{I}_K) \\ \hdashline -\mathbf{L}^T \otimes \mathbf{I}_K & \mathbf{CL} \otimes \boldsymbol{\Delta}_K \end{array}\right] \tag{5.13}$$

where $\mathbf{H} \equiv \mathrm{diag}[Q_1\mathbf{H}_1, \ldots, Q_N\mathbf{H}_N]$, $\mathbf{H}_i \equiv [h_i(s, t)]$ *is the Jacobian matrix of the choice probability function* $\mathbf{P}_i(\mathbf{v}_i(\mathbf{w}))$ *where the* (s, t) *element is* $h_i(s, t) \equiv \partial P_i(s)/\partial v_i(t)$, $(s = 1, \ldots, K; t = 1, \ldots, K)$.

In examining the properties of $\nabla\mathbf{G}(\mathbf{X})$, the following lemmas are useful.

Lemma 5.3(1). *The Jacobi matrix $\mathbf{H}_i(\mathbf{v}_i(\mathbf{w}))$ is negative semi-definite $\forall\mathbf{v}_i \in \mathbb{R}^K$ and is negative definite on the tangent space TX of $\{\mathbf{P}_i : \mathbf{1}^T\mathbf{P}_i = 1, \mathbf{P}_i \geq 0\}$.*

Proof. See for example, Sandholm (2010, Chap.6, pp.213).

Lemma 5.3(2). *The matrices $\mathbf{L}$ and $\boldsymbol{\Delta}_K$ are positive definite, and their eigenvalues are all equal to $+1$.*

Proof. $\mathbf{x}^T\mathbf{L}\mathbf{x} = \sum_{i=1}^N x_i \sum_{j=1}^N x_j = (1/2)\left(\sum_{j=1}^N x_j\right)^2 + (1/2)\sum_{j=1}^N x_j^2 > 0 \quad \forall\mathbf{x} \in \mathbb{R}^N \neq \mathbf{0}$

$$\mathbf{x}^T\boldsymbol{\Delta}_K\mathbf{x} = \sum_{k=1}^K (x_k - x_{k-1})x_k = (1/2)\sum_{k=1}^K (x_k - x_{k-1})^2 > 0 \qquad \forall\mathbf{x} \in \mathbb{R}^K \neq \mathbf{0}.$$

Because both of $\mathbf{L}$ and $\boldsymbol{\Delta}_K$ are lower triangular matrices with all the diagonal entries being 1, their eigenvalues also all equal to 1. **QED**.

Now, we can examine the positivity of the determinant of $\nabla\mathbf{G}(\mathbf{X})$.

Lemma 5.4A. $\det(\nabla\mathbf{G}(\mathbf{X})) = \det(\mathbf{A} - \mathbf{H})$, where $\mathbf{A} \equiv (\mathbf{L}^T)^{-1}\mathbf{C} \otimes \boldsymbol{\Delta}_K$.

Proof. $\det(\nabla\mathbf{G}(\mathbf{X})) = \det(\mathbf{C}\mathbf{L} \otimes \boldsymbol{\Delta}_K - (\mathbf{L}^T \otimes \mathbf{I}_K)\mathbf{H}(\mathbf{L} \otimes \mathbf{I}_K))$

$$= \det(\mathbf{L}^T \otimes \mathbf{I}_K) \cdot \det([\mathbf{L}^T \otimes \mathbf{I}_K]^{-1}[\mathbf{C}\mathbf{L} \otimes \boldsymbol{\Delta}_K][\mathbf{L} \otimes \mathbf{I}_K]^{-1} - \mathbf{H}) \cdot \det(\mathbf{L} \otimes \mathbf{I}_K)$$
$$= \det(\mathbf{L}^T \otimes \mathbf{I}_K) \cdot \det((\mathbf{L}^T)^{-1}\mathbf{C} \otimes \boldsymbol{\Delta}_K - \mathbf{H}) \cdot \det(\mathbf{L} \otimes \mathbf{I}_K).$$

It follows from Lemma 5.3(2) that $\det(\mathbf{L}^T \otimes \mathbf{I}_K) = \det(\mathbf{L} \otimes \mathbf{I}_K) = +1$, and hence,

$$\det(\nabla\mathbf{G}(\mathbf{X})) = \det((\mathbf{L}^T)^{-1}\mathbf{C} \otimes \boldsymbol{\Delta}_K - \mathbf{H}) = \det(\mathbf{A} - \mathbf{H}). \ \ \textbf{QED}.$$

Lemma 5.5A. *The determinant of $\nabla\mathbf{G}(\mathbf{X})$ is positive for all $\mathbf{X} \geq \mathbf{0}$.*

