

Trading mechanisms for bottleneck permits with multiple purchase opportunities[☆]

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Abstract

This paper extends the theory of tradable bottleneck permits system to cases with multiple period markets and designs its implementation mechanism. The multiple period markets can achieve more efficient resource allocation than a single period market when users' valuations of tradable permits change over time. To implement the multiple period trading markets, we propose an evolutionary mechanism that combines a dynamic auction with a capacity control rule that adjusts a number of permits issued for each market. Then, we prove that the proposed mechanism has the following desirable properties: (i) the dynamic auction is strategy-proof within each period and guarantees that the market choice of each user is optimal under a perfect information assumption of users, (ii) the mechanism maximizes the social surplus in a finite number of iterations. Finally, we show that the proposed mechanism may work well even for an incomplete information case.

Keywords: tradable bottleneck permits, multiple period markets, Benders decomposition, auction mechanism, capacity control

1. Introduction

1.1. Background and purpose

Transportation demand management schemes can be roughly divided into two types: “price-based regulation” and “quantity-based regulation.” As a representative of the former, congestion pricing is theoretically desirable for reducing traffic congestion in a distributed manner. However,

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in order to calculate the optimal toll levels, the road manager requires detailed and accurate demand information of all users. It is almost impossible for the road manager to obtain such private information due to an asymmetric information between road managers and road users. Therefore, it is difficult to guarantee the effect of the road pricing scheme.

As a representative of the latter scheme, we can take highway booking/reservation for examples (e.g., Akahane and Kuwahara, 1996; Wong, 1997; Akahane et al., 1996; Teodorovic and Edara, 2005). This type of schemes can achieve a quantitative policy target without requiring detailed user information. However, there may be cases in which road users cannot select their desired choice if the permits (or allocations) are assigned according to unrefined rules. Such an infringement on freedom of choice necessarily causes economic losses. However, with the progress of communication technology and the popularization of information and communication technology/intelligent transportation systems, it is not difficult to establish the mechanism in which the users can select his/her desired choice free.

As one possible way to eliminate bottleneck congestion and to resolve the above information and choice problems simultaneously, Akamatsu et al. (2006) proposed a novel system of “tradable bottleneck permits.” Their proposed scheme comprises two parts: (a) the road manager issues a right (bottleneck permits) that allows the permit holders to pass through a bottleneck during a pre-specified time interval, (b) a trading market is established for bottleneck permits that are differentiated on the basis of a pre-specified time. Under this scheme, the queuing congestion can be completely eliminated by setting the number of permits issued per unit time interval to be less than or equal to the bottleneck capacity. In addition, because of the part (b), users can select their desired permit freely. Furthermore, the equilibrium under the scheme is efficient and achieves Pareto improvement for both the road manager and all users. The properties of the scheme for general networks have been explored in Akamatsu (2007) and Akamatsu and Wada (2017).

In order to implement the trading markets for the bottleneck permits, Wada and Akamatsu (2010, 2011, 2013) designed an auction mechanism for single bottleneck/general networks. Then, they showed that (i) the bottleneck permits allocation achieved by the mechanism is efficient and (ii) the mechanisms are strategy-proof (a dominant strategy employed by each user is the truthful revelation of the value of the permits).

However, the previous studies do not explicitly treat that when road users would participate in trading markets or when the transaction would be established before their trips (they implicitly assumed that all of the users would gather in the markets on the day before making trips). This single purchase opportunity assumption would be reasonable for the first step of analyzing “efficient” implementation mechanisms of the trading markets. Meanwhile, the travel flexibility and convenience for an individual also have to be considered for implementations. One of the examples in this direction is to design the multiple period markets enhancing the flexibility of users’ decision making¹. Furthermore, if the users’ valuations for the permits change over periods for some reasons, the multiple period markets can achieve more efficient resource allocation than a single period market.

The purpose of this study is to design an implementation mechanism of tradable bottleneck permits scheme with multiple purchase opportunities for a single bottleneck network. Firstly, assuming that the users’ valuations for permits change depending on the purchase periods, we present the framework of the multiple periods scheme. Under this scheme, the road manager

¹Another example is to design cancelation rules (see Nagae and Gai, 2009 for example).

sets the number of permits in each market, while the users select the purchase period and time interval of permit (arrival time). We then propose a mechanism to implement the multiple periods markets. In this mechanism, the adjustment of the number of permits (adjustment phase) and permits allocation phase (auction phase) are repeated. We prove that the proposed evolutionary mechanism has the following desirable properties: (i) the dynamic auction for multiple period markets is strategy-proof within each period and guarantees that the market choice of each user is optimal under a perfect information assumption of users; (ii) the mechanism achieves the optimal social allocation in a finite number of iterations. Finally, we show that the proposed mechanism may work well even for an incomplete information case.

1.2. Literature review

The scheme considered in this research corresponds to introducing a reservation system to the conventional tradable bottleneck permits scheme. Reservation systems have been widely studied in the field of revenue management for many years (see Talluri and vanRyzin (2004); Chiang et al. (2007), for comprehensive reviews of the literature). Moreover, in the transportation field, much research has been performed on the theory and practice of reservation systems (e.g., airline seat reservations, see Kobayashi et al. (2008)). Almost all of the above studies have aimed at maximizing revenue or social surplus by market segmentation and discriminatory pricing. However, as we previously stated, it is difficult to determine an optimal price because there is an asymmetric information between suppliers (road managers) and buyers (road users).

One of the approaches to resolve the asymmetric information problem is to employ auction mechanisms. In the transportation field, for example, Teodorovic et al. (2008) proposed a concept of a new demand management scheme called auction-based congestion pricing. Lam (2016) and Hara and Hato (2018) found that the simple Vickrey-Clarke-Groves (VCG) mechanism (Vickrey, 1961; Clarke, 1971; Groves, 1973) is an effective approach to achieve the optimal resources allocation for the transportation system. In addition, some simulation studies have demonstrated the quantitative impacts of the auction-based tolling system such as revenue generation and total travel time saving (see Peng and Park (2015); Basar and Cetin (2017)). However, these studies did not deeply discuss the efficient trading mechanism under the multiple trading opportunities.

Recently, in the field of mechanism design/auction theory, much effort has been put into extending the theory to dynamic settings (Parkes (2007); Bergemann and Said (2011)). Cavallo et al. (2006) and Bergemann and Välimäki (2010) considered the time-varying users' valuations and generalized the static VCG mechanism to a dynamic setting (*dynamic pivot mechanism*). They proved that the mechanism achieves the efficient resource allocation and satisfies the ex-post incentive compatibility. However, they did not consider an adjustment problem of the number of items for each period (i.e., the number is fixed in advance). Hence, a trading mechanism that combines a dynamic auction and an adjustment rule is a major contribution of this study.

Another stream of research relevant to the tradable bottleneck permits is about the "tradable travel credit" scheme proposed by Yang and Wang (2011). Especially, Nie and Yin (2013), Tian et al. (2013) and Xiao et al. (2013) dealt with cases with bottleneck congestion as in the studies of tradable bottleneck permits. The tradable travel credit scheme may obtain some improvements in equity and social acceptability over the conventional road pricing schemes. However, it is different from the tradable bottleneck permits scheme because it requires detailed demand information of users to impose an optimal time-dependent credit charge. In this sense, we can see that the tradable travel credit is like a price-based regulation which cannot restrict the use of

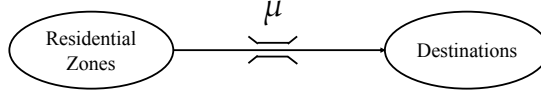


Figure 1: Road network

bottleneck capacity directly². To resolve the information issue, Wang and Yang (2013) proposed a modified bisection method for trial-and-error implementation of the credit charge which can let the road manager capture the social optimal state without a demand function. However, since the method was analyzed only for a single road and static congestion setting, it is largely unknown whether it can be applied to cases with bottleneck congestion. In addition to the tradable travel credit scheme, the tradable parking permits scheme is proposed and extended by some studies (see Zhang et al., 2011; Yang et al., 2013; Liu et al., 2014a for examples and Liu et al., 2014b for a review). However, these studies did not analyze the detailed trading mechanisms of the parking permits.

This paper is organized as follows. Section 2 outlines the tradable bottleneck permits scheme with multiple purchase opportunities. Section 3 formulates a system optimal bottleneck permits allocation problem. We also decompose the problem into two problems (the permits allocation problem, the number of permits adjustment problem), and present the design framework of an implementation mechanism. Section 4 shows the auction with multiple purchase opportunities for a given number of permits sold in each market and clarify the desired properties of it. Section 5 derives an adjustment rule of the number of permits for each market, and shows that the proposed mechanism can achieve the optimal social allocation in a finite number of iterations. Section 6 conducts numerical experiments. Section 7 presents some discussions and future research directions, and concludes the paper.

