

# Tradable network permits: A new scheme for the most efficient use of network capacity

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## Abstract

Akamatsu et al. (2006) proposed a new transportation demand management scheme called “tradable bottleneck permits” (TBP), and proved its efficiency properties for a single bottleneck model. This paper explores the properties of a TBP system for general networks. An equilibrium model is first constructed to describe the states under the TBP system with a single OD pair. It is proved that equilibrium resource allocation is efficient in the sense that the total transportation cost in a network is minimized. It is also shown that the “self-financing principle” holds for the TBP system. Furthermore, theoretical relationships between TBP and congestion pricing (CP) are discussed. It is demonstrated that TBP has definite advantages over CP when demand information is not perfect, whereas both TBP and CP are equivalent for the perfect information case. Finally, it is shown that the efficiency result also holds for more general demand conditions.

*Keywords:* bottleneck congestion, dynamic traffic assignment, time-space network, tradable permit, ITS

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## 1. Introduction

Recent advances in information and communication technology (ICT) have led to rapid changes in the “virtual” world represented by the internet. The increasing capabilities and decreasing cost of ICT is now becoming the impetus for changing the “real” world. The effects of ICT on transportation systems are no exception. The broadly defined “Intelligent Transportation Systems (ITS)” that exploits ICT has a large potential for dramatically improving efficiency of road transportation systems if the systems are implemented together with appropriate transportation demand management (TDM) schemes.

As an example of such futuristic TDM schemes making the most of ICT/ITS, Akamatsu et al. (2006) proposed the “tradable bottleneck permit” (TBP) system. Their proposed scheme is designed for resolving the problem of congestion during the morning rush hour at a single bottleneck, and consists of the following two parts:

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- a) the road manager issues a right that allows a permit holder to pass through the bottleneck at a pre-specified time period (“bottleneck permits”),
- b) a new trading market is established for bottleneck permits differentiated by a pre-specified time.

Note here that both parts *a*) and *b*) of this scheme are feasible for implementation from a technical point of view, even at the present time. The system for handling part *a*) may be constructed as an application of the dedicated short range communication (DSRC) system that is used in the current electric toll collection (ETC) system; the trading markets in part *b*) also can be realized inexpensively by using internet auction markets. It is, therefore, reasonable to assume that implementing this scheme will become technically easier when we take into account the future advances of ICT/ITS.

Part *a*) of this scheme is almost the same as the concept of “advance highway booking” (reservations or quotas) that is one of the *quantity-based regulation schemes* and has been previously proposed by several authors (e.g., [Akahane and Kuwahara, 1996](#); [Wong, 1997](#); [Teodorović and Edara, 2005](#); [Liu, Yang, and Yin, 2015](#)). Under this scheme, the arrival flow at a bottleneck at any time period is, from the definition of the scheme, equal to the number of permits issued for that time period. This implies that we can completely eliminate the occurrence of queuing congestion by setting the number of permits issued per unit time to be less than or equal to the bottleneck capacity. However, there may be cases in which road users cannot choose their desired time for using the road if the permits are assigned according to some unrefined rule (e.g., a simple “quota” scheme). Such an infringement on freedom of choice necessarily causes economic losses and should not be socially acceptable. In order to circumvent this problem that arises in employing only part *a*) of the scheme, we need to add an appropriate mechanism in which each user can choose his or her desired permit. It is part *b*) of the scheme that gives the foundation for this “choice mechanism” by a market system for buying and selling permits.

With the complementary properties of parts *a*) and *b*) in the combined scheme above, we can expect that this is the most efficient scheme of using the limited resource of road capacity. Indeed, for a departure-time choice equilibrium problem with a single bottleneck ([Vickrey, 1969](#)), [Akamatsu et al. \(2006\)](#) showed that the proposed system has the following desirable properties: (1) comparing equilibrium states with and without the proposed system, we can achieve Pareto improvement for both the road manager and all road users, (2) the equilibrium with the proposed system achieves the most efficient (i.e., Pareto optimal) resource allocation, (3) the “self-financing principle” holds for the equilibrium with the proposed system—that is, the total revenue (market value) of selling the permits is equal to the investment cost required for increasing the bottleneck capacity to a socially optimal level.

These properties of the tradable permit system are proved only for a road with a single bottleneck. Specifically, the proof is based on the isomorphism between the commuters’ departure-time choice equilibrium in a single bottleneck model and an equilibrium model of an urban residential location (see, for example, [Fujita, 1989](#)). Since such isomorphism cannot be extended to a case with multiple bottlenecks, the properties of the tradable permit system for general networks are largely unknown.

The purpose of this paper<sup>1</sup> is to explore some of the properties of a system of tradable bottleneck permits for general networks (we call this a system of “tradable network permits.”). To

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<sup>1</sup> This paper is an enhanced version of the working paper ([Akamatsu, 2007](#)).

attain this purpose in a clear manner, we consider ideal situations<sup>2</sup>; whereas practical and implementation issues for the tradable network permits are discussed accordingly. Specifically, after defining the system of a tradable network permit, we present a mathematical model that describes the equilibrium that arises under the tradable permit system. We then prove that the equilibrium resource allocation under the system is efficient in the sense that the total transportation cost in a network is minimized: formulating a dynamic system optimal assignment, we show that the equilibrium assignment coincides with the optimal assignment. We also prove that the “self-financing principle” holds not only for the single bottleneck case but also for the general network case. We further show the theoretical relationship between the tradable permit system and congestion pricing: we demonstrate the definite advantages of the tradable permit system over congestion pricing when the demand information is not perfect, whereas they are equivalent for the perfect information case.

The organization of this paper is as follows. In Section 2, we briefly review related works. In Section 3, we outline the framework of the tradable permits system analyzed in this paper. In Section 4, we present a model of the equilibrium under the tradable network permit system with a single OD pair. In Section 5, we analyze the efficiency of the equilibrium allocation. We also show that the self-financing principle holds for the tradable network permits system. In Section 6, we discuss the theoretical relationship between the tradable permit system and congestion pricing. Section 7 extends the model to the case with general demand conditions. Finally, Section 8 concludes the paper.

## 2. Related works

In general, there are two types of regulation schemes to eliminate economic inefficiency due to market externalities: *price-based* and *quantity-based* regulation schemes. For the choice of regulation schemes, “asymmetric information” between a regulation authority and firms is an essential issue that should be taken into account. The general discussion on this issue was given by Weitzman (1974) and Laffont (1977); a well-known application of this theory is the environmental policy choice between a tax and a quotas (or a tradable permits scheme<sup>3</sup>). The theory, however, cannot apply to traffic congestion problems directly because mechanisms of externalities for environmental problems are different from bottleneck congestion. For the dynamic traffic congestion problem, the detailed comparisons between congestion pricing and tradable network permits are given in Section 6.

Dynamic traffic assignment literature has been only focused on the price-based regulation scheme “dynamic congestion pricing” for eliminating bottleneck congestion. For simple networks such as single bottleneck or parallel link networks, the optimal congestion tolls have been well analyzed by several authors (e.g., Vickrey, 1969; Arnott et al., 1990, 1993; Kuwahara, 2007; Doan et al., 2011). However, no study has established a theory of dynamic congestion pricing for general networks in which queues can arise, although some attempts have been presented in the literature (e.g., Carey and Srinivasan, 1993; Yang and Meng, 1998; Ziliaskopoulos, 2000;

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<sup>2</sup>This is a natural first step when progressing the theory. For example, although the congestion pricing in the first best case (e.g., general networks in which congestion tolls are imposed on every link) may be impractical, the theory has been well systematized thanks to such an ideal assumption.

<sup>3</sup>See, for example, Montgomery (1972) and Tietenberg (1980), for tradable permits schemes for environmental protection.

Nie, 2011). Furthermore, despite its importance, few studies have addressed the asymmetric information problem.

For the asymmetric information problem, some studies have developed evolutionary (trial-and-error) implementation methods for congestion pricing for static settings (e.g., Sandholm, 2002; Yang et al., 2004; Sandholm, 2007; Han and Yang, 2009). These methods can set toll levels based on *realized* traffic flow patterns rather than the detailed demand information, which essentially rely on the property that there is a Beckmann-type potential function for a static user equilibrium. However, a dynamic user equilibrium cannot be reduced to an optimization problem in general. Therefore, it is not easy to generalize the methods to dynamic settings.

In recent years, a tradable permits scheme that is the generalization of a quantity-based regulation scheme has received much attention from the transportation community. Although the early studies (e.g., Verhoef et al., 1997) only discussed the possibilities of using tradable permits for managing traffic congestion, more recent studies begin to conduct quantitative analyses of various types of schemes (see, Fan and Jiang, 2013; Grant-Muller and Xu, 2014, for recent reviews). For managing bottleneck congestion, Akamatsu et al. (2006) first analyzed the tradable bottleneck permits scheme for the morning commuter problem (Vickrey, 1969). For the similar problems, Nie and Yin (2013), Tian et al. (2013) and Xiao et al. (2013) analyzed the “tradable travel credit scheme” proposed by Yang and Wang (2011). Note that the tradable travel credit scheme is different from the tradable bottleneck permits scheme. The credit scheme can be regarded as a price-regulation scheme, and thus it cannot eliminate bottleneck congestion unless imposing an optimal time-dependent credit charge. This also implies that the tradable travel credit scheme face the asymmetric information problem<sup>4</sup>. On the other hand, the tradable bottleneck permits scheme can resolve this problem (see, Section 6) in addition to eliminating bottleneck congestion directly by restricting the use of bottleneck capacity<sup>5</sup>.

