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Stochastic stability of dynamic user equilibrium in unidirectional networks: Weakly acyclic game approach☆

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ABSTRACT

The aim of this study is to analyze the stability of dynamic user equilibrium (DUE) with fixed departure times in unidirectional networks. Specifically, stochastic stability, which is the concept of stability in evolutionary dynamics subjected to stochastic effects, is herein considered. To achieve this, a new approach is developed by synthesizing the three concepts: the decomposition technique of DUE assignments, the weakly acyclic game, and the asymptotic analysis of the stationary distribution of perturbed dynamics. Specifically, we first formulate a DUE assignment as a strategic game (DUE game) that deals with atomic users. We then prove that there exists an appropriate order of assigning users for ensuring equilibrium in a unidirectional network. With this property, we establish the relationship between DUE games in unidirectional networks and weakly acyclic games. The convergence and stochastic stability of better response dynamics in the DUE games are then proved based on the theory of weakly acyclic games. Finally, we observe the properties of the convergence and stability from numerical experiments. The results show that the strict improvement of users' travel times by the applied evolutionary dynamics is important for ensuring the existence of a stochastically stable equilibrium in DUE games.

1. Introduction

There remain several issues regarding the theoretical properties of dynamic user equilibrium (DUE) (Iryo, 2013), such as existence, uniqueness, and stability. Stability is a particularly important property for equilibrium states to be realized in a transportation system. If the stability of an equilibrium state is not guaranteed, then the state cannot be preserved against small perturbations. As a result, the corresponding equilibrium flow pattern would just be an extreme state of rare occurrence (Beckmann et al., 1956).

In the analysis of the stability of equilibrium, we first consider evolutionary dynamics (also called day-to-day dynamics) that describes how users change their routes according to the current traffic state, such as the Smith dynamics (Smith, 1984),

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projection dynamics (Zhang and Nagurney, 1996), and others (see Sandholm, 2010). We then investigate the local and/or global behavior of the dynamics.[1] In particular, a powerful tool for the latter is a Lyapunov function for the dynamics. For example, Smith (1984) showed the convergence of the Smith dynamics to a set of equilibria in a static traffic assignment when the route travel time functions are monotonic. Moreover, if the functions are strictly monotonic, the unique equilibrium becomes convergent and stable[2] (i.e., asymptotically stable). For DUE problems with fixed departure times, Smith and Ghali (1990) showed the monotonicity of the route travel time functions in a network in which each route contains only one bottleneck (called single-bottleneck-per-route network). Utilizing this property, Mounce (2006) proved the existence of the Lyapunov function for the Smith dynamics and demonstrated the convergence of the dynamics to a set of equilibria in the network.

While the monotonicity is a desirable property for proving the existence of a Lyapunov function, it is not a sufficient condition for the DUE to be stable because the existence guarantees the convergence to a set of equilibria only. This means that small perturbations might lead a state away from an equilibrium state to another equilibrium state. Moreover, the monotonicity is not a general property of DUE problems. It is known that the monotonicity is not guaranteed even in simple cases, such as when a route has two or more bottlenecks (Kuwahara, 1990; Mounce and Smith, 2007). Thus, it may be difficult to apply the Lyapunov approach to DUE problems in networks other than the single-bottleneck-per-route networks; we need to develop a different approach to examine the stability of DUE in networks in which route travel time functions are not monotonic.

In this study, we analyze the stability of DUE with fixed departure times through a novel analytical approach synthesizing several concepts, each of which is developed in dynamic traffic assignment or game theory. This approach consists of the following three concepts: (A) the decomposition technique of DUE assignments, (B) the weakly acyclic game, which is a class of strategic games, and (C) the asymptotic analysis of the stationary distribution of perturbed dynamics, which is evolutionary dynamics subject to perturbations. More specifically, we study DUE problems in unidirectional networks (Iryo and Smith, 2018)[3] to which the concept (A) can apply as we will prove in this study. In the analysis, we first formulate a DUE assignment as a strategic game (we call it "DUE game") that deals with atomic users. We then prove the existence of appropriate order of assigning users one by one for ensuring equilibrium. With this ordering property, we further establish the relationship between DUE games in unidirectional networks and weakly acyclic games. We finally examine the stochastic stability of equilibrium (Young, 1993), which is the stability concept in perturbed dynamics, based on the theory of weakly acyclic games. Below, we outline our approach with brief reviews of each concept utilized in this study.

(A) The decomposition technique of DUE assignments, which is proposed by Kuwahara and Akamatsu (1993), is a concept for analyzing the equilibrium through the decomposition of a DUE assignment. This technique is applicable to a special class of networks, such as single-origin or single-destination networks. For example, in a single-origin network, the DUE assignment can be decomposed with respect to the departure time from the unique origin, i.e., we can solve the DUE assignment sequentially in the order of departure. This is because the equilibrium concept along with the first-in-first-out (FIFO) principle and the causality implies that users departing from the origin must arrive at any node not later than the others leaving the same origin at later departure times. In other words, the order of departure from the origin must be kept at any intermediate node.

This technique has been utilized to investigate the properties of equilibrium analytically (Akamatsu, 2000; Akamatsu et al., 2015; Wada et al., 2018) and to develop solution algorithms (Akamatsu, 2001; Waller and Ziliaskopoulos, 2006) for the model with continuum users. On the other hand, Iryo (2011) and Satsukawa and Wada (2017) explored the applicability of the technique to DUE games by using a slightly generalized ordering property from one for the continuum model. As we will show in this paper, a DUE game with this property can be viewed as a weakly acyclic game that is the basis of our analysis of the stochastic stability.

(B) The weakly acyclic game (Young, 1993; Marden et al., 2009) is a class of strategic games in which there exist better response paths (i.e., sequences of users' better responses) from every state to a (pure) Nash equilibrium. In this game, evolutionary dynamics always converges to a set of equilibria under mild conditions. The rest points of the dynamics play a central role in the analysis of the existence of a stochastically stable equilibrium.

(C) The analysis of the stationary distribution is useful for investigating the dynamical behavior of a transportation system subject to stochastic effects. In this analysis, we first consider perturbed dynamics, which is continually perturbed by time-independent small mutations or mistakes. This dynamics is a Markov chain on a finite state space and has a unique stationary distribution. Thus, we can investigate the long run behavior of the dynamical system regardless of the initial state. In the traffic assignment field, several studies investigated the stationary distribution of link/route flows, i.e., the mean and expected values of flows (e.g., Cascetta, 1989; Cantarella and Cascetta, 1995; Watling, 1998; Balijepalli and Watling, 2005; Cantarella and Watling, 2016).

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[1] Since stability properties are examined subject to the applied dynamics, they can be different under other dynamics.

[2] We explicitly distinguish convergence from stability based on the definitions in previous studies (e.g., Watling, 1998; Watling and Cantarella, 2013), although the former is sometimes called stability (e.g., Smith, 1984). Specifically, we define an equilibrium as stable if we can ensure that the state is arbitrarily close to the equilibrium for all times, and as convergent if the state will approach the equilibrium as time approaches infinity.

[3] Unidirectional networks are a more general class of networks than single-bottleneck-per-route networks in the sense that it includes multiple-bottleneck-per-route networks. However, there does not exist the simple inclusion relation between two classes.

Meanwhile, in game theory, the stability property of equilibrium has been investigated from the asymptotic behavior of the stationary distribution. Specifically, it has been shown that when the perturbations become small, the distribution becomes concentrated around particular states (Foster and Young, 1990; Young, 1993). These are called stochastically stable states, which will be observed frequently in the long run when the perturbations are small, and can be regarded as plausible states to be realized in the real world. These stable states are contained in a set of the rest points of the unperturbed version of the dynamics. This means that, in a weakly acyclic game, there always exists a stochastically stable equilibrium when the convergence of the unperturbed dynamics is guaranteed.

As a result, we can establish novel results on the stability of DUE by synthesizing these concepts. While each of the concepts is developed in each research field, this study presents several unique contributions. First, we establish the convergence and stability results of DUE in unidirectional networks whereas previous studies analyzing these properties have been limited to single-bottleneck-per-route networks. Second, we find that the strict improvement of users' travel times by the applied evolutionary dynamics is important for ensuring the existence of a stochastically stable equilibrium in DUE games. Third, we demonstrate the new role of the decomposition technique in the analysis of the theoretical properties of DUE, such as convergence, stochastic stability, and the relationship with game theory.

The remainder of this paper is organized as follows. In Section 2, we introduce the definition of the DUE game and a decomposition-based solution algorithm. Section 3 presents the proof of the ordering property of a DUE game in a unidirectional network with the decomposition technique. Section 4 establishes the relationship between the DUE game and the weakly acyclic game. Based on this relationship, we investigate the convergence and stochastic stability of DUE in this section and in Section 5. Section 6 observes these properties from the numerical experiments. Section 7 concludes the paper.

2. DUE game and decomposition solution algorithm

In this section, we first define a strategic game of a DUE assignment that deals with atomic users, which we call "DUE game." The DUE game consists of a road network, vehicles traveling through the network (atomic users), strategy sets (route sets), and utilities of vehicles (travel times of users). The Nash equilibrium of this game is considered as the user equilibrium of dynamic traffic assignments. Then, we describe a decomposition-based solution algorithm for the DUE game.

2.1. Definition

2.1.1. Network

A general road network with many-to-many origin-destination (OD) demands is herein considered. The network consists of a set of nodes $\mathcal{N}$ and a set of directed links $\mathcal{L}$. Sets of origin nodes and destination nodes are denoted by $\mathcal{N}_o$ and $\mathcal{N}_d$, respectively. A set of all acyclic routes from node a to node b is denoted by $\mathcal{R}(a, b)$. When these nodes are not connected, $\mathcal{R}(a, b) = \emptyset$. $\mathcal{N}(r)$ is a set of nodes included in route r.

2.1.2. Users and strategies

Each vehicle is considered as an atomic user of the game. A set of users is denoted by $\mathcal{P}$ and the number of the users is denoted by $|\mathcal{P}|$. The origin, destination, and departure time of user $i \in \mathcal{P}$ are denoted by o_i, d_i, and s_i, respectively. These are given exogenously. All users departing from the same origin have different departure times.

User i can choose any acyclic route r_i to travel between the user's origin and destination. A set of routes of user i is denoted by $\mathcal{R}_i$ $(= \mathcal{R}(o_i, d_i))$. This set includes the option ϕ_i when user i does not select any route (i.e., user i is not assigned onto the network). A user selecting this option is called a *non-assigned user*. We utilize this class of users in a decomposition-based solution algorithm. A route profile (strategy profile) of all users is denoted by $\mathbf{r} \equiv \{r_1, \dots, r_i, \dots, r_{|\mathcal{P}|}\} \in \mathcal{R}$ where $\mathcal{R} \equiv \mathcal{R}_1 \times \dots \times \mathcal{R}_{|\mathcal{P}|}$. In addition, the route profile of users other than user i is denoted by $\mathbf{r}_{-i} \equiv \{r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_{|\mathcal{P}|}\}$. With this notation, we sometimes represent a profile $\mathbf{r}$ as $(r_i, \mathbf{r}_{-i})$ when the route of a certain user should be clearly stated.