2. Settings

2.1. Networks

In this study, we consider discrete time dynamic traffic flows on a single bottleneck network where an origin (e.g., residential zones) is connected to a destination (see Figure 1). All of the road users must pass through a bottleneck to make trips. This bottleneck is presented by a point queue model with constant capacity μ . The time interval to which we assign the dynamic flow is divided into small intervals $k \in \mathcal{K}$.

In addition to the aforementioned within-day traffic assignment on a trip day, this study considers users' dynamic decision-making during the periods ($m \in \mathcal{M} \equiv \{1, 2, \dots, M\}$) leading up to the trip day. We assume that there are M periods for users to make a decision, the prior days $m = 1, 2, \dots, M - 1$ (hereafter, we call them "prior markets" simply) and the trip day $m = M$ (we call it "spot market" simply in the following).

²The comparison with tradable permits scheme (for emission control) and tradable travel credit scheme has been discussed in Nie (2012).

2.2. Agents

In this model, the road manager aims to alleviate traffic congestion in the network and maximize the social surplus. To achieve this, the manager regulates the traffic flow rates entering the bottleneck in the network using time-dependent bottleneck permits. The manager also establishes the prior and spot trading markets to sell the permits. The precise definition and setup of the bottleneck permits system with multiple purchase opportunities are described in Section 2.3.

Each (atomic) user $i \in \mathcal{I}$ makes, at most, a trip on day $m = M$ from the origin to the destination in the network. The user chooses a destination arrival time on the trip day and the decision-making period to maximize his/her utility. Moreover, under the system of tradable bottleneck permits, each user must purchase a permit corresponding to the chosen destination arrival time through trading markets. Therefore, the purchase of permits is conjunction with the two kinds of choice above. The detailed purchase method of permits is given in Section 2.3.

2.3. Tradable bottleneck permits with multiple purchase opportunities

“Tradable bottleneck permits” is the right that allows the permits holders to pass through a bottleneck during a pre-specified time interval. In this paper, the road manager issues the number of permits for the bottleneck in each time interval is equal to the bottleneck capacity μ . Under this setting, the arrival flow rate at a bottleneck at any time interval, from the definition of the scheme, is equal to (or less than) the number of permits. This implies that we can completely eliminate the occurrence of queuing congestion.

In this study, we consider the case in which bottleneck permits for the trip day are sold on period (or day) $m = 1, \dots, M$, although a few periods may be reasonable in practice. Therefore, the road manager needs to determine the number of permits sold for each period market $(\mu_k^m)_{\forall m}$ and ensures the total number of issued permits cannot exceed the μ (i.e., $\sum_{m \in \mathcal{M}} \mu_k^m \leq \mu$). Each user purchases a permit based on his/her destination arrival time in prior or spot market. In the trading markets, the prices and the allocation of time-dependent permits are determined through an auction³. Note that we assume that there is no resale and cancellation of the permits. The detail trading rules are given in Section 4.

3. System optimal allocation of bottleneck permits

In this section, we define the system optimal allocation of bottleneck permits (hereafter calls *social optimal state* in the following). First, we define the private valuation and utility of users and then formulate the social optimal allocation problem. In addition, we present an idea of an implementation mechanism by decomposing the problem into two problems.

3.1. User valuation and utility

We assume that each atomic user i has a valuation $v_{i,k}^m$ for arrival time k on the trip day and this valuation depends on the purchase period m . As an example of this type of situation, consider a case that a person has multiple activity plans at the destination that depend on purchase periods. Generally, the users’ valuations for permits are not the utility of the trip, but is the utility from

³We admit that the proposed scheme imposes the tedious trading procedures on users and seems unrealistic at first glance. However implementation of the scheme would become feasible with advanced vehicles in which an agent software is installed to automatically trade permits on based on users preferences (e.g., desired arrival time and willingness-to-pay). Further discussion on this issue can be found in Akamatsu and Wada (2017).

the activity with the trip. In this case, the valuation $v_{i,k}^m$ represents the utility of the activity corresponding to purchase period m . Note that the valuation of each user is private information that can not be observed by the road manager.

Each user is assumed to have a quasi-linear utility $u_{i,k}^m$. That is, the utility of each user who purchases a permit for destination arrival time k in market m is given by,

$$u_{i,k}^m = v_{i,k}^m - p_k^m \quad \forall i \in \mathcal{I}, \forall k \in \mathcal{K}, \forall m \in \mathcal{M}, \quad (1)$$

where p_k^m is the permit purchase cost that determined in an auction. We here do not model the time discount of financial payments in different periods because the proposed scheme is expected to apply the situation in which these financial transactions take place in a relatively short period (i.e., the interest rate is very small)⁴.

3.2. System optimal allocation problem

The purpose of the road manager is to achieve the traffic pattern which maximizes a social surplus. The social surplus is given as the sum of the users' valuations in every period⁵ because the users' valuations in each period represent the utility values obtained on the trip day. Thus, we formulate a problem [SO] to determine the system optimal allocation of bottleneck permits as follows.

$$SS \equiv \max_{y,z,\mu} \cdot \sum_{m \in \mathcal{M}} \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} v_{i,k}^m y_{i,k}^m, \quad (2)$$

subject to

$$\sum_{m \in \mathcal{M}} \mu_k^m \leq \mu \quad \forall k \in \mathcal{K}, \quad (3)$$

$$\sum_{i \in \mathcal{I}} y_{i,k}^m \leq \mu_k^m \quad \forall k \in \mathcal{K}, \forall m \in \mathcal{M}, \quad (4)$$

$$z_i^m - \sum_{k \in \mathcal{K}} y_{i,k}^m = z_i^{m+1} \quad \forall i \in \mathcal{I}, \forall m = 1, 2, \dots, M-1 \quad (5)$$

$$\sum_{k \in \mathcal{K}} y_{i,k}^M \leq z_i^M \quad \forall i \in \mathcal{I}, \quad (6)$$

$$y_{i,k}^m, z_i^m \in \{0, 1\} \quad \forall i \in \mathcal{I}, \forall k \in \mathcal{K}, \forall m \in \mathcal{M}, \quad (7)$$

$$\mu_k^m \geq 0 \quad \forall k \in \mathcal{K}, \forall m \in \mathcal{M}. \quad (8)$$

where $y_{i,k}^m$ is 1 if a bottleneck permit corresponding to destination arrival time k in market m is allocated to user i and 0 otherwise. A discrete variable z_i^m represents whether or not user i has an option to purchase bottleneck permits at the beginning of period m . Note that $z_i^1 \equiv 1$.

The problem [SO] finds an efficient permits allocation $\mathbf{y}^m \equiv (y_{i,k}^m)_{\forall i,k}$, the optimal purchase timing for each user $\mathbf{z}^m \equiv (z_i^m)_{\forall i}$, and the optimal number of permits to sell in each period market $\boldsymbol{\mu}^m \equiv (\mu_k^m)_{\forall k}$. More specifically, the objective function (2) represent the social surplus

⁴We can also interpret the model as the situation in which all financial transactions take place on the trip day $m = M$.

⁵The permit purchase cost that is transferred from users to road manager should not be included in the social surplus.

obtained in the prior and the spot markets. The first constraint (3) is the condition that the total number of permits sold in these markets does not exceed the bottleneck capacity. The second constraint (4) is the supply constraint for each market. The third and fourth constraints (5) and (6) result in the *unit-demand* condition: each user purchase at most one permit. The fifth and sixth constraints (7) and (8) are the 0-1 integer constraint and nonnegative constraint, respectively.

As is apparent from the above constraint conditions, the problem [SO] is a linear mixed-integer problem. Although the mixed-integer problems are difficult to solve in general, we obtain an optimal integer solution by solving a linear relaxation of the problem [SO], if the bottleneck capacity μ is integer-valued and a solution algorithm that produces extreme point solutions is used. This is because the constraint matrices satisfy *total unimodularity* (TU) (see Appendix A for the proof). Therefore, assuming that the bottleneck capacity is given as an integer, we can replace constraints (7) by $y_{i,k}^m \geq 0$ and $z_i^m \geq 0$, i.e., the problem reduces to a linear programming (LP) problem.

Note that we can investigate existence and uniqueness of the traffic pattern by examining those of a solution of the LP problem. That is, the solution is not unique in general according to the LP theory; the solution always exists because the potential users are allowed to not travel due to the capacity constraint (i.e., the unit demand assumption).