### 3. A system of tradable bottleneck permits in transportation networks

#### 3.1. Networks

In this paper, we consider dynamic traffic flows on a general network  $G$  (i.e., a transportation network with general topology). The network consists of a set  $N$  of nodes, and a set  $L$  of directed links. The node set  $N$  includes an origin node  $o$  from which users start their trips, and a destination node  $d$  at which users terminate their trips. To avoid notational complexity and to outline essential aspects of the theory, we first deal with networks with a single OD pair. The extensions of the theory (e.g., many-to-many OD pairs) are presented in Section 7. Each element of  $N$  (i.e., each node) is identified by a sequential natural number  $i$ , and each element of  $L$  (i.e., each link) is denoted by a pair  $(i, j)$  of the upstream node  $i$  and the downstream node  $j$ . The time interval  $[0, \mathcal{T}]$  is for which we assign the dynamic traffic flow is fixed. We assume that the travel demand  $Q$  that makes trips for the time interval  $[0, \mathcal{T}]$  is a given constant. The time interval  $[0, \mathcal{T}]$  is discretized into small intervals of length  $\Delta t$ : each time point is represented by  $t = m\Delta t$ , where

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<sup>4</sup>Nie (2012) noted this in the context of a comparison with the tradable permits scheme for emission control and the tradable travel credit scheme: “Suffice it to say here that the information that the government would need to run a mobility credit market is as much as the information required to operate a conventional pricing scheme. Therefore, the mobility credit market does not reduce the administrative burden of the government, unlike in the case of emission control.”

<sup>5</sup>Recently, the similar concept is applied to the parking management problem (Liu et al., 2014) and queue length control problem (Shirmohammadi and Yin, 2016).

$m = 0, 1, 2, \dots, M$ . Each time interval  $[t, t + \Delta t]$  is denoted  $t \in I$  and we call this interval time period  $t$ .

We also assume, without any loss of generality, that each link in a network consists of a free flow segment and a single bottleneck segment. The travel time to pass through the free flow segment of link  $(i, j)$  is a constant  $t_{ij}$  (i.e.,  $t_{ij}$  is independent of time and flow). We then assume that the travel time  $t_{ij}$  is represented by a natural multiplier of  $\Delta t$ . The bottleneck of each link is represented by a point queue model with constant capacity  $\mu_{ij} = \text{vehicles/time interval } \Delta t$ . Note that this modeling approach can deal with any number of bottlenecks in a road segment. When we wish to consider a segment without a bottleneck, we set the capacity as infinity; when we consider a segment with multiple bottlenecks, we just set up multiple links corresponding to each bottleneck.

### 3.2. Agents

The model presented in this paper has two types of agents: a road network manager and road network users. The road manager aims to restrain traffic congestion on the network and minimize the “social transportation cost.” To achieve this, the manager regulates the traffic flows entering into each bottleneck in the network by using “time-dependent bottleneck permits.” The precise definition and setup of the bottleneck permit system are described in Section 3.3 below.

Each user makes a single trip (for the time interval  $[0, \mathcal{T}]$ ) from an origin (e.g., residential zone) to a destination (e.g., central business district (CBD)) in the network. The user chooses a destination arrival time and a path between the origin and the destination so as to minimize his or her disutility (or “generalized transportation cost”). The detailed definition of disutility is mentioned in Section 4. Under the bottleneck permits system, each user must purchase a set of permits corresponding to a set of links included in the user’s chosen path. This implies that choosing a destination arrival time period and a path directly links to purchasing time-dependent bottleneck permits in the “permits markets.” More detailed explanations on the purchase scheme and the permit markets are given in Sections 3.3 and 3.4.

### 3.3. Bottleneck permits

A “time-dependent bottleneck permits” is a right that allows the permit holder to pass through a pre-specified bottleneck at a pre-specified time. In this paper, we assume that the road manager can issue time-dependent bottleneck permits for all bottlenecks (i.e., links) in the network. This implies that the traffic flow entering into link  $(i, j)$  at time period  $t$  consists of only users who have a “time period  $t$  permit for link  $(i, j)$ ,” and users without this permit cannot pass through this link at this time period.

Throughout this paper, we assume that the number of permits issued for each link for each unit time period is equal to or less than the traffic capacity of each link in the network. This means that queuing congestion never occurs in the network under this permits-issue scheme. This may be easily seen from this explanation of permits: the inflow of each link is equal to (or less than) the number of permits issued, and hence the inflow cannot exceed the traffic capacity of each link, which implies that queuing congestion at each link can never occur<sup>6</sup>.

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<sup>6</sup>It might be considered that the scheduled (available) time constraint of each bottleneck permit is unrealistic to follow. However, if we consider advanced vehicles such as autonomous vehicles, users would be able to follow the scheduled time periods (i.e., almost late or early arrivals can be eliminated). This is because, under non-congested states, such vehicles would be able to drive with free-flow travel time precisely.

### 3.4. Trading markets of bottleneck permits

For assigning the bottleneck permits to users, we can consider two representative schemes: *market selling scheme* and *free distribution scheme* (Akamatsu et al., 2006). In the market selling scheme, the road manager sells all the bottleneck permits to users in “bottleneck permits markets.” All sales from selling the permits result in revenue for the road manager in this scheme. In the free distribution scheme, the road manager distributes all the permits to users for free according to methods that consider the equity among users. In this scheme, the permits assigned for each user do not necessarily match one’s own desired arrival time and path. For that case, users can mutually trade a wide variety of permits in the bottleneck permits markets: each user can buy the necessary permits while selling the unnecessary permits. As a result of such trading activities, permits are re-distributed among users through a system of prices emerging in the market. Thus, all income transfers take place only among the users in this scheme.

In both schemes, there are as many markets for trading permits as there are links, and each market is dedicated for trading the permits for each link. The permits for each link are further distinguished by a specified time allowable to use the link. Under the bottleneck permits system, each user who would like to use a path must have a set of permits corresponding to a set of links included in the path before making a trip. To fulfill this requirement, each user is assumed to purchase the needed set of permits in the trading markets. The price of each permit is determined by an auction system, which implies that the price is adjusted so as to clear the excess demand for each type of permit. We also assume that the markets are perfectly competitive; that is, neither a monopoly nor oligopoly occurs and there are no transaction costs.

**Remark 1.** It must be admitted that the procedures for trading permits seem unrealistic at first glance, but implementation of these would become feasible with advanced vehicles in which an agent software is installed to automatically trade permits on based on users’ preferences (e.g., origin-destination, desired arrival time and willingness-to-pay). Further discussion on the implementation of the trading markets and the agent system can be found in Wada and Akamatsu (2013) and Section 8.

## 4. Equilibrium under a system of tradable network permits

This section provides a model of equilibrium that takes place after introducing the tradable network permits system. We first describe the conditions that should be satisfied by several dynamic transportation cost variables, and a model of users’ behaviors for choosing permits is shown. We then formulate the equilibrium under the tradable permits system.

### 4.1. Dynamic travel costs in general networks

The transportation cost for a single trip of a network user consists of the following three types of costs: *a)* “schedule cost,” *b)* “travel cost,” *c)* “permit purchase cost.”

*a)* The “schedule cost” for a user is the cost due to the difference between the user’s desired arrival time period  $s$  and the actual arrival time period  $t$ . The desired arrival time period is assumed to be the same for all users and is equal to  $s$ . Later, we will extend it to general cases. The schedule cost is represented by the function  $w(t)$  of destination arrival time period  $t$ , which is common to all users, and is assumed to be a strictly convex function with a minimum at  $t = s$ , following previous studies on the departure time equilibrium (e.g., Smith, 1984; Daganzo, 1985; Kuwahara, 1990; Akamatsu, Wada, and Hayashi, 2015).

b) The “travel cost” is the monetary equivalent of the travel time needed for a trip from the origin to the destination. The travel times are different among paths. The travel time of a path between the origin-destination pair is defined as the sum of travel times of the links included in the path. Note that the travel time of each link  $(i, j)$  is a constant  $t_{ij}$  at equilibrium under the permits system, in which no queuing occurs. Hence, at equilibrium, the travel time  $T_r$  of path  $r$  between the origin-destination pair arrival time is also constant:

$$T_r = \sum_{ij \in L} t_{ij} \delta_{ij,r(o,d)} \quad (1)$$

where  $\delta_{ij,r(o,d)}$  is a typical element of the path-link incidence matrix for node pair  $(o, d)$ : it is 1 if link  $(i, j)$  is on path  $r$  connecting OD pair  $(o, d)$ ; otherwise, it is zero.

c) The “permit purchase cost” is the total payment for purchasing a set of link permits required for going through a path from the origin to the destination. To put it another way, the permit purchase cost of a user is defined as the sum of permit prices of the links included in the path used. This cost varies depending on what path is taken and at what time because the permits for each link are further differentiated by the specified time and each permit is priced depending on the time and the link.

To see this cost more precisely, consider a user who uses path  $r$  and arrives at the destination at time period  $t$ . Suppose here that he or she uses a path  $r$  that contains link  $(i, j)$  and  $T_{i,r}$  is the travel time required for arriving at the destination from node  $i$ :

$$T_{i,r} = \sum_{kl \in L} t_{kl} \delta_{kl,r(i,d)} \quad (2)$$

where  $\delta_{kl,r(i,d)}$  is a typical element of the path-link incidence matrix for node-pair  $(i, d)$ : it is 1 if link  $(k, l)$  is on path  $r$  from node  $i$  to the destination  $d$ ; otherwise, it is zero. Then, the user should enter into the link at time  $t - T_{i,r}$ , which implies that this user has to obtain the time  $t - T_{i,r}$  permit for link  $(i, j)$ , whose price is  $p_{ij}(t - T_{i,r})$ . It follows from this that the permit purchase cost (i.e., the total payment for purchasing the set of link permits required for going through path  $r$  and arriving at the destination at time period  $t$ ) is given by

$$P_r(t) = \sum_{ij \in L} p_{ij}(t - T_{i,r}) \delta_{ij,r(o,d)} \quad (3)$$

We call the sum of the travel cost and permit purchase cost as the path *transportation cost*; that is, the transportation cost  $C_r(t)$  of path  $r$  for a user arriving at the destination at time period  $t$  is given by

$$C_r(t) = P_r(t) + \alpha T_r, \quad (4)$$

where  $\alpha$  is a coefficient that converts travel time to the monetary equivalent.

