Quantization of singular quaternionic nilpotent $K$-orbits

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Abstract
We study singular quaternionic representations of exceptional real simple Lie groups of real rank 4. These representations, constructed by Gross and Wallach, are closely related to singular orbits on certain prehomogeneous vector spaces arising from quaternionic structure of the Lie algebras. We show that these quaternionic representations are obtained by quantization of singular quaternionic nilpotent $K$-orbits.

1. Simple Lie groups of quaternionic type
Let $G_R$ be a connected, simply connected real simple Lie group of quaternionic type, and $K_R$ be a maximal compact subgroup of $G_R$. We denote by $g_R$, $k_R$ the Lie algebras of $G_R$, $K_R$ respectively. Take a compact Cartan subalgebra $t_R$ of $g_R$ contained in $k_R$. We write $g$, $k$ and $t$ for the complexification of real Lie algebras $g_R$, $k_R$ and $t_R$ respectively.

Consider the root system $\Delta$ for $(g; t)$. Then, there exists a positive system $\Delta^+$ of $\Delta$ with the following property: the $\mathbb{Z}$-gradation of $g$ defined by the highest root $g = \bigoplus_{j=0,\pm1,\pm2} g(j)$ with $g(j) = \{Z \in g | [H, Z] = jZ\}$, gives rise to a complexified Cartan decomposition $g = k \oplus p$ with $k = g(2) \oplus g(0) \oplus g(-2)$, $p = g(1) \oplus g(-1)$.

Here $H_\beta$ denotes the element of $t$ corresponding to the co-root $\beta^\vee = 2\beta/(\beta, \beta) \in t^*$ through the Killing form of $g$. Note that $g(\pm2)$ equals the root space for $\pm\beta$: $g(\pm2) = g_{\pm\beta}$.

Now, let $m$ be the semisimple part of $g(0)$. Then $k$ is a direct sum $k = k_1 \oplus m$ with $k_1 := g(2) \oplus CH_\beta \oplus g(-2) \simeq sl(2, \mathbb{C})$.
This implies that the complexification $K$ of $K_R$ is the direct product $K = K_1 \times M$, where $K_1 \simeq SL(2, \mathbb{C})$ and $M$ is the simply connected Lie group with Lie algebra $m$.

2. Orbit structure of prehomogeneous vector space $V$
In what follows, we assume that $g_R$ is one of four exceptional simple Lie algebras of real rank 4, namely $g_R = F_{4,4}$, $E_{6,4}$, $E_{7,4}$, $E_{8,4}$. We see $m = C_3$, $A_5$, $D_6$, $E_7$ respectively. Let $L$ be the centralizer of $H_\beta$ in $K$. Then $L = T_1 \times M$ with $T_1 = \exp CH_\beta$ is a connected Lie group with Lie algebra $g(0)$, and $L$ acts on the vector space $V := g(1)$ by the adjoint action. We thus have an irreducible reduced prehomogeneous vector space $(L, V)$ with relative invariant $f_4 \in \mathbb{C}[V]$ of degree 4. See (14), (5), (23) and (29) of [2, §7, Table I] for more details.

By Serge, Igusa and Harris, $V$ has exactly four nonzero $L$-orbits $O_i$ $(i = 1, 2, 3, 4)$. We can arrange them as $V = O_4 \supset O_3 \supset O_2 \supset O_1$, where $O_i$ denotes the closure of $O_i$. Note that $O_4$ is the open $L$-orbit and that $\overline{O_3}$ coincides with the hypersurface $f_4 = 0$. 
3. Singular quaternionic representations $\sigma_O$
We now focus our attention to three singular $L$-orbits $O := O_i$ for $i = 1, 2, 3$. The subgroup $M$ acts on $O$ transitively. Set $\kappa = 1, 2, 4, 8$ according as $g_\mathbb{R} = F_{4,4}, E_{6,4}, E_{7,4}, E_{8,4}$ respectively. We define a positive integer $k_O$ by $k_O = i\kappa + \delta_{i,3}$ with Kronecker’s $\delta_{i,3}$.

By using cohomological parabolic induction, Gross and Wallach [1] constructed an irreducible unitary representation $\sigma_O$ of $G_\mathbb{R}$ such that

$$\sigma_O|_K \simeq \bigoplus_{m=0}^{\infty} \tau_{m+k_O} \otimes \mathbb{C}^m[\mathcal{O}] \quad \text{as } K = K_1 \times M\text{-modules}. \quad (1)$$

Here $\tau_m$ denotes the irreducible representation of $K_1$ of dimension $m + 1$, and $\mathbb{C}^m[\mathcal{O}] = \mathbb{C}^m[V]|_{\mathcal{O}}$ is the $M$-module consisting of homogeneous polynomials on $V$ of degree $m$ restricted to the affine variety $\mathcal{O}$.

4. Quantization of nilpotent $K$-orbits $\tilde{\mathcal{O}}$
Let $\tilde{\mathcal{O}} = \text{Ad}(K)\mathcal{O}$ be the nilpotent $K$-orbit containing $\mathcal{O}$. Take an element $X \in \mathcal{O}$ and consider the isotropy subgroup $K(X) = Z_K(X)$ of $X$ in $K$. We write $\mathfrak{t}(X)$ for the Lie algebra of $K(X)$. We see $K(X) = L(X) \ltimes N_1$ with $L(X) = K(X) \cap L$ and $N_1 = \exp \mathfrak{g}(2)$. It is known that $K(X)$ is connected.

The following theorem, which is the main result of this paper, says that the representation $\sigma_O$ gives a quantization of nilpotent $K$-orbit $\tilde{\mathcal{O}}$ in the sense of [3].

**Theorem 1** (1) The square-root of coisotropy representation $\chi(y) = \det(\text{Ad}(y)|_{\mathfrak{t}(X)})^{-1/2}$ ($y \in K(X)$) gives a well-defined character of the group $K(X)$. Hence the nilpotent $K$-orbit $\tilde{\mathcal{O}}$ is admissible.

(2) One gets $\sigma_O \simeq \text{Ind}^K_{K(X)}(\chi)$ as $K$-modules. Here the induced representation on the right hand side is defined on the space of algebraic sections of the line bundle $K \times_{K(X)} \chi$ on $K/K(X)$.

Furthermore, we can show that $\chi$ coincides with the isotropy representation for $\sigma_O$, which is defined in connection with the associated cycle of Harish-Chandra modules (see [3], [4]).

The following proposition is crucial to prove Theorem 1.

**Proposition 2** (1) Choose an $\mathfrak{sl}_2$-triple $(X, H, Y)$ such that $H \in \mathfrak{t}$ and $X \in \mathcal{O}$. Set $\tilde{H} = 2(H - 2H_0)/(\beta(H) - 4)$. Then, $\tilde{H}$ is a nonzero central element of the reductive part of $\mathfrak{t}(X)$, and one gets $\text{tr}(\text{ad}(\tilde{H})|_{\mathfrak{t}(X)}) = 2k_O$.

(2) The affine coordinate ring $\mathbb{C}[\mathcal{O}] = \bigoplus_{m=0}^{\infty} \mathbb{C}^m[\mathcal{O}]$ is isomorphic to $\text{Ind}^M_{K(X) \cap M}(1)$ as an $M$-module.

References