2.1.3. Utility and Nash equilibrium

We assume that the disutility of each user is equal to the travel time of the user. Because the departure time of each user is fixed, the user's arrival time at the destination can be regarded as the user's disutility. In this regard, $g_i(r_i; \mathbf{r}_{-i})$ denotes the arrival time of user i when the route of the user is r_i and the route profile of the other users is $\mathbf{r}_{-i}$. Note that we set the arrival time of a non-assigned user to infinity: $g_i(\phi_i; \mathbf{r}_{-i}) = \infty$. We denote by $u_n(o_i, s_i, r_i; \mathbf{r}_{-i})$ the arrival time at node n when a user departing from $o_i \in \mathcal{N}_o$ at time s_i takes route r_i and the route profile of the others is $\mathbf{r}_{-i}$.

The node arrival times are determined by a dynamic loading model when the routes of all users are determined. Any appropriate model which satisfies a few natural conditions for dynamic loading can be employed. Note that the FIFO principle and the causality (Carey et al., 2003) must hold. Note also that a physical queue concept can be incorporated into the model (e.g., the car-following model by Newell, 2002).

Under the above settings, DUE is defined as (pure) Nash equilibrium of the game. In a Nash equilibrium, all users choose their routes to minimize their travel times. This corresponds to the situation where all users choose the best response strategy $\mathbf{r}^*$, which is mathematically described as

$$g_i(r_i^*; \mathbf{r}_{-i}^*) \le g_i(r_i; \mathbf{r}_{-i}^*), \quad \forall r_i \in \mathcal{R}_i, \forall i \in \mathcal{P}. \tag{1}$$

Furthermore, if Eq. (1) holds with strict inequality for all users, then $\mathbf{r}^*$ is called a *strict* Nash equilibrium.

2.2. Decomposition-based solution algorithm for atomic users

Iryo (2011) proposed a *decomposition-based solution algorithm* that obtains a Nash equilibrium of a DUE game by assigning users to a network one by one in appropriate order. Although the existence of such order for ensuring equilibrium is not expected in general networks, we will prove the existence in unidirectional networks in Section 3.

In this algorithm, we first consider the situation where all users are non-assigned users, i.e., we set the initial route of each user $i \in \mathcal{P}$ to ϕ_i. A set of non-assigned users for route profile $\mathbf{r}$ is denoted by $\overline{\mathcal{P}}(\mathbf{r})$. Then, the algorithm assigns each non-assigned user onto the user's shortest route one by one in order such that a user who is already assigned will not be overtaken by users who will be assigned later. This assignment order is not necessarily equal to the order of the departure times of users because a user may overtake those who have earlier departure times with different origins.[4] A Nash equilibrium is obtained when all users are assigned to the network (i.e., $r_i \neq \phi_i, \forall i \in \mathcal{P}$).

To determine the order mentioned above, this algorithm utilizes the concept of the *earliest non-assigned user*. The earliest non-assigned user is a non-assigned user that is not overtaken by the other non-assigned users when the user takes the shortest route. For the formal definition of the earliest non-assigned user, we define the earliest arrival time at a node and the shortest route of a user as follows:

Definition 1 (**Earliest arrival time at a node**). Consider a route profile $\mathbf{r} \in \mathcal{R}$ and user i who has departure time s_i from origin o_i. We denote by $\mathcal{N}_i$ a set of nodes included in at least one route in the route set of the user, i.e., $\mathcal{N}_i = \{n \mid n \in \mathcal{N}(r), r \in \mathcal{R}_i\}$. Then, the earliest arrival time of user i at node $n \in \mathcal{N}_i$ for the fixed route profile $\mathbf{r}_{-i}$ is denoted by $u_n^*(o_i, s_i; \mathbf{r}_{-i})$, and defined as follows:

$$u_n^*(o_i, s_i; \mathbf{r}_{-i}) = \min_{r \in \{r' \mid r' \in \mathcal{R}_i, n \in \mathcal{N}(r')\}} u_n(o_i, s_i, r; \mathbf{r}_{-i}). \tag{2}$$

Definition 2 (**Shortest route**). Consider a route profile $\mathbf{r} \in \mathcal{R}$. The shortest route of user i is denoted by r_i^* and is defined as follows:

$$r_i^* \in \arg\min_{r \in \mathcal{R}_i} g_i(r; \mathbf{r}_{-i}), \tag{3}$$

$$\text{s.t.} \quad u_n(o_i, s_i, r_i^*; \mathbf{r}_{-i}) = u_n^*(o_i, s_i; \mathbf{r}_{-i}), \quad \forall n \in \mathcal{N}(r_i^*). \tag{4}$$

Eq. (3) expresses the *best response* route of the user: when user i takes route r_i^*, the user arrives at the destination not later than when the user takes the other routes. Eq. (4) states that the user arrives at *any node* included in the route not later than when the user takes the other routes, i.e., dynamic programming (DP) principle holds. A set of the shortest routes of the user departing from origin o at s for the destination d is denoted by $\mathcal{SR}(o, s, d; \mathbf{r})$.

Note that the reason why Eq. (4) is explicitly included in the definition is that Eq. (4) is not automatically satisfied on the route satisfying Eq. (3) in networks where vehicle queues exist. Suppose that when a user takes the shortest route on which the DP principle holds, the user catches up with the tail end of the queue on a link included in the route. Then, the arrival times at the destination by taking other routes become the same with that of the shortest route if the user can catch up with the same queue (see, for example, the route passing link 2 in Fig. 3). This is because the departure time from the link is restricted by the queue dissipation time: these routes can satisfy Eq. (3) without satisfying Eq. (4). Since such routes causes some difficulty in defining the unidirectional network, we do not regard them as the shortest routes in this paper.[5]

We are now ready to define the earliest non-assigned user as follows:

Definition 3 (**Earliest non-assigned user**). Consider a route profile $\mathbf{r} \in \mathcal{R}$ such that $\overline{\mathcal{P}}(\mathbf{r}) \neq \emptyset$. A non-assigned user $i \in \overline{\mathcal{P}}(\mathbf{r})$ is the earliest non-assigned user when the user has the shortest route $\bar{r}_i^*$ satisfying the following condition[6]:

$$u_n(o_i, s_i, \bar{r}_i^*; \mathbf{r}_{-i}) \leq u_n^*(o_j, s_j; \mathbf{r}_{-j}), \quad \forall n \in \mathcal{N}(\bar{r}^*) \cap \mathcal{N}_j, \forall j \in \overline{\mathcal{P}}(\mathbf{r}) \setminus \{i\}. \tag{5}$$

Note that when $|\overline{\mathcal{P}}(\mathbf{r})| = 1$, the non-assigned user automatically becomes the earliest non-assigned user.

When earliest non-assigned user i takes the shortest route $\bar{r}_i^*(\mathbf{r})$ satisfying Eq. (5), the other non-assigned users cannot overtake user i. By combining this property with the FIFO principle and the causality of the dynamic loading model, it is guaranteed that the travel time of the shortest route $\bar{r}_i^*(\mathbf{r})$ is independent of route choices of the other non-assigned users. In addition, although the travel time of another route of the earliest non-assigned user may be dependent on the route choices, it does not become earlier than that of $\bar{r}_i^*(\mathbf{r})$. This is because the travel time is not decreased by assigning a user onto a network. This implies that, among the non-assigned users, the earliest non-assigned user should have the earliest assignment order for ensuring equilibrium. Therefore, by assigning the earliest non-assigned users to their shortest

[4] The solution algorithm in which the users are assigned to the network one by one as the order advances must be distinguished from a simulation in which the users enter the network in the order of their departure times and move simultaneously as time advances.

[5] When a DP based algorithm (such as Dijkstra's algorithm) is utilized to search for a route satisfying Eq. (3), Eq. (4) is automatically satisfied on the route. However, because the algorithm utilized in network loading procedures has no relation to the definition itself, Eq. (4) should be included.

[6] The node arrival time of a non-assigned user is constrained by assigned users only but not by the other non-assigned users because non-assigned users are not loaded to the network yet and do not physically interact with each other. Thus, the equality in Eq. (5) may hold.

routes one by one, all users come to choose the ex-post shortest routes, i.e., a Nash equilibrium (Eq. (1)) is obtained. The decomposition-based solution algorithm is now summarized as follows.

Decomposition-based solution algorithm for DUE games (Iryo, 2011)

0. **Initial setting**: Let the number of step $m = 0$ and all users are set as non-assigned users $\overline{\mathcal{P}}(\mathbf{r}^m) = \mathcal{P}$, i.e., $r_i^m = \phi_i, \forall i \in \mathcal{P}$, where r_i^m represents the route of user i at step m.
1. **Choose the earliest non-assigned user**: Find the earliest non-assigned user $i \in \overline{\mathcal{P}}(\mathbf{r}^m)$ according to Eq. (5) for the given route profile $\mathbf{r}^m$. If there exist multiple earliest non-assigned users, pick one of them.[7]
2. **Update the route profile and set of non-assigned users**: Assign user i to the shortest route $\overline{r}_i^*(\mathbf{r}^m)$ by using an appropriate dynamic loading model. Let $\mathbf{r}^{m+1} := (\overline{r}_i^*(\mathbf{r}^m), \mathbf{r}_{-i}^m)$ and $\overline{\mathcal{P}}(\mathbf{r}^{m+1}) := \overline{\mathcal{P}}(\mathbf{r}^m) \setminus \{i\}$.
3. **Judge the convergence**: If $\overline{\mathcal{P}}(\mathbf{r}^{m+1}) = \emptyset$, then terminate the algorithm; $\mathbf{r}^{m+1}$ is a Nash equilibrium. Otherwise, let $m := m+1$, and go back to Step 1.

As can be seen from the logic, the existence of the appropriate order for ensuring equilibrium corresponds to the existence of the earliest non-assigned user at every step. In the next section, we prove the existence of the earliest non-assigned user in a unidirectional network.

3. DUE game in unidirectional networks

3.1. Unidirectional network and its properties

The unidirectional network is a generalization of single-origin or single-destination networks in the sense that the decomposition technique is applicable to the DUE games. More specifically, for DUE games in these networks, the earliest arrival times at all nodes from all origins can be represented by functions of a *reference time* of a reference node. We call these functions *node potential functions*. For example, in a single-origin network, the earliest arrival times can be functions of a departure time from the origin, as shown in Kuwahara and Akamatsu (1993).

Besides the existence of such a (global) reference time for the DUE problems, there exists a monotonic relation between the reference time and the earliest arrival time at each node. That is, the earliest arrival time does not go back in time if the reference time advances. In a single-origin network, this ordering property is ensured because, as mentioned in Section 1, the equilibrium concept along with the FIFO principle and the causality implies that users departing from the origin must arrive at any node not later than the others leaving the same origin at later departure times. We thus can decompose the DUE problems with the ordering property in the order of the reference time.

To define the unidirectional network formally, we first introduce the possible dynamical link travel time profiles according to Iryo and Smith (2018). These profiles are utilized to guarantee the existence of the node potential functions independently of route choices and demand patterns. In other words, a unidirectional network has the node potential functions regardless of which patterns of the following possible dynamical link travel time profiles are given:

Definition 4 (**Possible dynamical link travel time profiles**). The possible dynamical link travel time profile on link l is a set of functions defined by

$$C_l^P := \left\{ c_l(t) \mid c_l(t) \in \left\{ \text{Any Lipschitz continuous function} \quad \text{s.t. } c_l(t) \geq c_l^{FT} \text{ and } \frac{c_l(t') - c_l(t)}{t' - t} \geq -1 \right\} \right\}. \tag{6}$$

where $c_l(t)$ is the travel time of link l for a user entering at time t, and c_l^{FT} is the free flow travel time of link l.