#### 4.2. Users' behaviors

Each user chooses an arrival time period  $t$  at the destination and a path  $r$  between the origin and the destination so as to minimize the *generalized transportation cost*, defined as the sum of the schedule cost and transportation cost. That is, the user solves the following problem:

$$\min_{t,r} w(t) + C_r(t) \quad (5)$$

This optimization problem can be solved by “backward induction.” That is, the optimal choice pair  $(t^*, r^*)$  can be obtained by solving a two-stage (hierarchical) choice problem, in which one chooses separately the destination arrival time period  $t$  (the upper-level choice) and the optimal path  $r$  (the lower-level choice). More specifically, we first solve the lower-level problem of the path choice for a given arrival time period  $t$ ; we then obtain the optimal route choice  $r^*(t)$  and the optimal value  $\pi(t)$  conditional on arrival time period  $t$ :

$$\pi(t) = \min_r .C_r(t) \quad (6a)$$

$$r^*(t) = \arg . \min_r .C_r(t) \quad (6b)$$

By using this optimal choice function, we can reduce the upper-level problem of the arrival time choice to

$$\min_t .w(t) + \pi(t) \quad (7a)$$

$$t^* = \arg . \min_t .w(t) + \pi(t) \quad (7b)$$

We should note here that each user’s behavior under both the market selling scheme and free distribution scheme is identical to Eq.(5). Under the free distribution scheme, although each user receives an initial endowment of some permits, the revenue from selling the permits (i.e., change of the budget) do not affect his or her choices. This is because the above user’s behavior principle corresponds to maximizing a quasi-linear utility function, which is linear in price and implies there is no income effect. More intuitively, without loss of generality, we can assume that each user buy the set of permits required for the trip after selling all the initial endowment in the perfectly competitive trading markets because the law of one price holds at equilibrium.

#### 4.3. Equilibrium conditions

For the settings above, we construct a model that describes the equilibrium that arises in the presence of the proposed system. At equilibrium, the following six conditions as well as the definitional equations (1)–(4) for time-dependent cost variables should hold.

##### 4.3.1. Flow conservations

1) *Flow conservation for path flows and OD flows:*

OD flow  $q(t)$  for users arriving at the destination at time period  $t$  should be the sum of the path flows:

$$\sum_{r \in R} f_r(t) = q(t), \quad \forall t \in I \quad (8)$$

where  $f_r(t)$  is the flow of path  $r$  arriving at the destination at time period  $t$ ;  $R$  is the set of paths connecting the origin-destination pair.

2) *Flow conservation for OD flows and OD travel demands:*

All OD travel demands  $Q$  have to be assigned to each time point (in terms of the arrival time at the destination) in the interval  $[0, \mathcal{T}]$ ; that is, the time-dependent OD flows  $\{q(t)\}$  should satisfy

$$\sum_{t \in I} q(t) = Q. \quad (9)$$



3) *Flow conservation for link flows and path flows:*

The inflow on each link at each time period should be consistent with the time-dependent path flows. A user going through path  $r$  and entering into link  $(i, j)$  at time period  $t$  arrives at the destination at time  $t + T_{i,r}$ . Therefore, the inflow  $y_{ij}(t)$  entering into link  $(i, j)$  at time period  $t$  is the sum of the flows on all paths going through that link and arriving at the destination at  $t + T_{i,r}$ :

$$y_{ij}(t) = \sum_{r \in R} f_r(t + T_{i,r}) \delta_{ij,r(o,d)} \quad \forall t \in I, \forall ij \in L \quad (10)$$

4.3.2. *Users' choices*

4) *Equilibrium conditions for the path choice:*

As mentioned in Section 4.2, the user's optimal choice problem can be solved by "backward induction." By first obtaining the solution of the path choice problem as a function of arrival time period  $t$ , we then solve the arrival time choice problem. To obtain the equilibrium condition for the first problem, consider a user arriving at the destination at time period  $t$ . At equilibrium, no user can improve his or her own cost by changing the path choice unilaterally (i.e., no user has the incentive to deviate from his or her strategy). That is, transportation cost  $C_r(t)$  of path  $r$  (from the origin to the destination) that is chosen at equilibrium is equal to the minimum path travel cost  $\pi(t)$ , and the costs of other (unused) paths are greater than  $\pi(t)$ :

$$\begin{cases} \pi(t) = C_r(t) & \text{if } f_r(t) > 0 \\ \pi(t) \leq C_r(t) & \text{if } f_r(t) = 0 \end{cases} \quad \forall t \in I, \forall r \in R \quad (11)$$

5) *Equilibrium conditions for destination arrival time choice:*

At equilibrium, no one can improve his or her own generalized transportation cost by changing the destination arrival time unilaterally. It follows from the path choice equilibrium condition in 4) that the generalized transportation cost for a user arriving at time period  $t$  is  $w(t) + \pi(t)$ , where  $w(t)$  is the schedule cost for a user arriving at the destination at time period  $t$ . Therefore, the equilibrium condition for the user's arrival time choice can be expressed as

$$\begin{cases} \rho = \pi(t) + w(t) & \text{if } q(t) > 0 \\ \rho \leq \pi(t) + w(t) & \text{if } q(t) = 0 \end{cases} \quad \forall t \in I \quad (12)$$

where  $\rho$  represents the minimum (equilibrium) generalized transportation cost.

6) *Demand-supply equilibrium (market clearing) conditions in each link permit market:*

As mentioned in Section 3, the permits for each link are distinguished by a specified time period at which each permit makes it allowable to use the link. Let  $p_{ij}(t)$  denote the price of the permit for link  $(i, j)$  with a specified allowance time period  $t$ . Since the trading markets are assumed to be perfectly competitive (i.e., neither a monopoly nor oligopoly occurs), the price  $p_{ij}(t)$  of each permit type is adjusted to clear the excess demand for each type of permit. More precisely, at equilibrium, if the price of a certain type of permit is positive, the quantities supplied and the quantities demanded for the permit are equal; for the permit whose supply quantity exceeds the quantity demanded, the price is zero. Note here that, for each link  $(i, j)$  and each

allowance time period  $t$ , the demand of the time period  $t$  permit of the link is equal to the inflow  $y_{ij}(t)$ . On the other hand, the maximum supply (upper bound) of the time period  $t$  permit of link  $(i, j)$  is given by the link capacity  $\mu_{ij}$ . Therefore, the demand-supply equilibrium condition for the permits market is represented as

$$\begin{cases} y_{ij}(t) = \mu_{ij} & \text{if } p_{ij}(t) > 0 \\ y_{ij}(t) \leq \mu_{ij} & \text{if } p_{ij}(t) = 0 \end{cases} \quad \forall t \in I, \forall ij \in L \quad (13)$$

#### 4.4. Arc-node formulation

The path-based formulation presented above is convenient for presenting the user's route choice behavior in a straightforward manner; however, this formulation is not necessarily convenient for analyzing the demand-supply equilibrium condition (13) at each link due to the complexity of the relationship (10) between time-dependent path flows and link flows. In order to alleviate this complexity, we transform the model above into a formulation expressed in terms of link-node variables, which enables us to analyze the efficiency of the equilibrium, as is presented in later sections. The arc-node formulation of the model can be summarized by the following five conditions.

##### 4.4.1. Flow conservations

###### 1a) Flow conservation at each node:

Conservation of the dynamic traffic flow in a network is represented as the equality of inflow and outflow at each node at each time period. To formalize this, let  $y_{ij}(t)$  be the flow arriving at link  $(i, j)$  at time period  $t$ , and  $z_{ij}(t)$  be the flow departing from link  $(i, j)$  at time period  $t$ . Then the flow conservation is represented as

$$\sum_{k \in NO(i)} y_{ik}(t) - \sum_{k \in NI(i)} z_{ki}(t) = -q(t)\delta_{id}, \quad \forall t \in I, \forall i \in N \quad (14)$$

where  $\delta_{id}$  is Kronecker's delta (i.e., 1 if  $i = d$ , zero otherwise);  $NO(i)$  is a set of downstream nodes of the links incident from node  $i$ ;  $NI(i)$  is a set of upstream nodes of the links incident to node  $i$ .

###### 1b) First-In-First-Out conditions on each link:

We assume that the dynamic traffic flow in our model should satisfy the First-In-First-Out (FIFO) property on each link (i.e., we assume that passing can be neglected). As shown in the literature (see, for example, [Kuwahara and Akamatsu, 1993](#); [Akamatsu and Kuwahara, 1994](#)), the FIFO condition for each link can be written as

$$A_{ij}(t) = D_{ij}(t + t_{ij}(t)), \quad (15)$$

where  $A_{ij}(t)$  and  $D_{ij}(t)$  are the cumulative numbers of vehicles entering into and leaving from link  $(i, j)$  at time  $t$ , respectively. Using the flow variables, we can equivalently rewrite this as

$$y_{ij}(t) = z_{ij}(t + t_{ij}(t)) \cdot (1 + dt_{ij}(t)/dt), \quad (16)$$

where  $t_{ij}(t)$  is the travel time of link  $(i, j)$  for a user entering into the link at time  $t$ . Note here that  $t_{ij}(t)$  is a constant regardless of the arrival time when there is no queue. Hence, at equilibrium

under the permits system (i.e., when there is no queue in the network), the FIFO condition (16) reduces to the following simpler representation:

$$y_{ij}(t) = z_{ij}(t + t_{ij}) \quad \forall t \in I, \forall ij \in L \quad (17)$$

2) *Flow conservation for OD flows and OD travel demand:*

This is the same as the conditions in 2 in Section 4.3, that is,

$$\sum_{t \in I} q(t) = Q \quad (18)$$

4.4.2. *Users' choices*

3) *Equilibrium conditions for path choice:*

Consider a user arriving at node  $i$  at time period  $t$ . If the user chooses link  $(i, j)$ , the arrival time at node  $j$  is  $t + t_{ij}$ . Hence, at equilibrium, link  $(i, j)$  should be on the minimum path for a user arriving at node  $j$  at time  $t + t_{ij}$  if there exists a user entering into link  $(i, j)$  at time period  $t$ . Denoting  $\pi_i(t)$  as the minimum path cost from the origin to node  $i$  for a user arriving at the node at time period  $t$ , we can represent this condition as

$$\begin{cases} \pi_j(t + t_{ij}) = c_{ij}(t) + \pi_i(t) & \text{if } y_{ij}(t) > 0 \\ \pi_j(t + t_{ij}) \leq c_{ij}(t) + \pi_i(t) & \text{if } y_{ij}(t) = 0 \end{cases} \quad \forall t \in I, \forall ij \in L \quad (19)$$

where  $c_{ij}(t)$  is the transportation cost for a user who enters into link  $(i, j)$  at time period  $t$ :

$$c_{ij}(t) \equiv p_{ij}(t) + \alpha t_{ij} \quad (20)$$

The condition (19) is equivalent to the minimum cost path choice condition (11) represented by the path variables since (19) can be derived by applying the dynamic programming principle to (11).