Proof. Since the matrix $\mathbf{A} - \mathbf{H}$ is a block upper triangular matrix, in which each diagonal block is $\mu_i\boldsymbol{\Delta}_K - Q_i\mathbf{H}_i$, the determinant of $\mathbf{A} - \mathbf{H}$ is given by $\prod_{i=1}^N \det(\mu_i\boldsymbol{\Delta}_K - Q_i\mathbf{H}_i)$. Note here that each of the matrix $\mu_i\boldsymbol{\Delta}_K - Q_i\mathbf{H}_i \ \forall i \in \mathcal{N}$ is positive definite because $\boldsymbol{\Delta}_K$ is positive definite (Lemma 5.3(2)), and $\mathbf{H}_i \ \forall i \in \mathcal{N}$ are negative semi-definite (Lemma 5.3 (1)), which implies that $\lambda_k(\mu_i\boldsymbol{\Delta}_K - Q_i\mathbf{H}_i) > 0 \ \forall k, i$, where $\lambda_k(\mathbf{M})$ denotes k th eigenvalue of a matrix $\mathbf{M}$. Thus,

$$\nabla\mathbf{G}(\mathbf{X}) = \prod_{i=1}^N \det(\mu_i\boldsymbol{\Delta}_K - Q_i\mathbf{H}_i) = \prod_{i=1}^N\prod_{k=1}^K \lambda_k(\mu_i\boldsymbol{\Delta}_K - Q_i\mathbf{H}_i) > 0. \ \ \textbf{QED}. \tag{5.14}$$

By extending the above analysis to the case of Jacobian matrices at equilibrium points, we obtain the uniqueness result for the One-to-Many model.

Theorem 5.2A. *For the One-to-Many DUE model with heterogeneous users, the equilibrium defined in (5.11) is unique.*

Proof. See Appendix A.

(*B*) *Many-to-One model*

The demand condition of the Many-to-One model can be represented in the same manner with that of the One-to-Many model, but the supply-side condition of the Many-to-One model is slightly different (see the discussions in Section 4), and it is given by

$$\mathbf{0} \leq \mathbf{w} \perp [\mathbf{CL}^T \otimes \mathbf{\Delta}_K]\mathbf{w} + \mathbf{C1} \otimes \mathbf{\Delta}\boldsymbol{\tau}_N - [\mathbf{L}^T \otimes \mathbf{I}_K]\mathbf{q} \geq \mathbf{0} \tag{5.15a}$$

$$\mathbf{\Delta}\boldsymbol{\tau}_N + (\mathbf{\Delta w}_N + \cdots + \mathbf{\Delta w}_1) = \mathbf{1}_K \text{ and } \mathbf{\Delta}\boldsymbol{\tau}_N \geq \mathbf{0}. \tag{5.15b}$$

Thus, if we impose Assumption 5.1B so that the consistency condition $\mathbf{\Delta}\boldsymbol{\tau}_N \geq \mathbf{0}$ is "automatically" satisfied at equilibrium, the problem [DUE-M] for the Many-to-One model with heterogeneous users is equivalent to the following mixed complementarity problem (MCP):

[DUE-M-MCP]: Find $\mathbf{X} \equiv [\mathbf{q}, \mathbf{w}]^T \in \mathbb{R}^{2NK}$ and $\mathbf{\Delta}\boldsymbol{\tau}_N$ such that

$$\mathbf{0} \leq \mathbf{X} \perp \mathbf{G}_S(\mathbf{q}, \mathbf{w}) - \mathbf{G}_D(\mathbf{w}) + \mathbf{G}_\tau(\mathbf{\Delta}\boldsymbol{\tau}_N) \geq \mathbf{0}, \tag{5.16a}$$

$$\mathbf{\Delta}\boldsymbol{\tau}_N + (\mathbf{\Delta w}_N + \cdots + \mathbf{\Delta w}_1) = \mathbf{1}_K, \tag{5.16b}$$

$$\text{where} \quad \mathbf{G}_S \equiv \left[\begin{array}{c:c} \mathbf{I} \otimes \mathbf{I}_K & \mathbf{0} \\ \hdashline -\mathbf{L}^T \otimes \mathbf{I}_K & \mathbf{CL}^T \otimes \mathbf{\Delta}_K \end{array}\right] \begin{bmatrix} \mathbf{q} \\ \mathbf{w} \end{bmatrix}, \quad \mathbf{G}_D \equiv \begin{bmatrix} \mathbf{QP(w)} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{G}_\tau \equiv \begin{bmatrix} \mathbf{0} \\ \mathbf{C1} \otimes \mathbf{\Delta}\boldsymbol{\tau}_N \end{bmatrix}.$$

To prove the uniqueness by using Lemma 5.1, it is convenient to convert the problem into a standard NCP. For this purpose, we convert the consistency condition (5.15b) into the following pair of complementarity conditions:

$$\mathbf{0} \leq \mathbf{x} \perp \mathbf{1}_K - (\mathbf{z} + \mathbf{\Delta}_K \mathbf{w}_1 + \cdots + \mathbf{\Delta}_K \mathbf{w}_N) \geq \mathbf{0} \text{ and } \mathbf{0} \leq \mathbf{z} \perp \mathbf{g}(\mathbf{x}, \mathbf{z}) \geq \mathbf{0} \tag{5.17}$$

where $\mathbf{z} \equiv \mathbf{\Delta}\boldsymbol{\tau}_N$, $\mathbf{g}(\mathbf{x}, \mathbf{z}) = [g(x(1), z(1)), \ldots, g(x(K), z(K))]^T$, and $g : \mathbb{R}_+^2 \to \mathbb{R}$ is a continuous and smooth function of $\mathbf{x}$ and $\mathbf{z}$ that satisfies

$$\partial g/\partial x > 0 \ \forall x \geq 0, \ \partial g/\partial z > 0 \ \forall z \geq 0, \ \arg_x .\{g(x, z) = 0\} > 0, \ \forall z \geq 0, \ \arg_z .\{g(x, z) = 0\} > 0, \forall x \geq 0.$$

A simple example of such a function is $g(x, z) \equiv f_1(z) - f_2(x)$, where $f_1(z) \equiv \ln(z + 1)$, $f_2(x) \equiv (1/x)$. These properties of g imply that $(x, z) > 0$ as well as the equality condition $(\mathbf{\Delta}_K \mathbf{w}_1 + \cdots + \mathbf{\Delta}_K \mathbf{w}_N) + \mathbf{z} = \mathbf{1}_K$ are always maintained in the complementarity system (5.17). Thus, we have the following equivalent NCP for the Many-to-One model.