3.3. Decomposition of the system optimal allocation problem

The problem [SO] optimizes three types of unknown variables, \mathbf{y}^m , \mathbf{z}^m , and $\boldsymbol{\mu}^m$, in a simultaneous manner. However, such a simultaneous optimization is difficult unless the manager accurately obtains users' private information $(\mathbf{v}^m)_{\mathcal{V}^m}$. Hence, we decompose the problem into the following two problems by applying the Benders decomposition principle to the problem [SO]: (1) sub-problem: a problem that determines the allocation of permits, \mathbf{y}^m (and \mathbf{z}^m); (2) master problem: a problem that adjusts the number of permits sold in each market, $\boldsymbol{\mu}^m$. We analyze the two problems in more detail in the following.

3.3.1. Bottleneck permits allocation problem under the fixed number of permits

Suppose that the number of permits sold in each period market is fixed $(\boldsymbol{\mu}^m)_{\mathcal{V}^m}$ and integer. Then a bottleneck permits allocation problem [SO_{sub-P}] that maximizes the social surplus is formulated as

$$\begin{aligned} \max_{\mathbf{y}, \mathbf{z} \geq 0} \quad & \sum_{m \in \mathcal{M}} \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} v_{i,k}^m y_{i,k}^m, & (9) \\ \text{subject to} \quad & \text{Eqs. (4), (5), and (6)}. \end{aligned}$$

This sub-problem [SO_{sub-P}] has two meanings. First, it is obvious that its optimal solution is equal to that of the problem [SO] if the number of permits for each market $\boldsymbol{\mu}^m$ is optimal. Second, the sub-problem [SO_{sub-P}] is equivalent to an optimization problem for a market equilibrium in which each user chooses both a destination arrival time and a purchase period.

To show the second point more precisely, we consider the Kuhn-Tucker conditions for the

sub-problem [SO_{sub}-P]:

$$\begin{cases} \sum_{i \in \mathcal{I}} y_{i,k}^m = \mu_k^m & \text{if } p_k^m > 0 \\ \sum_{i \in \mathcal{I}} y_{i,k}^m \leq \mu_k^m & \text{if } p_k^m = 0 \end{cases} \quad \forall k \in \mathcal{K}, \forall m \in \mathcal{M}, \quad (10)$$

$$\begin{cases} v_{i,k}^m - p_k^m = \pi_i^m & \text{if } y_{i,k}^m = 1 \\ v_{i,k}^m - p_k^m \leq \pi_i^m & \text{if } y_{i,k}^m = 0 \end{cases} \quad \forall i \in \mathcal{I}, \forall k \in \mathcal{K}, \forall m \in \mathcal{M}, \quad (11)$$

$$\begin{cases} \pi_i^{m+1} = \pi_i^m & \text{if } z_i^{m+1} = 1 \\ \pi_i^{m+1} \leq \pi_i^m & \text{if } z_i^{m+1} = 0 \end{cases} \quad \forall i \in \mathcal{I}, \forall m = 1, 2, \dots, M-1 \quad (12)$$

$$\begin{cases} \sum_{k \in \mathcal{K}} y_{i,k}^M - z_i^M = 0 & \text{if } \pi_i^M > 0 \\ \sum_{k \in \mathcal{K}} y_{i,k}^M - z_i^M \leq 0 & \text{if } \pi_i^M = 0 \end{cases} \quad \forall i \in \mathcal{I} \quad (13)$$

+ equality constraint (5),

where, $\mathbf{p}^m \equiv (p_k^m)_{\forall k}$, $\boldsymbol{\pi}^m \equiv (\pi_i^m)_{\forall i}$ are the optimal Lagrange multipliers for constraints Eq.(4) and Eq.(6), respectively. The optimality conditions, Eqs.(10)-(13), can be interpreted as the market equilibrium by regarding the Lagrange multipliers \mathbf{p}^m and $\boldsymbol{\pi}^m$ as equilibrium permits prices and option values in market m . Specifically, Eq.(10) represents the market-clearing condition, and Eqs.(11)-(13) are interpreted as the user equilibrium choice conditions of arrival time and purchase period when permits prices are given. Hence, the following proposition holds:

Proposition 1. *Assume that the number of permits sold in each period market is fixed and integer. We also assume that the trading markets are perfectly competitive. Then, the equilibrium resource allocation pattern that is realized under the tradable bottleneck permits with multiple purchase opportunities maximizes the social surplus defined by Eq.(9).*

Proof. We first confirm the demand-supply equilibrium condition for each destination arrival time in each period corresponding to the optimality condition Eq.(10). This correspondence is clear if the Lagrange multipliers $(\mathbf{p}^m)_{\forall m}$ are regarded as permits prices (competitive equilibrium prices) in the prior and the spot markets.

Then, we show that the user choice equilibrium conditions are equivalent to the optimality conditions Eqs.(11)-(13). For given permits prices, each user determines a destination arrival time and a purchase period so as to maximize his/her utility:

$$\max_{m \in \{1, 2, \dots, M+1\}} \cdot \max_{k \in \mathcal{K}} \{v_{i,k}^m - p_k^m\} \quad \forall i \in \mathcal{I}. \quad (14)$$

For convenience, we use $m = M + 1$ to show users that do not purchase any permits; their payoffs are zero. At this time, π_i^m can be viewed as the value function of the problem (14) in period $m \in \{1, 2, \dots, M + 1\}$:

$$\pi_i^m \equiv \max_{\tau \geq m} \cdot \max_{k \in \mathcal{K}} \{v_{i,k}^\tau - p_k^\tau\} \quad \forall i \in \mathcal{I}, \forall m = 1, \dots, M. \quad (15)$$

Applying the *Dynamic Programming* (DP) principle, we obtain the optimal decision-making at the beginning of each market. More specifically, the optimal choice pair (m^*, k^*) can be obtained by “backward induction.”

First, the choice problem of user i at $m = M$ (the trip day):

$$\pi_i^M = \max \left\{ \max_{k \in \mathcal{K}} \{v_{i,k}^M - p_k^M\}, \pi_i^{M+1} \right\}, \quad (16)$$

where $\pi_i^{M+1} = 0$. By using the optimal choice function (16), the optimal choice in period $m = M - 1$ is given by

$$\begin{cases} \max_{k \in \mathcal{K}} \{v_{i,k}^{M-1} - p_k^{M-1}\} \geq \pi_i^M \\ \Leftrightarrow \begin{cases} \text{purchase } z_i^M = 0 \text{ and } y_{i,k}^{M-1} = 1, \exists k \in \mathcal{K} & \text{if } z^{M-1} = 1 \\ \text{non-purchase } z_i^M = 0 \text{ and } y_{i,k}^{M-1} = 0, \forall k \in \mathcal{K} & \text{if } z^{M-1} = 0 \end{cases} \\ \max_{k \in \mathcal{K}} \{v_{i,k}^{M-1} - p_k^{M-1}\} < \pi_i^M \\ \Leftrightarrow \text{non-purchase } z_i^M = 1 \text{ and } y_{i,k}^{M-1} = 0, \forall k \in \mathcal{K} & \text{if } z^{M-1} = 1. \end{cases} \quad (17)$$

Then, the choice problem of user i at $m = M - 1$ is denoted as:

$$\pi_i^{M-1} = \max \left\{ \max_{k \in \mathcal{K}} \{v_{i,k}^{M-1} - p_k^{M-1}\}, \pi_i^M \right\}, \quad (18)$$

In the exactly same manner, for all $m = 1, 2, \dots, M - 1$, the choice problem of user i can be denoted as

$$\pi_i^m = \max \left\{ \max_{k \in \mathcal{K}} \{v_{i,k}^m - p_k^m\}, \pi_i^{m+1} \right\}, \quad (19)$$

Accordingly, the user choice equilibrium conditions can be denoted by (16) and (19).

Let us now confirm the equivalence between the user choice equilibrium conditions (16) and (19) and the optimality conditions (11)-(13) and (5). The equilibrium condition (19) in period m can be rewritten as

$$\pi_i^m \geq \pi_i^{m+1}, \quad (20)$$

$$\pi_i^m \geq v_{i,k}^m - p_k^m \quad \forall k \in \mathcal{K}, \quad (21)$$

but at least one of the conditions must hold with equality; then, an optimal choice is determined to correspond to one of such equality conditions. More specifically, like Eq.(17), there are two types of choices if the option remains (i.e., $z_i^m = 1$):

(i) purchase, i.e., $y_{i,k^*}^m = 1, z_i^{m+1} = 0$ and $\pi_i^m = v_{i,k^*}^m - p_{k^*}^m$

(ii) non-purchase or postpone the purchase, i.e., $y_{i,k}^m = 0 \forall k, z_i^{m+1} = 1$ and $\pi_i^m = \pi_i^{m+1}$;

otherwise, non-purchase because of no option (i.e., $z_i^m = 0, y_{i,k}^m = 0 \forall k, z_i^{m+1} = 0$). It is clear that these three conditions can be expressed by Eqs. (5), (11), and (12). In almost the same way, we can show that the equilibrium condition (16) in period M can be expressed by Eqs. (5), (11), and (13).