4) *Equilibrium conditions for destination arrival time choice:*

Minimum path cost  $\pi(t)$  defined in Section 4.3 is the same as  $\pi_d(t)$  defined in (19) above. Hence, in a similar manner to the conditions in 5) in Section 4.3, the equilibrium conditions for the user's choice of destination arrival times is given by

$$\begin{cases} \rho = \pi_d(t) + w(t) & \text{if } q(t) > 0 \\ \rho \leq \pi_d(t) + w(t) & \text{if } q(t) = 0 \end{cases} \quad \forall t \in I \quad (21)$$

5) *Demand-supply equilibrium (market clearing) conditions in each link permit market:*

This is the same as the conditions in 6) in Section 4.3, that is,

$$\begin{cases} y_{ij}(t) = \mu_{ij} & \text{if } p_{ij}(t) > 0 \\ y_{ij}(t) \leq \mu_{ij} & \text{if } p_{ij}(t) = 0 \end{cases} \quad \forall t \in I, \forall ij \in L \quad (22)$$

## 5. Efficiency of the equilibrium with tradable network permits

### 5.1. Dynamic system optimal assignment without queuing

In order to examine the efficiency of the equilibrium allocation patterns defined in (14)–(22), consider the following optimization problem [P-1]:

$$\min_{(\mathbf{q}, \mathbf{y}) \geq 0} .F_P(\mathbf{q}, \mathbf{y}) \equiv \sum_{t \in I} q(t)w(t) + \alpha \sum_{ij \in L} \sum_{t \in I} y_{ij}(t)t_{ij} \quad (23)$$

subject to

$$\sum_{t \in I} q(t) = Q \quad (24)$$

$$y_{ij}(t) \leq \mu_{ij} \quad \forall t \in I, \forall ij \in L \quad (25)$$

$$\sum_{k \in NO(i)} y_{ik}(t) - \sum_{k \in NI(i)} y_{ki}(t - t_{ki}) = -q(t)\delta_{id}, \quad \forall t \in I, \forall i \in N \quad (26)$$

This is the problem of finding an optimal dynamic traffic flow pattern<sup>7</sup> that minimizes the total generalized transportation cost in the network, subject to the physical constraints of flows representing the network performance. Specifically, the first term of the objective function  $F_P$  is the total schedule cost expended by all users, and the second term is the monetary equivalent of the total travel time paid by all users. The first constraint (24) is conservation of the OD demand, the second constraint (25) is the traffic capacity constraints on each link. The final constraint (26) is the conservation of dynamic link flows at each node (14) combined with the FIFO condition on each link (17).

This problem does not necessarily have a feasible solution (satisfying constraints (24), (25) and (26)) due to the capacity constraint (25) on each link. The most extreme case for the nonexistence of a solution is when the time interval  $[0, \mathcal{T}]$  for the assignment is limited to an extremely short interval. In this case, the OD travel demand  $Q$  cannot be distributed into a time space and is forced to be distributed into only a network space (i.e., paths) while satisfying all the link capacity constraints. This implies that no feasible solution exists if  $Q/\mathcal{T}$ , which can take an extremely large value, exceeds the maximum network capacity.

As shown later in [Appendix A](#), however, we can easily examine whether or not the problem [P-1] has a feasible solution for any network. Also, [P-1] always has feasible solutions (and an optimal solution) if the assignment time interval  $[0, \mathcal{T}]$  is large enough that we can make OD flows smaller than the maximum capacity of the underlying network. Thus, we concentrate our discussions on the relation between the optimal assignment and the equilibrium assignment. For the discussions in later sections, we note that any feasible solutions of [P-1] imply traffic flow patterns with no queuing congestion on the network.

Under this assumption, the most important property that characterizes the equilibrium assignment is that [P-1] is an equivalent optimization problem to the equilibrium conditions (14)–(22). That is,

<sup>7</sup>We use the word “optimal” in the conditional sense: a state is “optimal” in the sense that the total transportation cost is minimized under the no-queuing condition. Note, however, that we can consider a state in which the total transportation cost minimized under the condition that queuing is allowed to occur. The problem of finding such a state may be non-convex problem and is difficult to analyze in general. Thus, we will address this issue as a future work.

**Proposition 1.** *For any networks with a single OD pair in which [P-1] has feasible solutions, the equilibrium assignment under the system of time-dependent tradable link permits minimizes the “social transportation cost” defined by (23).*

**Proof .** See [Appendix B](#) for the proof.

### 5.2. Another interpretation of the efficiency of equilibrium

On the surface, Proposition 1 implies that both the market selling scheme and the free distribution scheme produce the same outcome. It is partly true because the equilibrium traffic patterns under both schemes always coincide with the optimal traffic patterns for [P-1]. However, as noted in Section 3.4, distributions of income across the users and road manager are different under two schemes.

In order to see the fact more clearly, it is convenient to consider the dual problem, [D-1], of the optimal assignment problem [P-1].

$$\max_{(\rho, \pi, \mathbf{p}) \geq 0} .F_D(\rho, \pi, \mathbf{p}) \equiv \rho Q - \sum_{ij \in L} \sum_{t \in I} p_{ij}(t) \mu_{ij} \quad (27)$$

subject to

$$\rho \leq w(t) + \pi_d(t) \quad \forall t \in I \quad (28)$$

$$\pi_j(t + t_{ij}) \leq \pi_i(t) + (\alpha t_{ij} + p_{ij}(t)) \quad \forall t \in I, \forall ij \in L \quad (29)$$

It is clear from the proof of Proposition 1 that *a*) the optimal Lagrange multipliers  $\{\mathbf{p}^*(t)\}$  for constraint (25) coincide with the time-dependent link permit prices at equilibrium, *b*) the optimal Lagrange multiplier  $\rho^*$  for constraint (24) gives the generalized transportation cost at equilibrium, and *c*) the optimal Lagrange multipliers  $\{\pi^*(t)\}$  for (26) yield the transportation cost from the origin to each node at equilibrium.

From the duality theorem, the optimal value of the objective function of [D-1],  $F_D^*$ , coincides with the optimal value of the objective function of [P-1],  $F_p^*$ , that is,

$$\sum_{t \in I} q^*(t) w(t) + \alpha \sum_{ij \in L} \sum_{t \in I} y_{ij}^*(t) t_{ij} = \rho^* Q - \sum_{ij \in L} \sum_{t \in I} p_{ij}^*(t) \mu_{ij}. \quad (30)$$

The left-hand side of this equation,  $F_p^*$ , is the social transportation cost at the equilibrium under the system of tradable permits:

$$[\text{total schedule cost}] + [\text{total travel time cost}]. \quad (31a)$$

On the other hand, from the identity relation of transportation costs defined in Section 4.1, this should coincide with the value of

$$[\text{total generalized transportation cost}] - [\text{total permits payments}] \quad (31b)$$

at equilibrium. Therefore, the right-hand side of (30),  $F_D^*$ , should be the value of (31b) at equilibrium. Indeed, we can easily verify that  $F_D^*$  represents the social transportation cost at equilibrium with the form of (31b) if we interpret the optimal solution  $(\rho^*, \pi^*(t), \mathbf{p}^*(t))$  of [D-1] as the equilibrium prices under the tradable permits system. More specifically, the first term of  $F_D^*$  on the right-hand side of (30) denotes the equilibrium generalized transportation cost multiplied by the

number of users, which is the total generalized transportation cost paid by users in the network. Note that this “cost” includes users’ payments to purchase permits, but these purchase costs should not be counted as the “social cost” because they are just income transfers between the users and road manager. The total amount of these income transfers is given by the total market values of all the permits (i.e., the sum of the number of each permit  $\mu_{ij}$  multiplied by its price  $p_{ij}(t)$  for all links and times), which is given by the second term of  $F_D^*$ . Thus, we see that  $F_D^*$  represents exactly the social transportation cost at the equilibrium with the form of (31b).

The only difference in the dual problem [D-1] between the market selling scheme and free distribution scheme is the interpretation of the second term of  $F_D^*$ . Under the former scheme, the second term means the revenue for the road manager, while under the latter scheme it means the sum of the users’ revenues from selling the initial endowments. Therefore, it is obvious that the total revenue created by introducing the tradable network permits system is identical for both schemes. However, the absolute value of each user’s utility (e.g., revenue minus generalized transportation cost) under the free distribution scheme is always more than (or equal to) that under the market selling scheme. Note that this fact may not imply that the free distribution scheme is the best for society. Under the market selling scheme, if the road manager redistributes the revenue by an appropriate way, the efficiency can improve as shown in the next subsection.

### 5.3. Self-financing principle

In order for the “market selling scheme” of the tradable network permits to be socially acceptable, it may be desirable to redistribute the revenue from selling the permits to the road users. As a benchmark scheme of the redistribution, it is significant to consider the case in which the revenue is used for financing the capacity expansion of the network. For this type of redistribution scheme in congestion pricing, the “self-financing principle” has been well known since the pioneering work of [Mohring and Harwitz \(1962\)](#): for a congestible facility with a constant long-run average cost, the revenue from the optimal congestion pricing exactly covers the cost of the optimal capacity. A considerable number of studies have been made on whether or not this principle holds for a variety of assumptions (see, for example, [Keeler and Small, 1977](#); [Arnott and Kraus, 1995, 1998](#); [Yang and Meng, 2002](#)). Accordingly, it is worthwhile to investigate whether this principle applies to the equilibrium revenue of tradable permits. As the first step of this investigation, [Akamatsu et al. \(2006\)](#) showed that the principle also applies for the tradable permit scheme for a single bottleneck. As we shall see below, this principle can be extended to the scheme of general networks.