Proposition 5.3B. (Many-to-One model) *If Assumption 5.1B holds, the problem [DUE-M] for the Many-to-One model with heterogeneous users is equivalent to the following standard NCP, and has a solution.*

[DUE-M-NCP]: Find $\mathbf{X} \equiv [\mathbf{q}, \mathbf{w}]^T \in \mathbb{R}^{2NK}$ and $\mathbf{Y} \equiv [\mathbf{x}, \mathbf{z}]^T \in \mathbb{R}^{2K}$ such that

$$\mathbf{0} \leq \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} \perp \mathbf{G}(\mathbf{X}, \mathbf{Y}) \equiv \begin{bmatrix} \mathbf{G}_S(\mathbf{q}, \mathbf{w}) - \mathbf{G}_D(\mathbf{w}) + \mathbf{G}_{12}\mathbf{Y} \\ \mathbf{G}_{21}\mathbf{X} + \mathbf{G}_2(\mathbf{Y}) \end{bmatrix} \geq \mathbf{0} \tag{5.18}$$

$$\text{where} \quad \mathbf{G}_2(\mathbf{Y}) \equiv \begin{bmatrix} \mathbf{1}_K - \mathbf{z} \\ \mathbf{g}(\mathbf{x}, \mathbf{z}) \end{bmatrix} : \mathbb{R}^{2K} \to \mathbb{R}^{2K}, \quad \mathbf{G}_{12} \equiv \left[\begin{array}{c:c} \mathbf{0} & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{C1} \otimes \mathbf{I}_K \end{array}\right] \in \mathbb{R}^{2NK \times 2K},$$

$$\mathbf{G}_{21} \equiv \left[\begin{array}{c:c} \mathbf{0} & \mathbf{1}_K \otimes \mathbf{\Delta}_K \\ \hdashline \mathbf{0} & \mathbf{0} \end{array}\right] \in \mathbb{R}^{2K \times 2NK}.$$

Note that if Assumption 5.1B does not hold, there is a possibility that the solution to the complementarity condition (5.17) cannot be satisfied, which implies that the solution to [DUE-M-NCP] does not exist even if the solution to [DUE-M-MCP] exists.

The positivity of the determinant of the Jacobian of the problem [DUE-M-NCP] can be examined by the similar manner to that in the One-to-Many model.

Lemma 5.2B. *The Jacobian matrix* $\nabla\mathbf{G}(\mathbf{X},\mathbf{Y})$ *of* $\mathbf{G}(\mathbf{X},\mathbf{Y})$ *in [DUE-M-NCP] is given by*

$$\nabla\mathbf{G} = \left[\begin{array}{c:c} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \hdashline \mathbf{G}_{21} & \mathbf{G}_{22} \end{array}\right], \text{ where } \mathbf{G}_{11} \equiv \left[\begin{array}{c:c} \mathbf{I}\otimes\mathbf{I}_K & -\mathbf{H}(\mathbf{L}\otimes\mathbf{I}_K) \\ \hdashline -\mathbf{L}^T\otimes\mathbf{I}_K & \mathbf{C}\mathbf{L}^T\otimes\boldsymbol{\Delta}_K \end{array}\right],\ \mathbf{G}_{22} \equiv \left[\begin{array}{c:c} \mathbf{0} & -\mathbf{I}_K \\ \hdashline \nabla_{\mathbf{x}}\mathbf{g} & \nabla_{\mathbf{z}}\mathbf{g} \end{array}\right].$$

Lemma 5.4B. $\det(\nabla\mathbf{G}(\mathbf{X},\mathbf{Y})) = \det(\hat{\mathbf{A}}-\mathbf{H}) = \det(\mathbf{B}_{11})\cdot\det(\mathbf{B}_{22})$

$$\text{where}\quad \hat{\mathbf{A}}-\mathbf{H} \equiv \left[\begin{array}{c:c} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \hdashline \mathbf{0} & \mathbf{B}_{22} \end{array}\right],\ \mathbf{B}_{11} \equiv \mathrm{diag}[\mu_{i+1}\boldsymbol{\Delta}_K - Q_i\mathbf{H}_i]_{i=1}^{N-1}, \tag{5.19}$$

$$\mathbf{B}_{22} \equiv \left[\begin{array}{c:c} \mu_N\boldsymbol{\Delta}_K - Q_N\mathbf{H}_N & \mathbf{0}\quad \mu_N\mathbf{I}_K \\ \hdashline -\boldsymbol{\Delta}_K & \\ 0 & \mathbf{G}_{22} \end{array}\right].$$

Lemma 5.5B. *The determinant of* $\nabla\mathbf{G}(\mathbf{X},\mathbf{Y})$ *is positive for all* $(\mathbf{X},\mathbf{Y})\geq\mathbf{0}$ *if an arbitrary pair of element* $(q_N(s), w_N(s))$ *of* $(\mathbf{q}_N,\mathbf{w}_N)$ *is fixed (or deleted from the unknown variables).*