This profile represents the feasible set of the link travel time functions with a dynamic loading model satisfying the FIFO principle. The vector of $c_l(t)$ for all links is denoted by $\mathbf{c}$. $\mathcal{C}^P = \prod_{l \in \mathcal{L}} C_l^P$ is used to specify the Cartesian product of C_l^P for all links.

Definition 5 (**Unidirectional network**). Given a network, choose an arbitrary origin as a reference node and denote it by $o_{REF} \in \mathcal{N}_o$. Then, the network is a unidirectional network if and only if for any $\mathbf{c} \in \mathcal{C}^P$, there exists the potential function $p_n(t; \mathbf{c})$ on node n as a function of reference time t as follows:

$$p_n(t; \mathbf{c}) = u_n^*(o, p_o(t; \mathbf{c}); \mathbf{c})), \quad \forall n \in \mathcal{N}(r^*), \exists r^* \in \mathcal{SR}(o, p_o(t; \mathbf{c}), d; \mathbf{c})), \forall o \in \mathcal{N}_o. \forall d \in \mathcal{N}_d. \tag{7}$$

$$\text{where} \quad p_{o_{REF}}(t; \mathbf{c}) = t. \tag{8}$$

where $u_n^*(o, p_o(t; \mathbf{c}); \mathbf{c}))$ is the earliest arrival time at node n for given link travel time profiles of $\mathbf{c}$.

For given link travel time profiles of $\mathbf{c}$, Eq. (7) shows that the potential function $p_n(t; \mathbf{c})$ on node n having reference time t is defined as the earliest arrival time at the node when a user departing from origin o at time $p_o(t; \mathbf{c})$ takes the shortest

[7] The resulting equilibrium state is dependent on how to select the earliest non-assigned user among them.

route r^* to destination d. To describe the potential function for a given route profile of $\mathbf{r}$, we represent the link travel time profile as a function of the route profile, $\mathbf{c}(\mathbf{r})$; as a result, the other variables can be defined as functions of $\mathbf{r}$ as follows: $u_n^*(o, s; \mathbf{r}) \equiv u_n^*(o, s; \mathbf{c}(\mathbf{r}))$ and $p_n(t; \mathbf{r}) \equiv p_n(t; \mathbf{c}(\mathbf{r}))$.

We then show the monotonic relation between the node potential and the reference time, which is the extension of Theorem 2 by Iryo and Smith (2018).[8]

Theorem 1. *Consider a unidirectional network with a route profile* $\mathbf{r}$ *and reference times* t *and* t'*. Then, the following relationship is obtained:*

$$t < t' \Rightarrow p_n(t; \mathbf{r}) \le p_n(t'; \mathbf{r}), \quad \forall n \in \mathcal{N}. \tag{9}$$

Proof. We prove the theorem by combining the following Lemmas 1 and 2. We here omit $\mathbf{c}(\mathbf{r})$ in the proofs for simplicity.

We first prove Lemma 1 that states the monotonic relation between the potential on each origin and the reference time.

Lemma 1. *Consider a unidirectional network with a route profile* $\mathbf{r}$ *and reference times* t *and* t'*. Then, the following relation holds:*

$$t < t' \Rightarrow p_o(t) \le p_o(t'), \quad \forall o \in \mathcal{N}_o. \tag{10}$$

Proof. The outline of the proof is described as follows. We first establish monotonic relations between potentials on the nodes that are included in the shortest routes from the same origin. Specifically, we first consider two shortest routes of users departing from an arbitrary origin $o \in \mathcal{N}_o$ at the potential $p_o(t)$ and $p_o(t')$ to arbitrary destinations that the users can reach. For the potentials on nodes o and n included in both of these shortest routes, we have

$$\begin{cases} p_o(t) \le p_o(t') \Rightarrow p_n(t) \le p_n(t'), \\ p_n(t) \le p_n(t') \Rightarrow p_o(t) \le p_o(t') \end{cases}. \tag{11}$$

Thus, considering the given relationship $t < t'$ on the reference node (origin), we obtain Eq. (10) by recursively applying Eq. (11) from the reference node to all origins and other nodes (see Appendix A for details). □

Next, we prove Lemma 2 from the definition of the unidirectional network, which states that the potentials can be defined on all nodes including those that are not on the shortest routes without violating Eq. (7).

Lemma 2. *Consider a unidirectional network with a route profile* $\mathbf{r}$*. Then, the potential* $p_n(t; \mathbf{r})$ *on node* n *having reference time* t *can be expressed as the minimum value of the earliest arrival times of users departing at the potentials having the same reference time on their origins. That is,*

$$p_n(t) = \min_{o \in \mathcal{N}_o} u_n^*(o, p_o(t)), \quad \forall n \in \mathcal{N}. \tag{12}$$

Proof. First, we consider the case when node n is not included in any shortest routes of users departing from their origins at potentials having reference time t. Because the potential on this type of the node is not defined in Definition 5, we define the potential as Eq. (12).

Next, we consider the case when node n is included in at least one of the shortest routes. We here divide origins into the two groups: (i) Group A includes the origins from which users departing at potentials have the shortest routes including node n (denoted by $\mathcal{N}_{o(n)}^A$), and (ii) Group B includes the other origins (denoted by $\mathcal{N}_{o(n)}^B$). For the origins in $\mathcal{N}_{o(n)}^A$, $p_n(t) = u_n^*(o, p_o(t)),\ \forall o \in \mathcal{N}_{o(n)}^A$ holds by the definition of the unidirectional network. Thus, it is sufficient for us to prove that the following equation holds for the origins in $\mathcal{N}_{o(n)}^B$:

$$p_n(t) \le u_n^*(o, p_o(t)), \quad \forall o \in \mathcal{N}_{o(n)}^B. \tag{13}$$

We will prove it by contradiction. Suppose that the earliest arrival time at node n of a user departing from origin o in $\mathcal{N}_{o(n)}^B$ is earlier than the potential, i.e., $u_n^*(o, p_o(t)) < p_n(t)$. Then, the user has route r_B that passes through the same nodes included in the way to destination d from node n along the shortest route of a user departing to d from an origin in $\mathcal{N}_{o(n)}^A$; and the arrival times at these nodes are not later than the potentials, e.g., $u_n(o, p_o(t), r_B) \le p_n(t)$ for node n. Because r_B is not the shortest route, there exists at least one node n' on which the DP principle does not hold at the downstream of node n along the route; the user can arrive at this node n' earlier by taking the shortest route r_B^* that satisfies the DP principle, i.e., $u_{n'}(o, p_o(t), r_B^*) < u_{n'}(o, p_o(t), r_B)$. Therefore, the following equation holds:

$$p_{n'}(t) = u_{n'}(o, p_o(t), r_B^*) < u_{n'}(o, p_o(t), r_B) \le p_{n'}(t). \tag{14}$$

This contradicts the definition of the unidirectional network. Hence, Eq. (13) is satisfied and Eq. (12) is obtained. □

Finally, by combining the results above, we have

$$t < t' \Rightarrow p_o(t) \le p_o(t'), \quad \forall o \in \mathcal{N}_o$$

[8] While Iryo and Smith (2018) showed a stronger relation, $t < t' \Rightarrow p_n(t; \mathbf{r}) < p_n(t'; \mathbf{r}),\ \forall n \in \mathcal{N}$, but it requires an additional strong condition named the *positive flow assumption*.

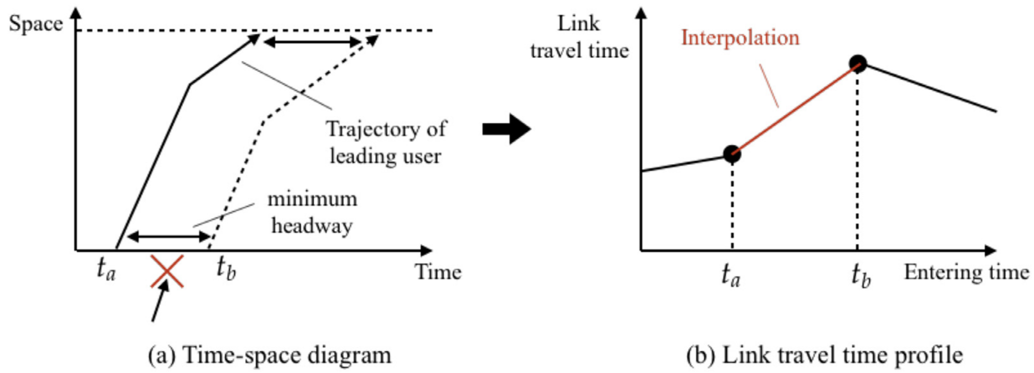


Fig. 1. The time-space diagram showing the relationship between trajectories of the leading user and following user and the link travel time profile on the link.

$$\Rightarrow \min_{o} \{u_n^*(o, p_o(t))\} \leq \min_{o} \{u_n^*(o, p_o(t'))\} \quad \forall n \in \mathcal{N}$$
$$\Rightarrow p_n(t) \leq p_n(t').$$

The first line of these equations is obtained from Lemma 1; the second line is a direct consequence of the first line, and the third line is obtained from Lemma 2. Therefore, Eq. (9) is obtained. □

While we implicitly assume that the node potential function can be defined for all reference times, these are not the cases for DUE problems with atomic users. On each node, there may exist some time intervals during which the node potential function cannot be defined. This is because there exist time intervals during which no user arrives at a node due to the following two reasons. First, a user cannot enter the link in the case that the user has less headway to its leading user than the minimum headway of the link (see Fig. 1(a)). In other words, there exist time intervals during which the link travel time function cannot be defined. Second, for the time interval during which no user departs from origins, Eq. (7) is not basically applicable because it is defined using the shortest routes of *users*. However, we can resolve these problems by introducing some appropriate interpolations. Specifically, we can resolve the first problem by introducing linear interpolations satisfying Eq. (6) (i.e., the FIFO principle) to the functions (see Fig. 1(b)). With regard to the second problem, what we should do is to assign virtual vehicles, who do not affect actual vehicles' behavior of the DUE game, to calculate node potentials. In both cases, we can define the potential function without affecting the travel times of the users.