The above discussion shows that both the equilibrium conditions and the optimality conditions have the exactly the same form. Therefore, equilibrium permits allocation pattern is equal to that obtained by solving the sub-problem [SO_{sub}-P]. \square

Proposition 1 states that the optimal solution of the problem [SO_{sub}-P] can be achieved as a result of distributed behavior of users. However, in order to hold this proposition, it is necessary to assume that users do not play strategic behaviors that affect prices (e.g., perfectly competitive markets). For example, we suppose that a user manipulates a permits price. This strategic behavior may decrease other users' utility, which results in a failure to achieve the system optimal state. Hence, we have to design a mechanism in which each user has no incentive to exhibit a strategic behavior. We will discuss such a mechanism in more detail in Section 4.

3.3.2. Adjustment of number of permits sold for each period market

The problem of adjusting the number of permits sold for each market is obtained as a Benders master problem. That is,

$$\max_{\mu \geq 0 \text{ and integer}} \cdot \sum_{m \in \mathcal{M}} \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} v_{i,k}^m y_{i,k}^m(\mu), \quad \text{subject to Eq. (3)}, \quad (22)$$

where $(\mathbf{y}^m(\mu))_{\forall m}$ is the optimal solution of the sub-problem [SO_{sub}-P] whose parameter is μ . In order to explore the explicit relationship between the master problem and sub-problem, we show the dual problem [SO_{sub}-D] of the sub-problem [SO_{sub}-P] is as follows:

$$\min_{\mathbf{p}, \boldsymbol{\pi} \geq 0} \cdot \sum_{m \in \mathcal{M}} \sum_{k \in \mathcal{K}} \mu_k^m p_k^m + \sum_{i \in \mathcal{I}} \pi_i^1, \quad (23)$$

subject to

$$\pi_i^m \geq v_{i,k}^m - p_k^m \quad \forall i \in \mathcal{I}, \forall k \in \mathcal{K}, \forall m \in \mathcal{M}, \quad (24)$$

$$\pi_i^m \geq \pi_i^{m+1} \quad \forall i \in \mathcal{I}, \forall m = 1, 2, \dots, M-1. \quad (25)$$

Then, by using duality theorem about the sub-problem, the value of the objective function of master problem is consistent with that of the sub-problem.

$$\sum_{m \in \mathcal{M}} \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} v_{i,k}^m y_{i,k}^m(\mu) = \sum_{m \in \mathcal{M}} \sum_{k \in \mathcal{K}} \mu_k^m p_k^m(\mu) + \pi^1(\mu). \quad (26)$$

where $(\mathbf{p}(\mu), \boldsymbol{\pi}^1(\mu))$ is the optimal solution for the [SO_{sub}-D] whose parameter is μ , and $\pi^1(\mu) \equiv \sum_{i \in \mathcal{I}} \pi_i^1(\mu)$. Therefore, the master problem can be transferred to the following optimization problem:

$$\max_{\mu \geq 0 \text{ and integer}} \cdot \sum_{m \in \mathcal{M}} \sum_{k \in \mathcal{K}} \mu_k^m p_k^m(\mu) + \pi^1(\mu), \quad \text{subject to Eq. (3)} \quad (27)$$

Furthermore, we can express $(\mathbf{p}(\mu), \boldsymbol{\pi}^1(\mu))$ by using the extreme point set:

$$S \equiv \{(\mathbf{p}(1), \boldsymbol{\pi}^1(1)), \dots, (\mathbf{p}(|S|), \boldsymbol{\pi}^1(|S|))\}$$

of convex feasible region which consists of the constraints (24), (25), and no-negativity condition. By using this set, the master problem finally reduces to

$$\max_{\mu \geq 0 \text{ and integer}} \cdot \left[\min_{(\mathbf{p}(s), \boldsymbol{\pi}^1(s)) \in S} \cdot \sum_{m \in \mathcal{M}} \sum_{k \in \mathcal{K}} \mu_k^m p_k^m(s) + \pi^1(s) \right], \quad \text{subject to Eq. (3)}. \quad (28)$$

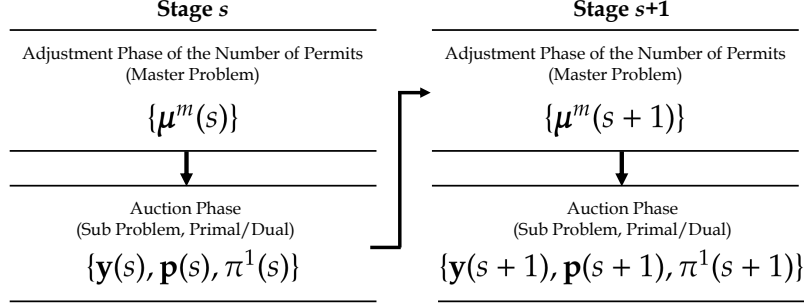


Figure 2: Procedures of the proposed mechanism

This problem is equivalent to the problem [SO] if we know the set S of extreme points. However, it is difficult to know the set in advance. Therefore, we explore a method which produces extreme points sequentially and converges to the optimal solution through the iterative calculation of a master problem and a sub-problem.

3.4. Design framework of the mechanism

From the above discussion, in order to implement the scheme, it is necessary to design a mechanism for solving the master problem and the sub-problem iteratively. More specifically, we have to design, (a) a dynamic auction to implement the user choice equilibrium; (2) an adjustment rule for the number of permits sold in each market to converge to the optimal solution. We call the former the “auction phase” and the latter the “adjustment phase” whereas one iteration of both phases is a “stage” (see Figure 2). Each stage is denoted by $s = 1, 2, \dots$. In addition, we assume that each user behaves myopically and makes his/her choice so as to maximize the utility defined at each stage s . In Section 4 and Section 5, we concretely design the auction and adjustment rules and clarify their desired properties.

4. Auction phase

Assuming that the number of permits sold in each period market $\mu^m(s)$ is fixed, we showed that the sub-problem [SO_{sub}] is equivalent to the market and the user choice equilibrium state (i.e., Proposition 1). However, the problem simultaneously determines the permits allocation variables of both the prior and spot markets: it does not represent an actual time sequence of the multiple period markets (the bottleneck permits allocation in the prior markets are determined before that in the spot market). Hence, we first show that the problem [SO_{sub}-P] can be further decomposed to be consistent with the actual sequence of the markets (or time) under a certain condition in Section 4.1. Section 4.2 then shows the auction mechanism which is strategy-proof.

4.1. Time decomposition of multiple period markets

Let us look into the sub-problem [SO_{sub}-D] in detail. In this problem, the interaction among the different markets arises only in the constraint (25), i.e., $\pi_i^m \geq \pi_i^{m+1}$, $\forall m = 1, 2, \dots, M-1$, which implies that the markets have the following structure. The decisions (\mathbf{p}^m, π^m) of each market m only depend on the variable π^{m+1} of the next market $m+1$, and thus we can solve

the sub-problem by sequentially determining the variables in the decreasing order, $m = M, M - 1, \dots, 1$. Meanwhile, we can rewrite the objective function (23) by using equilibrium option value in each market π_i^m as follows.

$$\sum_{m=1}^{M-1} \left[\sum_{k \in \mathcal{K}} \mu_k^m(s) p_k^m + \sum_{i \in \mathcal{I}} (\pi_i^m - \pi_i^{m+1}) \right] + \left[\sum_{k \in \mathcal{K}} \mu_k^M(s) p_k^M + \sum_{i \in \mathcal{I}} \pi_i^M \right]. \quad (29)$$

Therefore, each problem for the market $m = 1, 2, \dots, M$ of the above sequential optimization procedure can be formulated as follows.

[SO_{sub}-D^m]

$$\min_{\mathbf{p}^m \geq 0, \boldsymbol{\pi}^m} \cdot \sum_{k \in \mathcal{K}} \mu_k^m(s) p_k^m + \sum_{i \in \mathcal{I}} (\pi_i^m - \pi_i^{m+1}), \quad (30)$$

subject to

$$\pi_i^m \geq v_{i,k}^m - p_k^m \quad \forall i \in \mathcal{I}, k \in \mathcal{K}, \quad (31)$$

$$\pi_i^m \geq \pi_i^{m+1} \quad \forall i \in \mathcal{I}. \quad (32)$$

where $\boldsymbol{\pi}^{m+1}$ is given for the problem [SO_{sub}-D^m] and $\pi_i^{M+1} = 0$.