Proof. Because $\det(\nabla\mathbf{G}(\mathbf{X},\mathbf{Y})) = \det(\hat{\mathbf{A}}-\mathbf{H}) = \det(\mathbf{B}_{11})\cdot\det(\mathbf{B}_{22})$, we show that $\det(\mathbf{B}_{11}) > 0$ and $\det(\mathbf{B}_{22}) > 0$ in turn. The matrix $\mathbf{B}_{11}$ is a block upper triangular matrix, in which each diagonal block is a positive definite matrix $\mu_{i+1}\boldsymbol{\Delta}_K - Q_i\mathbf{H}_i$. Hence, $\det(\mathbf{B}_{11})$ reduces to the products of eigenvalues $\lambda_k(\mu_{i+1}\boldsymbol{\Delta}_K - Q_i\mathbf{H}_i) > 0\ \forall k, i$:

$$\det(\mathbf{B}_{11}) = \textstyle\prod_{i=1}^{N-1}\det(\mu_{i+1}\boldsymbol{\Delta}_K - Q_i\mathbf{H}_i) = \prod_{i=1}^{N-1}\prod_{k=1}^{K}\lambda_k(\mu_{i+1}\boldsymbol{\Delta}_K - Q_i\mathbf{H}_i) > 0.$$

As for the matrix $\mathbf{B}_{22}$, a simple calculation yields

$$\det(\mathbf{B}_{22}) = g_x\cdot\det(-Q_N\mathbf{H}_N)$$

where $g_x \equiv \prod \partial g(x,z)/\partial x|_{x=x_i(s),z=z_i(s)} > 0\ \forall(\mathbf{x},\mathbf{z})\geq\mathbf{0}$. Since the demand side Jacobian $\mathbf{H}_N$ is negative definite *on the tangent space* TX (see Lemma 5.3 (2)), deleting a single row and column of $\mathbf{H}_N$ guarantees that $\det(\mathbf{B}_{22}) = g_x\cdot\det(-Q_N\mathbf{H}_N) > 0$. **QED**.

Comparing these lemmas and those for the One-to-Many model, we see first that the main difference is concerning the equilibrium conditions for the most upstream bottleneck. Specifically, the submatrix $\mathbf{B}_{22}$ of the Jacobian $\nabla\mathbf{G}(\mathbf{X},\mathbf{Y})$ stems from the equilibrium condition for the most upstream (N th) bottleneck/Origin, which corresponds to the last (N th) diagonal block, $\mu_N\boldsymbol{\Delta}_K - Q_N\mathbf{H}_N$, of the matrix $\mathbf{A}-\mathbf{H}$ in the One-to-Many model. Second, the determinant of the matrix $\mathbf{B}_{22}$ for the Many-to-One model reduces to the determinant of the demand side matrix $-Q_N\mathbf{H}_N$ only, while the determinant of $\mu_N\boldsymbol{\Delta}_K - Q_N\mathbf{H}_N$ for the One-to-Many model is decided from two terms stemming from the supply-side condition ($\mu_N\boldsymbol{\Delta}_K$) and the demand-side condition ($-Q_N\mathbf{H}_N$). This reflects the fact that the equilibrium condition of the Many-to-One model is lacking in the queuing condition for the most upstream bottleneck. This fact together

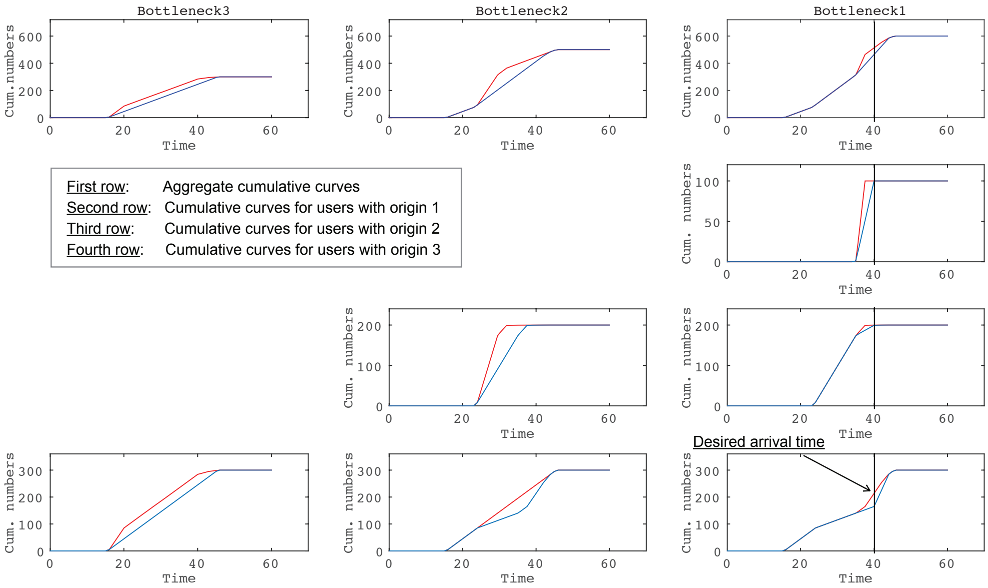

Figure 5: An equilibrium flow pattern for the Many-to-One model with the desired arrival time $t_w = 40$ (first row: aggregate cumulative curves; second-fourth rows: disaggregate cumulative curves for each origin).