3.2. Existence of earliest non-assigned user in unidirectional network

We are now ready to prove the existence of the earliest non-assigned user through the decomposition technique as follows:

Theorem 2. *Consider a route profile* $\mathbf{r} \in \mathcal{R}$ *such that* $\overline{\mathcal{P}}(\mathbf{r}) \neq \emptyset$ *in a unidirectional network. There exists at least one earliest non-assigned user.*

Proof. We first consider the reference time t_j of user j such that $p_{o_j}(t_j) = s_j,\ \forall j \in \overline{\mathcal{P}}(\mathbf{r})$. Suppose that user i has the minimum reference time t_i. Then, there exists the shortest route $\bar{r}_i^*$ of user i such that

$$\begin{aligned} u_n^*(o_i, s_i; \mathbf{c}(\mathbf{r}_{-i})) &= p_n(t_i; \mathbf{c}(\mathbf{r}_{-i})), \\ &\leq p_n(t_j; \mathbf{c}(\mathbf{r}_{-j})), \\ &\leq u_n^*(o_j, s_j; \mathbf{c}(\mathbf{r}_{-j})), \quad \forall n \in \mathcal{N}(\bar{r}_i^*) \cap \mathcal{N}_j, \quad \forall j \in \overline{\mathcal{P}}(\mathbf{r}) \setminus \{i\}. \end{aligned}$$

The first line of these equations means that user i takes the shortest route satisfying Eq. (7); the second line is obtained from Theorem 1, and the third line is a direct consequence of Eq. (12). This means that user i arrives at the nodes on route $\bar{r}_i^*$ not later than any other non-assigned user, i.e., user i has the shortest route satisfying Eq. (5). Thus, user i is the earliest non-assigned user. □

4. Weakly acyclic games and convergence of DUE game in unidirectional network

In this section, we establish the relationship between the DUE game and the *weakly acyclic game* (Young, 1993). Specifically, we first introduce the definition and the concept of the weakly acyclic game. We then prove that a DUE game in a unidirectional network is a weakly acyclic game. After the proof, we investigate the convergence of the DUE game as the basis for investigating the stability in the next section.

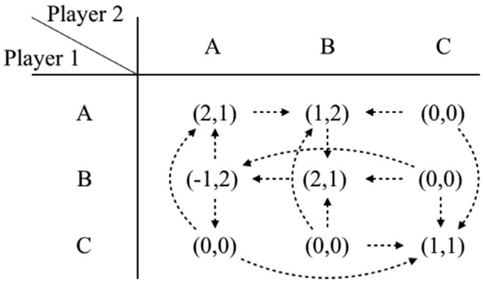


Fig. 2. Example of a two-player weakly acyclic game (we refer to Marden et al., 2009).

Note that, hereinafter, we consider the situation that a DUE game is repeated on a day-to-day basis. Let $\tau = 1, 2, \ldots$ denote the successive time steps (e.g., day). The route and the utility of user $i \in \mathcal{P}$ at day τ are denoted by r_i^τ and $g_i(r_i^\tau; \mathbf{r}_{-i}^\tau)$, respectively. At each day, one user is randomly selected (we call this user a *selected user*) and the user can change the current route to another route according to a behavioral rule which is common to all users (we specify it later).

4.1. Definition of weakly acyclic games

The weakly acyclic game is characterized by the following *better response path*. Consider any game with a set of finite strategy profiles $\mathcal{R}$. A better response path is a sequence of strategy profiles $\mathbf{r}^1, \mathbf{r}^2, \ldots$ such that, for each successive pair $\mathbf{r}^\tau, \mathbf{r}^{\tau+1}$, there is exactly one user i that satisfies the following condition:

$$\begin{cases} r_i^\tau \neq r_i^{\tau+1}, & \text{s.t. } g_i(r_i^{\tau+1}; \mathbf{r}_{-i}^{\tau+1}) < g_i(r_i^\tau; \mathbf{r}_{-i}^\tau), \\ r_j^\tau = r_j^{\tau+1}, & \forall j \in \mathcal{P} \setminus \{i\}. \end{cases} \tag{15}$$

In other words, at each day, only one user changes the route to improve the arrival time at the destination.

The weakly acyclic game is then defined as follows (see also, Young, 1993; Marden et al., 2009):

Definition 6 (**Weakly acyclic game**). A game is weakly acyclic if, and only if from any strategy profile $\mathbf{r} \in \mathcal{R}$, there exists a better response path starting at $\mathbf{r}$ and ending at a (pure) Nash equilibrium of the game.

Note that whether a game is weakly acyclic or not is independent of the evolutionary dynamics applied to the game, i.e., only dependent on the structure of the game that determines the existence of such a better response path in the game. The weakly acyclic game is a generalized concept of *potential games* (Monderer and Shapley, 1996). Specifically, potential games are a special class of weakly acyclic games, which do not have cycles in better response paths. However, both the weakly acyclic game and the potential game have better response paths ending at a Nash equilibrium.

An example of a two-player (user) weakly acyclic game is illustrated in Fig. 2. This figure shows the payoff matrix of the game and the dotted lines from a profile represent the better responses from the profile. As can be seen from the figure, there exist better response paths ending at the Nash equilibrium of the game (C, C) from every profile. For example, from profile (B, B), there exists a better response path $(B, B) \to (B, A) \to (C, A) \to (C, C)$. Note that a better response path including a cycle may exist in a weakly acyclic game (e.g., $(B, B) \to (B, A) \to (A, A) \to (A, B) \to (B, B)$ in the figure). However, a Nash equilibrium is reachable from any profile in the cycle through the appropriate better response path in the weakly acyclic game.

4.2. A DUE game in a unidirectional network is a weakly acyclic game

Now we establish the relationship between a DUE game in a unidirectional network and a weakly acyclic game:

Theorem 3. *A DUE game in a unidirectional network is a weakly acyclic game.*

Proof. We prove the existence of a better response path ending at a Nash equilibrium from every route profile in a constructive manner. Specifically, we construct an algorithm that is guaranteed to converge to a Nash equilibrium from an arbitrary initial route profile by changing users' routes so that they improve the arrival times at the destinations (i.e., the better response).

We first introduce some definitions. The users of the game are divided into the two groups: (i) Group A includes users who take ex-post best response routes, and (ii) Group B includes users who do not. The sets of users in Group A and Group B are denoted by $\mathcal{P}_A$ and $\mathcal{P}_B$, respectively. For a given route profile $\mathbf{r}$, we also denote by $\mathbf{r}_A$ and $\mathbf{r}_B$ the route profiles of Group A and Group B, respectively (i.e., $\mathbf{r} = (\mathbf{r}_A, \mathbf{r}_B)$). We then propose the following algorithm that finds a Nash equilibrium by transferring the users from $\mathcal{P}_B$ to $\mathcal{P}_A$ one by one:

0. **Initial setting**: Let $m = 0$, $(\mathcal{P}_A^m, \mathcal{P}_B^m) = (\emptyset, \mathcal{P})$ and $\mathbf{r}^m = (\mathbf{r}_A^m, \mathbf{r}_B^m) = (\emptyset, \mathbf{r}^0)$. Here $\mathbf{r}^0$ is an initial route profile.

1. **Find the earliest non-assigned user**: Consider a new DUE game with only users in $\mathcal{P}_B^m$ but route profile $\mathbf{r}_A^m$ is given and fixed as a constraint. Then, regard the users in $\mathcal{P}_B^m$ as the non-assigned users and search for the earliest non-assigned user $i \in \mathcal{P}_B^m$ according to the criterion described in Section 2. We denote by g_i^* and $\bar{r}_i^*$ the earliest arrival time at the destination and the shortest route of user i, respectively.
2. **Update the route profile through better responses**: Compare g_i^* with the arrival time $g_i(r_i^m; \mathbf{r}_{-i}^m)$ at the destination of user i with the current route r_i^m. Then, there are exhaustive two cases:
   (a) If $g_i^* < g_i(r_i^m; \mathbf{r}_{-i}^m)$, then update the route of user i to $\bar{r}_i^*$, i.e., $r_i^{m+1} := \bar{r}_i^*$. The other users keep their current routes: $r_j^{m+1} := r_j^m$ for all $j \in \mathcal{P}_B^m \setminus \{i\}$.
   (b) If $g_i^* = g_i(r_i^m; \mathbf{r}_{-i}^m)$, first search for a route profile $\mathbf{r}'_B$ of the users in $\mathcal{P}_B^m$ that satisfies the following conditions: (i) $\mathbf{r}'_B$ satisfies $g_i(r_i^m; \mathbf{r}_{-i}^m) < g_i(r'_i; \mathbf{r}'_{-i})$ where $\mathbf{r}' = (\mathbf{r}_A^m, \mathbf{r}'_B)$, and (ii) $\mathbf{r}'_B$ is reachable from $\mathbf{r}_B^m$ through better responses of the users in $\mathcal{P}_B^m$. If there exists such a route profile, then update the route profile in the following way: $r_i^{m+1} := \bar{r}_i^*$ and $r_j^{m+1} := r'_j$ for all $j \in \mathcal{P}_B^m \setminus \{i\}$ (Case (b)-1). If such a route profile does not exist, then $r_j^{m+1} := r_j^m$ for all $j \in \mathcal{P}_B^m$ (Case (b)-2).
3. **Update the sets of users and judge the convergence**: Let $\mathcal{P}_A^{m+1} := \mathcal{P}_A^m \cup \{i\}$ and $\mathcal{P}_B^{m+1} := \mathcal{P}_B^m \setminus \{i\}$. If $\mathcal{P}_B^{m+1} \neq \emptyset$, let $m := m+1$ and go back Step 1. If $\mathcal{P}_B^{m+1} = \emptyset$, then terminate the algorithm; $\mathbf{r}^{m+1} = \mathbf{r}_A^{m+1}$ is a Nash equilibrium.

The basic idea of the algorithm is the same with the decomposition-based algorithm in the sense that this algorithm finds a Nash equilibrium through changing the earliest non-assigned user's route to their ex-post best response route one by one. However, these are different in that this algorithm changes their routes through better responses of users from an arbitrary initial route profile whereas the decomposition-based algorithm assigns users who do not select any route to their ex-post shortest routes (i.e., the algorithm finds the equilibrium through best responses of the users from the fixed route profile). The role of each step in this algorithm is explained below.

In Step 1, the earliest non-assigned user is searched for from $\mathcal{P}_B^m$. Note that the earliest non-assigned user always exists in a unidirectional network (see Theorem 2). Because the shortest route $\bar{r}_i^*$ satisfies Eq. (5), regardless of the route profile of users in $\mathcal{P}_B^m$, the destination arrival time of user i by taking the route g_i^* remains the same, and the destination arrival times of the other routes of the user does not become earlier than g_i^*, as mentioned in Section 2. Therefore, the route of user i becomes the ex-post best response route when user i can change the current route r_i^m to $\bar{r}_i^*$ by a better response.

Step 2 checks whether the current route r_i^m of the selected user i is an ex-post best response route or not; if not, then the algorithm changes the route of i to the ex-post shortest route $\bar{r}_i^*$ through better responses. In case (a), it is clear that user i does not take an ex-post best response route since the arrival time by taking r_i^m is later than that by taking $\bar{r}_i^*$; thus the route of user i is changed to $\bar{r}_i^*$ through a better response of the user.

In case (b), it is not obvious that r_i^m is an ex-post best response route although the destination arrival time by taking the route is the same with that by taking $\bar{r}_i^*$. This is because the arrival time of user i by taking r_i^m may increase due to overtaking by other users in $\mathcal{P}_B^m$ through their better responses to a route profile $\mathbf{r}'_B$. Therefore, the algorithm checks the existence of such a better response path.[9] If it exists (Case (b)-1), then r_i^m is not an ex-post best response route. In this case, first, the routes of users in $\mathcal{P}_B^m$ are changed to $\mathbf{r}'_B$; then, because the arrival time of user i increases due to the better responses, the route of user i is changed to the ex-post shortest route $\bar{r}_i^*$ through the better response of the user. If not (Case (b)-2), it is guaranteed that r_i^m is the ex-post best response route because its arrival time is already minimized (i.e., $g_i^* = g_i(r_i^m; \mathbf{r}_{-i}^m)$), and independent of better responses of the users in $\mathcal{P}_B^m$; there is no need to change the routes of users in $\mathcal{P}_B^m$.