Although the sequential optimization procedure is the opposite direction of time, it tells us a condition that each market can be treated independently. Namely, if each user can know his/her own option value in the next market, the multiple period markets can be decomposed to be consistent with time sequence. From now on, we discuss the auction mechanism, assuming the condition holds. We will discuss how to relax this condition and will numerically test the proposed mechanism for the relaxed situation in Section 6.

Let us introduce a new variable $\hat{v}_{i,k}^m \equiv v_{i,k}^m - \pi_i^{m+1}$; it represents the “net valuation” (a truthful valuation minus the option value). Then, the problem [SO_{sub}-D^m] is equal to the following problem with a new unknown variable $\hat{\pi}_i^m \equiv \pi_i^m - \pi_i^{m+1}$:

$$\min_{\mathbf{p}^m, \hat{\boldsymbol{\pi}}^m \geq 0} \cdot \sum_{k \in \mathcal{K}} \mu_k^m(s) p_k^m + \sum_{i \in \mathcal{I}} \hat{\pi}_i^m, \quad (33)$$

subject to

$$\hat{\pi}_i^m \geq \hat{v}_{i,k}^m - p_k^m \quad \forall i \in \mathcal{I}, k \in \mathcal{K}. \quad (34)$$

We, finally, obtain an independent assignment problem for each period market as the primal problem [SO_{sub}-P^m] of the dual problem (33).

$$\max_{\mathbf{y}^m \geq 0} \cdot \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} \hat{v}_{i,k}^m y_{i,k}^m \quad (35)$$

subject to

$$\sum_{i \in \mathcal{I}} y_{i,k}^m \leq \mu_k^m(s) \quad \forall k \in \mathcal{K}, \quad (36)$$

$$\sum_{k \in \mathcal{K}} y_{i,k}^m \leq 1 \quad \forall i \in \mathcal{I}, \quad (37)$$

where the constraint (37) is the unit-demand condition. The discussions indicate that the sub-problem [SO_{sub}-D] can be solved to be consistent with time sequence if each user reports *net valuations* \hat{v}^m truthfully in each market (i.e., we can solve the allocation problem [SO_{sub}-D^{m+1}] after solving the [SO_{sub}-D^m]).

4.2. Auction mechanism for multiple period markets

Because the problem [SO_{sub}-P^m] is the standard assignment problem, we can apply the various incentive compatible auction mechanisms (e.g., the VCG mechanism) to it. Now, let us employ the proxy DGS auction (shown in Appendix B and Wada and Akamatsu, 2013) for implementing the multiple period markets. Then, the following proposition holds:

Proposition 2. *Assume that the number of permits sold in each period market is fixed and integer. We also assume that, in each market, each user knows his/her own option value realized in the next market. Then the proxy DGS auction mechanism for each period market is strategy-proof and achieves an efficient bottleneck permits allocation.*

Proof. Each market can be treated independently when each user knows his/her own option value realized in each market. Therefore, each market is strategy-proof from Demange et al. (1986) and Parkes and Ungar (2000). The allocation of the bottleneck permits of each market is the optimal solution of the decomposed sub-problem [SO_{sub}-P^m]. On the other hand, the (undecomposed) sub-problem [SO_{sub}-P] maximizes the social surplus under the condition that the number of permits sold for each market is fixed. Because the undecomposed sub-problem [SO_{sub}-P] and decomposed sub-problems [SO_{sub}-P^m] are equivalent, the bottleneck permits allocation achieved by the DGS auction also maximizes the social surplus. \square

Furthermore, we reveal that the user's market choice is optimal by using the Proposition 2. Because each prior market is strategy-proof, each user's allocation of permits is given by

$$\begin{cases} \hat{v}_{i,k}^m - p_k^m = \hat{\pi}_i^m & \text{if } y_{i,k}^m = 1 \\ \hat{v}_{i,k}^m - p_k^m \leq \hat{\pi}_i^m & \text{if } y_{i,k}^m = 0 \end{cases} \quad \forall i \in \mathcal{I}, \forall m \in \mathcal{M}. \quad (38)$$

Thus, for all users,

$$\begin{aligned} \hat{\pi}_i^m &= \max \left\{ \max_{k \in \mathcal{K}} \{ \hat{v}_{i,k}^m - p_k^m \}, 0 \right\} \\ \Leftrightarrow \pi_i^m - \pi_i^{m+1} &= \max \left\{ \max_{k \in \mathcal{K}} \{ v_{i,k}^m - \pi_i^{m+1} - p_k^m \}, 0 \right\} \\ \Leftrightarrow \pi_i^m &= \max \left\{ \max_{k \in \mathcal{K}} \{ v_{i,k}^m - p_k^m \}, \pi_i^{m+1} \right\} \end{aligned}$$

holds. This equation is equal to the optimal market choice condition (19) in each market m . That is, truthful reporting of the net valuations in each market means choosing a purchase period so as to maximize the utility of each user. Then, the following corollary holds:

Corollary 1. *Assume that, in each market, each user knows his/her own option value realized in the next market. Then, the market choice of every user that participate in the multiple period markets is optimal.*

5. Adjustment phase of the number of permits

5.1. Adjustment rule

In the adjustment phase, the road manager generates a new extreme point $(\mathbf{p}(s), \pi^1(s))$ from the information obtained in the multiple period markets, and then determines the number of permits sold in each period market in the next stage. The prices $(\mathbf{p}^m)_{m \in \mathcal{M}}$ can be obtained directly in the auction phase for each period market. The total payoff $\pi^1(s)$, on the other hand, is computed in an indirect way. In the proxy DGS auction, because each user reports true net valuations to the proxy agent about the permits that they are interested in, the road manager can obtain his/her winning valuations $(\hat{v}_{i,k^*}^{m^*})_{i \in \mathcal{I}}$ through the agent. By using this information, the manager calculate the total payoff from the duality theorem (see [Appendix C](#)).

$$\pi^1(s) = \sum_{i \in \mathcal{I}} \hat{v}_{i,k^*}^{m^*} - \sum_{m \in \mathcal{M}} \sum_{k \in \mathcal{K}} \mu_k^m(s) p_k^m. \quad (39)$$

Note here that the extreme point $(\mathbf{p}(s), \pi^1(s))$ consists of the aggregate information of all users (is not the information of each user).

After generating the extreme point, the road manager considers the set of extreme points until stage s :

$$S' \equiv \{(\mathbf{p}(1), \pi^1(1)), \dots, (\mathbf{p}(s), \pi^1(s))\} \subseteq S.$$

Then, the road manager adjusts the number of permits sold in each period market by solving the following optimization problem:

$$\max_{\mu \geq 0 \text{ and integer}} \left[\min_{(\mathbf{p}(s), \pi^1(s)) \in S'} \sum_{m \in \mathcal{M}} \sum_{k \in \mathcal{K}} \mu_k^m p_k^m(s) + \pi^1(s) \right], \quad \text{subject to Eq. (3)}. \quad (40)$$

The solution $\mu(s+1)$ of this problem is the number of permits sold in the each market at the next stage $s+1$. Unlike the master problem (Eq. (28)), this problem use a subset of the extreme points, which produce an upper bound on the optimal value of the problem [SO]. Moreover, the problem can be reduced to the following linear program:

$$\max_{\theta, \mu \geq 0 \text{ and integer}} \theta, \quad (41)$$

subject to Eq. (3),

$$\theta \leq \sum_{m \in \mathcal{M}} \sum_{k \in \mathcal{K}} \mu_k^m p_k^m(s) + \pi^1(s) \quad \forall (\mathbf{p}(s), \pi^1(s)) \in S'. \quad (42)$$

Thus, the problem can be solved in a very efficient way.

5.2. Convergence of whole mechanism

The proposed mechanism corresponds to Benders decomposition algorithm. The algorithm terminates (converges to an optimal solution) when the upper bound of master problem is equal to the lower bound (optimal objective value) of the sub-problem. In other words, the social surplus is maximized when $\sum_{i \in \mathcal{I}} \hat{v}_{i,k^*}^{m^*}$ (by the auction) is equivalent to θ . Otherwise, when $\sum_{i \in \mathcal{I}} \hat{v}_{i,k^*}^{m^*} < \theta$, the manager should adjust the number of permits for the next stage. Furthermore, a new extreme point is generated in the auction phase before the procedure terminates, and the set of extreme points is finite. Thus, the following proposition holds:

Proposition 3. *The proposed mechanism that combines the auction and adjustment phases maximizes the social surplus in a finite number of iterations.*

Proof. Because the decomposed sub-problem [SO_{sub}-D^m] is bounded (from below), an extreme point always is generated at every stage. Therefore, the proposed mechanism corresponds to the Benders decomposition algorithm excluding the step for the case where extreme rays are generated. For a complete proof see, for example, Lasdon (1970) and Tone (2007). □

Two remarks on the proposed mechanism are in order. First, the standard Benders decomposition algorithm for a LP problem does not require an integer solution of the master problem. However, in the proposed mechanism, an integer solution is needed for the auction phase (or for ensuring the subproblem’s solution integer). Nevertheless, the Benders decomposition algorithm with this special treatment converges because the linear relaxation of the problem [SO] has an optimal integer solution as discussed in Section 3.2. Second, the number of permits for some time intervals and some markets can become zero. Although the proxy DGS auction (Appendix B) can produce prices for such permits if users input their valuations into the proxy agents, this may be impractical. To avoid this situation, it is enough to add the minimum permits number constraint, $\mu_k^m \geq \mu_{\min} (> 0)$, to the master problem (41). This constraint may reduce the social surplus for some cases, but the reduction becomes zero if there exists at least one assigned user at an optimum for every time interval and market or the number of permits is sufficiently large compared to the number of users. Simple examples in Appendix D illustrate the second remark concretely as well as the proposed mechanism itself.