Consider the following minimization problem [P-K]:

$$\min_{(\mathbf{q}, \mathbf{y}, \boldsymbol{\mu}) \geq 0} .K(\boldsymbol{\mu}) + F_p(\mathbf{q}, \mathbf{y}) \quad (32)$$

subject to (24), (25) and (26).

This problem is almost the same as [P-1] except that *a*) the unknown (control) variables in [P-K] are not only the flow pattern  $(\mathbf{q}, \mathbf{y})$  but also the link capacity pattern  $\boldsymbol{\mu}$  that is assumed to be constant in [P-1], *b*) the “investment cost”  $K$ , which is precisely the sum of amortized investment cost and daily management cost, as a function of  $\boldsymbol{\mu}$  is added into the objective function  $F_p$  of [P-1]. This means that the social cost we would like to minimize is defined as the sum of the total transportation cost and the costs required for the increases in network capacity. The cost

function  $K$  is assumed to be homogeneous of degree 1 with respect to  $\boldsymbol{\mu}$ ; that is,

$$K(\boldsymbol{\mu}) = \sum_{ij \in L} \frac{\partial K(\boldsymbol{\mu})}{\partial \mu_{ij}} \mu_{ij} \quad (33)$$

The optimality conditions of [P-K] are similar to those of [P-1]. The only difference is in the addition of the following condition for the optimality of link capacity  $\boldsymbol{\mu}$ :

$$\mu_{ij}^* \frac{\partial L}{\partial \mu_{ij}} = 0, \quad \frac{\partial L}{\partial \mu_{ij}} \geq 0, \quad \mu_{ij}^* \geq 0 \quad \forall ij \in L \quad (34)$$

where  $L$  is the Lagrangean function of [P-K], and the derivative with respect to  $\boldsymbol{\mu}$  is given as

$$\frac{\partial L}{\partial \mu_{ij}} = \frac{\partial K(\boldsymbol{\mu})}{\partial \mu_{ij}} - \sum_{t \in I} p_{ij}(t). \quad (35)$$

Substituting (35) into (34), and summing the first equation of (34) for all links, we obtain

$$K(\boldsymbol{\mu}^*) = \sum_{ij \in L} \left\{ \mu_{ij}^* \sum_{t \in I} p_{ij}^*(t) \right\}. \quad (36)$$

where we use the homogeneity property (33) of function  $K$ . The left-hand side of (36) is the investment cost required for achieving the optimal link capacity pattern. The right-hand side of (36) is the total market value of the link permits at equilibrium under the tradable network permit system; this can be seen from the fact that, for any capacity pattern, the optimal Lagrange multiplier  $\{\mathbf{p}^*(t)\}$  of [P-1] coincides with the equilibrium permit prices, as shown in Section 5.2. Therefore, (36) means that the “self-financing principle” holds for the tradable network permit scheme. This can be summarized as follows:

**Proposition 2.** *Consider any networks with a single OD pair in which [P-1] has feasible solutions, and further suppose that the investment cost function  $K(\boldsymbol{\mu})$  is homogeneous of degree 1 in link capacities  $\boldsymbol{\mu}$ . Let the optimal investment cost be the cost required for increasing the link capacities so as to minimize the social cost defined in (32). Then the optimal investment cost is equal to the total market value of bottleneck permits (i.e., total revenue of the road manager); that is, the “self-financing principle” holds for the tradable network permits system.*

## 6. Congestion pricing vs. tradable permits

### 6.1. Perfect information case

The equilibrium permit prices formulated in Section 4 can be interpreted as the optimal toll levels for a congestion pricing scheme. To see this, consider a “dynamic congestion pricing scheme” (e.g., [Yang and Meng, 1998](#)) in which the road manager imposes a time-dependent toll (congestion tax) for each link in the network. Denoting by  $p_{ij}(t)$  the toll of link  $(i, j)$  for a user arriving at the link at time period  $t$ , we easily see that all transportation costs required for a user to make a trip have precisely the same form as the transportation costs under the permit system in Section 4. Consequently, it is obvious that the equilibrium states for the dynamic congestion pricing with a toll pattern  $\{\mathbf{p}(t)\}$  satisfy the conditions (14)–(21). In other words, we can interpret

the conditions (14)–(21) as those for defining the equilibrium flow pattern  $\{\mathbf{y}(t)\}$  that arises when the road manager sets a dynamic toll pattern  $\{\mathbf{p}(t)\}$ . With this in mind, suppose that the road manager wishes to prevent the occurrence of queuing congestion in the network by setting an appropriate toll pattern  $\mathbf{p}^*(t)$  such that the equilibrium inflow  $y_{ij}(t)$  of each link never exceeds the capacity  $\mu_{ij}$  (i.e., queuing congestion never occurs). Then, it follows that the conditions for achieving this goal are given by the same form as (22). That is, condition (22), which represents the market clearing (demand-supply equilibrium) condition for the tradable permit scheme, is now the condition for the road manager to achieve the optimal flow pattern under the congestion pricing scheme. Thus, the system of equations (14)–(22) can be viewed as the conditions for obtaining the optimal toll pattern when the equilibrium flow pattern does not cause queuing congestion.

As we have seen in Proposition 1, the system of equations (14)–(22) are also equivalent to the optimal assignment problem [P-1], and therefore, the pair  $\{\mathbf{p}^*(t), \mathbf{y}^*(t)\}$  of the optimal toll pattern and the resulting equilibrium flow pattern exist if [P-1] has feasible solutions. Therefore, we obtain the following proposition on the equivalence of the tradable permit scheme and the congestion pricing scheme:

**Proposition 3.** *Consider any networks with a single OD pair in which [P-1] has feasible solutions, and further suppose that the road manager has perfect information about the transportation demands. Then, the equilibrium assignment under the system of time-dependent tradable link permits coincides with the equilibrium assignment under the system of optimal dynamic congestion pricing.*

### 6.2. Imperfect information cases: Economic losses due to congestion pricing

The proposition above states that a congestion pricing scheme *under a certain condition* attains the same ideal state as does the tradable permit scheme. However, we should pay attention to the fact that the required conditions for attaining the ideal state in the tradable permit scheme are largely different from those in the congestion pricing scheme. The differences lie in the amount and accuracy of the information needed for the authority (i.e., the road manager) to implement the regulation. In the tradable permit scheme, what the road manager should know is just the traffic capacity of each link (and queuing congestion can be eliminated just by issuing the number of permits for each link that is equal to or less than the link capacity). In the congestion pricing scheme, on the other hand, it is not just traffic capacity that the road manager is required to know it is obvious from the discussion above that the road manager cannot calculate an appropriate toll pattern that prevents congestion without having accurate information on the users' behaviors (i.e., precise demands).

Considering the differences between the two schemes, the desirable transportation demand management scheme must be found. In generalized terms, this becomes the problem of comparing between “quantity-based regulation” and “price-based regulation.” In the field of economics, there have been many studies concerning the comparison of the two regulation schemes. According to the standard theory (see, example, [Weitzman, 1974](#); [Laffont, 1977](#)), quantity-based regulation produces more efficient outcomes than price-based regulation if a regulation authority has only imperfect information on the demand side conditions (i.e., demand functions) while having perfect information on the supply side conditions (i.e., supply functions). For the problem in this paper, we obtain a similar conclusion, although the underlying assumptions of our problem are different from those in conventional economic theory.



We show this fact below in more concrete terms. In our problem, what corresponds to “supply side conditions” is the network performance (traffic capacity of each link), for which information can be obtained accurately with relatively small effort. On the other hand, the information on the “demand side conditions” is the OD demand  $Q$ , the schedule cost function  $w(t)$ , the travel time pattern  $t$  and the value of time  $\alpha$ . It is not easy for the road manager to obtain this information accurately.

It is therefore natural to think of the events that happen in the dynamic congestion pricing scheme if the demand side information is not perfect. As a simple example, suppose that the road manager estimates the users’ value of time as  $\beta$ , which is different from the true value of time  $\alpha$ . Under this assumption, the road manager solves (14)-(22), and then obtains a toll pattern  $\{\mathbf{p}'(t)\}$  that is different from the true optimal toll pattern  $\{\mathbf{p}^*(t)\}$ . It is obvious that the traffic pattern  $\{\mathbf{y}'(t)\}$  arising under the incorrect toll pattern  $\{\mathbf{p}'(t)\}$  does not minimize the objective function of [P-1] (i.e., the sum of the schedule cost and travel cost), which causes economic loss in the social transportation cost. Worse still, there is no guarantee that the capacity constraint in each link is satisfied in the traffic flow patterns  $\{\mathbf{y}'(t)\}$ ; that is, queuing congestion may occur in the network. The reason is as follows: the manager believes that a traffic flow pattern  $\{\mathbf{y}''(t)\}$  should arise assuming that the users’ value of time is  $\beta$  and the toll pattern is  $\{\mathbf{p}'(t)\}$ , which is determined such that  $\{\mathbf{y}''(t)\}$  satisfies capacity constraint (22); but the flow pattern arising in reality is  $\{\mathbf{y}'(t)\}$  (i.e., the flow pattern that arises when the users’ value of time is  $\alpha$  and the toll pattern is  $\{\mathbf{p}'(t)\}$ ), and it follows that there is no necessity that  $\{\mathbf{y}'(t)\}$  satisfies (22). Note here that the social transportation cost in the optimal assignment problem [P-1] is defined under the assumption that queuing congestion never occurs. Consequently, in the congestion pricing scheme, any incorrect toll pattern  $\{\mathbf{p}'(t)\}$  not only minimizes the sum of the schedule cost and travel cost but also causes additional economic loss due to the occurrence of queuing congestion.