Finally, in Step 3, the algorithm transfers user i from $\mathcal{P}_B$ to $\mathcal{P}_A$. In this algorithm, once the user is transferred to Group A, the route of the user always remains the ex-post best response route as described so far: regardless of the route profile of users in $\mathcal{P}_B$, the destination arrival times of the users in $\mathcal{P}_A^m$ remain in their earliest destination arrival times.

As a result, this algorithm can find a Nash equilibrium at exactly $|\mathcal{P}|$ iterations through better responses of users when the existence of the earliest non-assigned user is guaranteed. The existence in a DUE game in a unidirectional network is proved in Theorem 2. Therefore, there exists a better response path ending at a Nash equilibrium from an arbitrary initial route profile in the DUE game. □

Note that this algorithm is applicable to a DUE game in which the existence of the earliest non-assigned user is guaranteed even when the network is not unidirectional. Thus, the following corollary is obtained:

Corollary 1. *A DUE game in which the earliest non-assigned user exists is a weakly acyclic game.*

4.3. Convergence of evolutionary dynamics in a DUE game

In this subsection, we investigate the convergence of evolutionary dynamics in a DUE game in a unidirectional network. Although the convergence is not a sufficient condition for equilibrium to be stable, this property is useful for investigating

[9] Note that the algorithm shown here is a pseudo algorithm, and an additional procedure have to be specified when this algorithm is implemented. However, it is possible to conduct this step in principle since the state space $\mathcal{R}$ is finite. Thus, this does not matter in this proof.

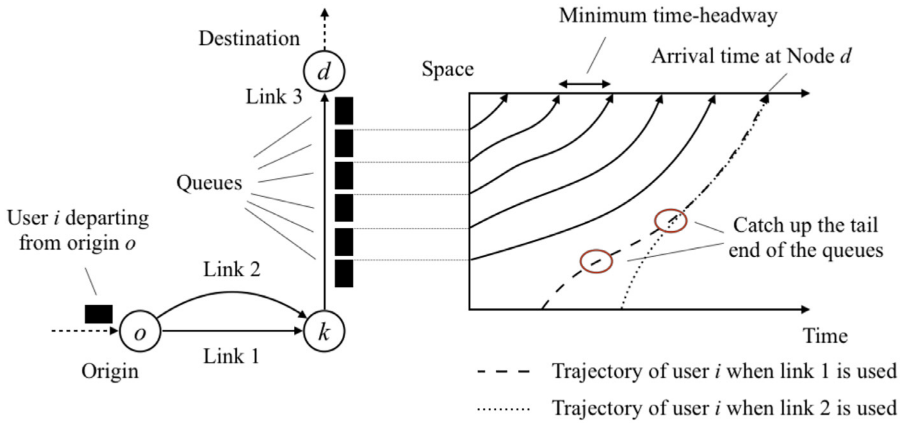


Fig. 3. Illustrated network and time-space diagram showing the trajectories of the user on link having queues (the route passing link 1 is the best response and shortest route; the route passing link 2 is the best response route but not shortest one).

the stochastic stability as we will show in Section 5. Here, we consider the following two representative dynamics: *better response dynamics* (Hart and Mas-Colell, 2000) and *best response dynamics* (Blume, 1993).

4.3.1. Better response dynamics

We first show the convergence of the better response dynamics. At each day τ, one user $i \in \mathcal{P}$ is randomly chosen and allowed to change the current route. The other users must repeat their current routes at the day. Then, user i changes route r_i^τ to $r_i^{\tau+1}$ such that the user arrives at the destination strictly earlier than the previous day. If the user does not have a route satisfying the condition, the user does not change the route. The set of better responses of user i when the current route profile is $\mathbf{r}^\tau$ is denoted by $D_i(\mathbf{r}^\tau)$, and

$$D_i(\mathbf{r}^\tau) := \left\{ r_i^* \mid r_i^* \in \mathcal{R}_i \text{ s.t. } g_i(r_i^*, \mathbf{r}_{-i}^\tau) < g_i(r_i^\tau, \mathbf{r}_{-i}^\tau) \right\}. \tag{16}$$

It is obvious that a Nash equilibrium becomes a rest point of the dynamics. This property and Eq. (16) correspond to Nash stationarity and positive correlation property, respectively, which are natural behavioral properties in evolutionary game stated by Sandholm (2010).

We can obtain the convergence of this dynamics in a DUE game in a unidirectional network as a direct consequence of Theorem 3 as follows: because the DUE game is a weakly acyclic game, there exists at least one better response path ending to a Nash equilibrium from every profile. As $\tau \to \infty$, the probability that such a better response path is selected becomes 1. Thus, the following proposition is established:

Proposition 1. *In a DUE game in a unidirectional network, route profile* $\mathbf{r}$ *generated by the better response dynamics from an arbitrary initial profile converges almost surely to a Nash equilibrium as* $\tau \to \infty$.

Proof. Young (2004) proved the convergence of the better response dynamics in weakly acyclic games. □

4.3.2. Best response dynamics

We then introduce the best response dynamics (Blume, 1993). In this dynamics, the selected user i chooses a route that has the minimum arrival time at the destination. In other words, the user i chooses a route randomly from the following route set $B_i(\mathbf{r}^\tau)$:

$$B_i(\mathbf{r}^\tau) := \left\{ r_i^* \mid r_i^* \in \mathcal{R}_i \text{ s.t. } g_i(r_i^*, \mathbf{r}_{-i}^\tau) \le \min_{r \in \mathcal{R}_i} g_i(r, \mathbf{r}_{-i}^\tau) \right\}. \tag{17}$$

In the best response dynamics, a *strict* Nash equilibrium becomes a rest point of the dynamics (Alós-Ferrer and Netzer, 2017). The reason why a *non-strict* Nash equilibrium does not become the rest point is that even when a user takes a best response strategy, the user is allowed to change it to another best response strategy whose utility is the same. Such a best response could cause the following "ripple effect" even though the current state is a Nash equilibrium. A change in a user's strategy could affect utilities of others and make their strategies into non-best responses; then, strategy profiles generated by further best responses of them may go away from the Nash equilibrium. On this point, this dynamics is widely different from the better response dynamics in which the user is not allowed to change its strategy to another whose utility is the same. Thus, to prove the convergence of the best response dynamics, the DUE game should have a strict Nash equilibrium.

However, in a DUE game, the strictness of a Nash equilibrium cannot be guaranteed in general. This is because the uniqueness of the best response strategy of a user may be lost when there exist queues in a network. For example, consider the network shown in Fig. 3. This network has one origin and one destination and there exist two routes passing link 1 or 2. Suppose that in an equilibrium state, link 3 has queues and a user can catch up the tail end of the queue before the queue dissipates by using either route. Suppose also that users departing later than this user do not overtake this user. Then, we can see that the destination arrival times of both routes become the same because the earliest possible departure time from link 3 is restricted to the departure time of the leading user as shown in the figure. This means that multiple best response strategies exist in Eq. (1) for the user, and the strictness of the equilibrium is lost. Consequently, we can conclude that the convergence of the best response dynamics in DUE games is not guaranteed in general.

From the discussion in the previous and this sections, it is evident that the property of the dynamics that a user improves its utility strictly or the strictness of equilibrium plays a central role in guaranteeing the convergence. Furthermore, because a strict Nash equilibrium may not exist in a DUE game due to queues, the better response dynamics that has the former property is more desirable than the best response dynamics in that the convergence is guaranteed in general. This is an interesting effect of queues in that they could affect the convergence of equilibrium under certain dynamics. Moreover, the difference in convergence properties between these dynamics will affect the stochastic stability of the dynamics, as will be shown in the next section.

5. Stochastic stability of DUE game

5.1. Definition of stochastic stability

Young (1993) introduced the concept of stochastic stability to investigate the asymptotic behavior of evolutionary dynamics as follows: *over the long run, states that are not stochastically stable will be observed infrequently compared to states that are stable, provided that the probability of mistakes ϵ is small.* To introduce the formal definition, we first consider the finite state Markov chain over the state space $\mathcal{R}$ generated by evolutionary dynamics without perturbation (e.g., better response dynamics), referred as the *unperturbed Markov chain*. Let $\mathbf{P}^0$ be the transition matrix of the Markov chain. $P^0_{\mathbf{rr}'}$ is the transition probability from state $\mathbf{r}$ to state $\mathbf{r}'$. Then, consider the *regular perturbed Markov chain* generated by the perturbed dynamics, which is continually perturbed by small mutations or mistakes. In this Markov chain, the selected user randomly chooses the strategy from the set of all available strategies with positive probability characterized by parameter ϵ. Let $\mathbf{P}^\epsilon$ be the transition matrix of the perturbed Markov chain. As the perturbation becomes small, such a choice probability becomes zero and the transition matrix of the perturbed Markov chain becomes the same with the unperturbed one, i.e., $\lim_{\epsilon\to 0}\mathbf{P}^\epsilon = \mathbf{P}^0$.

For any $\epsilon > 0$, the perturbed Markov chain is aperiodic and irreducible, i.e., the Markov chain is ergodic. This means that the perturbed Markov chain has a unique stationary distribution $\boldsymbol{\mu}^\epsilon$ satisfying $\boldsymbol{\mu}^\epsilon\mathbf{P}^\epsilon = \boldsymbol{\mu}^\epsilon$. This stationary distribution gives the observation probability of each state when the process with the perturbation runs for a very long time. In addition, it has been shown that such a stationary distribution converges to one of the stationary distribution of the unperturbed Markov chain as $\epsilon \to 0$ (Young, 1993). Then, stochastic stability (Young, 1993) is defined as follows:

Definition 7 (**Stochastic stability**). A state $\mathbf{r} \in \mathcal{R}$ is *stochastically stable* relative to a perturbed Markov chain if $\lim_{\epsilon\to 0}\mu^\epsilon_{\mathbf{r}} > 0$.

5.2. Stochastic stability of DUE game in unidirectional network

We will now investigate the stochastic stability of a DUE game in a unidirectional network. Here, we consider the following two perturbed dynamics: *perturbed better response dynamics* and *perturbed best response dynamics*. Particularly in the latter, we discuss the stability of the logit response dynamics (Blume, 1993; Marden and Shamma, 2012; Alós-Ferrer and Netzer, 2017), which is one of the perturbed best response dynamics and is widely used in traffic assignments (e.g., Miyagi et al., 2013).

5.2.1. Perturbed better response dynamics

In the perturbed better response dynamics, one user $i \in \mathcal{P}$ is randomly chosen and allowed to change the current route at each day, as with the case without perturbation. However, with the probability $p_i(\epsilon)$ characterized by ϵ, the selected user chooses a route randomly from $\mathcal{R}_i$ instead of the better response. Then, the transition matrix from state $\mathbf{r} = (r_i, \mathbf{r}_{-i})$ to $\mathbf{r}' = (r'_i, \mathbf{r}_{-i})$ of this dynamics is described as follows:

$$P^\epsilon_{\mathbf{rr}'} = (1 - p_i(\epsilon))P^0_{\mathbf{rr}'} + \frac{1}{|\mathcal{P}|}\cdot p_i(\epsilon)\cdot q_i(r'_i;\mathbf{r}_{-i}) \tag{18}$$

where $P^0_{\mathbf{rr}'}$ represents the transition probability in the better response dynamics without perturbation. $q_i(r'_i;\mathbf{r}_{-i})$ is the transition probability to route r'_i such that $\sum_{r'_i\in\mathcal{R}_i} q_i(r'_i;\mathbf{r}_{-i}) = 1$. $p_i(\epsilon)$ and $q_i(r'_i;\mathbf{r}_{-i})$ are dependent on the applied perturbation. For example, when we assume the perturbation such that the user randomly chooses a route regardless of the destination arrival time, then these probabilities are determined as follows: $p_i(\epsilon) = \epsilon,\ q_i(r'_i;\mathbf{r}_{-i}) = 1/|\mathcal{R}_i|$.