6. Numerical experiments

6.1. Settings

We conduct two types of numerical experiments, one to illustrate the convergence process of the proposed mechanism (Section 6.2.1) and the other to examine the mechanism in the case of relaxing the perfect information assumption of users (Section 6.2.2). Here, we use the two-period model to illustrate our theory as simply as possible, although we considered the model with general M markets. In all the experiments, we set the bottleneck capacity at 1800 (veh/h), the number of destination arrival times as $|K| = 6$, and the time interval of destination arrival time as $\Delta k = 10$ (min). During the one hour, 2000 potential users want to travel through the bottleneck. The distribution of the desired arrival times k^* to the destination is shown in Figure 3. In this setting, 200 users cannot travel due to the capacity constraint in each stage.

The users’ valuations for the prior market are defined as the willingness-to-pay minus a schedule delay cost. The former is set randomly within the range of [50,70] (yuan), and the latter cost for each arrival time is given as

$$\begin{cases} \beta(k^* - k) & \text{if } k \leq k^* \\ \gamma(k - k^*) & \text{if } k > k^* \end{cases}$$

where $\beta = 5$ (yuan/10 min) and $\gamma = 10$ (yuan/10 min) are the coefficients of converting schedule delays to monetary costs. In addition, we set the valuations for the spot market by randomly changing those for the prior market, at a certain percentage (5%, 10%, 20%, 30%, 40%, 50%). We call this percentage the “valuation ratio.”

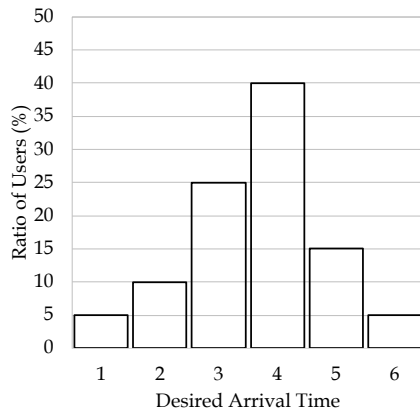


Figure 3: Distribution of the desired destination arrival times

Table 1: Comparison on the ratio of optimal social surplus between the single-period and the two-period markets (100 samples for each variation ratio)

Variation ratio	5%	10%	20%	30%	40%	50%
Mean	97.3%	94.7%	89.9%	85.4%	81.5%	77.6%
Standard deviation ($\times 10^{-4}$)	1.9	1.8	5.3	5.5	4.4	2.6

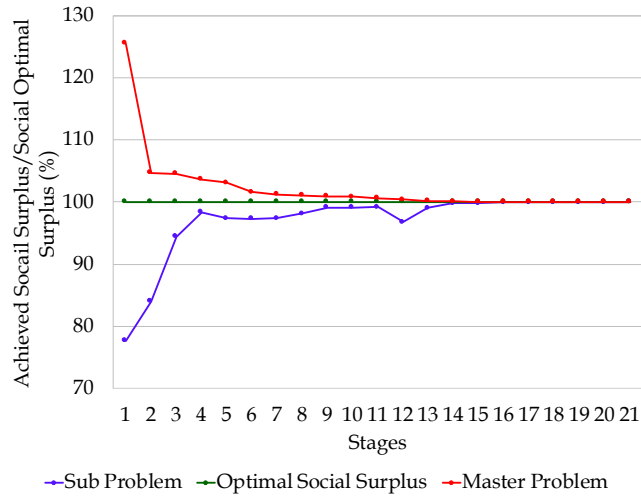
6.2. Results and analysis

In Section [6.2.1](#), we show the numerical results for a case with complete information, where users know their own option values in the spot market. The results for a case with incomplete information, where users predict the option values based on some rules, are presented in Section [6.2.2](#).

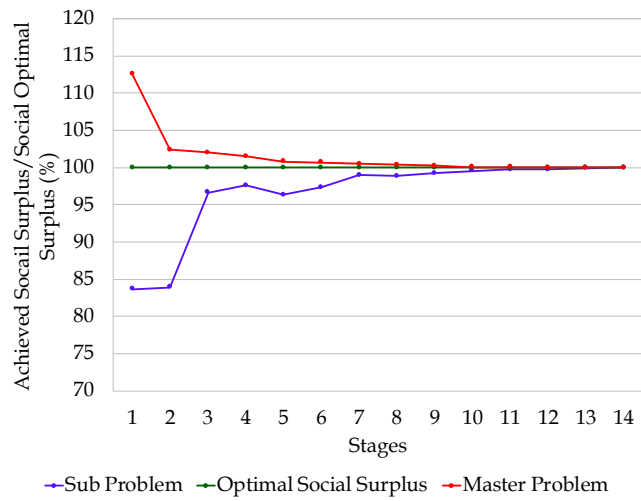
6.2.1. Complete information case

We conduct a Monte-Carlo simulation for each valuation ratio to evaluate how efficient the two period markets are, when compared to the single period setting. In the single period setting, we use the users' valuations for the prior market of the two period markets. The mean and standard deviation of the ratio of optimal social surplus between the single and the two period markets are summarized in Table [1](#). On average, the relative difference of the optimal social surplus (normalized by the value of the two period markets) is half the valuation ratio, which implies that the two period markets become more efficient with an increase of the variation ratio. Furthermore, this property may be independent of the detailed setting of users' valuations, because the standard deviation of the ratio is quite small.

Figure [4](#) shows some convergence processes (sample paths) of the proposed mechanism with different initial distributions of bottleneck permits for the prior and spot markets. The horizontal axis is the number of stages, and the vertical axis represents the ratio between the achieved social surplus and the optimal social surplus at each stage. From these figures, we see that (i) in a few iterations, the objective values of the master problem (red line) and sub-problem (blue line) get drastically closer to an optimal value (green line), although the final convergence is slow; (ii) the initial setting may not significantly affect the speed of convergence.

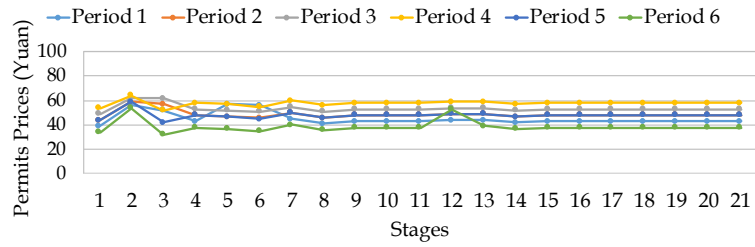


(a) All permits are initially sold in prior market

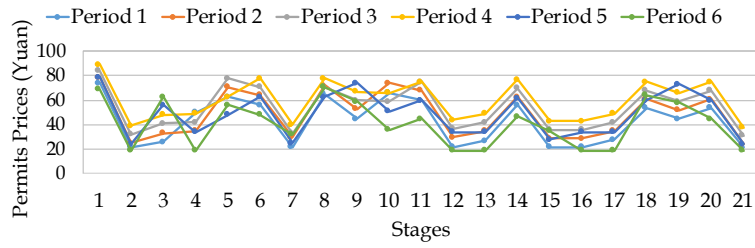


(b) All permits are initially sold evenly for any time period in prior and spot markets

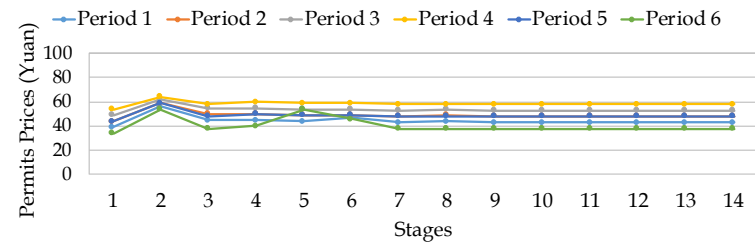
Figure 4: Convergence processes of the proposed mechanism under different initial distributions of bottleneck permits for the prior and spot markets (valuation ratio: 50%)