with the positive *semi*-definiteness of $-\mathbf{H}_N$ (i.e., its rank is $K-1$) also leads to the indeterminacy of the $(\mathbf{q}_N, \mathbf{w}_N)$, and the requirement of deleting a single pair of elements $(q_N(s), w_N(s))$ from $(\mathbf{q}_N, \mathbf{w}_N)$ in Lemma 5.5B; in the One-to-Many model, the possible indeterminacy from the *semi*-definiteness of $\mathbf{H}_N$ is recovered by the positive definiteness of the supply-side term $\mu_N \mathbf{\Delta}_K$. Finally, the possible indeterminacy of the $(\mathbf{q}_N, \mathbf{w}_N)$ in the Many-to-One model may not be a serious problem in practice because it can be easily avoided by taking the assignment time period $\mathcal{S}$ long enough, and setting, say, $q_N(1) = w_N(1) = 0$; furthermore, it cannot occur if there is at least a single time period s at which the most upstream bottleneck is not active (i.e., $w_N(s) = 0$) at equilibrium, which may be often the case.

Now, we are in a position to state the uniqueness theorem for the Many-to-One model.

Theorem 5.2B. *For the Many-to-One DUE model with heterogeneous users, suppose that Assumption 5.1B holds. Then the equilibrium $(\mathbf{q}^*, \mathbf{w}^*)$ defined in (5.18) is unique, if an arbitrary pair of element $(q_N(s), w_N(s))$ of $(\mathbf{q}_N, \mathbf{w}_N)$ is fixed (or deleted from the unknown variables).*

Proof. The difference between the Jacobians of the One-to-Many model and the Many-to-One model is just the block-submatrix $\mathbf{B}_{22}$ discussed above. Since this submatrix cannot be affected by the operations deleting the rows and columns of the Jacobian matrix corresponding to $B(\mathbf{x}) = \{i \in \overline{N} \,|x_i > 0\}$ in Lemma 5.1 (recall that $(\mathbf{x}, \mathbf{z})$ cannot be zero from the definition of the function $\mathbf{g}$), the proof is almost the same as that of Theorem 5.1A, and omitted here.

6. Numerical Examples

This section presents numerical examples of an equilibrium flow pattern for each model by solving the problems [DUE-E-NCP] and [DUE-M-MCP]. For the computation of the equilib-

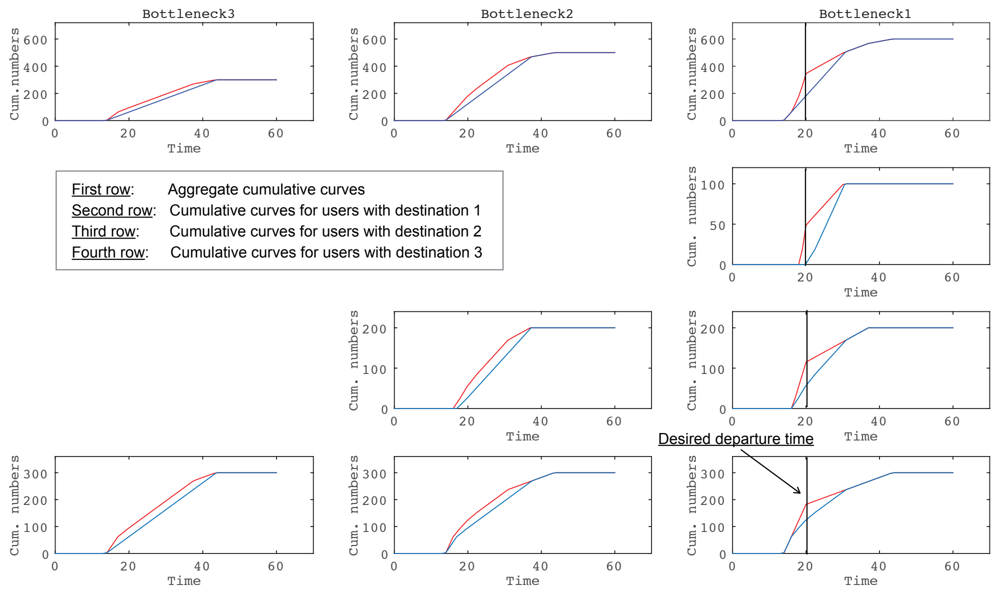

Figure 6: An equilibrium flow pattern for the One-to-Many model with the desired departure time $t_w = 20$ (first row: aggregate cumulative curves; second-fourth rows: disaggregate cumulative curves for each destination).

rium, we use the ReSNA (Regularized Smoothing Newton Algorithm) [3] that was originally developed for solving the second-order cone complementarity problems (SOCCP) (Hayashi et al., 2005). Because the SOCCP involves the NCP as a subclass, ReSNA is also applicable to NCP in a direct manner. In Hayashi (2015), the global convergence of the algorithm is proved under the (Cartesian) P_0 assumption, which is a natural extension of P_0 assumption for NCP or MCP (a weaker condition than monotonicity)[4]. Although our problems may not be P_0, our experiments have successfully obtained equilibrium solutions in most cases when the logit parameter θ is not so high.