Regarding this dynamics, the stochastic stability of DUE is presented as follows:

Proposition 2. *Consider a DUE game in a unidirectional network. Then, there exists a stochastically stable equilibrium of the perturbed better response dynamics.*

Proof. Young (1993) showed the following theorem that provides a criterion for determining the stochastically stable states:

Theorem 4 (**Young, 1993, Theorem 4**). *Consider an unperturbed Markov chain on the finite state space $\mathcal{R}$ with recurrent communication classes $r_1, \ldots, r_J$, and a regular perturbation of the Markov chain that has a unique stationary distribution $\boldsymbol{\mu}^\epsilon$ for every small positive ϵ. Then, as $\epsilon \to 0$, $\boldsymbol{\mu}^\epsilon$ converges to a stationary distribution $\boldsymbol{\mu}^0$ of the unperturbed Markov chain. In addition, the stochastically stable states are contained in the recurrent classes with the minimum stochastic potential.*[10]

Then, the proposition can be proved by utilizing this theorem and the property of the better response dynamics in a weakly acyclic game as follows. Because a DUE game in a unidirectional network is a weakly acyclic game, the recurrent classes of the unperturbed Markov chain correspond one on one to the rest points of the better response dynamics. In addition, the rest points correspond to Nash equilibria of the game. Thus, from Young's theorem, it follows that the stochastically stable states are contained in the set of Nash equilibria. Therefore, there exists a stochastically stable equilibrium of the perturbed better response dynamics. □

Note that the proposed approach can be applied to the dynamics under multiple users change their routes simultaneously. This is because for any game that is a weakly acyclic game, the rest points of such a synchronous better response dynamics are precisely a set of Nash equilibria; then, we can prove the existence of a stochastically stable equilibrium of the perturbed synchronous better response dynamics by utilizing Young's theorem (Marden and Shamma, 2012).

5.2.2. Perturbed best response dynamics

Next, we investigate the stochastic stability of the perturbed best response dynamics. One of the most known perturbed best response dynamics is logit response dynamics. In this dynamics, the transition matrix from state $\mathbf{r} = (r_i, \mathbf{r}_{-i})$ to $\mathbf{r}' = (r_i', \mathbf{r}_{-i})$ is described as follows:

$$P^{\beta}_{\mathbf{rr'}} = \frac{\exp(-\beta g_i(r_i', \mathbf{r}_{-i}))}{\sum_{r \in \mathcal{R}_i} \exp(-\beta g_i(r, \mathbf{r}_{-i}))}. \tag{19}$$

where $0 < \beta < \infty$ measures the degree of noise in the best response. The logit response dynamics converges to the best response dynamics when $\beta \to \infty$.

The stochastic stability of this dynamics is proved in Theorem 1 by Alós-Ferrer and Netzer (2017) as follows: *Consider a weakly acyclic game in which the best response dynamics converges to a strict Nash equilibrium. Then the set of stochastically stable states of the logit response dynamics is contained in the set of strict Nash equilibria*. This is the application of Young's theorem, which utilizes the fact that the rest point of the best response dynamics is the strict Nash equilibrium. However, as mentioned above, the existence of a strict Nash equilibrium in a DUE game is not guaranteed in general due to the existence of queues. Thus, we can conclude that the existence of the stochastically stable equilibrium of the logit response dynamics is not guaranteed in general, either.

6. Numerical experiments

We finally conduct numerical experiments to observe the convergence and stochastic stability of a DUE game in a unidirectional network. Specifically, we first examine the convergence speed of the better response dynamics through comparing a physical-queue model and point-queue model. We then observe stationary distributions of the perturbed better response dynamics and logit response dynamics to examine properties of the stochastic stability.

6.1. Settings

We consider the unidirectional network with many-to-many OD shown in Fig. 4, which is a modification of the Nguyen–Dupuis network. Nodes $\{o_1, o_2\}$ are origins, and nodes $\{d_1, d_2\}$ are destinations. The physical condition of each link (e.g., free-flow travel time and capacity) are summarized in Table 1. Each link has a bottleneck section with a bottleneck capacity at the end of the link. The network has 4 routes for each destination, respectively. These routes are numbered as shown in the figure. The number of users departing from each origin for each destination is 50, and the total number of users is 200. Each user from each origin departs with a fixed time headway (we will specify later).

Traffic dynamics within the network is simulated using a mesoscopic LWR model proposed by Leclercq and Bécarie (2012). This simulator provides the event time when each user crosses the specific points of the network (e.g., nodes); then according to the event time, the trajectory of each user is calculated based on the dynamic loading model. We employ the simplified car following model by Newell (2002) as the dynamic loading model.

6.2. Results

6.2.1. Convergence speed of better response dynamics

In the experiments, we will see whether the difference in the dynamic loading model affects the convergence speed or not. Here, we consider the different queue models, the physical queue and point queue, by changing a parameter in the

[10] We omit the definition of the stochastic potential here because this is not related to the discussion in this paper (see Young, 1993 for details).

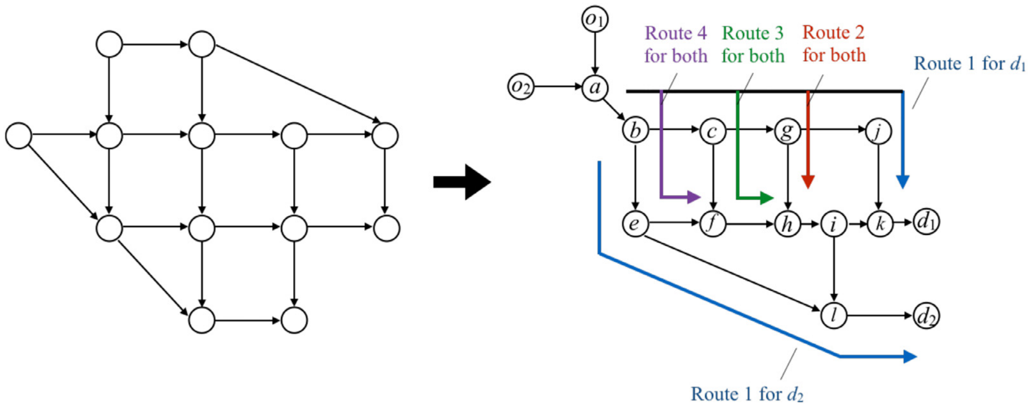


Fig. 4. Left: Original Nguyen Dupuis network. Right: Modification to the unidirectional network.

Table 1
Physical condition of links (FFTT: Free Flow Travel Time, BC: Bottleneck Capacity, SF: Saturation Flow) .

Link	FFTT [sec]	BN [veh/sec]	SF [veh/sec]	Link	FFTT [sec]	BN [veh/sec]	SF [veh/sec]
(o_1, a)	14	2	2	(f, h)	12	1.5	1.5
(o_2, a)	18	2	2	(g, h)	18	0.17	0.5
(a, b)	6	2	2	(g, j)	6	0.2	0.5
(b, c)	6	2	2	(h, i)	9	0.67	83
(b, e)	18	1.5	1.5	(i, k)	12	0.27	0.67
(c, f)	26	1.5	1.5	(i, l)	6	0.2	0.5
(c, g)	10	1.5	1.5	(j, k)	6	0.67	1
(e, f)	20	1.5	1.5	(k, d_1)	6	0.67	1
(e, l)	42	0.27	0.5	(l, d_2)	6	0.67	1

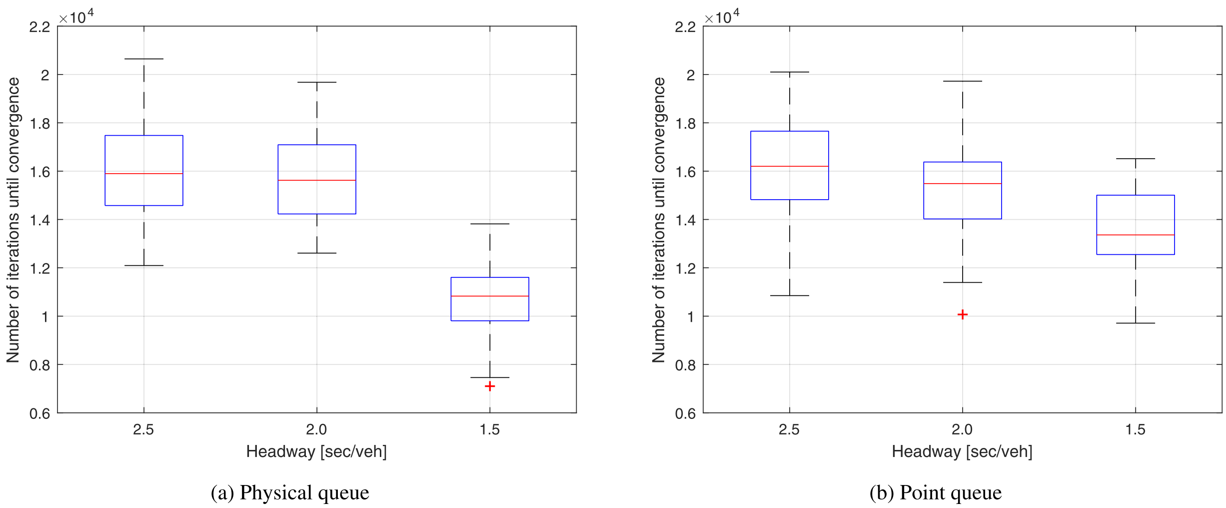


Fig. 5. Number of iterations of the better response until convergence to a Nash equilibrium.

car-following model. We also consider the three demand levels by changing the headway: with 2.5[sec], 2.0[sec], 1.5[sec]. For each case, we iterate the better response until convergence to a Nash equilibrium and repeat this process 50 times; we then compare the number of iterations until convergence between the cases. Note that the initial route of each user is set as the shortest-distance route.

Fig. 5 displays the comparison result of the convergence speed. The horizontal axis represents the demand level, and the vertical axis represents the number of iterations of the better response until convergence. From this figure, we observe that the number of the better response required to reach a Nash equilibrium increases as the demand level reduces, and the point queue model is employed, i.e., a large number of iterations are required to fix the users' routes to their ex-post best response routes. This implies that queue spillbacks reduce the number of candidates of the better response in this particular scenario.