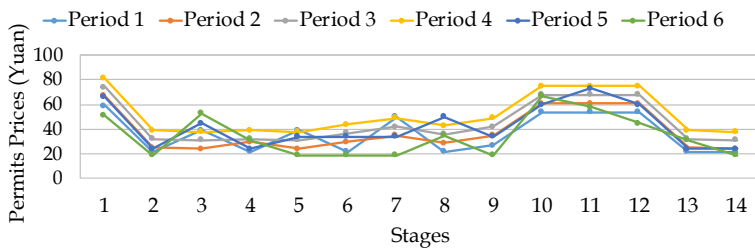
(a) Prior market in Figure 4(a)



(b) Spot market in Figure 4(a)



(c) Prior market in Figure 4(b)



(d) Spot market in Figure 4(b)

Figure 5: Dynamics of the permits prices for prior and spot markets

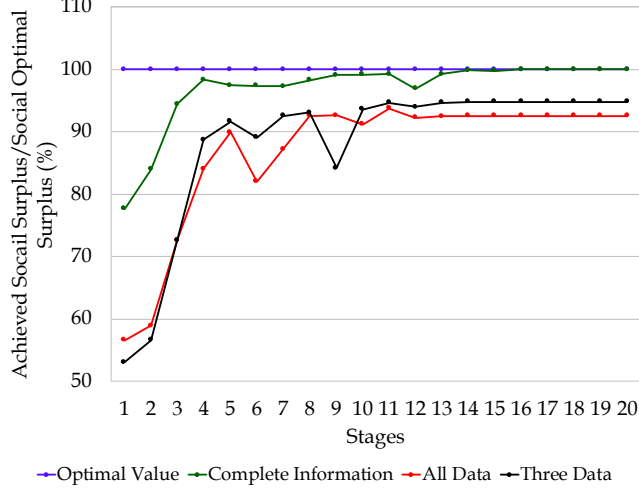


Figure 6: Comparison on convergence process of the proposed mechanisms under complete and incomplete information conditions

Figure 5 shows the dynamics of permits prices during the above convergence processes. The horizontal axis is the number of stages, the vertical axis is the price value, and the different color represents different arrival times. We see that the permits prices of Figures 5(a) and (c) (Figures 5(b) and (d)) converge to the same values although the convergence processes are different. The permits prices for arrival time 4 are the highest, because the number of users who want to arrive at this time is the largest (see Figure 3). The reason why the permits prices for the spot market vary more than those for the prior market is due to the assumption of users' valuations (i.e., the valuations for the spot market vary more than they do for the prior market).

6.2.2. Incomplete information case

We finally investigate the case where users do not know their option values in the spot market completely, but they are assumed to predict their option values based on historical data of permits prices that are provided by the road manager. Specifically, we assume two simple prediction behaviors in which users calculate the option value at stage s using Eq. (18) with the mean value of the permits prices for all the previous days $\{1, 2, \dots, s-1\}$ (we call it "all data pattern"), or the last three days $\{s-3, s-2, s-1\}$ (we call it "three data pattern") in the spot market. Note that we here assume that the variation ratio is 50%.

The results of the Monte-Carlo simulations for different cases (100 samples for each pattern) are summarized in Table 2. From this table, we have the following observations: (i) the social surplus of the proposed mechanism with the simple prediction behaviors is very high (i.e., 92.5% for the all data pattern, 94.3% for the three data pattern); and (ii) the social surplus achieved for the three data pattern is higher than that for the all data pattern, although the convergence speeds and their variations are larger for the three data pattern. The first observation indicates that the proposed mechanism may work well even for a case of incomplete information if the appropriate information provision on the permits prices are provided by the road manager. The second observation may be explained by the fact that the all data pattern includes the permits

Table 2: Comparison on the ratio of achieved social surplus under different purchase opportunity and prediction behaviors (100 samples for each pattern)

Pattern	Single-period	All data	Three data
Mean value of ratio	77.6%	92.5%	94.3%
Standard deviation of ratio	2.6×10^{-4}	0.4	0.7
Mean value of iteration	-	14.8	19.2
Standard deviation of iteration	-	2.0	5.1

prices at the beginning of the convergence process, and these prices are far from the optimal ones. The sample paths of the proposed mechanism with different user’s behavioral models are shown in Figure 6. Although the final values of the social surplus for the prediction models (black and red lines) are less than that for case of perfect information (green line), the desirable convergence property of the proposed mechanism may hold, i.e., the high social surplus is achieved in a few iterations.

7. Concluding remarks

This study considered a situation where bottleneck permits for a trip day are sold in multiple period markets and designed a trading mechanism of these markets. We first showed that the system optimal permits allocation for a fixed permits issue pattern is equivalent to the equilibrium in perfectly competitive markets. This enabled us to decompose the system optimal allocation problem into sequential sub-problems. We then constructed the mechanism for implementing each sub-problem independently, and proved that the proposed mechanisms have the following desirable properties: (i) the dynamic auction is strategy-proof within each period and guarantees that the market choice of each user is optimal under the perfect information assumption on the option values in the future market; (ii) the whole mechanism combining the auction and the adjustment rules achieves the optimal permits allocation pattern in a finite number of iterations. Finally, we numerically showed the convergence process of the proposed mechanism and analyzed the case of relaxing the information assumption.

While we made the perfect information assumption for the theory, the thought behind it is that the user has a prediction formation mechanism based on some learning dynamics (e.g., [Fudenberg and Levine, 1998](#)). Section 6.2.2 considered a prediction rule based on the mean values of the historical permits prices. The numerical results suggested that the proposed mechanism works well even under such a simple rule.

Another situation in which the proposed mechanism may work well is that markets participants change but the aggregate distribution of valuations is stable over stages. This is because the adjustment rule of the number of permits sold in each market only requires the aggregate information, as mentioned in Section 5.1. The numerical experiment that supports this argument can be found in Section 5.6 of [Wada \(2013\)](#). As an example of this situation might be road bottlenecks in sightseeing areas, where the trip is non-recurrent for each user but the congestion may occur recurrently.

Note however that, for modeling non-recurrent trips, it would be important to extend our model to incorporate the users’ dynamic decision-making under uncertainty. Since the proposed mechanism considers important aspects of dynamic allocation problems (the users’ dynamic decision-making and irresistibility of resource allocation), it seems applicable under uncertainty.

Thus, generalizing the proposed mechanism to handle uncertainty situations is an important topic for the future work. Although we restricted our attention to finite purchase opportunities, an infinite (or continuous time) setting may be suitable in some situations. In this case, the road manager may have to make a decision in real-time, thereby implying that analyzing the permits allocation problem could become more of a challenge. Nevertheless, establishing mechanisms for the situation seems a fruitful topic for future research work.

Appendix A. Proof of total unimodularity of problem [SO]

A totally unimodular (TU) matrix is defined as follows.

Definition 1. An integer matrix \mathbf{A} is totally unimodular if any subdeterminant of \mathbf{A} is 0 or ± 1 .

Then, if a constraint matrix \mathbf{A} is a TU matrix, the following theorem holds:

Theorem 1. Let \mathbf{A} be totally unimodular. Then, for any integer vector \mathbf{b} , extreme points of the following polyhedron: $\{\mathbf{x} : \mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ are integers.

Therefore, a bounded linear program in which the constraint matrix is a TU matrix always produces integer solutions if a solution algorithm that produces extreme point solutions is used (e.g., simplex method). Well-known problems that have such a constraint matrix are weighted matching problems and network flow problems (e.g., the maximum flow problem, the minimum cost flow problem).