On the other hand, the tradable permit system never produces the above economic loss in theory, if the road manager just has correct information on the link capacities. Of course, there may be the possibility in reality that appropriate (theoretically correct) permit prices do not occur in the tradable permits market. This corresponds to the possibility of setting the incorrect toll pattern  $\{\mathbf{p}'(t)\}$  in the congestion pricing scheme. It may be said from conventional wisdom that the degree of mispricing in the markets<sup>8</sup> is likely to be lower than that of the authority’s mispricing. Furthermore, even if the degrees of mispricing are at a comparable level, there is a significant difference in the resulting effects of “distorted prices” between the two schemes. The effect of the distorted prices in the tradable permit system is confined to the loss in only the permit trading markets. They are irrelevant to the occurrence of queuing in the network, unlike the congestion pricing scheme. This is because the definition of the permit system ensures that the traffic flow never exceeds the number of permits issued, no matter how “incorrect” the permit prices are; thus, the tradable permit system never produces additional economic losses due to the occurrence of queuing, as observed in the congestion pricing system. This also implies that, even if an evolutionary (trial-and-error) implementation method is established for a dynamic congestion pricing scheme<sup>9</sup>, it may not fill the abovementioned gap between congestion pricing and tradable permit scheme. That is, like the mispricing case, the disequilibrium price during

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<sup>8</sup>This may occur if users reveal their information in the markets incorrectly (or untruthfully). Such an information problem can be addressed by the auction theory (Milgrom, 2004; Cramton et al., 2006). For the trading markets of bottleneck permits, Wada and Akamatsu (2010, 2013) showed efficient auction mechanisms that can induce truth-telling.

<sup>9</sup>Establishing such an evolutionary implementation method itself may be difficult due to the reason mentioned in Section 2.

an adjustment process to the equilibrium state inevitably produce economic losses due to the occurrence of queuing.

Still, one might argue that there may be the possibility in reality that queuing congestion occurs in the tradable permit system if the users' arrival times at each link are not accurate<sup>10</sup>. However, the possibility that such a phenomenon occurs as well as the resulting losses are all the same in dynamic congestion pricing, too. Hence, this can be ignored from the standpoint of the relative comparison of these two schemes. Thus, the discussion above can be summarized by the following proposition:

**Proposition 4.** *Consider any networks with a single OD pair in which [P-1] has feasible solutions, and further suppose that the road manager does **not** have perfect information about the transportation demands. Then, the equilibrium assignment under a system of dynamic congestion pricing incurs a larger total transportation cost defined as the sum of cost (23) and queuing delay cost than that in the equilibrium under a system of time-dependent tradable link permits.*

## 7. Extensions

### 7.1. The case with heterogenous user groups

The results of Section 4–Section 6 can be extended to the case with heterogeneous user groups only by differentiating all variables by the groups. Here we consider the case that many-to-many OD demands exist and desired arrival time periods are distributed. In this case, *a*) the arrival flow and the departure flow at each link are replaced by variables  $y_{ij}^o(t), z_{ij}^o(t)$  with respect to origin  $o$ , *b*) the minimum path travel cost from the origin to each node is also replaced by variables with respect to the origin, *c*) the variables with respect to OD pair are replaced by those with respect to origin-destination pair  $(o, d)$  and desired arrival time period  $s$ .

At the equilibrium under the tradable permits system with many-to-many OD pairs and general distribution of desired arrival time periods, the following conditions 1)–5) should hold.

1a) *Flow conservation at each node:*

At each node, the inflow and outflow with respect to origin should satisfy

$$\sum_{k \in NO(i)} y_{ik}^o(t) - \sum_{k \in NI(i)} z_{ki}^o(t) = -q_{od}(t)\delta_{id}, \quad \forall t \in I, \forall o \in O, \forall i \in N \quad (37)$$

where  $q_{od}(t)$  is the OD flow for a OD pair  $(o, d)$  arriving at the destination  $d$  at time period  $t$ ;  $O$  is the set of origins.

1b) *Firs-In-First-Out conditions at each link:*

At each link, the inflow and outflow with respect to origin should satisfy

$$y_{ij}^o(t) = z_{ij}^o(t + t_{ij}) \quad \forall t \in I, \forall o \in O, \forall ij \in L \quad (38)$$

<sup>10</sup>For that case, various practical treatments can be considered. One of reasonable treatments we assume here is to allow users who have permits but fail to reach a bottleneck within the scheduled time period to pass through that bottleneck.

2) *Flow conservation for OD flows and OD travel demands:*

Since the OD travel demand of OD pair  $(o, d)$  with desired arrival time  $s$  for the time interval is given as  $Q_{od}(s)$ , the following condition should hold for all OD pairs:

$$\sum_{t \in I} q_{od}(t, s) = Q_{od}(s), \quad \forall s \in S, \forall od \in W \quad (39)$$

where  $S, W$  are the set of desired arrival time periods and the set of origin-destination pairs, respectively;  $q_{od}(t, s)$  is the OD flow for a OD pair  $(o, d)$  with desired arrival time period  $s$  arriving at destination  $d$  at time period  $t$ . Further, the OD flow  $q_{od}(t, s)$  has to be consistent with the OD flow irrespective of desired arrival time  $q_{od}(t)$  that arises in the condition (37). Thus, the following conditions must hold:

$$q_{od}(t) = \sum_{s \in S} q_{od}(t, s) \quad \forall t \in I, \forall od \in W. \quad (40)$$

3) *Equilibrium conditions for path choice:*

Denoting  $\pi_i^o(t)$  as the minimum path cost from origin  $o$  to node  $i$  for a user arriving at the node at time period  $t$ , we can represent equilibrium conditions for path choice with respect to origin as

$$\begin{cases} \pi_j^o(t + t_{ij}) = c_{ij}(t) + \pi_i^o(t) & \text{if } y_{ij}^o(t) > 0 \\ \pi_j^o(t + t_{ij}) \leq c_{ij}(t) + \pi_i^o(t) & \text{if } y_{ij}^o(t) = 0 \end{cases} \quad \forall t \in I, \forall o \in O, \forall ij \in L \quad (41)$$

where  $c_{ij}(t)$  is the same definition as transportation cost in 3) of Section 4.4, that is,

$$c_{ij}(t) \equiv p_{ij}(t) + \alpha t_{ij} \quad (42)$$

4) *Equilibrium conditions for destination arrival time choice:*

Let  $w_d(t, s)$  be the schedule cost for a user with desired arrival time period  $s$  arriving at the destination  $d$  at time period  $t$ , and  $\rho_{od}(s)$  be the equilibrium generalized transportation cost for OD pair  $(o, d)$  with desired arrival time period  $s$ . Then the equilibrium conditions for destination arrival time choice with respect to OD pair and desired arrival time period as

$$\begin{cases} \rho_{od}(s) = \pi_d^o(t) + w_d(t, s) & \text{if } q_{od}(t, s) > 0 \\ \rho_{od}(s) \leq \pi_d^o(t) + w_d(t, s) & \text{if } q_{od}(t, s) = 0 \end{cases} \quad \forall t \in I, \forall s \in S, \forall od \in W \quad (43)$$

5) *Demand-supply equilibrium (market clearing) conditions in each link permit market:*

As in the case of single OD pair, each trading market is dedicated for trading the permits for each link. Note that the permits are differentiated by the specified time period. Thus the demand of the permit of link  $(i, j)$  for each unit time is equal to the inflow irrespective of origin  $o$ :

$$y_{ij}(t) = \sum_{o \in O} y_{ij}^o(t) \quad \forall t \in I, \forall ij \in L \quad (44)$$

The number of permits issued for each link is also equal to the traffic capacity of each link irrespective of origin  $o$ . Therefore, the demand-supply equilibrium condition for the permits market is represented as

$$\begin{cases} y_{ij}(t) = \mu_{ij} & \text{if } p_{ij}(t) > 0 \\ y_{ij}(t) \leq \mu_{ij} & \text{if } p_{ij}(t) = 0 \end{cases} \quad \forall t \in I, \forall ij \in L \quad (45)$$

Note here that the price of each permit needs not be differentiated by OD pairs.

In order to examine the efficiency of the equilibrium allocation patterns defined in (37)–(45), consider the following optimization problem [P-H] that is extended version of the problem [P-1]:

$$\min_{(\mathbf{q}, \mathbf{y}) \geq 0} J_P^H(\mathbf{q}, \mathbf{y}) \equiv \sum_{o \in O} \sum_{d \in D} \sum_{s \in S} \sum_{t \in I} q_{od}(t, s) w_d(t, s) + \alpha \sum_{ij \in L} \sum_{t \in I} y_{ij}(t) t_{ij} \quad (46)$$

subject to

$$\sum_{t \in I} q_{od}(t, s) = Q_{od}(s) \quad \forall s \in S, \forall od \in W \quad (47)$$

$$q_{od}(t) = \sum_{s \in S} q_{od}(t, s) \quad \forall t \in I, \forall od \in W \quad (48)$$

$$y_{ij}(t) = \sum_{o \in O} y_{ij}^o(t) \quad \forall t \in I, \forall ij \in L \quad (49)$$

$$y_{ij}(t) \leq \mu_{ij} \quad \forall t \in I, \forall ij \in L \quad (50)$$

$$\sum_{k \in NO(i)} y_{ik}^o(t) - \sum_{k \in NI(i)} y_{ki}^o(t - t_{ki}) = -q_{od}(t) \delta_{id}, \quad \forall t \in I, \forall i \in N, \forall o \in O \quad (51)$$

The objective function of this problem is the total generalized transportation cost (the sum of schedule cost and travel time) expended by all users in the network. The constraints, same as [P-1], are flow conservation and the traffic capacity constraints on each link. The only difference is that flow conservation condition corresponds to the case of many-to-many OD pairs and distribution of desired arrival time period. It is easy to imagine, from the above discussion, the problem [P-H] is an equivalent optimization problem to the equilibrium conditions (37)–(45). That is,

**Proposition 5.** *For any networks with many-to-many OD pairs in which [P-H] has feasible solutions, and further suppose that the desired arrival time periods are different among users. Then the equilibrium assignment under the system of time-dependent tradable link permits minimizes the “social transportation cost” defined by (46).*

**Proof .** We omit the proof of Proposition 5 because it is similar to the proof of Proposition 1: we should just confirm that the optimality conditions of the problem [P-H] coincides with the equilibrium conditions (37)–(45).