We considered a corridor network with the number of bottlenecks $N = 3$ and employed a piecewise linear schedule delay function with early [late] penalty parameter 0.5 and late [early] penalty parameter 2.0 for the Many-to-One model [One-to-Many model]. We used the logit choice function (i.e., the discrete time version of (3.12)) with the parameter $\theta = 60$. The free flow travel times, capacities of bottlenecks, and total demands were give by $\mathbf{c} = \mathbf{0}$, $\boldsymbol{\mu} = [30, 20, 10]^T$, and $\mathbf{Q} = [100, 200, 300]^T$, respectively. The number of assignment intervals was 60, which was sufficiently long for all the demands to be served in assignment period $\mathcal{S}$.

The results for these numerical examples are shown in Figs.5 and Fig.6. These figures illustrate the aggregate (first rows) and disaggregate cumulative curves (second-fourth rows). Each disaggregate curve represents the origin-specific (or destination-specific) cumulative number of users at each bottleneck. The aggregate curve at each bottleneck is the sum of the disaggregate curves at the same bottleneck. The red and blue lines represent the cumulative arrival and

[3] Website of ReSNA: `http://www.plan.civil.tohoku.ac.jp/opt/hayashi/ReSNA/`.

[4] In Hayashi et al. (2005), the quadratic convergence of the algorithm is also proved under monotonicity assumption.

departure curves, respectively. The vertical lines in the figures for bottleneck 1 represent the desired arrival [departure] time at the destination [origin], which is common to all users in the Many-to-One [One-to-Many] model.

From these figures, it is clear that equilibrium flow patterns for the two models have different characteristics even though the situations are only the reverse of each other. Specifically, for the Many-to-One model, the disaggregate arrival and departure patterns among different origins are very different, but the patterns for the One-to-Many model are not significantly different for different destinations. This seems to be related to the difference between two models discussed in Section 4.3. Future work should investigate this in detail, along with other characteristics of the equilibrium flow patterns.

7. Concluding remarks

This paper considered a departure-time choice equilibrium for a corridor problem where discrete multiple bottlenecks exist along a freeway. We first transformed the equilibrium conditions for Many-to-One and One-to-Many models from the conventional Eulerian coordinate system to a *Lagrangian-like coordinate system*. This approach allowed us to achieve a deep understanding of the mathematical structure of the problem. Specifically, we analyzed the equilibrium for the homogeneous users case and revealed the following properties: (1) the supply (queuing) sub-model of the Many-to-One model has a slightly more complicated structure than that of the One-to-Many model, although the demand sub-models of both models have same structure; (2) the mappings of the equivalent linear complementarity problems of both models are not monotone in general, which implies that the uniqueness of the equilibria is not necessarily guaranteed for the homogeneous users case. We then examined the existence and uniqueness of equilibria for heterogeneous users case and obtained the following results: (3) equilibria exist in both models for the case of homogeneous users and heterogeneous users case, but the Many-to-One model requires an assumption on the schedule penalty function; (4) the equilibria in both of the models for the case of the heterogeneous users are unique. Finally, we gave numerical examples of the equilibrium flow pattern of each model.

Although we focused on the theoretical properties of equilibria for the corridor problem due to space limitations, a systematic numerical experiment is required to characterize the equilibrium flow and cost patterns. The algorithm used in Section 6 does not give guaranteed convergence, but our uniqueness result supports the case for proceeding with such a numerical experiment because we should, at most, find one equilibrium solution. Such an experiment is currently being conducted by the authors, and the outcome will be reported in the near future.

Appendix A. Proof of Theorem 5.2A

We prove the uniqueness of the solution of [DUE-E-NCP] by using Lemma 5.1. Since the equilibrium $\mathbf{q}^*$ of [DUE-E-NCP] is always an interior solution, it is sufficient for us to consider the set of time periods during which equilibrium queuing delay $\mathbf{w}^*$ is positive. For this, we first define a subset $\mathcal{S}_i$ of the index set $\mathcal{S} \equiv \{1, \ldots, K\}$ by $\mathcal{S}_i(\mathbf{w}_i) = \{s \in \mathcal{S} \mid w_i(s) > 0\} (i = 1, \ldots, N)$, and the associated matrix $\mathbf{I}(\mathcal{S}_i(\mathbf{w}_i), \mathcal{S}) \in \mathbb{R}^{K_i \times K}$ that is a submatrix of $\mathbf{I}_K$ consisting of the set of rows with indices in $\mathcal{S}_i(\mathbf{w}_i)$, where $K_i \equiv |\mathcal{S}_i|$. Let $B(\mathbf{w})$ be the union of $\{\mathcal{S}_i(\mathbf{w}_i) : i = 1, \ldots, N\}$, $B(\mathbf{w}) = \bigcup_{i=1}^{N} \mathcal{S}_i(\mathbf{w}_i)$, its associated matrix $\mathbf{B}(\mathbf{w})$ be

$$\mathbf{B}(\mathbf{w}) \equiv \mathrm{diag}[\mathbf{I}(\mathcal{S}_1(\mathbf{w}_1), \mathcal{S}), \ldots, \mathbf{I}(\mathcal{S}_N(\mathbf{w}_N), \mathcal{S})] \in \mathbb{R}^{m \times n}, \text{ where } m \equiv \textstyle\sum_{i=1}^{N} K_i, \; n \equiv NK,$$

and $\mathbf{J}_B(\mathbf{G}(\mathbf{q}^*,\mathbf{w}^*))$ be the principal submatrix of the Jacobian of $\mathbf{G}(\mathbf{q}^*,\mathbf{w}^*)$ corresponding to the indices of $B(\mathbf{w})$. We also use the following abbreviation of the notation for the simplicity of exposition:

$$\mathbf{I}(i,*) \equiv \mathbf{I}(\mathcal{S}_i(\mathbf{w}_i),\mathcal{S}),\ \mathbf{I}(*,i) \equiv \mathbf{I}(i,*) = \mathbf{I}(\mathcal{S},\mathcal{S}_i(\mathbf{w}_i)),$$
$$\mathbf{I}(i,j) \equiv \mathbf{I}(\mathcal{S}_i(\mathbf{w}_i),\mathcal{S})\ \mathbf{I}(\mathcal{S},\mathcal{S}_j(\mathbf{w}_j)) \equiv \mathbf{I}(i,*)\ \mathbf{I}(*,j).$$