Because the problem [SO] is different from the typical problems, we prove that the constraint matrix of the problem is a TU matrix by using the following sufficient condition (Heller and Tompkins, 1956):

Theorem 2. (Heller and Tompkins, 1956) Let \mathbf{A} be a $0, \pm 1$ matrix with at most two nonzero entries per column. Then, \mathbf{A} is totally unimodular if there is a partition of rows such that (1) if two nonzero entries in a column have the same sign, then the rows are partitioned into disjoint sets T_1, T_2 ; (2) if two nonzero entries in a column have opposite sign, then the rows are in the same set (T_1 or T_2);

Let us confirm that the constraint matrices of the problem [SO] satisfy the sufficient condition. We first transpose the unknown variables of the constraints to the left-hand side and partition the constraints as follows:

$$T_1 = \begin{cases} \sum_{i \in \mathcal{I}} y_{i,k}^m - \mu_k^m \leq 0 & \forall k \in \mathcal{K}, \forall m \in \mathcal{M} \\ \sum_{m \in \mathcal{M}} \mu_k^m \leq \mu & \forall k \in \mathcal{K} \\ z_i^m - \sum_{k \in \mathcal{K}} y_{i,k}^m - z_i^{m+1} = 0 & \forall i \in \mathcal{I}, \forall m = 1, 2, \dots, M-1 \\ z_i^M - \sum_{k \in \mathcal{K}} y_{i,k}^M \leq 0 & \forall i \in \mathcal{I} \end{cases}$$

$$T_2 = \emptyset$$

We let \mathbf{A} be the coefficient matrices of the left-hand side. Then, every entry of \mathbf{A} is 0 or ± 1 , and \mathbf{A} has two nonzero entries in every column. In addition, two nonzero entries have opposite signs. Then all of the rows are in T_1 ; the set T_2 is empty. Thus, the constraint matrix of the problem [SO] is totally unimodular.

Appendix B. Ascending proxy auction

The proxy DGS auction that combines the ascending auction proposed by Demange et al. (1986) and the proxy agent system (semi-autonomous proxy bidding agent) proposed by Parkes and Ungar (2000) is described as follows.

Step 0 Round $R = 0$. Set $\mathbf{p}^{m,0} = \mathbf{0}$ for all permits. Each user reports the (not necessarily true) valuations \mathbf{b}_i^m to one's proxy agent for subset of permits.

Step 1 In round R , each proxy agent submits "time intervals" of permits that can maximize utility of the user under current price $\mathbf{p}^{m,R}(e)$ (we call it a *demand set*). If each user can be allocated a permit from his/her demand set, then stop. Otherwise, go to **Step 2**.

Step 2 The manager chooses a *minimal overdemanded set* and raises the prices of permits in the set by one unit price. $R = R + 1$, go to **Step 1**.

Here, an overdemanded set is a set of permits for which the number of users demanding only the permits in that set exceeds the number of permits sold in the auction, and the minimal overdemanded set is an overdemanded set of permits with no proper overdemanded subset.

Allocation \mathbf{y}^m and price \mathbf{p}^m are obtained based on the procedure above, furthermore, we obtain the bidding b_{i,k^*}^{m*} of users through proxy agent.

This auction is corresponding to solving the decomposed subproblem (35) by a primal-dual algorithm. If a user reports his/her truthful valuation to proxy agent ($\mathbf{b}_i^m = \hat{\mathbf{v}}_i^m$), the prices under the above procedure converge to the those induce truthful reporting of the valuations of users to their proxy agents (i.e., VCG prices).

Appendix C. Derivation of Eq.(39)

We here derive the total payoff $\pi^1(s)$ by exploiting information, permits prices $(\mathbf{p}^{m*})_{\mathcal{M}}$ and winning valuation (or bids) $(\hat{v}_{i,k^*}^{m*})_{\mathcal{I}}$, which are obtained in the auction phase at each stage s . Note that a single asterisk (*) indicates the optimal value of each variable at each stage (i.e., the value achieved through the auction mechanism).

The social surplus achieved by the auction mechanism is represented as

$$\sum_{m \in \mathcal{M}} \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} \hat{v}_{i,k}^m y_{i,k}^{m*} = \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} \hat{v}_{i,k^*}^{m*}. \quad (\text{C.1})$$

Form the duality theorem, the optimal value of the decomposed sub-problem $[\text{SO}_{\text{sub}}\text{-P}^m]$ coincides with that of the dual problem $[\text{SO}_{\text{sub}}\text{-D}^m]$:

$$\sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} \hat{v}_{i,k}^m y_{i,k}^{m*} = \sum_{i \in \mathcal{I}} \hat{\pi}_i^{m*} + \sum_{k \in \mathcal{K}} \mu_k^m(s) p_k^{m*}. \quad (\text{C.2})$$

By substituting this equation into Eq.(C.1), we have

$$\sum_{i \in \mathcal{I}} \hat{v}_{i,k^*}^{m*} = \sum_{m \in \mathcal{M}} \sum_{i \in \mathcal{I}} \hat{\pi}_i^{m*} + \sum_{m \in \mathcal{M}} \sum_{k \in \mathcal{K}} \mu_k^m(s) p_k^{m*}. \quad (\text{C.3})$$

We here recall the definitions $\hat{\pi}_i^m \equiv \pi_i^m - \pi_i^{m+1}$, $\hat{\pi}_i^M \equiv \pi_i^M$. Then, Eq.(C.3) reduces to the Eq.(39):

$$\sum_{i \in \mathcal{I}} \hat{v}_{i,k^*}^{m*} = \pi^1(s) + \sum_{m \in \mathcal{M}} \sum_{k \in \mathcal{K}} \mu_k^m(s) p_k^{m*}. \quad (\text{C.4})$$

Table C.3: Users's valuations $v_{i,k}^m$ in simple numerical examples

(m, k)	$i = 1$	$i = 2$	$i = 3$	$i = 4$
(1, 1)	10	16	30	18
(1, 2)	20	26	20	8
(2, 1)	15	24	15	9
(2, 2)	30	39	10	4

Table C.4: Convergence process of the proposed mechanism in Case 1

Stage	μ_1^1	μ_2^1	μ_1^2	μ_2^2	p_1^1	p_2^1	p_1^2	p_2^2	π_1^1	π_2^1	π_3^1	π_4^1	Sub	Master
$s = 1$	1	1	4	4	9	0	0	0	30	39	21	9	108	144
$s = 2$	5	0	0	5	0	0	0	0	30	39	30	18	117	117

Appendix D. Simple numerical examples of the proposed mechanism

We show simple numerical examples to illustrate the proposed mechanism. The setting is as follows: the number of periods is $M = 2$, the bottleneck capacity is 5 (veh/unit time), the number of destination arrival times is $|K| = 2$, and the number of users is 4. The users' valuations for these bottleneck permits are shown in Tab.C.3. Under this setting, the maximum social surplus is $SS = 117$.

We show two convergence processes in Tabs.C.4 (Case 1) and C.5 (Case 2). In the latter case, the minimum permits number constraint, $\mu_k^m \geq \mu_{\min} = 1$ is added to the master problem. In both cases, we set the initial number of permit as $\mu(1) = (1, 1, 4, 4)$ at stage $s = 1$. Then, in the proxy DGS auction, each user bids its net valuation, $\hat{v}_{i,k}^m = v_{i,k}^m - \pi_i^{m+1}$, that can be calculated based on the known option value (perfect information assumption) in the next market, $\pi^2 = (30, 39, 0, 9)$ and $\pi^3 = \mathbf{0}$. In the prior market, the all net valuations of users $i = 1, 2$ are negative and thus they do not attend the auction; since those of users $i = 3, 4$ are $\hat{v}_3^1 = (30, 20)$ and $\hat{v}_4^1 = (9, -1)$, user $i = 3$ gets the permit for interval $k = 1$ with price $p_1^1 = 9$ and $\pi_3^1 = 30 - 9 = 21$. In the spot market, since the number of permits is enough, the remained users can get their preferred permits with prices 0: users $i = 1, 2$ are assigned to interval $k = 2$ and user $i = 4$ is assigned to interval $k = 1$.

We next consider the adjustment phase or master problem. At stage $s = 1$, the problem (40) is given as

$$\max_{\mu \geq 0 \text{ and integer}} .9\mu_1^1 + 99, \quad \text{subject to Eq.(3)}.$$

From this problem, we see that the solution is not necessarily unique. In Case 1, all 5 permits should be assigned to the prior market for interval $k = 1$ and the value of the objective function becomes 144. In Case 2, since at least one permit should be assigned for all intervals and markets due to the minimum number constraint, 4 permits are assigned to the prior market for interval $k = 1$ and the value of the objective function becomes 135. Note that permits allocation for the other intervals and markets are arbitrary subject to the constraints. In both cases, the mechanism proceeds to the next stage because the values of the objective functions of the master- and sub-problems are not equivalent. At stage $s = 2$, in the same way above, the auction phase and adjustment phase are conducted and the mechanism converges to the optimal state in both cases.

Table C.5: Convergence process of the proposed mechanism in Case 2

Stage	μ_1^1	μ_2^1	μ_1^2	μ_2^2	p_1^1	p_2^1	p_1^2	p_2^2	π_1^1	π_2^1	π_3^1	π_4^1	Sub	Master
$s = 1$	1	1	4	4	9	0	0	0	30	39	21	9	108	135
$s = 2$	4	1	1	4	0	0	0	0	30	39	30	18	117	117

As we mentioned before, in the current setting, the minimum permits number constraint does not affect the optimal state because the number of permits is sufficiently large compared to the number of users.

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