As in the case of single OD pair, link permit prices at equilibrium are can be obtained as the

solution to the dual problem, [D-H], of the problem [P-H]:

$$\max_{(\rho, \pi, \mathbf{p}) \geq 0} \cdot \sum_{s \in S} \sum_{od \in W} \rho_{od}(s) Q_{od}(s) - \sum_{ij \in L} \sum_{t \in I} p_{ij}(t) \mu_{ij} \quad (52)$$

subject to

$$\rho_{od}(s) \leq w_d(t, s) + \pi_d^o(t) \quad \forall t \in I, \forall s \in S, od \in W \quad (53)$$

$$\pi_i^o(t + t_{ij}) \leq \pi_i^o(t) + (\alpha t_{ij} + p_{ij}(t)) \quad \forall t \in I, \forall ij \in L, \forall o \in O \quad (54)$$

It is obvious that the objective function of this problem is, same as [D-1], the social transportation cost represented as [total generalized transportation costs]–[total permits payments].

### 7.2. The case with elastic demands

So far, we found that the equilibrium state under the tradable permits system is efficient when the OD demands  $\{Q_{od}(s)\}$  are assumed to be constant independent of the disutility between OD pairs. This result also holds when  $\{Q_{od}(s)\}$  are elastic with respect to the generalized transportation costs.

The OD demand is assumed to be monotone decreasing function  $Q_{od}(\rho_{od}(s), s)$  with respect to the generalized transportation cost  $\rho_{od}(s)$ . Then, the equilibrium state under the tradable permits system is the solution to the system of equations that are obtained by replacing only Eq. (39) in the equilibrium conditions in Section 7.1 by the following equation:

$$\sum_{t \in I} q_{od}(t, s) = Q_{od}(\rho_{od}(s), s) \quad \forall s \in S, \forall od \in W \quad (55)$$

Alternatively, for the case that the inverse function  $Q_{od}^{-1}(Q_{od}(s), s)$  (i.e., inverse demand function) of  $Q_{od}(\rho_{od}(s), s)$  is given, we can rewrite the Eq. (39) as

$$\begin{aligned} \sum_{t \in I} q_{od}(t, s) &= Q_{od}(s) & \forall s \in S, \forall od \in W \\ \begin{cases} \rho_{od}(s) = Q_{od}^{-1}(Q_{od}(s), s) & \text{if } Q_{od}(s) > 0 \\ \rho_{od}(s) \geq Q_{od}^{-1}(Q_{od}(s), s) & \text{if } Q_{od}(s) = 0 \end{cases} & \forall s \in S, \forall od \in W \end{aligned} \quad (56)$$

The equilibrium conditions coincide with the following optimization problem [P-E]:

$$\max_{(\mathbf{Q}, \mathbf{q}, \mathbf{y}) \geq 0} \cdot \sum_{s \in S} \sum_{od \in W} \int_0^{Q_{od}(s)} Q_{od}^{-1}(\omega, s) d\omega - F_P^H(\mathbf{q}, \mathbf{y}) \quad (57)$$

subject to (47), (48), (49), (50) and (51)

or its dual problem [D-E]:

$$\min_{(\rho, \pi, \mathbf{p}) \geq 0} \cdot \sum_{s \in S} \sum_{od \in W} \int_{\rho_{od}(s)}^{+\infty} Q_{od}(v, s) dv + \sum_{ij \in L} \sum_{t \in I} p_{ij}(t) \mu_{ij} \quad (58)$$

subject to (53) and (54)

The problem (57) is a consumer surplus maximization problem. More specifically, the first term of the objective function is total willingness to pay for making a trip (i.e., gross consumer

surplus), the second term is the total generalized transportation cost. Thus, the problem [P-E] is the problem of finding a dynamic traffic flow pattern that maximizes the consumer surplus represented as the gross consumer surplus minus the total generalized transportation cost. Since the equilibrium condition under the tradable permits system is equivalent to this problem, we conclude that the following proposition:

**Proposition 6.** *For any networks with many-to-many OD pairs in which the optimal allocation problem [P-E] has feasible solutions, and further suppose that the desired arrival time periods are different among users and the OD demand is a monotone decreasing function with respect to the generalized transportation cost. Then the equilibrium assignment under the system of time-dependent tradable link permits coincides with the optimal resource allocation maximizes the consumer surplus defined by (57).*

**Proof.** We omit the proof of Proposition 6 because it is trivial: we should only confirm that the optimality condition with respect to  $\mathbf{Q}$  in problem [P-E] (or  $\rho$  in problem [D-E]) coincides with Eq. (56) (Eq. (55)).

The propositions presented in this section imply that Proposition 3 and Proposition 4 in Section 6 hold for more general demand conditions. The equilibrium flow pattern and permit prices for the general demand case can be obtained by solving the equivalent problem [P-E] (or [P-H]). As in the case of single OD pair (Proposition 3) in Section 6, this equilibrium permit prices coincide with optimal toll levels under the optimal dynamic congestion pricing scheme. Therefore, in principle, the road manager can obtain an optimal toll pattern by solving the problem [P-E]. However, from the fact that the problem requires precise demand information, it would be clear that calculating an optimal toll pattern becomes more difficult when the demand conditions are generalized. On the other hand, in the tradable network permits scheme, the equivalent procedure that solves the optimization problem is conducted by users' trading in the permit markets in a decentralized manner. Thus, what the road manager should do is the same in all the cases discussed in Section 6 and this section. That is, the manager should know the traffic capacity of each link and issue the permits that is equal to its capacity. From the discussion, we can conclude that the properties in Proposition 4 is strengthened (i.e., *congestion pricing scheme becomes more disadvantageous*) in addition to being applicable to more general demand cases.

## 8. Concluding remarks

This paper considered a system of tradable bottleneck permits for general networks ("tradable network permits"). We first provided a model that describes time-dependent flow patterns at equilibrium under the proposed system. We then revealed that the equilibrium coincides with the optimal assignment pattern that minimizes the social transportation cost (i.e., the equilibrium under the proposed system is efficient). We also proved that the self-financing principle holds in the proposed scheme (i.e., the total market value of the permits is equal to the investment cost required for increasing the network capacity up to a socially optimal level). We further showed the theoretical relationship between the tradable permit system and congestion pricing: we demonstrated the definite advantages of the tradable permit system over the congestion pricing system when the demand information is not perfect, whereas they are equivalent for the perfect information case. Finally, we showed that the efficiency property of the proposed scheme can be extended to the more general cases such as networks with many-to-many OD pairs, heterogeneous users with different schedule delay function, and users with elastic trip demands.

We showed the efficiency of the tradable network permits system from the viewpoint of aggregated social costs (i.e., minimization of total transportation cost). This result, however, does not necessarily mean that the introduction of the proposed system always leads to Pareto improvement. In a single bottleneck case, the Pareto improvement can be proved since there is a one-to-one correspondence between the queuing delay (at equilibrium without the system) and the permit price (at equilibrium with the system). For networks with many bottlenecks, there are complex interactions among queuing delays at equilibrium without the system (see, for example, [Kawahara and Akamatsu, 1993](#); [Akamatsu, 2001](#); [Akamatsu and Heydecker, 2003](#)); the correspondences between the equilibrium queuing delays and the permit prices are not straightforward. Further exploration on the Pareto improvement property of tradable network permits is one of the important issues that should be addressed in future research. Note that this complex equilibrium may arise even under the system when considering a second best situation where the road manager can issue bottleneck permits for a limited number of bottlenecks.

In addition to the extensions on the “demand side” conditions presented in this paper, we can also generalize the theory to include “supply side” conditions: the equilibrium permit prices can be exploited for obtaining dynamic capacity allocation policies (e.g., signal control policies) that minimize the total transportation cost. More discussion of this issue can be found in [Wada and Akamatsu \(2012\)](#).

Although this paper analyzed the tradable network permits system from the theoretical point of view, there remain important issues for implementing the system. In particular, the cumbersome procedures for trading network permits should be alleviated. As we mentioned in Remark 2 in Section 3.4, this requirement may be met with a “multi-agent system” in which the vehicle-installed agent software automatically trades permits on behalf of each user. The components that determine essential properties of such a system are (i) a trading rule for the trading markets and (ii) a path/arrival time choice rule for each agent; these should be designed to make a traffic flow dynamics, which is realized by the rules, to reach the efficient equilibrium. The design of (off-line) trading rules was dealt with in [Wada and Akamatsu \(2010, 2013\)](#) and the modeling of agents’ behaviors was dealt with in [Wada et al. \(2008\)](#). Furthermore, as well as the (system) efficiency, travel flexibility and convenience for an individual have to be considered for the implementation rules. Designing dynamic/online trading rules and cancellation rules enhancing the flexibility of users’ decision making is one of the examples in this direction (e.g., [Nagae and Gai, 2009](#); [Wada et al., 2012](#); [Wada, 2013](#)).

## Appendix A. Feasibility of the assignment without queuing

We can examine whether or not the problem [P-1] has a feasible solution for any networks by simply constructing “time-space networks.” The concept of “time-space network” may be best explained by the example of a train diagram. Denoting  $x_i$  as the position of the  $i$ -th train in a 1-dimensional guideway as a function of time period  $t$ , we can depict the trajectories of trains,  $\{x_i(t)\}$ , in a 2-dimensional (time, space) plane. For a set of appropriate time points  $\{t_k\}$ , the trajectories provide a set of points  $\{t_k, x_i(t_k)\}$  in the plane; regarding these points as a set  $N$  of “nodes,” we can define a set  $L$  of “links” by treating each segment of trajectories as a link connecting two nodes in  $N$ . The two sets  $N$  and  $L$  thus obtained constitute a time-space network  $G(N, L)$  for the train diagram.

By applying a procedure similar to that for the train diagram above, we can construct the time-space network for analyzing dynamic flows in a general network. This is a powerful approach for solving a range of network problems with constant link travel times (for more detailed examples

of this approach, see [Ieda et al., 1988](#); [Cascetta, 2001](#)), although it is difficult to apply this to the problems with state-dependent link travel times. We can take advantage of this approach to check the feasibility of [P-1] since we do not need to consider queuing for the feasible solution of [P-1], which implies that the link travel times in [P-1] are constant.