Under this setting, the Jacobian matrix $\mathbf{J}_B(\mathbf{G}(\mathbf{q}^*,\mathbf{w}^*))$ can be written as

$$\begin{aligned}
\mathbf{J}_B(\mathbf{G}(\mathbf{q}^*,\mathbf{w}^*)) &\equiv \begin{bmatrix} \mathbf{I}_K & \\ & \mathbf{B} \end{bmatrix} \nabla\mathbf{G}(\mathbf{q}^*,\mathbf{w}^*) \begin{bmatrix} \mathbf{I}_K & \\ & \mathbf{B} \end{bmatrix}^T \\
&= \begin{bmatrix} \mathbf{I}_K & \\ & \mathbf{B} \end{bmatrix} \left[\begin{array}{c:c} \mathbf{I}\otimes\mathbf{I}_K & -\mathbf{H}(\mathbf{w}^*)(\mathbf{L}\otimes\mathbf{I}_K) \\ \hdashline -\mathbf{L}^T\otimes\mathbf{I}_K & \mathbf{C}\mathbf{L}\otimes\mathbf{\Delta}_K \end{array}\right] \begin{bmatrix} \mathbf{I}_K & \\ & \mathbf{B} \end{bmatrix}^T \equiv \left[\begin{array}{c:c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hdashline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array}\right],
\end{aligned}$$

where $\mathbf{A}_{22} = \mathbf{B}(\mathbf{C}\mathbf{L}\otimes\mathbf{\Delta}_K)\mathbf{B}^T$, $\mathbf{A}_{21}\mathbf{A}_{12} = \mathbf{B}(\mathbf{L}^T\otimes\mathbf{I}_K)\mathbf{H}(\mathbf{L}\otimes\mathbf{I}_K)\mathbf{B}^T$. Hence, $\det(\mathbf{J}_B)$ is given by

$$\begin{aligned}
\det(\mathbf{J}_B) &= \det(\mathbf{A}_{11})\det(\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}) = \det(\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{12}) \\
&= \det(\mathbf{B}(\mathbf{C}\mathbf{L}\otimes\mathbf{\Delta}_K)\mathbf{B}^T - \mathbf{B}(\mathbf{L}^T\otimes\mathbf{I}_K)\mathbf{H}(\mathbf{L}\otimes\mathbf{I}_K)\mathbf{B}^T) \\
&\equiv \det(\mathbf{J}_B^{(S)} - \mathbf{J}_B^{(D)})
\end{aligned}$$

where $\mathbf{J}_B^{(S)} \equiv \mathbf{B}(\mathbf{C}\mathbf{L}\otimes\mathbf{\Delta}_K)\mathbf{B}^T$, and $\mathbf{J}_B^{(D)} \equiv \mathbf{B}(\mathbf{L}^T\otimes\mathbf{I}_K)\mathbf{H}(\mathbf{L}\otimes\mathbf{I}_K)\mathbf{B}^T$. In order to simplify the demand-side matrix $\mathbf{J}_B^{(D)}$, we first represent the principal submatrix of the Jacobian $\mathbf{H}(\mathbf{w}^*)$ corresponding to $\mathcal{S}_i(\mathbf{w}_i)$ as $\hat{\mathbf{H}}_i \equiv \mathbf{I}(i,*)\mathbf{H}_i\mathbf{I}(*,i)$ $(i = 1,\dots,N)$. Then the matrix $\mathbf{J}_B^{(D)}$ can be written as

$$\mathbf{J}_B^{(D)} \equiv \mathbf{B}(\mathbf{L}^T\otimes\mathbf{I}_K)\mathbf{H}(\mathbf{L}\otimes\mathbf{I}_K)\mathbf{B}^T = \hat{\mathbf{L}}^T\mathbf{S}\mathbf{H}\mathbf{S}^T\hat{\mathbf{L}} = \hat{\mathbf{L}}^T\hat{\mathbf{H}}\hat{\mathbf{L}},$$

where $\hat{\mathbf{L}}^T$ is a block upper triangular matrix in which (i,j) block is $\mathbf{I}(i,j)$ and $\mathbf{S} \equiv \mathrm{diag}[\mathbf{I}(1,*),\dots,\mathbf{I}(N,*)]$. We used the identity $\mathbf{I}(i,i)\mathbf{I}(i,*) = \mathbf{I}(i,*)$, $\mathbf{I}(i,i)\mathbf{I}(i,j) = \mathbf{I}(i,j)$, and $\mathbf{I}(i,j)\mathbf{I}(j,*) = \mathbf{I}(i,*)$ if $\mathcal{S}_i \subseteq \mathcal{S}_j$. Note here that the inverse of the block upper triangular matrix $\hat{\mathbf{L}}^T$ can be given as an block upper triangular (and double diagonal) matrix in which each diagonal block is $\mathbf{I}(i,i)$ and (i,j) block is $-\mathbf{I}(i,j)$ if $j = i+1$, zero otherwise, because $\mathbf{I}(i,j-1)\mathbf{I}(j-1,j) = \mathbf{I}(i,j)\mathbf{I}(j,j)$ $(i < j)$ always hold. By using this, we can diagonalized the matrix $\mathbf{J}_B^{(D)}$ as follows:

$$(\hat{\mathbf{L}}^T)^{-1}\mathbf{J}_B^{(D)}\hat{\mathbf{L}}^{-1} = \mathbf{S}\mathbf{H}\mathbf{S}^T = \hat{\mathbf{H}},$$

which implies that $\det(\mathbf{J}_B^{(S)} - \mathbf{J}_B^{(D)})$ is simplified as

$$\begin{aligned}
\det(\mathbf{J}_B^{(S)} - \mathbf{J}_B^{(D)}) &= \det(\hat{\mathbf{L}}^T[(\hat{\mathbf{L}}^T)^{-1}(\mathbf{J}_B^{(D)} - \mathbf{J}_B^{(S)})\hat{\mathbf{L}}^{-1}]\hat{\mathbf{L}}) \\
&= \det(\hat{\mathbf{L}}^T)\det((\hat{\mathbf{L}}^T)^{-1}\mathbf{J}_B^{(S)}\hat{\mathbf{L}}^{-1} - \hat{\mathbf{H}})\det(\hat{\mathbf{L}}) \\
&= \det((\hat{\mathbf{L}}^T)^{-1}\mathbf{J}_B^{(S)}\hat{\mathbf{L}}^{-1} - \hat{\mathbf{H}}).
\end{aligned}$$

Here the last equality follows from the fact $\det(\hat{\mathbf{L}}) = \det(\hat{\mathbf{L}}^T) = +1$ because $\hat{\mathbf{L}}^T$ is a block upper triangular matrix with all diagonal entries equal to $+1$. Corresponding to this diagonalization, we needs to calculate $(\hat{\mathbf{L}}^T)^{-1}\mathbf{J}_B^{(S)}\hat{\mathbf{L}}^{-1}$ to evaluate $\det(\mathbf{J}_B^{(S)} - \mathbf{J}_B^{(D)})$. By defining $\mathbf{\Delta}(i, j) \equiv \mathbf{I}(i, *)\mathbf{\Delta}_K\mathbf{I}(*, j)$, we see that the supply-side matrix $\mathbf{J}_B^{(S)} \equiv \mathbf{B}(\mathbf{CL}\otimes\mathbf{\Delta}_K)\mathbf{B}^T$ reduces to a block lower triangular matrix whose (i, j) block is $\mu_i\mathbf{\Delta}(i, j)$. Then, a straightforward calculation reveals that $(\hat{\mathbf{L}}^T)^{-1}\mathbf{J}_B^{(S)}\hat{\mathbf{L}}^{-1} - \hat{\mathbf{H}}$ is also a block lower triangular matrix in which each diagonal block is $\mu_i\mathbf{\Delta}(i, i)$. Therefore, each diagonal block of $(\hat{\mathbf{L}}^T)^{-1}\mathbf{J}_B^{(S)}\hat{\mathbf{L}}^{-1} - \hat{\mathbf{H}}$ is $\mu_i\mathbf{\Delta}(i, i) - Q_i\hat{\mathbf{H}}_i = \mathbf{I}(i, *)(\mu_i\mathbf{\Delta}_K - Q_i\mathbf{H}_i)\mathbf{I}(*, i)$, and hence, $\det(\mathbf{J}_B^{(S)} - \mathbf{J}_B^{(D)})$ is given by the products of $\det(\mathbf{I}(i, *)(\mu_i\mathbf{\Delta}_K - \mathbf{H}_i)\mathbf{I}(*, i))$. Note here that each of the matrix $\mathbf{I}(i, *)(\mu_i\mathbf{\Delta}_K - Q_i\mathbf{H}_i)\mathbf{I}(*, i)$ is positive definite for any choice of the set $\mathcal{S}_i$ because $\mathbf{\Delta}_K$ is positive definite (Lemma 5.3(2)), and $\mathbf{H}_i$ $(i = 1, \ldots, N)$ are negative semi-definite (Lemma 5.3 (1)), and these properties are inherited to their principal submatrices. This implies that all the eigenvalues of $\mathbf{I}(i, *)(\mu_i\mathbf{\Delta}_K - \mathbf{H}_i)\mathbf{I}(*, i)$ are positive. Thus, we can conclude that for $B(\mathbf{w})$ corresponding to all solutions of [DUE-E-NCP],

$$\det(\mathbf{J}_B(\mathbf{G}(\mathbf{q}^*, \mathbf{w}^*))) = \det(\mathbf{J}_B^{(S)} - \mathbf{J}_B^{(D)}) = \textstyle\prod_{i=1}^{N} \det(\mathbf{I}(i, *)(\mu_i\mathbf{\Delta}_K - Q_i\mathbf{H}_i)\mathbf{I}(*, i)) > 0,$$

and Lemma 5.1 guarantees that the solution is unique. **QED**.

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