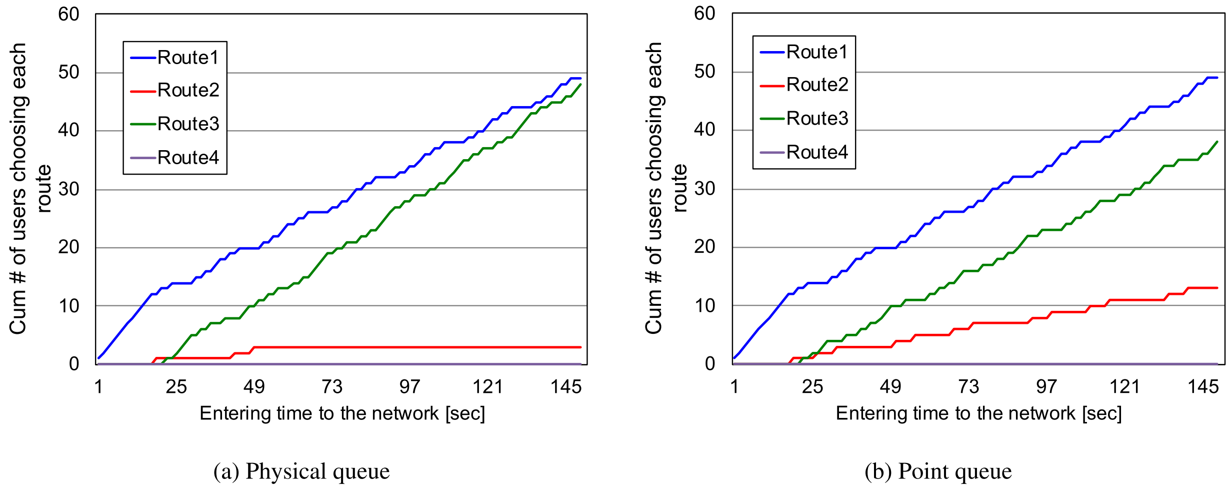


Fig. 6. Cumulative number of users whose destination is d_1 choosing each route.

Table 2
Resulting total number of mistakes in route choices under each perturbed dynamics.

Perturbation level	High ($\epsilon = 0.005,\ \beta = 5$)	Middle ($\epsilon = 0.001,\ \beta = 70$)	Low ($\epsilon = 0.0001,\ \beta = 160$)
Perturbed better response	726	146	12
Logit response	719	135	13

To investigate the mechanism, we examine the route choice pattern at equilibrium. Fig. 6 shows the cumulative number of users choosing each route for d_1 with the physical and point queue models. First, from Fig. 6(a), we observe that route 2 is unused early in the simulation with the physical queue model. This is because the queues on link (j, g) spillback to link (c, g), and the travel time of route 2 increases. This implies that the candidates of the better response for most of the users until convergence are route 1 and 3. On the other hand, from Fig. 6(b), we observe that users continue using route 2 in all times of the simulation. This is because there is no queue spillback and route 2 remains to be the best response route. As a result, there are three candidates for the better response. Hence, in this particular scenario, we observe that the probability of selecting the ex-post best response route becomes large due to queue spillbacks, i.e., the queue spillover can reduce the number of better responses required to reach a Nash equilibrium. However, the effect of the queue and the convergence speed may largely depend on the initial conditions and numerical settings. A systematic numerical experiment should be conducted for different types of network settings and the effect should be investigated.

6.2.2. State distribution of perturbed dynamics

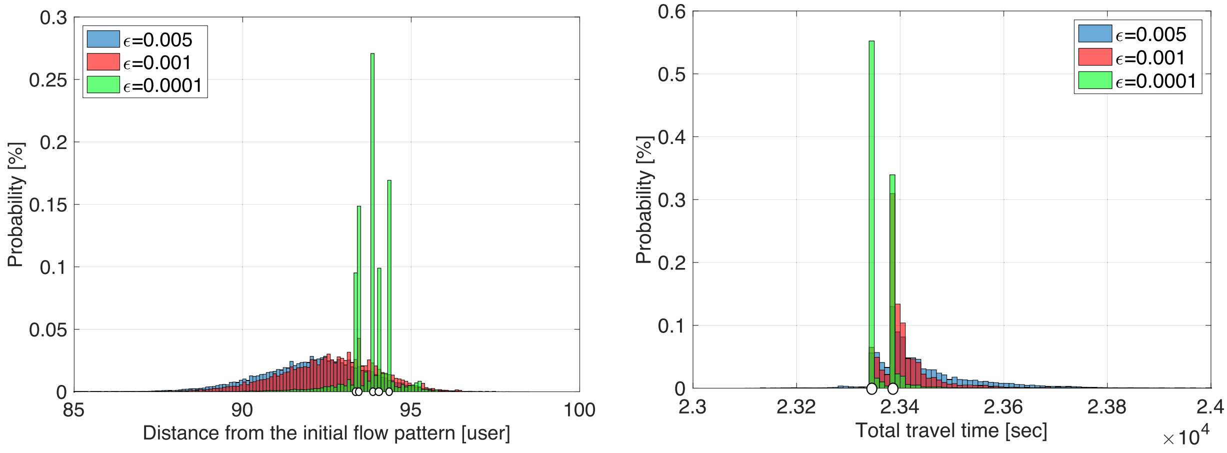
Next, we observe state distributions of the perturbed better response dynamics and logit response dynamics to examine properties of the stochastic stability, i.e., whether the distributions tend to converge to a Nash equilibrium or not as the perturbation levels become small. In this experiment, we consider the case where the headway is 2.5[sec] and the physical queue model is employed. The perturbation of the better response dynamics is set such that $p_i(\epsilon) = \epsilon$ and $q_i(r_i'; \mathbf{r}_{-i}) = 1/|\mathcal{R}_i|$ for user i (i.e., the selected user randomly chooses the route regardless of the arrival time). Each response is iterated 200,000 times with the three perturbation levels: for the perturbed better response, $\epsilon = 0.005, 0.001$, and 0.0001; for the logit response, $\beta = 5, 70$, and 160. We set the parameters so that the resulting total number of mistakes in route choices (i.e., changes to routes which are not included in the set of routes of unperturbed dynamics) is almost the same between these dynamics (see Table 2 for the resulting number of mistakes).

For each dynamics with each perturbation level, we obtain two empirical state distributions of the users' route choice pattern and of the total travel time. In the former, the state of each day $z(\tau)$ is defined as the Euclidean distance from the initial flow pattern (the shortest-distance route choice pattern) as follows:

$$z(\tau) = \left[\sum_{r \in \mathcal{R}} (f^r(\tau) - f^r(0))^2 \right]^{1/2}, \tag{20}$$

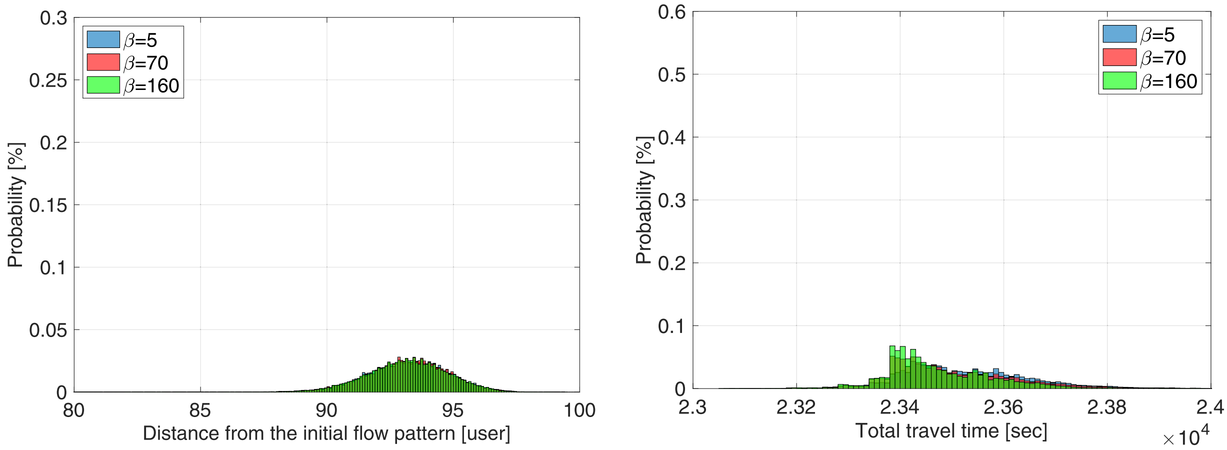
where $f^r(\tau)$ is the number of users who choose route r in day τ.

Fig. 7 shows the state distributions of the perturbed better response dynamics. In each figure, the horizontal axis shows states, and the vertical axis shows the frequency of these states. From Fig. 7(a), we observe that the distribution tends to converge to multiple Nash equilibria according to the decrease in the perturbation level. Note that, we observed one Nash equilibrium in the dynamics with $\epsilon = 0.001$, and five Nash equilibria with $\epsilon = 0.0001$. Meanwhile, from Fig. 7(b), we also



(a) Route choice patterns (b) Total travel times

Fig. 7. State distributions of the perturbed better response dynamics (white circles show Nash equilibria).



(a) Route choice patterns (b) Total travel times

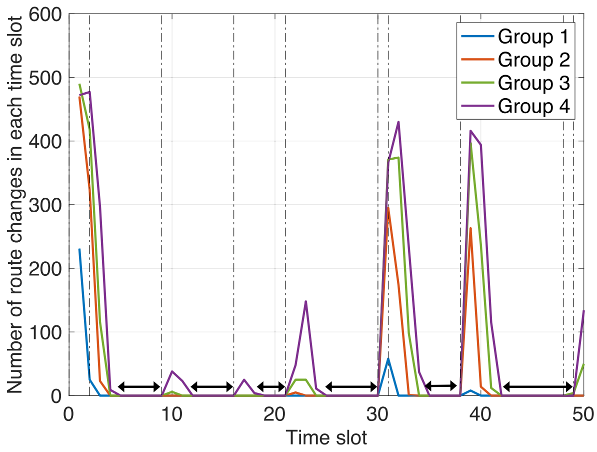
Fig. 8. Stationary distributions of the logit response dynamics.

observe that the total travel times tend to converge to almost the same value although there is a slight difference. This implies that the link travel time pattern is also almost the same and expected to converge to the same value as the number of users becomes large.

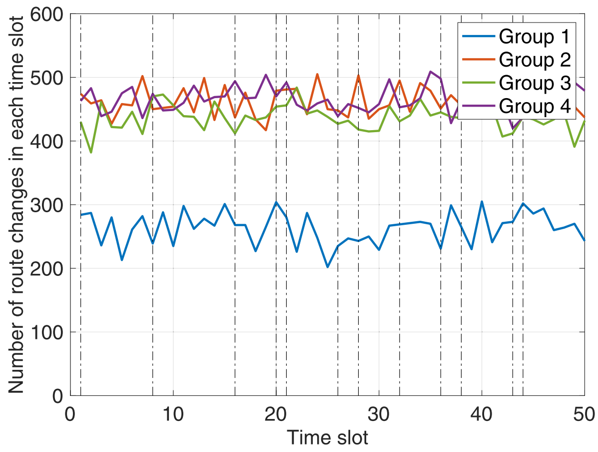
Fig. 8(a) and (b) show the distributions of the logit response dynamics. From these figures, we see that the shapes of the distributions become almost the same despite of the decrease in the perturbation level. We also observe that a Nash equilibrium is not seen during the iterations: the candidate of the stochastically stable equilibrium is not seen.