The “time-space network” is constructed by the following procedure:

**Step 1-a.** Replicates  $M$  copies of each node in the original network  $G(N, L)$ .

**Step 1-b.** For the expanded node set, add a single dummy node for the origin in  $G$ , and similarly, add a single new dummy node for the destination in  $G$ .

The set of nodes created by *Step 1-a* and *Step 1-b* is the node set  $N^*$  of the time-space network. To identify each node in  $N^*$ , we denote by  $i(m)$  the node in  $N^*$  that represents node  $i$  at time  $m\Delta t$ . We denote by  $o^*$  the dummy node in  $N^*$  for the origin  $o$ , and denote by  $d^*$  in  $N^*$  the dummy node for the destination  $d$ .

**Step 2-a.** For every  $m$ , create a link going from node  $i(m)$  toward node  $j(m + n_{ij})$  if there exists a link with travel time  $t_{ij} = n_{ij}\Delta t$  in the original network  $G(N, L)$ . The attributes (i.e., travel cost and capacity) of each link created above is the same as those of the corresponding original link in  $L$ . Repeat this procedure for all links in  $L$ .

**Step 2-b.** For every  $m$ , create a link going from dummy node  $o^*$  toward origin copy node  $o(m)$  in  $N^*$ . For these new links, the travel cost is set to zero, and the capacity is infinity. For every time point  $m$ , create a link incident from destination copy node  $d(m)$  toward dummy node  $d^*$ . For these new links, the travel cost is set to  $w_d(m\Delta t)$ , and the capacity is infinity.

A set of links created by *Step 2-a* and *Step 2-b* is the link set  $L^*$  of the time-space network.

The dynamic system optimal assignment problem [P-1] (or its equivalent of the equilibrium problem (14)-(22)) reduces to a “static” minimum cost flow problem defined on the time-space network  $G^*(N^*, L^*)$  constructed by the procedure above. More specifically, consider a static minimum cost flow problem defined on  $G^*$ , in which the origin node is given by the dummy node  $o^*$ , the destination node is the dummy node  $d^*$ , and the OD travel demand is given as  $Q$ . We then regard the flow on link  $(i(m), j(m + n_{ij}))$  in  $G^*$  as the dynamic link flow  $y_{ij}(m\Delta t)$  in the original network  $G$ , and the flow on link  $(d(m), d^*)$  in  $G^*$  as the dynamic OD flow  $q(m\Delta t)$  in  $G$ . For this setting, the optimal solution of [P-1] is given by the link flow pattern in  $G^*$  obtained as the solution of the minimum cost flow problem above.

To understand this fact, it is sufficient to check that all conditions of [P-1] are represented as the conditions for the minimum cost flow problem defined on  $G^*$ . We first see that conservation of the dynamic flow in [P-1] is automatically represented by conservation of the static flow at each node on  $G^*$ . Constraint (26) is satisfied by the flow conservation at each node  $i(m)$  on  $G^*$ ; constraint (24) is satisfied by the flow conservation at each dummy node  $d$  in *Step 2-a* of the procedure to construct  $G^*$ . We also see that the capacity constraint of each link (25) for each time period corresponds to the capacity constraint of each link on  $G^*$ . Furthermore, the objective function of [P-1] is consistently represented by the total travel cost in the minimum cost flow problem defined for  $G^*$ : the first term of the objective function is represented by the sum of travel costs for the set of links  $(d(m), d^*)$  constructed in *Step 2-b*, and the second term is the sum of travel costs for the set of links constructed in *Step 2-a*.

Thus, the dynamic optimal assignment problem [P-1] is reduced to solving a static minimum cost flow problem on  $G^*$ . The static minimum cost flow problem has been well studied in computer science (see, for example, [Kennington and Helgason, 1980](#); [Ahuja et al., 1993](#)), and very



efficient (polynomial order) algorithms are available for solving this problem. Therefore, we see that the feasibility of [P-1] can be easily verified by using these algorithms.

For more general demand (heterogeneous user groups and elastic demands) cases presented in Section 7, we can also examine the feasibility of the problems [P-H] and [P-E] by using time-space networks. Specifically, for the problem [P-H] (heterogeneous user groups case), *Step 1-b* and *Step 2-b* should be replaced with the following procedures:

**Step 1-b’.** For the expanded node set, add a dummy node  $o_o^*$  for each origin  $o \in O$  in original network, and add a new dummy node  $d_d^*(s)$  for each destination-desired arrival time period pair  $(d, s) \in D \times S$ .

**Step 2-b’.** For every  $m$ , create a link going from each dummy node  $o_o^*$  toward origin copy node  $o(m)$ . For these new links, the travel cost is set to zero, and the capacity is infinity. For every time point  $m$ , create a link incident from destination copy node  $d(m)$  toward each dummy node  $d_d^*(s)$ . For these new links, the travel cost is set to  $w_d(m\Delta t, s)$ , and the capacity is infinity.

For this time-space network, the travel demand for each OD pair  $(o, d)$  with desired arrival time  $s$  is given as  $Q_{od}(s)$ . We then regard the flow with respect to origin  $o$  on  $(i(m), j(m + n_{ij}))$  as the  $y_{ij}^o(m\Delta t)$ , and flow with respect to origin  $o$  on link  $(d(m), d_d^*(s))$  as the dynamic OD flow  $q_{od}(m\Delta t, s)$ . For this setting, from the same discussion of the problem [P-1], the problem [P-H] reduces to a (static) minimum cost multicommodity flow problem defined on the time-space network explained above, and its feasibility can be easily verified.

In the time-space network for the problem [P-H], the problem [P-E] (elastic demands case) can be regarded as a static traffic assignment problem with elastic demands<sup>11</sup>. It is well-known that the elastic-demand static traffic assignment problem can be transformed into a fixed-demand problem in an expanded network (see, for details of several transformation methods, [Gartner, 1980](#)). Therefore, by using such a transformation method, the problem [P-E] can be reduced to a fixed-demand traffic assignment problem defined on the slightly modified time-space network. We can easily verify the feasibility of [P-E] by using existing efficient algorithms for the static traffic assignment problem.

## Appendix B. Proof of Proposition 1

We will show that a necessary and sufficient condition for the optimality of the optimization problem [P-1] coincides with the equilibrium conditions (14)-(22). To derive the optimality conditions, we first define Lagrangean function  $L$  for the problem [P-1]:

$$\begin{aligned}
L(\mathbf{q}, \mathbf{y}, \rho, \boldsymbol{\pi}, \mathbf{p}) &\equiv F_P(\mathbf{q}, \mathbf{y}) & (B.1) \\
&+ \rho \left\{ Q - \sum_{t \in I} q(t) \right\} + \sum_{ij \in L} \sum_{t \in I} p_{ij}(t) \{ y_{ij}(t) - \mu_{ij} \} \\
&+ \sum_{i \in N} \sum_{t \in I} \pi_i(t) \left\{ q(t) \delta_{id} + \sum_{k \in NO(i)} y_{ik}(t) - \sum_{k \in NI(i)} y_{ki}(t - t_{ki}) \right\},
\end{aligned}$$

<sup>11</sup>Note that each link cost is independent of link flow in this problem.

where function  $F_P$  is the objective function of [P-1] defined in (23);  $\rho$ ,  $\{\mathbf{p}(t)\}$ , and  $\{\boldsymbol{\pi}(t)\}$  are Lagrange multipliers corresponding to constraints (24), (25), and (26), respectively. Then, the necessary and sufficient conditions for the optimality of [P-1] are given by the following Kuhn-Tucker conditions:

$$\begin{cases} \partial L / \partial q^*(t) = 0 & \text{if } q^*(t) > 0 \\ \partial L / \partial q^*(t) \geq 0 & \text{if } q^*(t) = 0 \end{cases} \quad \forall t \in I \quad (\text{B.2})$$

$$\begin{cases} \partial L / \partial y_{ij}^*(t) = 0 & \text{if } y_{ij}^*(t) > 0 \\ \partial L / \partial y_{ij}^*(t) \geq 0 & \text{if } y_{ij}^*(t) = 0 \end{cases} \quad \forall t \in I, \forall ij \in L \quad (\text{B.3})$$

$$\partial L / \partial \rho^* = 0 \quad (\text{B.4})$$

$$\begin{cases} \partial L / \partial p_{ij}^*(t) = 0 & \text{if } p_{ij}^*(t) > 0 \\ \partial L / \partial p_{ij}^*(t) \leq 0 & \text{if } p_{ij}^*(t) = 0 \end{cases} \quad \forall t \in I, \forall ij \in L \quad (\text{B.5})$$

$$\partial L / \partial \pi_i^*(t) = 0 \quad \forall t \in I, \forall i \in N \quad (\text{B.6})$$

It can be easily seen that conditions (B.4), (B.5), and (B.6) reduce to the equilibrium conditions (18), (22), and (14), respectively. To examine conditions (B.2) and (B.3), we calculate the partial derivatives of the Lagrangean function:

$$\partial L / \partial q^*(t) = w(t) + \pi_d(t) - \rho \quad (\text{B.7})$$

$$\partial L / \partial y_{ij}^*(t) = (\alpha t_{ij} + p_{ij}(t)) + \pi_i(t) - \pi_j(t + t_{ij}) \quad (\text{B.8})$$

Substituting (B.7) into (B.2), we have the same form of conditions as in equilibrium condition (21); similarly, we see that (B.3) reduces to the equilibrium condition (19). Thus, Lagrange multipliers  $\mathbf{p}^*(t)$ ,  $\boldsymbol{\pi}^*(t)$ , and  $\rho^*$  in the optimality conditions (B.2)-(B.6) coincide with the equilibrium link permit prices, equilibrium minimum path costs, and the equilibrium generalized transportation costs in equilibrium conditions (14)-(22); the optimal flow patterns  $(\mathbf{q}^*(t), \mathbf{y}^*(t))$  also coincide with the equilibrium flow patterns. *Q.E.D.*

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