To investigate the reason why such differences occur between these perturbed dynamics, let us observe how a route profile evolves in detail. For this purpose, we divide the users of the DUE game into the four groups (each group contains 50 users) in the order obtained from the decomposition-based algorithm. For example, Group 1 consists of 50 earliest users. Then, we aggregate the number of route changes in each group for every time slot (4000 iterations).

Fig. 9(a) and (b) show that the evolution of the number of route changes of users in each group under the dynamics with lowest perturbation levels. In these figures, the horizontal black arrows indicate that Nash equilibria are being achieved, and the vertical dotted black lines indicate the time slots when mistakes occur. Fig. 9(a) shows that, in the perturbed better response dynamics, the number of route changes decreases to zero earlier in a group which contains users having earlier order during time slots when mistakes do not occur. This result can be interpreted as follows. In congested networks, because of the FIFO principle and the causality of a dynamic loading model, a route change of a user affects route travel times of users having the later order, but not vice versa, as discussed in Sections 2 and 3. By this temporal asymmetry of the interaction, users having earlier order become likely to choose their ex-post best response routes earlier. In addition, once users choose their ex-post best response routes, they do not change their routes unless mistakes occur because of the



(a) Perturbed better response dynamics



(b) Logit response dynamics

Fig. 9. The number of route changes of users in each group for every time slot.

strict improvement property of the better response dynamics mentioned in Section 4. Thus, the dynamics goes toward Nash equilibria almost monotonically.

Fig. 9(b) shows that the number of route changes in Group 1 are smaller than that in the other groups in the logit response dynamics: this comes from the asymmetric interaction in the DUE game as in the case with the perturbed better response dynamics. However, unlike Fig. 9(a), the number of route changes does not decrease to zero even during time slots when mistakes do not occur. The reason is that, in the logit response dynamics, switching a user's route between best response strategies is allowed, and such a route switch acts like a perturbation; this could cause the ripple effect as mentioned in Section 4. From these results, we can see that the strict improvement property is important for fixing route choices to the best response routes, and thus ensuring the convergence of distributions in DUE games.

7. Conclusion

In this study, we examined the stability of DUE in a unidirectional network. The presented approach consists of the three key concepts: the decomposition technique of DUE assignments, the weakly acyclic game, and the asymptotic analysis of the stationary distribution of perturbed dynamics. This synthesizing approach enables us to examine the theoretical properties of DUE without requiring the monotonicity of the route travel time function. Specifically, after formulating a DUE assignment as a strategic game, we first proved the existence of the earliest non-assigned user in a unidirectional network, which implies that there exists an order of assigning users one by one to the network for ensuring an equilibrium. With this ordering property, we proved that a DUE game in a unidirectional network is a weakly acyclic game, which guarantees the convergence of better response dynamics to a Nash equilibrium. We then established the existence of the stochastically stable equilibrium of the better response dynamics. Finally, we observed the convergence and stochastic stability of the DUE in a unidirectional network through the numerical experiments. The results showed that the strict improvement of users' travel times by the applied evolutionary dynamics is important for ensuring the existence of a stochastically stable equilibrium in DUE games.

While we showed the stability of DUE in unidirectional networks, how the flow and cost patterns arise at the stable equilibrium is largely unknown. In particular, if there are multiple equilibria, it is important to examine which equilibrium is selected by the dynamics (i.e., an equilibrium selection) and how to characterize this equilibrium. Also, if there are efficient and inefficient equilibria, it is interesting to design an incentive scheme to make the resulting dynamics to converge to the former equilibrium (or system optimal assignments). These analyses may be challenging even for DUE games in unidirectional networks because the state space is large and there are complex interactions among physical queues. Nevertheless, these seem fruitful topics for establishing a distributed control of dynamic transportation networks. Furthermore, we are interested in exploring the applicability of the proposed approach to the other types of networks and to DUE assignments with departure time choices.

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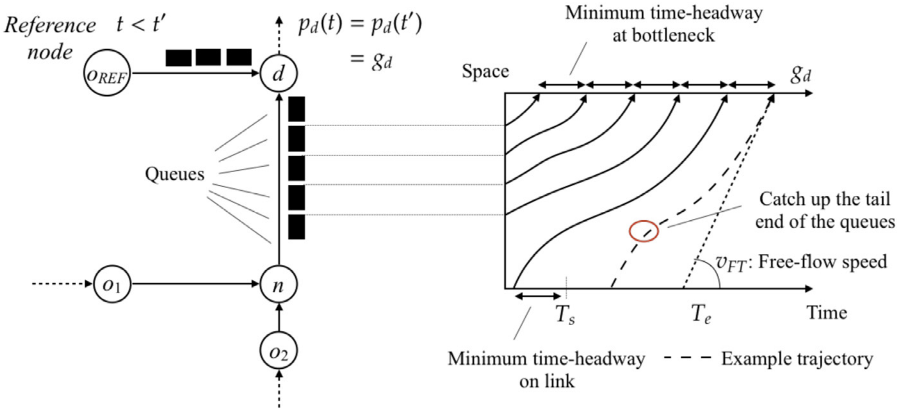

Fig. A.10. Example network that has a zero-flow congested link and the time-space diagram showing the trajectories of users on link (n, d).

Appendix A. Proof of Lemma 1

Consider an origin $o \in \mathcal{N}_o$ and the shortest routes $r^*(t)$ and $r^*(t')$ departing from this origin at potentials $p_o(t)$ and $p_o(t')$ to arbitrary destinations that the users can reach. Consider also node $n \in \mathcal{N}(r^*(t)) \cap \mathcal{N}(r^*(t'))$. Hereafter, we first prove Eq. (11) and then obtain Eq. (10).

The first line of Eq. (11) is simply obtained from the FIFO principle of a dynamic loading model, as also shown by Iryo and Smith (2018). Specifically, the following relationships hold because of the FIFO principle (i.e., no overtaking on the same route):

$$\begin{cases} p_o(t) < p_o(t') \Rightarrow p_n(t) \le p_n(t') \\ p_o(t) = p_o(t') \Rightarrow p_n(t) = p_n(t') \end{cases} . \tag{A.1}$$

This means that, for given order of potentials on the origin, the order of the potentials on node n is determined in a *forward manner* along traffic flows on the shortest routes.

We next prove the second line of Eq. (11) in a *backward manner* along traffic flows. First, if there is a strict order between two potentials on node n, it is apparent that the order between two potentials on node o is also strict owing to the FIFO principle. That is,

$$p_n(t) < p_n(t') \Rightarrow p_o(t) < p_o(t'). \tag{A.2}$$

Eq. (A.2) shows that a user arriving at a node earlier must depart from the origin earlier. This suggests that the order of potentials on the origin can be determined if there exists at least one node satisfying the following property: the node is included in the shortest routes from the origin, and the order of the potentials on the node is strict.

On the other hand, however, when the potentials on all nodes that are included in the shortest routes from the origin are the same (i.e., $p_n(t) = p_n(t')$), there are cases where the order between $p_o(t)$ and $p_o(t')$ is not determined uniquely. The cases are caused by the existence of a *zero-flow congested link*. The zero-flow congested link is the link that has queues with the time interval in which there are no incoming flows. On the link, the change in the travel time function per unit time is equal to -1 during a certain time interval, i.e., $\frac{c_l(t')-c_l(t)}{t'-t} = -1$. This means that when a user enters the link during the interval, the leaving time from the link becomes the same regardless of the entering time.

We demonstrate this issue using the example shown in Fig. A.10. The left-hand side of the figure shows the network with three origin $\{o_1, o_2$, and $o_{REF}\}$ and one destination d. o_{REF} is chosen as the reference node. Links (o_{REF}, d) and (n, d) are the zero-flow congested links, which have queues represented as black rectangles. The right-hand side of the figure shows the time-space diagram on link (n, d). This diagram shows that when a user enters the link during the time interval $[T_s, T_e]$, the leaving time becomes g_d regardless of the entering time due to the existence of queues. Also, the link (o_{REF}, d) is a zero-flow congested link and then the potentials on node d having different reference times t and t' can be the same with g_d, i.e., $p_d(t) = p_d(t') = g_d$. In this situation, we cannot uniquely determine the order between $p_n(t)$ and $p_n(t')$ in addition to the value of each potential, as shown in the left-hand side of Fig. A.11. However, the physical condition that should be satisfied here is that $p_n(t), p_n(t') \in [T_s, T_e]$; as long as this condition is satisfied, the correspondence between the reference times and the potential times at node n is arbitrary. In other words, in the left-hand side of Fig. A.11, we can draw an arbitrary line on the area surrounded by the dotted lines. Thus, without loss of generality, we can set the correspondence such that satisfies the following condition:

$$t < t' \wedge p_n(t) = p_n(t') \Rightarrow p_o(t) \le p_o(t'). \tag{A.3}$$

Eq. (A.3) shows the condition for ensuring that the monotonic relation between the reference times and the potentials. An example of the correspondence that satisfies this condition is shown in the right-hand side of Fig. A.11.

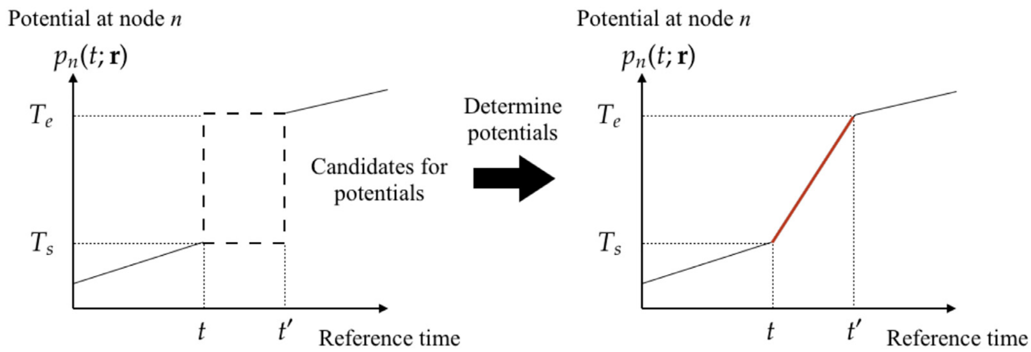

Fig. A.11. Arbitrary relation between potentials on node n and one example of determining the relation.

As a result, summarizing Eqs. (A.2) and (A.3), for $t < t'$, we obtain the second line of Eq. (11):

$$p_n(t) \le p_n(t') \Rightarrow p_o(t) \le p_o(t'), \quad \forall o \in \mathcal{N}_o, \forall n \in \mathcal{N}(r^*(t)) \cap \mathcal{N}(r^*(t')). \tag{A.4}$$

Now, we are in a position to prove Eq. (10). First, for $t < t'$, we can determine the order of the potentials on node n that is included in the shortest routes from the reference node by using Eq. (A.1) as follows: $t < t' \Rightarrow p_n(t) \le p_n(t')$. Then, we can also determine the order of the potentials on origin o from which the shortest routes include this node n by using Eq. (A.4) as follows: $p_n(t) \le p_n(t') \Rightarrow p_o(t) \le p_o(t')$. Therefore, by recursively applying Eqs. (A.1) and (A.4), we can obtain

$$t < t' \Rightarrow p_o(t) \le p_o(t'), \quad \forall o \in \mathcal{N}_o. \tag{A.5}$$

Thus, Eq. (10) is proved. □

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